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IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **34** (2021) 1288–1315 https://doi.org/10.1088/1361-6544/abbd85

Revisiting the Christ–Kiselev's multi-linear operator technique and its applications to Schrödinger operators

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Received 29 March 2020, revised 18 September 2020
Accepted for publication 1 October 2020
Published 11 February 2021

Abstract

Based on Christ–Kiselev's multi-linear operator techniques, we prove several spectral results of perturbed periodic Schrödinger operators, including WKB type solutions, sharp transitions of preservation of absolutely continuous spectra, criteria of absence of singular spectra, and sharp bounds of the Hausdorff dimension of singular continuous spectra.

Keywords: Christ–Kiselev lemma, Floquet theory, WKB solutions, periodic Schrodinger operators

Mathematics Subject Classification numbers: Primary: 34L40, Secondary: 81Q10;34E20.

1. Introduction and main results

In quantum mechanics, the time evolution of the state ψ is described by the Schrödinger equation

$$i\,\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H\psi(t), \tag{1}$$

where H is a self-adjoint operator on a Hilbert space. The solution of (1) is given by

$$\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\ \psi(0).$$

Let H be the one dimensional Schrödinger operators on $L^2(\mathbb{R})$,

$$Hu = -u'' + (V + V_0)u, \tag{2}$$

where $V_0(x)$ is 1-periodic and $V(x)$ is a decaying perturbation. When $V \equiv 0$, we have the 1-

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periodic Schrödinger operators,

$$H_0 u = -u'' + V_0 u. \tag{3}$$

Periodic Schrödinger operators have been studied intensively in both mathematics and physics. We refer readers to a recent survey [19] for details.

The spectral measure associated with the state $\psi(0)$ is closely related to the dynamical behavior of the time evolution $\psi(t)$ governed by the Schrödinger equation. A classical example would be the RAGE theorem [[41], chapter 5]. See [21] and references therein for more examples. In this paper, we focus on the spectral types: absolutely continuous spectrum, pure point spectrum (eigenvalues) and singular continuous spectrum of Schrödinger operators.

It is known that the spectrum of these periodic operators H_0 consists of a union of closed intervals $\cup_n [a_n, b_n]$, which we refer to as 'bands' Moreover, H_0 has purely absolutely continuous spectrum. Naturally, we would ask the following two questions:

Q1 What are the criteria of perturbations V that leave the absolutely continuous spectrum unchanged?

Q2 In the case that H_0 and H have the same absolutely continuous spectrum, what are the criteria of perturbations V that there are no eigenvalues and/or singular continuous spectrum embedded into the absolutely continuous spectrum?

The study of those problems has a long history going back to Weidmann in the 1960s [43]. Stimulated by several conjectures posted by Barry Simon at ICMP in 1994 and 2000 [37], the theory of Schrödinger operators with decaying potentials has seen significant progress in the past 25 years through the work of Christ, Deift, Denisov, Killip, Kiselev, Last, Molchanov, Remling, Simon, Stolz and among others. We refer readers to two survey articles [6, 11] and references therein for details.

When the background periodic potential V_0 is zero, the dynamical properties of (2) are well understood. In particular, the two aforementioned questions are solved when the perturbed functions V are in L^p spaces or algebraically decaying ($V(x) = O(|x|^{-\alpha})$, $\alpha > 0$). However, those problems are less understood when the periodic functions V_0 are non-zero. The goal of this paper is to address some of the remaining problems.

For simplicity, we only consider the equation on the half-line $\mathbb{R}_+$. All the results can be generalized to the whole line $\mathbb{R}$.

For any $p \geqslant 1$, denote by $\ell^p(L^1)(\mathbb{R}_+)$ the Banach space of all measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$ with the norm

$$\|f\|_{\ell^p(L^1)} = \left(\sum_{k=0}^{\infty}\left(\int_k^{k+1} |f(x)|\,\mathrm{d}x\right)^p\right)^{1/p}.$$

This Banach space contains $L^1 + L^p$. If $p \leqslant q$, then $\ell^p(L^1) \subset \ell^q(L^1)$. For simplicity, we always assume V_0 is in $L^1[0, 1]$ and periodic.

Let $S = \cup_{n=0}^{\infty}[a_n, b_n]$ be the spectrum of the operator (3) and $\varphi(x, E)$ be the Floquet solution.

Theorem 1.1. *If the potential $V \in \ell^p(L^1)$ for some $1 \leqslant p < 2$, then S is an essential support of the absolutely continuous spectrum of the operator $H = H_0 + V$ with any boundary condition at zero. Moreover, for almost every $E \in S$, there exists a solution $u(x, E)$ of the equation*

$$-u'' + (V_0(x) + V(x))u = Eu \tag{4}$$

with the asymptotic behavior

$$u(x,E)=\varphi(x,E)\exp\left(\frac{i}{2\,\Im(\varphi\overline{\varphi}')}\int_0^x V(t)|\varphi^2(t,E)|\ \mathrm{d}t\right)(1+o(1)) \tag{5}$$

as $x\to\infty$.

Let p' be the conjugate number to p, namely $\frac{p}{p-1}$ for $p>1$.

Theorem 1.2. *Suppose* $|x|^\gamma V\in\ell^p(L^1)$ *for some* $1<p\leqslant 2$, $\gamma>0$ *with* $\gamma p'\leqslant 1$. *Then for every* $E\in S$, *there exists a solution* $u(x,E)$ *of* $Hu=Eu$ *satisfying the asymptotic behavior* (5), *except for a set of values of* E *in* S *with Hausdorff dimension less than or equal to* $1-\gamma p'$.

As a corollary, we have

Corollary 1.3. *Suppose* $V(x)=\frac{O(1)}{1+x^\alpha}$ *with* $\alpha\in[\frac{1}{2},1]$. *Then for every* $E\in S$, *there exists a solution* $u(x,E)$ *of* $Hu=Eu$ *satisfying the asymptotic behavior* (5), *except for a set of values of* E *in* S *with Hausdorff dimension less than or equal to* $2(1-\alpha)$.

Now we want to talk about the proof and history of theorems 1.1 and 1.2. Note that the second part of theorem 1.1 implies the first part (e.g. [35, 40]). There are two main approaches to study the spectral theory of Schrödinger operators with decaying potentials. The first approach is to study the spectral theory of Schrödinger operators via establishing the WKB type eigensolutions, namely eigensolutions with asymptoics (5). If $V_0\equiv 0$, theorems 1.1, 1.2 and corollary 1.3 have been proved by Christ–Kiselev [1–4] and Remling [31, 32] with some partial results [13, 14]. If $V_0\equiv 0$, Remling [33] and Kriecherbauer–Remling [18] constructed examples which show that $2(1-\alpha)$ in corollary 1.3 is the best bound to be achieved. Under a stronger assumption on the potential V, that is $|x|^\gamma V\in\ell^p(L^1)$ for some $\gamma>0$, theorem 1.1 was proved by Christ and Kiselev [1]. For $p>2$, it is known that theorem 1.1 is not true even for the case $V_0\equiv 0$ (see [16] for example).

Another approach beginning with Deift and Killip aims to study the absolutely continuous spectrum directly. This approach can handle the critical case $p=2$ without asymptotics of the eigensolutions. For $p=2$, the first part of theorem 1.1 was proved by Deift–Killip for the case $V_0\equiv 0$ [5] and Killip [10] for non-zero functions V_0.

It is widely believed that for $p=2$, the second part of theorem 1.1 holds, which is open even for $V_0\equiv 0$.

The proof of theorems 1.1 and 1.2 is largely inspired by Christ–Kiselev's arguments in [2–4]. In [2–4], Christ and Kiselev developed a scheme, referred to as the multi-linear operator technique, to establish the WKB type solutions, which turned out to be a robust approach. We want to mention that the multilinear operator technique is not extendable to tackle the case $p=2$ [28].

The multi-linear operator technique to establish the WKB type eigensolutions is based on writing the differential equation in an integral form and seeking a formal series solution. Each term of the series is defined by a multi-integral operator. The difficulty lies in a rigorous definition of improper integrals in a suitable topology, showing the convergence of the series in a proper measure space, and verifying that the formal series solution is an actual solution. We establish a more general version of the Christ–Kiselev's multi-linear operator technique than that appearing in [2–4] (see section 2). The main scheme of our proofs is definitely developed from Christ–Kiselev. However, several important technical improvements have been added to Christ–Kiselev's scheme. Firstly, we strengthen several conclusions while requiring fewer assumptions. For example, we proved the conclusion in lemma 2.8 as it appears n [3, 4, 12] without any lower bound assumptions on the second and third derivatives (see (28)). Secondly, we simplify Christ–Kiselev's original arguments. We define the multi-linear operators as iterations. This is different from Christ–Kiselev's plan (see the proof of lemma 4.2 in [4]). Our approach is more natural and makes it easier to verify that the formal series solution is an actual solution. The price we need to pay is to show the existence of a stronger limit in the definition of multi-integral operators. See remark 2.6. Thirdly, we give two ways to establish the WKB type solutions. One way closely follows from Christ–Kiselev's approach. The new way allows us to avoid using maximal operators and simplifies the previous proof. See section 5. Our new proof is based on modifications of norms of a family of Banach spaces.

Now we move to the second question. The sharp transition for (dense) embedded eigenvalues was recently obtained by the author and Ong [26] with some partial results in the past [17, 20, 27, 29, 34], so we only focus on the singular continuous spectrum in the following. Let us review the results for the case $V_0 \equiv 0$ first.

(a) If $V(x) = \frac{o(1)}{1+x}$, $H_0 + V$ does not have any positive eigenvalues [9].
(b) Wigner–von Neumann type functions imply that there exist potentials $V(x) = \frac{O(1)}{1+x}$ such that $H_0 + V$ has positive eigenvalues [42].
(c) For any given positive function $h(x)$ tending to infinity as $x \to \infty$, there exist potentials $V(x)$ such that $|V(x)| \leqslant \frac{h(x)}{1+x}$ and $H_0 + V$ has dense embedded eigenvalues [30, 36].
(d) If $V(x) = \frac{O(1)}{1+x}$, $H_0 + V$ does not have any singular continuous spectrum [15].
(e) For any given positive function $h(x)$ tending to infinity as $x \to \infty$, there exist potentials $V(x)$ such that $|V(x)| \leqslant \frac{h(x)}{1+x}$ and the singular continuous spectrum of $H_0 + V$ is non-empty [15].

Clearly, the above statements from (a) to (e) imply the criteria for the absence of singular spectra (eigenvalues and singular continuous spectra). It is natural to expect that corresponding criteria are true for any non-zero periodic function V_0. For embedded eigenvalues, cases (a)–(c) have been proved for any function V_0 [17, 26]. For the singular continuous spectrum, we conjecture that (d) and (e) hold for any periodic function V_0. In this paper, we are able to prove half of the conjecture.

Theorem 1.4. *Suppose $V(x) = \frac{O(1)}{1+x}$. Then the singular continuous spectrum of $H = H_0 + V$ with any boundary condition is empty.*

Theorem 1.4 and the case (a) for any function V_0 imply

Corollary 1.5. *Suppose $V(x) = \frac{o(1)}{1+x}$. Then the spectral measure of $H_0 + V$ with any boundary condition at zero is purely absolutely continuous in S.*

Define P as

$$P = \{E \in \mathbb{R} : -u'' + (V(x) + V_0(x))u = Eu \text{ has an } L^2(\mathbb{R}_+) \text{ solution} \}, \tag{6}$$

It has been proved that $P \cap S$ is a countable set provided $V(x) = \frac{O(1)}{1+x}$ [24]. Therefore, theorem 1.4 also implies

Corollary 1.6. *Suppose $V(x) = \frac{O(1)}{1+x}$. Then except for countably many boundary conditions at zero, the spectral measure of $H_0 + V$ is purely absolutely continuous in S.*

The proof of theorem 1.4 is inspired by [15], where the case $V_0 \equiv 0$ was treated. Under the assumption of theorem 1.4, the first observation is that the singular component of the spectral measure is supported on a set of zero Hausdorff dimension (by corollary 1.3). Following the strategy in [15], four additional steps are needed to prove theorem 1.4. Step 1: establish the quantitative almost orthogonality among Prüfer angles. Step 2: control the total number of 'separate energies'[1] based on step 1. Step 3: establish spectral measures of Schrödinger operators with eventually zero potentials. Step 4: use spectral measures of eventually zero potentials to make approximations.

In our case, periodic potentials are involved so the problem is a lot more complicated, the steps 1 and 3 in particular. Let us mention that the almost orthogonality is between $\frac{\theta(x,E_1)}{1+x}$ and $\frac{\theta(x,E_2)}{1+x}$ in Hilbert spaces $L^2([0,B],(1+x)\mathrm{d}x)$ for large B, where $\theta(x,E_1)$ (resp. $\theta(x,E_2)$) is the (generalized) Prüfer angle with respect to energy E_1 (resp. E_2). In [15], Kiselev established sharp bounds of the almost orthogonality for perturbed free Schrödinger operators (step 1). In our case, rather than using the standard Prüfer variables, we have to instead use the generalized Prüfer variables. The almost orthogonality of general cases was proved recently in [26] without quantitative estimates, which is a key ingredient to construct embedded eigenvalues. However, in order to study the singular continuous spectrum, the quantitative bounds are essential, in particular, we need to control the blowup when E_1 approaches E_2. In [26], one of the innovations is the use of Fourier expansions to ensure that some key terms decay sufficiently quickly. The rest of the terms can be controlled by using oscillatory integration techniques. Even though we use the full strength of Fourier expansions and oscillatory integral techniques from [26] in a quantitative way, the bounds are not enough. We overcome the difficulty by splitting the frequencies into high and low ones, where frequencies come from the quasimomenta of Floquet solutions. For high frequencies, we quantify the oscillatory integral techniques in [26] in a sharp way. For low frequencies, we combine Fourier expansions in [26] with the techniques in [15] to establish the sharp bounds.

In the end, we remark that the spectral theory of perturbed periodic operators in higher dimensions is much more difficult. We refer readers to [19, 25] for recent progress.

2. Christ–Kiselev's multi-linear operator techniques

Since we only consider operators on the half-line $\mathbb{R}_+$, all the functions are defined on $\mathbb{R}_+$. Let us introduce the multilinear operator M_n, acting on n functions g_k, $k=1,2,\ldots,n$, by

$$M_n(g_1,g_2,\ldots,g_n)(x,x') = \int_{x\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant x'}\prod_{k=1}^{n} g_k(t_k)\mathrm{d}t_k \tag{7}$$

$$= \int_{x\leqslant t_1\leqslant\cdots\leqslant t_n<\infty}\prod_{k=1}^{n} g_k(t_k)\chi_{[0,x']}(t_k)\mathrm{d}t_k, \tag{8}$$

where χ is the characteristic function. If there is a single function g such that $g_k\in\{g,\overline{g}\}$, $k=1,2,\ldots,n$, by abusing the notation, we write $M_n(g_1,g_2,\ldots,g_n)(x,x')$ by $M_n(g)(x,x')$. Although we use $M_n(g)$ for all possible $(g_1,g_2,\ldots,g_n)\in\{g,\overline{g}\}^n$, there should be no ambiguity since $(g_1,g_2,\ldots,g_n)$ is fixed in our proof.

A collection of subintervals $E_j^m\subset\mathbb{R}_+$, $1\leqslant j\leqslant 2^m$ and $m\in\mathbb{Z}_+$ is called a martingale structure [4] if the following is true:

- $\mathbb{R}_+=\cup_j E_j^m$ for every m.
- $E_j^m\cap E_i^m=\emptyset$ for every $i\neq j$.
- If $i<j, x\in E_i^m$ and $x'\in E_j^m$, then $x<x'$.
- For every m, $E_j^m=E_{2j-1}^{m+1}\cup E_{2j}^{m+1}$.

[1] See the definition in section 8.

Denote by $\chi_j^m = \chi_{E_j^m}$. Let $\mathfrak{B}_s$ be the Banach space consisting of all complex-valued sequences $a = a(m, j)$ indexed by $1 \leqslant m < \infty$ and $1 \leqslant j \leqslant 2^m$, for which

$$\|a\|_{\mathfrak{B}_s} = \sum_{m\in\mathbb{Z}_+} m^s \left(\sum_{j=1}^{2^m} |a(m, j)|^2\right)^{1/2} < \infty.$$

Denote by $\mathfrak{B} = \mathfrak{B}_1$. For any function g on $\mathbb{R}_+$, we define a sequence with index $m \in \mathbb{Z}_+$ and $1 \leqslant j \leqslant 2^m$,

$$\left\{\int_{E_j^m} g(x)\mathrm{d}x\right\} = \left\{\int_{\mathbb{R}_+} g(x)\chi_j^m \,\mathrm{d}x\right\}.$$

By abusing the notation, denote by

$$\|g\|_{\mathfrak{B}^s} = \left\|\left\{\int_{E_j^m} g(x)\mathrm{d}x\right\}\right\|_{\mathfrak{B}^s} = \sum_{m}^{\infty} m^s \left(\sum_{j=1}^{2^m} \left|\int_{E_j^m} g(x)\mathrm{d}x\right|^2\right)^{\frac{1}{2}}. \tag{9}$$

Define

$$M_n^*(g_1, g_2, \ldots, g_n) = \sup_{0<x\leqslant x'<\infty} |M_n(g_1, g_2, \ldots, g_n)(x, x')|, \tag{10}$$

and

$$M_n^*(g) = \sup_{0<x\leqslant x'<\infty} |M_n(g)(x, x')|. \tag{11}$$

Theorem 2.1 [2]. *For any martingale structure $E_j^m \subset \mathbb{R}_+$, $1 \leqslant j \leqslant 2^m$ and $m \in \mathbb{Z}_+$, the following estimates hold,*

$$M_n^*(g_1, g_2, \ldots, g_n) \leqslant C^n \prod_{i=1}^{n} \|g_i\|_{\mathfrak{B}}, \tag{12}$$

and

$$M_n^*(g) \leqslant C^n \frac{\|g\|_{\mathfrak{B}}^n}{\sqrt{n!}}, \tag{13}$$

where C is an absolute constant.

A martingale structure $E_j^m \subset \mathbb{R}_+$, $1 \leqslant j \leqslant 2^m$ and $m \in \mathbb{Z}_+$ is said to be adapted in $\ell^p(L^1)$ to a function f if for all possible m, j,

$$\|f\chi_j^m\|_{\ell^p(L^1)}^p \leqslant 2^{-m}\|f\|_{\ell^p(L^1)}^p. \tag{14}$$

Since all the functions are in $\ell^p(L^1)$, we omit 'adapted' in the rest of this paper.

Lemma 2.2 (p 433, [4]). *For any function $f \in \ell^p(L^1)$, there exists a martingale structure $\{E_j^m \subset \mathbb{R}_+ : m \in \mathbb{Z}_+, 1 \leqslant j \leqslant 2^m\}$ to f.*

Let P be a linear or sublinear bounded operator from $\ell^p(L^1)$ to $L^q(J)$, where $J \subset \mathbb{R}$ is a closed interval. For $s > 0$, denote by

$$G^{(s)}_{P(f)(\lambda)} = \|\{P(f\chi^m_j)(\lambda)\}\|_{\mathfrak{B}^s}$$
$$= \sum_{m=1}^{\infty} m^s \left(\sum_{j=1}^{2^m} |P(f\chi^m_j)(\lambda)| \right)^{1/2}.$$

Remark 2.3. In the case that P has an integral kernel $p(x, \lambda)$, $G^{(s)}_{P(f)(\lambda)} = \|p(x, \lambda)f(x)\|_{\mathfrak{B}^s}$, which is the norm of $\{\int_{E^m_j} p(x, \lambda)g(x)\mathrm{d}x\}$ in $\mathfrak{B}^s$.

The following statement is from [4]. We include a proof here for completeness.

Theorem 2.4 [[4], *proposition 3.3*]. *Given a function $f \in \ell^p(L^1)$, fix a martingale structure to f. Suppose P is a linear or sublinear bounded operator from $\ell^p(L^1)$ to $L^q(J)$, where $1 \leqslant p < 2 < q$ and $J \subset \mathbb{R}$ is a closed interval. Then*

$$\|G^{(s)}_{P(f)(\lambda)}\|_{L^q(J)} \leqslant C(p, q, s, \|P\|)\|f\|_{\ell^p(L^1)}.$$

Proof. Let

$$t_m(\lambda) = \left(\sum_{j=1}^{2^m} |P(f\chi^m_j)(\lambda)| \right)^{1/2}.$$

By the definition,

$$G^{(s)}_{P(f)(\lambda)} = \sum_{m=1}^{\infty} m^s \left(\sum_{j=1}^{2^m} |P(f\chi^m_j)(\lambda)| \right)^{1/2}$$
$$= \sum_{m=1}^{\infty} m^s t_m(\lambda). \tag{15}$$

Let us give an inequality first, for $\gamma \geqslant 1$

$$\left(\sum_{i=1}^{N} a_i \right)^{\gamma} \leqslant N^{\gamma-1} \sum_{i=1}^{N} |a_i|^{\gamma}. \tag{16}$$

Direct computations imply

$$\int_J t^q_m(\lambda)\mathrm{d}\lambda = \int_J \left(\sum_{j=1}^{2^m} |P(f\chi^m_j)(\lambda)|^2 \right)^{q/2} \mathrm{d}\lambda$$
$$\overset{\text{by (16)}}{\leqslant} 2^{m(q/2-1)} \int_J \sum_{j=1}^{2^m} |P(f\chi^m_j)(\lambda)|^q \ \mathrm{d}\lambda$$
$$\leqslant C2^{m(q/2-1)} \sum_{j=1}^{2^m} \|f\chi^m_j\|^q_{\ell^p(L^1)}$$
$$\overset{\text{by (14)}}{\leqslant} C2^{m(q/2-1)} \sum_{j=1}^{2^m} 2^{-m\frac{q}{p}} \|f\|^q_{\ell^p(L^1)}$$
$$\leqslant C\|f\|^q_{\ell^p(L^1)} 2^{m\frac{q}{2} - m\frac{q}{p}}, \tag{17}$$

where the second inequality holds by the boundedness of P.

Finally, we have

$$\begin{aligned}\|G^{(s)}_{S(f)(\lambda)}\|_{L^q(J)} &\overset{\text{by (15)}}{=} \|\sum_{m=1}^{\infty} m^s t_m(\lambda)\|_{L^q(J)} \\ &\leqslant \sum_{m=1}^{\infty} m^s \|t_m(\lambda)\|_{L^q(J)} \\ &\overset{\text{by (17)}}{\leqslant} C\sum_{m=1}^{\infty} m^s 2^{m/2-m/p}\|f\|_{\ell^p(L^1)} \\ &\leqslant C\|f\|_{\ell^p(L^1)}.\end{aligned}$$

□

Denote by

$$B_n(g_1, g_2, \dots, g_n)(x) = \int_x^\infty \int_{t_1}^\infty \dots \int_{t_{n-1}}^\infty \prod_{j=1}^n g_j(t_j)\mathrm{d}t_1\ \mathrm{d}t_2 \dots \mathrm{d}t_n. \tag{18}$$

If there is a single function g such that $g_k \in \{g, \bar{g}\}$, $k = 1, 2, \dots, n$, we write $B_n(g_1, g_2, \dots, g_n)(x)$ by $B_n(g)(x)$.

Theorem 2.5. *Assume that g_j, $j = 1, 2, \dots, n$ is locally integrable. Suppose for $j = 1, 2, \dots, n$,*

$$\limsup_{M\to\infty} \|g_j\chi_{[M,\infty)}\|_{\mathfrak{B}} = 0, \tag{19}$$

and there is a constant C (does not depend on I) such that for any closed interval $I \subset \mathbb{R}_+$,

$$\|g_j\chi_I\|_{\mathfrak{B}} \leqslant C. \tag{20}$$

Then (18) is well defined as the limit

$$B_n(g_1, g_2, \dots, g_n)(x) = \lim_{y_1,\dots,y_n\to\infty} \int_x^{y_1} \int_{t_1}^{y_2} \dots \int_{t_{n-1}}^{y_n} \prod_{j=1}^n g_j(t_j)\mathrm{d}t_1\ \mathrm{d}t_2 \dots \mathrm{d}t_n, \tag{21}$$

and

$$\lim_{x\to\infty} B_n(g_1, g_2, \dots, g_n)(x) = 0. \tag{22}$$

Moreover, for almost every x

$$\frac{\mathrm{d}B_n(g_1, g_2, \dots, g_n)(x)}{\mathrm{d}x} = -g_1(x)B_{n-1}(g_2, \dots, g_n)(x). \tag{23}$$

Proof. In order to prove the existence of the limit, it suffices to show that

$$\begin{aligned}\lim_{\substack{y_k,z_k\to\infty\\k=1,2,\dots n}} \sup_x\ &|B_n(g_1\chi_{[0,y_1]}, g_2\chi_{[0,y_2]}, \dots, g_n\chi_{[0,y_n]})(x) \\ &- B_n(g_1\chi_{[0,z_1]}, g_2\chi_{[0,z_2]}, \dots, g_n\chi_{[0,z_n]})(x)| = 0.\end{aligned} \tag{24}$$

Assume $M < y_k < z_k$, $k = 1, 2, \ldots, n$. By telescoping techniques,

$$\begin{aligned}
&|B_n(g_1\chi_{[0,y_1]}, g_2\chi_{[0,y_2]}, \ldots, g_n\chi_{[0,y_n]})(x) - B_n(g_1\chi_{[0,z_1]}, g_2\chi_{[0,z_2]}, \ldots, g_n\chi_{[0,z_n]}(x)| \\
&\quad \leqslant \sum_{k=1}^{n} |B_n(g_1\chi_{[0,y_1]}, \ldots, g_{k-1}\chi_{[0,y_{k-1}]}, g_k\chi_{[y_k,z_k]}, g_{k+1}\chi_{[0,z_{k+1}]}, \ldots, g_n\chi_{[0,z_n]})(x)| \\
&\quad \leqslant C\sum_{k=1}^{n} \|g_1\chi_{[0,y_1]}\|_{\mathfrak{B}} \ldots \|g_{k-1}\chi_{[0,y_{k-1}]}\|_{\mathfrak{B}} \|g_k\chi_{[y_k,z_k]}\|_{\mathfrak{B}} \|g_{k+1}\chi_{[0,z_{k+1}]}\|_{\mathfrak{B}} \ldots \|g_n\chi_{[0,z_n]}\|_{\mathfrak{B}} \\
&\quad \leqslant C\sum_{k=1}^{n} \|g_k\chi_{[y_k,z_k]}\|_{\mathfrak{B}} \\
&\quad \leqslant C\sum_{k=1}^{n} \|g_k\chi_{[M,\infty)}\|_{\mathfrak{B}},
\end{aligned} \tag{25}$$

where the second inequality holds by (12) and (8), and the third inequality holds by (20). Now (24) follows from (19).

By (21), one has

$$\begin{aligned}
&\lim_{x\to\infty} |B_n(g_1, \ldots, g_n)(x)| \\
&= \lim_{x\to\infty} \lim_{x'\to\infty} \int_x^{x'} \int_{t_1}^{x'} \ldots \int_{t_{n-1}}^{x'} \prod_{j=1}^{n} g_j(t_j)\mathrm{d}t_1\ \mathrm{d}t_2 \ldots \mathrm{d}t_n \\
&= \lim_{x\to\infty} \lim_{x'\to\infty} \int_x^{x'} \int_{t_1}^{x'} \ldots \int_{t_{n-1}}^{x'} \prod_{j=1}^{n} g_j(t_j)\chi_{[x,\infty)}\mathrm{d}t_1\ \mathrm{d}t_2 \ldots \mathrm{d}t_n \\
&\overset{\text{by (12)}}{\leqslant} C \lim_{x\to\infty} \prod_{k=1}^{n} \|g_k\chi_{[x,\infty)}\|_{\mathfrak{B}} \\
&\overset{\text{by (19)}}{=} 0.
\end{aligned}$$

This completes the proof of (22). Direct computations imply

$$\begin{aligned}
&\lim_{y\to x-} \frac{B_n(g_1, \ldots, g_n)(y) - B_n(g_1, \ldots, g_n)(x)}{y - x} \\
&= \lim_{y\to x-} \frac{1}{y-x} \int_y^x g_1(t_1)\mathrm{d}t_1 \int_{t_1}^{\infty} \ldots \int_{t_{n-1}}^{\infty} \prod_{j=2}^{n} g_j(t_j)\mathrm{d}t_2 \ldots \mathrm{d}t_n \\
&= -g_1(x) \int_x^{\infty} \ldots \int_{t_{n-1}}^{\infty} \prod_{j=2}^{n} g_j(t_j)\mathrm{d}t_2 \ldots \mathrm{d}t_n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\lim_{y\to x+} \frac{B_n(g_1, \ldots, g_n)(y) - B_n(g_1, \ldots, g_n)(x)}{y - x} \\
&\qquad = -g_1(x) \int_x^{\infty} \ldots \int_{t_{n-1}}^{\infty} \prod_{j=2}^{n} g_j(t_j)\mathrm{d}t_2 \ldots \mathrm{d}t_n.
\end{aligned}$$

The last two equalities imply (23). □

Remark 2.6. In [[4], proposition 4.1], Christ and Kiselev proved the existence of a weaker limit $\tilde{B}_n(g_1, g_2, \ldots, g_n)(x)$, where

$$\tilde{B}_n(g_1, g_2, \ldots, g_n)(x) = \lim_{y\to\infty} \int_x^y \int_{t_1}^y \ldots \int_{t_{n-1}}^y \prod_{j=1}^n g_j(t_j)\mathrm{d}t_1\ \mathrm{d}t_2 \ldots \mathrm{d}t_n.$$

Let $p(x, \lambda)$ be a measurable function on $\mathbb{R}_+ \times J$. Define the integral operator P:

$$P(f)(\lambda) = \int_{\mathbb{R}_+} p(x, \lambda) f(x)\mathrm{d}x,$$

and the maximal operator P^*:

$$P^*(f)(\lambda) = \sup_{y\in\mathbb{R}_+} \left| \int_y^\infty p(x, \lambda) f(x)\mathrm{d}x \right|.$$

Lemma 2.7 [[2], Christ–Kiselev lemma]. *Let $1 \leqslant p < q < \infty$. Suppose P is a bounded operator from $\ell^p(L^1)$ to $L^q(J)$. Then P^* is also a bounded operator from $\ell^p(L^1)$ to $L^q(J)$.*

In our situation (see next section), $s(x, \lambda) = w(x, \lambda)\mathrm{e}^{-\mathrm{i}h(x,\lambda)}$, where h is a real-valued function. We obtain two operators

$$S(f)(\lambda) = \int_{\mathbb{R}_+} w(x, \lambda)\mathrm{e}^{-\mathrm{i}h(x,\lambda)} f(x)\mathrm{d}x, \tag{26}$$

and

$$S^*(f)(\lambda) = \sup_{y\in\mathbb{R}_+} \left| \int_y^\infty w(x, \lambda)\mathrm{e}^{-\mathrm{i}h(x,\lambda)}\ f(x)\mathrm{d}x \right|. \tag{27}$$

Lemma 2.8. *Assume $1 \leqslant p \leqslant 2$. Suppose there exist a constant C and a closed interval $\tilde{J}$ such that $J \subset \mathrm{Int}\ \tilde{J}$ and for any $\lambda \in \mathrm{Int}\ \tilde{J}$*

$$|\partial_\lambda[h(x, \lambda) - h(y, \lambda)]| \geqslant \frac{|x - y|}{C} \tag{28}$$

and

$$|\partial_\lambda^i[h(x, \lambda) - h(y, \lambda)]| \leqslant C|x - y|, \quad i = 1, 2, 3, \tag{29}$$

provided $|x - y| \geqslant C$. Suppose

$$\sup_{x\in\mathbb{R}_+, \lambda\in\tilde{J}} \sum_{i=1}^2 |\partial_\lambda^i w(x, \lambda)| \leqslant C.$$

Let $p' = \frac{p}{p-1}$ be the conjugate exponent to p ($p' = \infty$ when $p = 1$). Then

$$\|Sf\|_{L^{p'}(J,\mathrm{d}\lambda)} \leqslant O(1)\|f\|_{\ell^p(L^1)},$$

and

$$\|S^*f\|_{L^{p'}(J,\mathrm{d}\lambda)} \leqslant O(1)\|f\|_{\ell^p(L^1)},$$

where $O(1)$ depends on $C, J, \tilde{J}$ and p.

Proof. By lemma 2.7, we only need to prove the boundedness of S. By the interpolation theorem, it suffices to prove the cases $p = 1$ and $p = 2$. The case $p = 1$ is trivial since h is a real-valued function, so we only need to consider the case $p = 2$. Let $\xi(\lambda)$ be a positive function so that $\xi \equiv 1$ on J and supp $\xi \subset$ Int $\tilde{J}$. Then one has

$$
\begin{aligned}
\|Sf\|^2_{L^2(J,\mathrm{d}\lambda)} &= \int_J \left| \int_{\mathbb{R}_+} w(x,\lambda)\mathrm{e}^{-\mathrm{i}h(x,\lambda)}\ f(x)\mathrm{d}x \right|^2 \mathrm{d}\lambda \\
&\leqslant \int_{\tilde{J}} \left| \int_{\mathbb{R}_+} w(x,\lambda)\mathrm{e}^{-\mathrm{i}h(x,\lambda)}\ f(x)\mathrm{d}x \right|^2 \xi(\lambda)\mathrm{d}\lambda \\
&= \int_{\tilde{J}} \left[\int_{\mathbb{R}_+} w(x,\lambda)\mathrm{e}^{-\mathrm{i}h(x,\lambda)}\ f(x)\mathrm{d}x \right] \left[\int_{\mathbb{R}_+} \bar{w}(y,\lambda)\mathrm{e}^{\mathrm{i}h(y,\lambda)}\ \bar{f}(y)\mathrm{d}y \right] \xi(\lambda)\mathrm{d}\lambda \\
&= \int_{\mathbb{R}^2_+} f(x)\bar{f}(y)\mathrm{d}x\ \mathrm{d}y \int_{\tilde{J}} \mathrm{e}^{-\mathrm{i}h(x,\lambda)+\mathrm{i}h(y,\lambda)} w(x,\lambda)\bar{w}(y,\lambda)\xi(\lambda)\mathrm{d}\lambda. \qquad (30)
\end{aligned}
$$

Multiplying $-i\partial_\lambda(h(x,\lambda) - h(y,\lambda))$, dividing $-i\partial_\lambda(h(x,\lambda) - h(y,\lambda))$ and integrating by part twice, we have for $|x - y| \geqslant C$,

$$
\begin{aligned}
&\int_{\tilde{J}} \mathrm{e}^{-\mathrm{i}h(x,\lambda)+\mathrm{i}h(y,\lambda)} w(x,\lambda)\bar{w}(y,\lambda)\xi(\lambda)\mathrm{d}\lambda \\
&= \int_{\tilde{J}} \frac{-\mathrm{i}\partial_\lambda(h(x,\lambda) - h(y,\lambda))}{-\mathrm{i}\partial_\lambda(h(x,\lambda) - h(y,\lambda))}\ \mathrm{e}^{-\mathrm{i}h(x,\lambda)+\mathrm{i}h(y,\lambda)}\ w(x,\lambda)\bar{w}(y,\lambda)\xi(\lambda)\mathrm{d}\lambda \\
&= \int_{\tilde{J}} \mathrm{e}^{-\mathrm{i}h(x,\lambda)+\mathrm{i}h(y,\lambda)} \partial_\lambda \left(\frac{w(x,\lambda)\bar{w}(y,\lambda)\xi(\lambda)}{-\mathrm{i}\partial_\lambda(h(x,\lambda) - h(y,\lambda))} \right) \mathrm{d}\lambda \\
&= \int_{\tilde{J}} \mathrm{e}^{-\mathrm{i}h(x,\lambda)+\mathrm{i}h(y,\lambda)} \partial_\lambda \left[\frac{1}{-\mathrm{i}\partial_\lambda(h(x,\lambda) - h(y,\lambda))} \partial_\lambda \left(\frac{w(x,\lambda)\bar{w}(y,\lambda)\xi(\lambda)}{-\mathrm{i}\partial_\lambda(h(x,\lambda) - h(y,\lambda))} \right) \right] \mathrm{d}\lambda \\
&= \frac{O(1)}{|x-y|^2}, \qquad (31)
\end{aligned}
$$

where the last equality holds by the assumptions of lemma 2.8.

By (30) and (31), we have

$$
\begin{aligned}
\|Sf\|^2_{L^2(J,\mathrm{d}\lambda)} &\leqslant \int_{|x-y|>C} f(x)\bar{f}(y)\mathrm{d}x\ \mathrm{d}y \int_{\tilde{J}} \mathrm{e}^{-\mathrm{i}h(x,\lambda)+\mathrm{i}h(y,\lambda)}\ w(x,\lambda)\bar{w}(y,\lambda)\xi(\lambda)\mathrm{d}\lambda. \\
&\quad + \int_{|x-y|\leqslant C} f(x)\bar{f}(y)\mathrm{d}x\ \mathrm{d}y \int_{\tilde{J}} \mathrm{e}^{-\mathrm{i}h(x,\lambda)+\mathrm{i}h(y,\lambda)}\ w(x,\lambda)\bar{w}(y,\lambda)\xi(\lambda)\mathrm{d}\lambda \\
&= O(1) \int_{\mathbb{R}^2_+} \frac{|f(x)f(y)|}{1 + |x-y|^2}\ \mathrm{d}x\ \mathrm{d}y \\
&= O(1)\|f\|_{\ell^2(L^1)}, \qquad (32)
\end{aligned}
$$

where the last equality holds by direct calculation (for convenience, we include the details in the appendix A). This completes the proof. □

Remark 2.9. The formulation and proof of lemma 2.8 closely follow from the corresponding parts appearing in [3, 4, 12].

3. Technical preparations

We set up the basics in this section. By the Floquet theory, $\varphi(x, E)$ has the form

$$\varphi(x, E) = J(x, E)\mathrm{e}^{\mathrm{i}k(E)x}, \quad \text{or} \quad \varphi(x, E) = \bar{J}(x, E)\mathrm{e}^{-\mathrm{i}k(E)x}, \tag{33}$$

where $k(E) \in [0, \pi]$ is the quasimomentum, and $J(x, E)$ is 1-periodic.

Without loss of generality, assume $\varphi(x, E) = J(x, E)\mathrm{e}^{\mathrm{i}k(E)x}$. Since $\varphi(x, E)$ and $\overline{\varphi}(x, E)$ are two linearly independent solutions of $-u' + V_0u = Eu$, the Wronskian $W(\overline{\varphi}, \varphi)$ is a non-zero constant and

$$W(\overline{\varphi}, \varphi) = \overline{\varphi}(x)\varphi'(x) - \overline{\varphi}'(x)\varphi(x) = 2i\Im[\overline{\varphi}(x)\varphi'(x)]. \tag{34}$$

Let us study the solutions of the equation

$$-u'' + (V_0(x) + V(x))u = Eu.$$

We rewrite this equation as a linear system

$$u_1' = \begin{pmatrix} 0 & 1 \\ V_0 + V - E & 0 \end{pmatrix} u_1,$$

where u_1 is the vector $\begin{pmatrix} u \\ u' \end{pmatrix}$. Introduce

$$u_1 = \begin{pmatrix} (x, E) & \overline{\varphi}(x, E) \\ \varphi'(x, E) & \overline{\varphi}'(x, E) \end{pmatrix} u_2.$$

Then

$$u_2' = \frac{i}{2\Im(\varphi\overline{\varphi}')} \begin{pmatrix} V(x)|\varphi(x, E)|^2 & V(x)\overline{\varphi}(x, E)^2 \\ -V(x)\varphi(x, E)^2 & -V(x)|\varphi(x, E)|^2 \end{pmatrix} u_2. \tag{35}$$

In the following discussion, E is always the energy. For a two-variable function $f(x, E)$, denote by f' the derivative of f with respect to the non-energy variable, namely $f'(x, E) = \partial_x f(x, E)$. Define

$$p(x, E) = \frac{1}{2\Im(\varphi\overline{\varphi}')} \int_0^x V(y)|\varphi(y, E)|^2 \, \mathrm{d}y.$$

Let us apply another transformation,

$$u_2 = \begin{pmatrix} \exp(\mathrm{i}p(x, E)) & 0 \\ 0 & \exp(-\mathrm{i}p(x, E)) \end{pmatrix} u_3.$$

We obtain the equation for u_3:

$$u_3' = \frac{\mathrm{i}}{2\Im(\varphi\overline{\varphi}')} \begin{pmatrix} 0 & V(x)\overline{\varphi}(x, E)^2 \, \exp(-2\mathrm{i}p(x, E)) \\ -V(x)\varphi(x, E)^2 \, \exp(2\mathrm{i}p(x, E)) & 0 \end{pmatrix} u_3. \tag{36}$$

Lemma 3.1. *Suppose there exists a solution of* (36) *satisfying*

$$u_3(x,E) = \begin{pmatrix}1\\0\end{pmatrix} + o(1)$$

as $x \to \infty$. *Then there exists a solution* $u(x,E)$ *of* (2) *satisfying* (5).

Proof. The proof is straightforward by substitutions. □

Let $Y = u_3$. Let $\phi(x,E)$ be so that $\mathrm{e}^{\mathrm{i}\phi(x,E)} = \varphi(x,E)$. We note that $\phi(x,E)$ is a complex-valued function.

Denote by

$$w(x,E) = \frac{i}{2\Re\phi'}, \tag{37}$$

and

$$h(x,E) = 2\Re\phi - \int_0^x \frac{V(t)}{\Re\phi'(t,E)}\,\mathrm{d}t. \tag{38}$$

In the following, w and h are always given by (37) and (38) respectively. The operators S and S^* are given by (26) and (27) respectively. Denote by

$$\mathcal{F}(x,E) = w(x,E)\mathrm{e}^{-\mathrm{i}h(x,E)}\,V(x). \tag{39}$$

Under this notation and following the calculations in p 249 and p 250 in [3], (36) becomes

$$Y' = \begin{pmatrix}0 & w\,\mathrm{e}^{-\mathrm{i}h}V\\ \bar{w}\,\mathrm{e}^{\mathrm{i}h}V & 0\end{pmatrix} Y = \begin{pmatrix}0 & \mathcal{F}\\ \bar{\mathcal{F}} & 0\end{pmatrix} Y. \tag{40}$$

For convenience, we include a verification of (40) in the appendix A.

Denote by

$$D = \begin{pmatrix}0 & \mathcal{F}\\ \bar{\mathcal{F}} & 0\end{pmatrix}.$$

The linear equation (40) becomes $Y' = DY$. We are going to find a solution of

$$Y(x) = \begin{pmatrix}1\\0\end{pmatrix} - \int_x^\infty D(y)Y(y)\mathrm{d}y, \tag{41}$$

and we obtain a series solution by iterations

$$Y(x) = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{k=1}^{\infty}(-1)^k \int\cdots\int_{x\leqslant t_1\leqslant t_2\ldots\leqslant t_k<\infty} \times D(t_1)D(t_2)\ldots D(t_k)\begin{pmatrix}1\\0\end{pmatrix}\mathrm{d}t_k\,\mathrm{d}t_{k-1}\ldots\mathrm{d}t_2\,\mathrm{d}t_1. \tag{42}$$

Let

$$T_n(\mathcal{F})(x,x',E) = M_n(\mathcal{F}(\cdot,E))(x,x').$$

Under the above notation, one has

$$\int \cdots \int_{x \leqslant t_1 \leqslant t_2 \ldots \leqslant t_{2k} \leqslant x'} D(t_1)D(t_2)\ldots D(t_{2k}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}t_{2k} \ldots \mathrm{d}t_2 \ \mathrm{d}t_1$$
$$= \begin{pmatrix} T_{2k}(\mathcal{F})(x, x', E) \\ 0 \end{pmatrix},$$

and

$$\int \cdots \int_{x \leqslant t_1 \leqslant t_2 \ldots \leqslant t_{2k+1} \leqslant x'} D(t_1)D(t_2)\ldots D(t_{2k+1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}t_{2k+1} \ldots \mathrm{d}t_2 \ \mathrm{d}t_1$$
$$= \begin{pmatrix} 0 \\ T_{2k+1}(\mathcal{F})(x, x', E) \end{pmatrix}.$$

The series solution (42) becomes

$$Y(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \sum_{m=1}^{\infty} T_{2m}(\mathcal{F})(x, \infty, E) \\ -\sum_{m=0}^{\infty} T_{2m+1}(\mathcal{F})(x, \infty, E) \end{pmatrix}. \tag{43}$$

We will show (43) is well defined and gives an actual solution of (40).

4. Proof of theorem 1.1

Fix a martingale structure $\{E_j^m \subset \mathbb{R}_+ : m \in \mathbb{Z}_+, 1 \leqslant j \leqslant 2^m\}$ to the potential V. Choose a spectral band (a_n, b_n) and let $K \subset (a_n, b_n)$ be an arbitrary closed interval. We will apply theorem 2.5 to complete our proof.

Lemma 4.1. *For any $E \in (a_n, b_n)$, there exists a constant $C = C(E)$ (depends on E uniformly in any compact subset of (a_n, b_n)) such that*

$$|\partial_E[h(x, E) - h(y, E)]| \geqslant \frac{|x - y|}{C} \tag{44}$$

and for $i = 1, 2, 3$

$$|\partial_E^i[h(x, E) - h(y, E)]| \leqslant C|x - y| \tag{45}$$

provided $|x - y| \geqslant C$.

Proof. We will prove (44) first. By the definition of ϕ and (33), one has

$$\Re\ \phi = k(E)x + \Im\ \log\ J(x, E), \tag{46}$$

with $k(E) \in (0, \pi)$. By the Floquet theory,

$$\frac{\mathrm{d}k(E)}{\mathrm{d}E} \neq 0. \tag{47}$$

By the fact that $J(x, E)$ is 1-periodic, one has

$$\Im\ \log\ J(x + 1, E) - \Im\ \log\ J(x, E) = 2q\pi, \tag{48}$$

for some $q \in \mathbb{Z}$. It implies

$$\partial_E(\Im \ \log \ J(x+1,E) - \Im \ \log \ J(x,E)) = 0. \tag{49}$$

By (47) and (49), we have

$$|\Re \ \phi(x,E) - \Re \ \phi(y,E)| \geqslant \frac{|x-y|}{C}. \tag{50}$$

Since $V(x) \in \ell^p(L^1)$, one has that $\partial_E \int_N^{N+1} \frac{V(t)}{\Re \ \phi'(t,E)} \, \mathrm{d}t$ goes to zero as $N \to \infty$. It implies

$$\left|\partial_E \int_x^y \frac{V(t)}{\Re \ \phi'(t,E)} \, \mathrm{d}t\right| = o(y-x) + O(1), \tag{51}$$

as $y-x$ goes to ∞. Now (44) follows from (50) and (51). The proof of (45) can be completed in a similar way. □

Lemma 4.2. *Let p' be the number conjugate to p with $1 \leqslant p < 2$. Then*

$$G^{(s)}_{S^*(V)(E)} \in L^{p'}(K, \mathrm{d}E).$$

In particular ($s = 1$),

$$G_{S^*(V)(E)} \in L^{p'}(K, \mathrm{d}E).$$

Proof. By lemmas 4.1 and 2.8, and applying $P = S^*$ in theorem 2.4, we have

$$G^{(s)}_{S^*(V)(E)} \in L^{p'}(K, \mathrm{d}E). \tag{52}$$

□

Corollary 4.3. *Let p' be the number conjugate to p with $1 \leqslant p < 2$. Then for almost every $E \in K$ and any closed interval I,*

$$\|\mathcal{F}(\cdot,E)\chi_I\|_{\mathfrak{B}} \leqslant C(E), \tag{53}$$

and

$$\limsup_{M\to\infty} \|\mathcal{F}(\cdot,E)\chi_{[M,\infty)}\|_{\mathfrak{B}} = 0. \tag{54}$$

Proof. By direct computation, one has

$$\begin{aligned}
\|\mathcal{F}(\cdot,E)\chi_I\|_{\mathfrak{B}} &= \left\|\left\{\int_{E_j^m} \mathcal{F}(x,E)\chi_I(x)\mathrm{d}x\right\}\right\|_{\mathfrak{B}} \\
&= \left\|\left\{\int_{E_j^m} w(x,E)\mathrm{e}^{\mathrm{i}h(x,E)} \ V(x)\chi_I(x)\mathrm{d}x\right\}\right\|_{\mathfrak{B}} \\
&= \left\|\left\{\int_I w(x,E)\mathrm{e}^{\mathrm{i}h(x,E)} \ V(x)\chi_j^m \ \mathrm{d}x\right\}\right\|_{\mathfrak{B}} \\
&\leqslant 2\| \left\{S^*(V\chi_j^m)(E)\right\} \|_{\mathfrak{B}} \\
&= 2G_{S^*(V)(E)}.
\end{aligned}$$

Now (53) follows since $G_{S^*(V)(E)} \in L^q(K, \mathrm{d}E)$ by lemma 4.2. Applying $P = S$ and $f = V(x)\chi_{[M,\infty)}$ in theorem 2.4 and recalling that

$$\|\mathcal{F}(\cdot, E)\chi_{[M,\infty)}\|_{\mathfrak{B}} = G_{S(V\chi_{[M,\infty)})},$$

one has

$$\begin{aligned}\limsup_{M\to\infty}\left(\int_J \|\mathcal{F}(\cdot, E)\chi_{[M,\infty)}\|_{\mathfrak{B}}^q \mathrm{d}E\right)^{1/q} &= \limsup_{M\to\infty} \|G_{S(V\chi_{[M,\infty)})}\|_{L^q(J)} \\ &\leqslant O(1)\limsup_{M\to\infty} \|V\chi_{[M,\infty)}\|_{\ell^p(L^1)} \\ &= 0. \end{aligned} \tag{55}$$

This implies that for almost every $E \in K$,

$$\limsup_{M\to\infty} \|\mathcal{F}(\cdot, E)\chi_{[M,\infty)}\|_{\mathfrak{B}} = 0.$$

This leads to (54). □

Proof of theorem 1.1. Under the assumption of theorem 1.1, $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0)$ [22, 39]. This yields that $\sigma_{\mathrm{ac}}(H) \subset S$. It is well known that the boundedness of the eigensolution implies purely absolutely continuous spectrum (e.g. [35, 40]). Then the second part of theorem 1.1 implies the first part. If $p = 1$ ($V \in L^1(\mathbb{R}_+)$), one has that for every $E \in K$, $\frac{\mathrm{i}V}{2\,\Re\,\phi'}\,\mathrm{e}^{\mathrm{i}h}$ given by (40) is in L^1. In this case, it is well known (see [7] for example) that (40) has a solution $Y(x)$ satisfying

$$Y(x) = \begin{pmatrix}1\\0\end{pmatrix} + o(1),$$

as $x \to \infty$. By lemma 3.1, theorem 1.1 is true for $p = 1$. So we assume $1 < p < 2$.

By corollary 4.3 and theorem 2.5, for almost every $E \in K$ the following limit is well defined,

$$T_{2m}(\mathcal{F})(x, \infty, E) = \lim_{x'\to\infty} T_{2m}(\mathcal{F})(x, x', E).$$

By (13) and (53), we have

$$|T_{2m}(\mathcal{F})(x, \infty, E)| \leqslant \frac{C(E)^{2m}}{\sqrt{(2m)!}}. \tag{56}$$

Thus

$$\sum_{m=1}^{\infty} T_{2m}(\mathcal{F})(x, \infty, E) \tag{57}$$

is absolutely convergent for almost every $E \in K$. Similarly,

$$\sum_{m=0}^{\infty} T_{2m+1}(\mathcal{F})(x, \infty, E) \tag{58}$$

is absolutely convergent for almost every $E \in K$. Therefore, the series in (43) is well defined for almost every $E \in K$. Based on (23), it is easy to check that for almost every $E \in K$, the series in (43) actually gives a solution of (40). The WKB behavior (5) follows from (22) and lemma 3.1. □

5. An alternative proof of theorems 1.1 and 1.2

We will give a new proof of theorem 1.1. By the arguments in the previous section, it suffices to prove corollary 4.3. We will give a proof without using the maximal operator.

Lemma 5.1 [2]. *Let $\{E_j^m \subset \mathbb{R}_+ : m \in \mathbb{Z}_+, 1 \leqslant j \leqslant 2^m\}$ be a martingale structure. Then there exists an absolute constant C such that for any closed interval I,*

$$\|g\chi_I\|_{\mathfrak{B}^s} \leqslant C\|g\|_{\mathfrak{B}^{s+1}}.$$

In particular,

$$\|g\chi_I\|_{\mathfrak{B}} \leqslant C\|g\|_{\mathfrak{B}^2}. \tag{59}$$

A new proof of corollary 4.3 without using S^*. The proof of (54) does not use S^*, so we keep it. We only need to show that (53) is true for almost every $E \in K$. Applying $P = S$ and $f = V(x)$ in theorem 2.4 and recalling that

$$\|\mathcal{F}(\cdot, E)\|_{\mathfrak{B}^2} = G^{(2)}_{S(V)(E)},$$

one has

$$\|\mathcal{F}(\cdot, E)\|_{\mathfrak{B}^2} \in L^q(J). \tag{60}$$

Now (53) follows from (59) and (60). □

Suppose the assumptions of theorem 1.2 hold for some $1 < p \leqslant 2$ and $\gamma > 0$. Let β be any positive number bigger than $1 - p'\gamma$. Denote by $\mathcal{H}^\beta$ the β-dimensional Hausdorff measure. Let

$$\Lambda_c = \{E \in K : \|\mathcal{F}(\cdot, E)\chi_{[N,\infty)}\|_{\mathfrak{B}^2} \geqslant c \quad \text{for every } N \geqslant 0\}.$$

Lemma 5.2. *For any $c > 0$, we have*

$$\mathcal{H}^\beta(\Lambda_c) = 0.$$

Proof. The lemma follows from the arguments in [3]. Actually, lemma 5.2 is a particular case of what was studied in section 8 of [3]. □

Proof of theorem 1.2. By lemma 5.2, we have for every E in K except for a set of $\mathcal{H}^\beta$ measure zero and any $c > 0$, there exists $N > 0$ such that

$$\|\mathcal{F}(\cdot, E)\chi_{[N,\infty)}\|_{\mathfrak{B}^2} \leqslant c. \tag{61}$$

Fix such E. Let N_0 be such that (61) holds for $c = 1$. By changing x to $x - N_0$, we can assume $N_0 = 0$. Therefore, by (59), one has

$$\sup_{I \subset \mathbb{R}_+} \|\mathcal{F}(\cdot, E)\chi_I\|_{\mathfrak{B}} \leqslant C.$$

For any $\varepsilon > 0$, let $N(\varepsilon)$ be large enough so that (61) holds for $c = \varepsilon$. For any $M > N(\varepsilon)$, by (59) again,

$$\begin{aligned}\|\mathcal{F}(\cdot, E)\chi_{[M,\infty)}\|_{\mathfrak{B}} &\leqslant \|\mathcal{F}(\cdot, E)\chi_{[N(\varepsilon),\infty)}\|_{\mathfrak{B}} + \|\mathcal{F}(\cdot, E)\chi_{[N(\varepsilon),M)}\|_{\mathfrak{B}} \\ &\leqslant \|\mathcal{F}(\cdot, E)\chi_{[N(\varepsilon),\infty)}\|_{\mathfrak{B}} + C\|\mathcal{F}(\cdot, E)\chi_{[N(\varepsilon),\infty)}\|_{\mathfrak{B}^2} \\ &\leqslant C\varepsilon.\end{aligned}$$

This implies (54). Now the rest of the proof of theorem 1.2 follows from the proof of theorem 1.1. □

6. Sharp estimates for almost orthogonality among generalized Prüfer angles

In this section, we always assume that for some $B > 0$,

$$|V(x)| \leqslant \frac{B}{1+x}. \tag{62}$$

Without loss of generality, we only consider the Dirichlet boundary condition.

For any spectral band $[a_n, b_n]$, let c_n be the unique number such that $k(c_n) = \frac{\pi}{2}$. Let I be a closed interval in (a_n, c_n) or (c_n, b_n). All the energies E in this section are in I and the estimates are uniform with respect to $E \in I$.

For $z \in \mathbb{C}\backslash\mathbb{R}$, denote by $\tilde{v}_1(x,z)$ ($\tilde{v}_2(x,z)$) the solution of $H_0 + V$ with boundary conditions $\tilde{v}_1(0,z) = 1$ and $\tilde{v}_1'(0,z) = 0$ ($\tilde{v}_2(0,z) = 0$ and $\tilde{v}_2'(0,z) = 1$). The Weyl m-function $m(z)$ (well defined on $z \in \mathbb{C}\backslash\mathbb{R}$) is given by the unique complex number $m(z)$ so that $\tilde{v}_1(x,z) + m(z)\tilde{v}_2(x,z) \in L^2(\mathbb{R}_+)$. The spectral measure μ on $\mathbb{R}$, is given by the following formula, for $z \in \mathbb{C}\backslash\mathbb{R}$

$$m(z) = C + \int \left[\frac{1}{x-z} - \frac{x}{1+x^2}\right] \mathrm{d}\mu(x),$$

where C is a constant.

Denote by μ_{sc} the singular continuous component of μ. It is well known that $\sigma_{\mathrm{sc}}(H_0 + V) = \emptyset$ if and only if $\mu_{\mathrm{sc}} = 0$.

Recall that $\varphi(x,E) = J(x,E)\mathrm{e}^{\mathrm{i}k(E)x}$ and define a continuous function $\gamma(x,E)$ such that

$$\varphi(x,E) = |\varphi(x,E)|\ \mathrm{e}^{\mathrm{i}\gamma(x,E)}. \tag{63}$$

By [[17], proposition 2.1], we know that there exists some constant $C > 0$ such that

$$\frac{1}{C} \leqslant \gamma'(x,E) \leqslant C, \text{ or } \frac{1}{C} \leqslant -\gamma'(x,E) \leqslant C. \tag{64}$$

Let $u(x,E)$ be an arbitrary solution of $-u'' + V_0 u + Vu = Eu$ and define $\rho(x,E) \in \mathbb{C}$ by

$$\begin{pmatrix} u(x,E) \\ u'(x,E) \end{pmatrix} = \frac{1}{2i}\left[\rho(x,E)\begin{pmatrix} \varphi(x,E) \\ \varphi'(x,E) \end{pmatrix} - \overline{\rho}(x,E)\begin{pmatrix} \overline{\varphi}(x,E) \\ \overline{\varphi}'(x,E) \end{pmatrix}\right]. \tag{65}$$

Define $R(x,E)$ and $\theta(x,E)$ by

$$R(x,E) = |\rho(x,E)|; \theta(x,E) = \gamma(x,E) + \mathrm{Arg}(\rho(x,E)). \tag{66}$$

Proposition 6.1 **[17].** *Suppose u is a real solution of (2). Then the real functions $R(x) > 0$ and $\theta(x)$ satisfy*

$$[\ln\ R(x,E)]' = \frac{V(x)}{2\gamma'(x,E)}\ \sin\ 2\theta(x,E) \tag{67}$$

and

$$\theta(x,E)' = \gamma'(x,E) - \frac{V(x)}{2\gamma'(x,E)}\ \sin^2\ \theta(x,E). \tag{68}$$

Before we establish the almost orthogonality among Prüfer angles, some preparation is necessary.

Lemma 6.2 [15]. *Suppose the function $G(x)$ satisfies $|G'(x)| = \frac{O(1)}{1+x}$ and $\gamma \neq 0$. Then*

$$\left| \int_0^L \frac{\sin(\gamma x + G(x))}{1+x} \, \mathrm{d}x \right| \leqslant O(1) \log \ |\gamma^{-1}| + O(1). \tag{69}$$

We remark that $O(1)$ in (69) and also throughout the following proof does not depend on L.

Lemma 6.3 [26]. *Suppose the function $G(x)$ satisfies $|G'(x)| = \frac{O(1)}{1+x}$ and $\gamma \neq 0$. Then*

$$\left| \int_0^L \frac{\sin(\gamma x + G(x))}{1+x} \, \mathrm{d}x \right| \leqslant \frac{O(1)}{|\gamma|} + O(1).$$

Lemma 6.4. *Suppose $0 < \gamma < 2\pi$ and the function $G(x)$ satisfies $|G'(x)| = \frac{O(1)}{1+x}$. Then we have for $k = -1, 0, 1$,*

$$\left| \int_0^L \mathrm{e}^{2\pi \mathrm{i} kx} \ \frac{\sin(\gamma x + G(x))}{1+x} \, \mathrm{d}x \right| \leqslant O(1) \log \ \gamma^{-1} + O(1) \log (2\pi - \gamma)^{-1} + O(1),$$

and for $k \in \mathbb{Z} \backslash \{-1, 0, 1\}$,

$$\left| \int_0^L \mathrm{e}^{2\pi \mathrm{i} kx} \ \frac{\sin(\gamma x + G(x))}{1+x} \, \mathrm{d}x \right| = O(1).$$

Proof. By the trigonometric identity, one has

$$\begin{aligned} 2 \ \mathrm{e}^{2\pi \mathrm{i} kx} \ \sin(\gamma x + G(x)) &= \sin(\gamma x + 2\pi kx + G(x)) + \sin(\gamma x - 2\pi kx + G(x)) \\ &\quad + \mathrm{i} \ \cos(2\pi k - \gamma x - G(x)) - \mathrm{i} \ \cos(2\pi k + \gamma x + G(x)) \end{aligned}$$

Now the proof follows from lemmas 6.2 and 6.3. □

Denote by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Theorem 6.5. *Suppose $f \in L^2(\mathbb{T})$. Then the following estimates hold*

$$\left| \int_0^L f(x) \frac{\cos \ 4\theta(x, E)}{1+x} \, \mathrm{d}x \right| = O(1), \tag{70}$$

and

$$\left| \int_0^L f(x) \frac{\sin \ 2\theta(x, E_1) \ \sin \ 2\theta(x, E_2)}{1+x} \, \mathrm{d}x \right| = O(1) \log \ \frac{1}{|E_1 - E_2|} + O(1), \tag{71}$$

where $O(1)$ only depends on I, B, f and V_0.

Proof. We give the proof of (71) first. By (68) and (62), we obtain the differential equations of $\theta(x, E_1)$ and $\theta(x, E_2)$,

$$\theta'(x, E_1) = \gamma'(x, E_1) + \frac{O(1)}{1+x}, \tag{72}$$

and

$$\theta'(x, E_2) = \gamma'(x, E_2) + \frac{O(1)}{1+x}. \tag{73}$$

By (33) and (63), we have

$$\gamma(x,E) = k(E)x + \eta(x,E), \tag{74}$$

where $\eta(x,E) \bmod 2\pi$ is a function that is 1-periodic in x.

By basic trigonometry,

$$-2\ \sin\ 2\theta(x,E_1)\sin\ 2\theta(x,E_2) = \cos(2\theta(x,E_1)+2\theta(x,E_2)) - \cos(2\theta(x,E_1)-2\theta(x,E_2)), \tag{75}$$

it suffices to bound

$$\int_0^L f(x)\frac{\cos(2\theta(x,E_1)\pm 2\theta(x,E_2))}{1+x}\,\mathrm{d}x.$$

Without loss of generality, we only bound

$$\int_0^L f(x)\frac{\cos(2\theta(x,E_1)-2\theta(x,E_2))}{1+x}\,\mathrm{d}x. \tag{76}$$

By (72)–(74), we have

$$\frac{\mathrm{d}}{\mathrm{d}x}([\theta(x,E_1)-\eta(x,E_1)]-[\theta(x,E_2)-\eta(x,E_2)] = k(E_1)-k(E_2)+\frac{O(1)}{1+x}. \tag{77}$$

Let

$$\tilde{\theta}(x,E) = \theta(x,E) - \eta(x,E).$$

By trigonometry again, one has

$$\begin{aligned}\cos(2\theta(x,E_1)-2\theta(x,E_2)) &= \cos(2\tilde{\theta}(x,E_1)-2\tilde{\theta}(x,E_2)+2\eta(x,E_1)-2\eta(x,E_2))\\ &= \cos(2\eta(x,E_1)-2\eta(x,E_2))\cos(2\tilde{\theta}(x,E_1)-2\tilde{\theta}(x,E_2))\\ &\quad - \sin(2\eta(x,E_1)-2\eta(x,E_2))\sin(2\tilde{\theta}(x,E_1)-2\tilde{\theta}(x,E_2)).\end{aligned}$$

Thus

$$\begin{aligned}&\int_0^L f(x)\frac{\cos(2\theta(x,E_1)-2\theta(x,E_2))}{1+x}\,\mathrm{d}x\\ &= \int_0^L f(x)\frac{\cos(2\eta(x,E_1)-2\eta(x,E_2))\cos(2\tilde{\theta}(x,E_1)-2\tilde{\theta}(x,E_2))}{1+x}\,\mathrm{d}x\\ &\quad - \int_0^L f(x)\frac{\sin(2\eta(x,E_1)-2\eta(x,E_2))\sin(2\tilde{\theta}(x,E_1)-2\tilde{\theta}(x,E_2))}{1+x}\,\mathrm{d}x.\end{aligned}$$

Without loss of generality, we only give the estimate of

$$\int_0^L f(x)\frac{\sin(2\eta(x,E_1)-2\eta(x,E_2))\sin(2\tilde{\theta}(x,E_1)-2\tilde{\theta}(x,E_2))}{1+x}\,\mathrm{d}x. \tag{78}$$

We proceed by Fourier expansion of $f(x)\sin(2\eta(x,E_1)-2\eta(x,E_2))$ (1-periodic function) and obtain that

$$f(x)\sin(2\eta(x,E_1)-2\eta(x,E_2))=\frac{c_0}{2}+\sum_{k=1}^{\infty}c_k\ \cos(2\pi kx)+d_k\ \sin(2\pi kx). \tag{79}$$

By (79) and (78), we obtain

$$\begin{aligned}(78)=&\int_0^L\frac{c_0}{2}\ \frac{\sin(2\tilde{\theta}(x,E_1)-2\tilde{\theta}(x,E_2))}{1+x}\ \mathrm{d}x\\&+\sum_{k=1}^{\infty}c_k\ \cos(2\pi kx)\frac{\sin(2\tilde{\theta}(x,E_1)-2\tilde{\theta}(x,E_2))}{1+x}\ \mathrm{d}x\\&+\sum_{k=1}^{\infty}d_k\ \sin(2\pi kx)\frac{\sin(2\tilde{\theta}(x,E_1)-2\tilde{\theta}(x,E_2))}{1+x}\ \mathrm{d}x.\end{aligned} \tag{80}$$

Since $k(E_1),k(E_2)\in(0,\frac{\pi}{2})$ or $k(E_1),k(E_2)\in(\frac{\pi}{2},\pi)$ depending on either $I\subset(a_n,c_n)$ or $I\subset(c_n,b_n)$, and $k(E_1)\neq k(E_2)$, we have

$$0<|k(E_1)-k(E_2)|<\frac{\pi}{2}. \tag{81}$$

Since $f\in L^2(\mathbb{T})$, one has $\sum c_k^2+d_k^2<\infty$. Now (71) follows from lemma 6.4, (80) and (81).

The proof of (70) is similar to the estimate of (76). We omit the details.

□

7. Spectral analysis of Schrödinger operators with eventually periodic potentials

In this section, we establish the spectral measure with eventually periodic potentials in terms of the Prüfer variables, which is likely known. However, we did not find this in the literature. We thus present a calculation. See [8] for a calculation for the eventually periodic Jacobi operators.

For $L>0$, let $V_L(x)=V(x)\chi_{[0,L]}(x)$. Let μ_L be the spectral measure of the operator $-D^2+V_0+V_L$.

Theorem 7.1. *Let $u(x,E)$ be the solution of $-u''+V_0u+Vu=Eu$ with initial conditions $u(0)=0$ and $u'(0)=1$. Then the following formula holds,*

$$\frac{\mathrm{d}\mu_L(E)}{\mathrm{d}E}=\frac{2}{\pi|W(\overline{\varphi},\varphi)|}\frac{1}{R^2(L,E)} \tag{82}$$

for $E\in S$.

Proof. For $E\in S$ and $\varepsilon\geqslant 0$, let $z=E+i\varepsilon$. By the Floquet theory, $-u'+V_0u=zu$ has two linearly independent solutions:

$$\varphi_1(x,z)=J_1(x,z)\mathrm{e}^{\mathrm{i}(\tilde{k}(z)+\mathrm{i}\tau(z))x},\qquad \varphi_2(x,z)=J_2(x,z)\mathrm{e}^{-\mathrm{i}(\tilde{k}(z)+\mathrm{i}\tau(z))x}, \tag{83}$$

where $J_1(x,z)$ and $J_2(x,z)$ are 1-periodic, $\tilde{k}(z)\in[0,\pi]$ and $\tau(E+i\varepsilon)\neq 0$ for any $\varepsilon>0$.

If $\tau(E+i\varepsilon)>0$ for any $\varepsilon>0$, then $\tilde{k}(E)=k(E)$, $\varphi_1(x,E)=\varphi(x,E)$ (up to a constant) and $J_1(x,z)=J(x,E)$ (up to a constant).

If $\tau(E+i\varepsilon)<0$ for any $\varepsilon>0$, then $\tilde{k}(E)=-k(E)$, $\varphi_1(x,E)=\bar{\varphi}(x,E)$ (up to a constant) and $J_1(x,z)=\bar{J}(x,E)$ (up to a constant).

Without loss of generality, assume

$$\Im\tau(E+i\varepsilon)>0 \quad \text{for any } \varepsilon>0.$$

Define $\tilde{u}(x,z)=J_1(x,z)\mathrm{e}^{\mathrm{i}(\tilde{k}(z)+\mathrm{i}\tau(z))x}=\varphi_1(x,z)$ for $x\geqslant L$ and extend $\tilde{u}(x,z)$ to $0\leqslant x\leqslant L$ by solving equation

$$-\tilde{u}''(x,z)+(V_0(x)+V_L(x)-z)\tilde{u}(x,z)=0.$$

Since $\tilde{u}(x,z)\in L^2(\mathbb{R}_+)$, by basic facts of spectral theory (we refer the readers to [38] and references therein for details), we have

$$m(z)=\frac{\tilde{u}'(0,z)}{\tilde{u}(0,z)},$$

and

$$\frac{\mathrm{d}\mu_L}{\mathrm{d}E}=\frac{1}{\pi}\lim_{\varepsilon\to 0+}\Im m(E+i\varepsilon). \tag{84}$$

Let $T(z)$ be the transfer matrix of H_0+V_L from 0 to L, that is

$$T(z)\begin{pmatrix}\phi(0)\\ \phi'(0)\end{pmatrix}=\begin{pmatrix}\phi(L)\\ \phi'(L)\end{pmatrix}$$

for any solution ϕ of $(-D^2+V_0+V_L)\phi=z\phi$.

Denote by

$$T(z)=\begin{pmatrix}a(z) & b(z)\\ c(z) & d(z)\end{pmatrix}.$$

Clearly,

$$\begin{aligned}\begin{pmatrix}\tilde{u}(0,z)\\ \tilde{u}'(0,z)\end{pmatrix}&=\begin{pmatrix}a(z) & b(z)\\ c(z) & d(z)\end{pmatrix}^{-1}\begin{pmatrix}\tilde{u}(L,z)\\ \tilde{u}'(L,z)\end{pmatrix}\\ &=\begin{pmatrix}d(z) & -b(z)\\ -c(z) & a(z)\end{pmatrix}\begin{pmatrix}\tilde{u}(L,z)\\ \tilde{u}'(L,z)\end{pmatrix}.\end{aligned}$$

Direct computation implies that (using $ad-bc=1$)

$$\begin{aligned}\lim_{\varepsilon\to 0+}\Im\, m(E+i\varepsilon)&=\Im\,\frac{a\varphi_1'(L,E)-c\varphi_1(L,E)}{d\varphi_1(L,E)-b\varphi_1'(L,E)}\\ &=\Im\,\frac{a\varphi'(L,E)-c\varphi(L,E)}{d\varphi(L,E)-b\varphi'(L,E)}\\ &=\frac{A_1\ \sin\ B_1}{(d-bA_1\ \cos\ B_1)^2+(bA_1\ \sin\ B_1)^2},\end{aligned} \tag{85}$$

where $A_1>0$ and B_1 are defined by

$$\frac{\varphi'(L)}{\varphi(L)}=A_1\ \mathrm{e}^{\mathrm{i}B_1}. \tag{86}$$

By the assumption of theorem 7.1, one has

$$\begin{aligned}\begin{pmatrix} u(L,E) \\ u'(L,E) \end{pmatrix} &= T(E)\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} \\ &= T(E)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.\end{aligned} \tag{87}$$

Let

$$\rho(L)\varphi(L) = A_2\ \mathrm{e}^{\mathrm{i}B_2} \quad \text{and} \quad A_2 > 0. \tag{88}$$

By (65) and (87), we have

$$\begin{aligned}\begin{pmatrix} b \\ d \end{pmatrix} &= \Im A_2\ \mathrm{e}^{\mathrm{i}B_2}\begin{pmatrix} A_1\ \mathrm{e}^{\mathrm{i}B_1} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} A_2\ \sin\ B_2 \\ A_1A_2\ \sin(B_1+B_2) \end{pmatrix}.\end{aligned} \tag{89}$$

Therefore,

$$b = A_2\ \sin\ B_2; d = A_1A_2\ \sin(B_1+B_2). \tag{90}$$

By (85) and (90),

$$\lim_{\varepsilon\to 0+} \Im m(E+i\varepsilon) = \frac{1}{A_1A_2^2\ \sin\ B_1}. \tag{91}$$

It is easy to see that (see p 295 in [17] for example)

$$|\varphi|^2\ \Im\frac{\varphi'}{\varphi} = \frac{|W(\overline{\varphi},\varphi)|}{2}. \tag{92}$$

By (86), (88), (91) and (92), one has

$$\begin{aligned}\lim_{\varepsilon\to 0+}\Im m(E+i\varepsilon) &= \frac{1}{|\rho(L)\varphi(L)|^2\ \Im\left(\frac{\varphi'(L)}{\varphi(L)}\right)} \\ &= \frac{2}{|W(\overline{\varphi},\varphi)|}\frac{1}{R(L,E)^2}.\end{aligned} \tag{93}$$

Now the theorem follows from (84) and (93). □

8. Proof of theorem 1.4

In this section, we indicate the dependence of parameters explicitly except for V_0, since V_0 is fixed all the time.

Let $L = \epsilon^{-1-\sigma}$ with $\sigma > 0$. Let $C_1 = C_1(B, I)$, which will be determined later.

We say a subset $A \subset I$ is (ϵ, N) separate if the following two conditions hold:

For any $E \in A$,

$$\left| \int_0^L V(x) \frac{\sin\ 2\theta(x, E)}{\gamma'(x, E)} \, \mathrm{d}x \right| \geqslant (1 - \beta) C_1(B, I) \log\ \epsilon^{-1}. \tag{94}$$

For any $E_1, E_2 \in A$ and $E_1 \neq E_2$,

$$|k(E_1) - k(E_2)| \geqslant \epsilon^{1/N^2}. \tag{95}$$

Lemma 8.1 [[16], lemma 4.4]. *Let $\{e_i\}_{i=1}^N$ be a set of unit vectors in a Hilbert space $\mathcal{H}$ so that*

$$\alpha = N \sup_{i \neq j} |\langle e_i, e_j \rangle| < 1.$$

Then for any $g \in \mathcal{H}$,

$$\sum_{i=1}^N |\langle g, e_i \rangle|^2 \leqslant (1 + \alpha) \|g\|^2. \tag{96}$$

Theorem 8.2. *There exist $\epsilon_1(B, I, \sigma, \beta) > 0$ and $C(B, I, \sigma, \beta)$ such that for any $\epsilon < \epsilon_1$ and $N \geqslant C(B, I, \sigma, \beta)$, any (ϵ, N) separate set A satisfies $\#A \leqslant N$.*

Proof. We consider the Hilbert space

$$\mathcal{H} = L^2((0, L), (1 + x)\mathrm{d}x).$$

In $\mathcal{H}$, by (62) we have

$$\|V\|_{\mathcal{H}}^2 \leqslant B^2\ \log(1 + L). \tag{97}$$

Let

$$e_i(x) = \frac{1}{\sqrt{A_i}} \frac{\sin\ 2\theta(x, E_i)}{\gamma'(x, E_i)(1 + x)} \chi_{[0,L]}(x), \tag{98}$$

where A_i is chosen so that e_i is an unit vector in $\mathcal{H}$. Direct computation implies

$$\begin{aligned} A_i &= \int_0^{B_j} \frac{\sin^2\ 2\theta(x, E_i)}{|\gamma'(x, E_i)|^2 (1 + x)} \, \mathrm{d}x \\ &= \int_0^L \frac{1}{2|\gamma'(x, E_i)|^2 (1 + x)} \mathrm{d}x - \int_0^L \frac{\cos\ 4\theta(x, E_i)}{|\gamma'(x, E_i)|^2 (1 + x)} \, \mathrm{d}x. \end{aligned} \tag{99}$$

By (70), one has

$$\left| \int_0^L \frac{\cos\ 4\theta(x, E_i)}{|\gamma'(x, E_i)|^2 (1 + x)} \, \mathrm{d}x \right| = O(1). \tag{100}$$

Direct computation shows that

$$\begin{aligned} \int_0^L \frac{1}{|\gamma'(x, E_i)|^2 (1 + x)} \, \mathrm{d}x &= O(1) + \sum_{n=0}^{L-1} \int_n^{n+1} \frac{1}{|\gamma'(x, E_i)|^2 (1 + n)} \mathrm{d}x \\ &= O(1) + \Gamma(E_i) \log\ L, \end{aligned} \tag{101}$$

where $\Gamma(E) = \int_n^{n+1} \frac{1}{|\gamma'(x,E)|^2}\, \mathrm{d}x$ (does not depend on n, e.g., [[17], proposition 2.1]).

By (99)–(101), we have

$$A_i = \frac{1}{2}\Gamma(E_i)\log\ L + O(1). \tag{102}$$

We should mention that $O(1)$ in (100)–(102) only depend on B and I.

By (71) and (98), we have

$$|\langle e_i, e_j\rangle \leqslant \frac{2}{1+\sigma}C(I,B)N^{-2} + \frac{C(I,B)}{\log\ \epsilon^{-1}}. \tag{103}$$

The first condition (94) implies

$$|\langle V, e_i\rangle|^2 \geqslant \frac{C_1^2}{1+\sigma}\ \log\ \epsilon^{-1} - C(I,B). \tag{104}$$

By (96) and (103), one has

$$\sum_{i=1}^{N} |\langle V, ei\rangle_{\mathcal{H}}|^2 \leqslant \left(1 + \frac{2}{1+\sigma}C(I,B)N^{-1} + \frac{NC(I,B)}{\log\ \epsilon^{-1}}\right)\|V\|_{\mathcal{H}}. \tag{105}$$

By (97), (104) and (105), we have

$$N\left(\frac{C_1^2(1-\beta)^2}{1+\sigma}\ \log\ \epsilon^{-1} - C(I,B)\right) \leqslant \left(1 + \frac{2}{1+\sigma}C(I,B)N^{-1} + \frac{NC(I,B)}{\log\ \epsilon^{-1}}\right) \times B^2(1+\sigma)\log\ \epsilon^{-1}.$$

This implies theorem 8.2. □

Proof of theorem 1.4. Once we have theorems 7.1 and 8.2, theorem 1.4 can be proved by the arguments in [15] (also see [23]). We omit the details here. □

Acknowledgments

I would like to thank Barry Simon and Gunter Stolz for some useful discussions. I also want to thank the referee and editor, whose comments led to improvements of the text. This research was supported by NSF DMS-1700314/2015683, DMS-2000345 and the Southeastern Conference (SEC) Faculty Travel Grant 2020–2021.

Appendix A

Proof of (40). By (36) and (40), it suffices to show that

$$\frac{\mathrm{i}}{2\Im(\varphi\overline{\varphi}')}V\overline{\varphi}^2\ \exp(-2\mathrm{i}p) = \frac{-\mathrm{i}V}{2\ \Re\phi'}\ \mathrm{e}^{-\mathrm{i}h}. \tag{106}$$

By the definition, one has

$$\phi' = -i\frac{\varphi'}{\varphi}.$$

Direct computation implies

$$\frac{|\varphi|^2}{\Im(\varphi\bar{\varphi}')} = -\frac{1}{\Re\phi'}. \tag{107}$$

By the definitions of h and p, we have

$$\begin{aligned}\frac{\mathrm{i}V}{2\,\Re\phi'}\,\mathrm{e}^{-\mathrm{i}h} &= \frac{\mathrm{i}V}{2\,\Re\phi'}\ \exp\left(-\mathrm{i}2\,\Re\phi + \mathrm{i}\int_0^x \frac{V(t)}{\Re\,\phi'(t,E)}\mathrm{d}t\right)\\ &= -\mathrm{i}V\frac{|\varphi|^2}{2\,\Im(\varphi\bar{\varphi}')}\ \exp(-\mathrm{i}2\,\Re\,\phi)\exp\left(-\mathrm{i}\int_0^x \frac{V(t)|\varphi(t,E)|^2}{\Im(\varphi\bar{\varphi}')}\mathrm{d}t\right)\\ &= -\mathrm{i}V\frac{|\varphi|^2}{2\,\Im(\varphi\bar{\varphi}')}\ \exp(-\mathrm{i}2\,\Re\,\phi)\ \exp(-2\mathrm{i}p)\\ &= -\mathrm{i}V\frac{\bar{\varphi}^2}{2\,\Im(\varphi\bar{\varphi}')}\ \exp(-2\mathrm{i}p).\end{aligned}$$

It implies (106) and hence (40). □

Proof of (32). For $k \in \mathbb{Z}_+$, denote by

$$f_k = \int_{k-1}^{k} |f(x)|\ \mathrm{d}x.$$

Then

$$\|f\|^2_{\ell^2(L^1)} = \sum_{k=1}^{\infty} f_k^2. \tag{108}$$

Direct computations imply

$$\begin{aligned}\int_{\mathbb{R}^2_+} \frac{|f(x)f(y)|}{1+|x-y|^2}\ \mathrm{d}x\ \mathrm{d}y &= O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{f_m f_n}{1+|m-n|^2}\\ &= O(1)\sum_{n=1}^{\infty}|f_n|^2,\end{aligned} \tag{109}$$

where the second equality holds by the Young's convolution inequality. Now (32) follows from (108) and (109). □

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References

[1] Christ M and Kiselev A 1998 Absolutely continuous spectrum for one-dimensional Schrödinger operators with slowly decaying potentials: some optimal results *J. Am. Math. Soc.* **11** 771–97

[2] Christ M and Kiselev A 2001 Maximal functions associated to filtrations *J. Funct. Anal.* **179** 409–25
[3] Christ M and Kiselev A 2001 WKB and spectral analysis of one-dimensional Schrödinger operators with slowly varying potentials *Commun. Math. Phys.* **218** 245–62
[4] Christ M and Kiselev A 2001 WKB asymptotic behavior of almost all generalized eigenfunctions for one-dimensional Schrödinger operators with slowly decaying potentials *J. Funct. Anal.* **179** 426–47
[5] Deift P and Killip R 1999 On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials *Commun. Math. Phys.* **203** 341–7
[6] Denisov S A and Kiselev A 2007 Spectral properties of Schrödinger operators with decaying potentials *Spectral Theory and Mathematical Physics: a Festschrift in Honor of Barry Simon's 60th Birthday* (*Proceedings of Symposium Pure Mathematics* vol 76) (Providence, RI: American Mathematical Society) pp 565–89
[7] Dollard J D and Friedman C N 1978 Asymptotic behavior of solutions of linear ordinary differential equations *J. Math. Anal. Appl.* **66** 394–8
[8] Kaluzhny U and Shamis M 2012 Preservation of absolutely continuous spectrum of periodic Jacobi operators under perturbations of square-summable variation *Constr. Approx.* **35** 89–105
[9] Kato T 1959 Growth properties of solutions of the reduced wave equation with a variable coefficient *Commun. Pure Appl. Math.* **12** 403–25
[10] Killip R 2002 Perturbations of one-dimensional Schrödinger operators preserving the absolutely continuous spectrum *Int. Math. Res. Not.* 2029–61
[11] Killip R 2007 Spectral theory via sum rules *Spectral Theory and Mathematical Physics: a Festschrift in Honor of Barry Simon's 60th Birthday* (*Proceedings Symposium Pure Mathematics* vol. 76) (Providence, RI: American Mathematical Society) pp 907–30
[12] Kim A and Kiselev A 2009 Absolutely continuous spectrum of discrete Schrödinger operators with slowly oscillating potentials *Math. Nachr.* **282** 552–68
[13] Kiselev A 1996 Absolutely continuous spectrum of one-dimensional Schrödinger operators and Jacobi matrices with slowly decreasing potentials *Commun. Math. Phys.* **179** 377–99
[14] Kiselev A 1998 And a.e. convergence of integral operators *Duke Math. J.* **94** 619–46
[15] Kiselev A 2005 Imbedded singular continuous spectrum for Schrödinger operators *J. Am. Math. Soc.* **18** 571–603
[16] Kiselev A, Last Y and Simon B 1998 Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators *Commun. Math. Phys.* **194** 1–45
[17] Kiselev A, Remling C and Simon B 1999 Effective perturbation methods for one-dimensional Schrödinger operators *J. Diff. Equ.* **151** 290–312
[18] Kriecherbauer T and Remling C 2001 Finite gap potentials and WKB asymptotics for one-dimensional Schrödinger operators *Commun. Math. Phys.* **223** 409–35
[19] Kuchment P 2016 An overview of periodic elliptic operators *Bull. Am. Math. Soc.* **53** 343–414
[20] Kurasov P and Naboko S 2007 Wigner–von Neumann perturbations of a periodic potential: spectral singularities in bands *Math. Proc. Camb. Phil. Soc.* **142** 161–83
[21] Last Y 1996 Quantum dynamics and decompositions of singular continuous spectra *J. Funct. Anal.* **142** 406–45
[22] Last Y and Simon B 2006 The essential spectrum of Schrödinger, Jacobi, and CMV operators *J. Anal. Math.* **98** 183–220
[23] Liu W 2019 Absence of singular continuous spectrum for perturbed discrete Schrödinger operators *J. Math. Anal. Appl.* **472** 1420–9
[24] Liu W 2019 The asymptotical behaviour of embedded eigenvalues for perturbed periodic operators *Pure Appl. Funct. Anal.* **4** 589–602
[25] Liu W 2020 Irreducibility of the Fermi variety for discrete periodic Schrödinger operators and embedded eigenvalues (arXiv:2006.04733)
[26] Liu W and Ong D C Sharp spectral transition for eigenvalues embedded into the spectral bands of perturbed periodic operators *J. Anal. Math.* (at press)
[27] Lukic M and Ong D C 2015 Wigner–von Neumann type perturbations of periodic Schrödinger operators *Trans. Am. Math. Soc.* **367** 707–24
[28] Muscalu C, Tao T and Thiele C 2003 A counterexample to a multilinear endpoint question of Christ and Kiselev *Math. Res. Lett.* **10** 237–46
[29] Naboko S and Simonov S 2012 Zeroes of the spectral density of the periodic Schrödinger operator with Wigner–von Neumann potential *Math. Proc. Camb. Phil. Soc.* **153** 33–58

[30] Naboko S N 1986 On the dense point spectrum of Schrödinger and Dirac operators *Teoret. Mat. Fiz.* **68** 18–28

[31] Remling C 1998 The absolutely continuous spectrum of one-dimensional Schrödinger operators with decaying potentials *Commun. Math. Phys.* **193** 151–70

[32] Remling C 2000 Bounds on embedded singular spectrum for one-dimensional Schrödinger operators *Proc. Am. Math. Soc.* **128** 161–71

[33] Remling C 2000 Schrödinger operators with decaying potentials: some counterexamples *Duke Math. J.* **105** 463–96

[34] Rofe-Beketov F S 1964 A finiteness test for the number of discrete levels which can be introduced into the gaps of the continuous spectrum by perturbations of a periodic potential *Dokl. Akad. Nauk SSSR* **156** 515–8

[35] Simon B 1996 Bounded eigenfunctions and absolutely continuous spectra for one-dimensional Schrödinger operators *Proc. Am. Math. Soc.* **124** 3361–9

[36] Simon B 1997 Some Schrödinger operators with dense point spectrum *Proc. Am. Math. Soc.* **125** 203–8

[37] Simon B 2000 Schrödinger operators in the twenty-first century *Mathematical Physics* (London: Imperial College Press) pp 283–8

[38] Simon B 2004 Analogs of the *m*-function in the theory of orthogonal polynomials on the unit circle *J. Comput. Appl. Math.* **171** 411–24

[39] Stolz G 1991 On the absolutely continuous spectrum of perturbed Sturm–Liouville operators *J. Reine Angew. Math.* **416** 1–23

[40] Stolz G 1992 Bounded solutions and absolute continuity of Sturm–Liouville operators *J. Math. Anal. Appl.* **169** 210–28

[41] Teschl G 2014 *Mathematical Methods in Quantum Mechanics* (*Graduate Studies in Mathematics* vol. 157) 2nd edn (Providence, RI: American Mathematical Society) With applications to Schrödinger operators

[42] von Neumann J and Wigner E 1929 Uber merkwürdige diskrete Eigenwerte; Uber das Verhalten von Eigenwerten bei adiabatischen Prozessen *Phys. Zeitschrift* **30** 467470

[43] Weidmann J 1967 Zur spektraltheorie von Sturm–Liouville–Operatoren *Math. Z.* **98** 268–302