Isoperimetry, Scalar Curvature, and Mass in Asymptotically Flat Riemannian 3-Manifolds

OTIS CHODOSH
University of Cambridge

MICHAEL EICHMAIR
University of Vienna

YUGUANG SHI
Peking University

AND

HAOBIN YU
Peking University

We dedicate this paper to Richard Schoen on the occasion of his 70^{th} birthday.

Abstract

Let (M, g) be an asymptotically flat Riemannian 3-manifold with nonnegative scalar curvature and positive mass. We show that each leaf of the canonical foliation of the end of (M, g) through stable constant mean curvature spheres encloses more volume than any other surface of the same area.

Unlike all previous characterizations of large solutions of the isoperimetric problem, we need no asymptotic symmetry assumptions beyond the optimal conditions for the positive mass theorem. This generality includes examples where global uniqueness of the leaves of the canonical foliation as stable constant mean curvature spheres fails dramatically.

Our results here resolve a question of G. Huisken on the isoperimetric content of the positive mass theorem.

1 Introduction

A complete Riemannian 3-manifold (M, g) is said to be *asymptotically flat* if there is a nonempty compact subset $K \subset M$ and a diffeomorphism

$$M \setminus K \cong \{x \in \mathbb{R}^3 : |x| > 1/2\} \tag{1.1}$$

with

$$g_{ij} = \delta_{ij} + \sigma_{ij} \quad \text{where } \partial_I \sigma_{ij} = O(|x|^{-\tau-|I|}) \text{ as } |x| \to \infty \tag{1.2}$$

for some $\tau > 1/2$ and all multi-indices I of order $|I| \leq 2$. We also require that the scalar curvature of (M, g) be integrable. If the boundary of M is nonempty,

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we require that it be minimal. We also require that there be no closed minimal surfaces in the interior of M. Given $\rho > 1$, we use S_ρ to denote the surface in M that corresponds to the centered coordinate sphere $\{x \in \mathbb{R}^3 : |x| = \rho\}$ in the *chart at infinity* (1.1). We let B_ρ denote the bounded open region in M that is enclosed by S_ρ.

The ADM-mass (after R. Arnowitt, S. Deser, and C. W. Misner [1]) of such an asymptotically flat manifold (M, g) is given by

$$m_{\text{ADM}} = \lim_{\rho\to\infty} \frac{1}{16\pi\rho} \int_{\{|x|=\rho\}} \sum_{i,j=1}^{3} (\partial_i g_{ij} - \partial_j g_{ii}) x^j$$

where integration is with respect to the Euclidean metric. R. Bartnik [2] has shown that this quantity is independent of the particular choice of chart at infinity (1.1). The fundamental *positive mass theorem*, proven by R. Schoen and S.-T. Yau [39] using minimal surface techniques and then by E. Witten [44] using spinors, asserts that for (M, g) asymptotically flat with nonnegative scalar curvature, we have that $m_{\text{ADM}} \geq 0$ with equality if and only if (M, g) is flat Euclidean space.

Let $V > 0$. Consider

$$\mathcal{R}_V = \{\Omega : \Omega \subset M \text{ is a compact region with } \partial M \subset \partial\Omega \text{ and } \text{vol}(\Omega) = V\}$$

and let

$$A(V) = -\operatorname{area}(\partial M) + \inf\{\operatorname{area}(\partial\Omega) : \Omega \in \mathcal{R}_V\}. \tag{1.3}$$

When the scalar curvature of (M, g) is nonnegative, a result of the third-named author [40] combined with an observation in appendix K of [7] shows that there is a region $\Omega_V \in \mathcal{R}_V$ that achieves the infimum in (1.3). The proof that such *isoperimetric regions* exist in (M, g) is indirect and offers no clue as to the position of these regions. The main result of this paper is to show that if (M, g) is not Euclidean space and if $V > 0$ is sufficiently large, then Ω_V is bounded by the horizon ∂M and a stable constant mean curvature surface that belongs to the *canonical foliation* (see Appendix D) of the end of M through stable constant mean curvature spheres.

THEOREM 1.1. *Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat at rate $\tau > 1/2$ and has nonnegative scalar curvature and positive mass. There is $V_0 > 0$ with the following property. Let $V \geq V_0$. There is a region $\Omega_V \in \mathcal{R}_V$ such that*

$$\operatorname{area}(\partial\Omega_V) \leq \operatorname{area}(\partial\Omega)$$

for all $\Omega \in \mathcal{R}_V$, with equality only when $\Omega = \Omega_V$. The boundary of Ω_V consists of the horizon ∂M and a leaf of the canonical foliation of the end of M.

In particular, the solution of the isoperimetric problem in (M, g) is *unique* for large volumes.

We have included the assumption that there are are no closed minimal surfaces in the interior of (M, g) in the definition of asymptotically flat. One could omit this

assumption first by replacing (M, g) with the region outside of all such closed minimal surfaces as in [25, lemma 4.1] and observing that the centering mechanisms obtained in the proof of Theorem 1.1 imply that (M, g) satisfies the conclusion of Theorem 1.1 as well.

Our proof of Theorem 1.1 shows that the outer boundary of a large isoperimetric region Ω_V is close to a centered coordinate sphere S_ρ where $V \sim 4\pi\rho^3/3$. From this, uniqueness of Ω_V follows from characterization results for the leaves of the *canonical foliation* that we discuss in Appendix D.

The special case of Theorem 1.1 where (M, g) is asymptotic to Schwarzschild with positive mass, i.e., where instead of (1.2) we have

$$g_{ij} = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} + O(|x|^{-2}) \quad \text{as } |x| \to \infty \tag{1.4}$$

for some $m > 0$ was conjectured by H. Bray [5, p. 44] and G. Huisken and finally settled in joint work [16] by J. Metzger and the second-named author. Their proof develops an ingenious idea of H. Bray [5] for the exact Schwarzschild metric and uses the spherical symmetry in the asymptotic expansion (1.4) in a crucial way. It carries over to higher dimensions [17] and makes no assumption on the scalar curvature.

We recall from, e.g., [32] that the value of the scalar curvature at a given point can be characterized by the isoperimetric deficit of small geodesic balls. Qualitatively, in order to enclose a small *given amount of volume* by a geodesic sphere, *less area* is needed when the sphere is centered at a point of *larger scalar curvature*. In Appendix C, we discuss how the isoperimetric deficit of large solutions of the isoperimetric problem detects the mass of (M, g). Theorem 1.1 expresses the positive mass theorem as a *local to global* transfer of isoperimetry in the small to isoperimetry in the large in a precise way. More importantly, it adds to the short list of geometries and the even shorter list of geometries with no exact symmetries (see appendix H in [17] for an overview), where we can describe the solutions of the isoperimetric problem exactly.

The uniqueness of isoperimetric regions of a given volume in Theorem 1.1 is in stark contrast to the nonuniqueness of stable constant mean curvature spheres of a given area. This nonuniqueness is particularly dramatic in the following example constructed by A. Carlotto and R. Schoen in [8, p. 561].

Example 1.2 ([8]). There is an asymptotically flat Riemannian metric

$$g = g_{ij}\, dx^i \otimes dx^j \quad \text{on } \mathbb{R}^3$$

with nonnegative scalar curvature and positive mass so that $g_{ij} = \delta_{ij}$ on $\mathbb{R}^2 \times (0, \infty)$.

We mention that there are examples of (M, g) asymptotic to Schwarzschild where there are other large stable constant mean curvature spheres than the leaves of the canonical foliation; cf. [6, 12].

We emphasize that the examples constructed in [8] are asymptotically flat of rate $\tau < 1$. They clearly contain an abundance of sequences of *stable constant mean spheres* that drift off in (M, g) while their area diverges. On the way of proving Theorem 1.1, we show in Section 2 that, quite generally, the isoperimetric defect along any such sequence tends to 0.

THEOREM 1.3. *Let (M, g) be a Riemannian 3-manifold with nonnegative scalar curvature that is asymptotically flat of rate $\tau > 3/4$. There is $\rho_0 > 1$ with the following property. For $k = 1, 2, \ldots$, let $\Sigma_k = \partial\Omega_k$ be connected stable constant mean curvature spheres such that $\Omega_k \cap B_{\rho_0} = \varnothing$ and $\operatorname{area}(\Sigma_k) \to \infty$ as $k \to \infty$. Then*

$$\lim_{k\to\infty} \frac{2}{\operatorname{area}(\Sigma_k)} \left(\operatorname{vol}(\Omega_k) - \frac{\operatorname{area}(\Sigma_k)^{3/2}}{6\sqrt{\pi}} \right) = 0.$$

1.1 Outline of Our Proofs of Theorem 1.1

We now give two independent proofs of Theorem 1.1 that have different merits.

The Hawking mass of stable constant mean curvature spheres in maximal initial data (M, g) for spacetimes that satisfy the dominant energy condition has been affirmed by D. Christodoulou and S.-T. Yau [13] as a *quasi-local* measure of the gravitational field. Indeed, they prove that the Hawking mass of such surfaces is nonnegative in this case. Recall that stable constant mean curvature surfaces arise as boundaries of isoperimetric regions. The potential for the development of quasi-local mass of the *isoperimetric defect* from Euclidean space

$$m_{\mathrm{iso}}(\Omega) = \frac{2}{\operatorname{area}(\partial\Omega)} \left(\operatorname{vol}(\Omega) - \frac{\operatorname{area}(\partial\Omega)^{3/2}}{6\sqrt{\pi}} \right)$$

of compact regions $\Omega \subset M$ has been observed by G. Huisken; cf., e.g., [23, 24]. For example, the ADM-mass of the initial data (and thus the spacetime evolving from it) is encoded in the isoperimetric profile of (M, g). In fact, as we recall in Appendix C,

$$m_{\mathrm{ADM}} = \lim_{V\to\infty} \frac{2}{A(V)} \left(V - \frac{A(V)^{3/2}}{6\sqrt{\pi}} \right).$$

In particular, the isoperimetric defect $m_{\mathrm{iso}}(\Omega_V)$ of isoperimetric regions Ω_V of large volume $V > 0$ *must be* close to m_{ADM}. Now, as we recall in Appendix B, large isoperimetric regions Ω_V in (M, g) look like Euclidean unit balls $B_1(\xi) \subset \mathbb{R}^3$ with center at $\xi \in \mathbb{R}^3$ when scaled by their volume in the chart at infinity (1.1).

When $|\xi| > 1$, we can use a delicate integration by parts inspired by the work of X.-Q. Fan, P. Miao, L.-F. Tam, and the third-named author in [18] to relate the isoperimetric defect of Ω_V to the *mass integral* of its boundary. Using that the scalar curvature is integrable, one sees that the isoperimetric defect of such a region is close to 0 rather than m_{ADM}, a contradiction.

When $|\xi| = 1$, the argument becomes much harder. First, we use the solution of a conjecture of R. Schoen by the first- and the second-named authors and discussed here in Appendix B to ensure that either Ω_V encloses the center of (M, g)

or that the unique large component Ω_V^∞ of Ω_V is far from the center, with the distance from the center diverging as $V \to \infty$. In the latter case, we combine the Christodoulou-Yau Hawking mass estimate with the monotonicity of the Hawking mass towards m_{ADM} proven by G. Huisken and T. Ilmanen [25] to obtain a crucial analytic estimate for $\partial\Omega_V^\infty$. This estimate allows us to compare the isoperimetric deficit of Ω_V^∞ with that of a large outlying coordinate sphere to conclude as before that it is *too* Euclidean to be an isoperimetric region in (M, g).

The case where $|\xi| < 1$ is covered by the uniqueness of the leaves of the canonical foliation.

The proof of Theorem 1.1 outlined above is carried out in Sections 2 and 3. However, it only works when we impose the stronger decay assumptions (3.1) and (3.2) on (M, g). Incidentally, the decay assumptions stated in Theorem 1.1 are optimal for the positive mass theorem. We obtain Theorem 1.1 in the stated generality from a completely different line of argument that we develop in Section 4.

In this second proof of Theorem 1.1, we study the mean curvature flow of large isoperimetric surfaces. We prove that, upon appropriate rescaling, the level set flow of such large isoperimetric surfaces converges to the Euclidean flow

$$\{S_{\sqrt{1-4t}}(\xi)\}_{t\in[0,\frac{1}{4})}$$

of $S_1(\xi)$ in $\mathbb{R}^3$. When $\xi \neq 0$, part of this flow will be in a shell-like region that avoids the center of the manifold, which corresponds to the origin in the rescaled picture. We show that the Hawking mass of the surfaces forming this shell is close to 0. Using this, we apply the monotonicity of the *isoperimetric defect from Schwarzschild* discovered by G. Huisken in *two* steps to obtain a contradiction. First, we compare with Schwarzschild of mass m_{ADM} until the time when the surfaces have jumped across the center of (M, g). Then, we compare with Schwarzschild of mass $o(1)m_{\mathrm{ADM}}$ until the surfaces have all but disappeared. In this argument, we only need a very weak characterization of the leaves of the canonical foliation to conclude uniqueness in Theorem 1.1.

This second proof of Theorem 1.1 is effective in that it provides an explicit estimate for the isoperimetric deficit of general large outward area-minimizing regions that are close to balls $B_1(\xi)$ on the scale of their volume. The more analytic, first proof is delicately tuned to large stable constant mean curvature spheres for which it provides very precise information. Furthermore, it allows us to prove Theorem 1.3 on the isoperimetric defect of large, outlying stable constant mean curvature spheres.

After his work was finished, a third proof of Theorem 1.1 was discovered by the fourth-named author [45]. In fact, it is shown in [45] that Theorem 1.1 also holds when the assumption of nonnegative scalar curvature is replaced by a decay condition.

2 Isoperimetric Deficit of Large Outlying Stable CMC Spheres

Throughout this section, we consider a complete Riemannian 3-manifold (M, g) that is asymptotically flat at rate $\tau = 1$ and has nonnegative scalar curvature and positive mass $m_{\mathrm{ADM}} > 0$. We also require the additional decay assumption

$$\partial_I \sigma_{ij} = O(|x|^{-1-|I|}) \quad \text{as } |x| \to \infty$$

for all multi-indices I of order $|I| \leq 3$. The control of the third-order derivatives is used in the proof of Proposition 2.3. This could be avoided there by adapting an important observation of J. Metzger from [31]—the derivatives of the ambient curvature enter Simons' identity in divergence form. We also mention that the results in this section could be generalized to the slower decay $\tau > 3/4$. (This is the threshold for the proof of the key estimate (2.10).) However, as a step in the proof of Theorem 1.1, we require the full strength of the results by S. Ma in [30], which in turn needs the even stronger assumptions (3.1) and (3.2). This is why we restrict the exposition to the present case.

The results proven here will be applied in Section 3 to the study of large isoperimetric regions. Since the methods apply equally well to large stable constant mean curvature surfaces, we consider this more general setting here.

Let $\Sigma = \partial\Omega \subset M$ be a connected stable constant mean curvature surface so that $\Omega \cap B_{\rho_0} = \varnothing$ where $\rho_0 > 1$ is large. The error terms in this section are all with respect to $\mathrm{area}(\Sigma) \to \infty$.

The following result is proven by the first- and second-named authors in section 2 of [11].[1]

LEMMA 2.1. *When* $\rho_0 > 1$ *and* $\mathrm{area}(\Sigma)$ *are sufficiently large, then* Σ *is homeomorphic to a sphere.*

Let $r > 0$ denote the area radius of Σ defined by

$$\mathrm{area}(\Sigma) = 4\pi r^2.$$

We use $H > 0$ to denote the mean curvature of Σ. The second fundamental form of Σ and its trace-free part are denoted by h and $\mathring{h}$, respectively. Any quantity computed with respect to the reference Euclidean metric as opposed to with respect to g will have a bar over it.

Using (E.2) and Corollary E.3, we find

$$r \int_\Sigma |\mathring{h}|^2 d\mu \leq 48\pi(1 + o(1)) m_{\mathrm{ADM}} \tag{2.1}$$

[1] In the original version of this argument, see https://arxiv.org/pdf/1606.04626v2.pdf, the analysis in this section used an a priori assumption that Σ was spherical. This assumption was subsequently justified by combining the results for spherical isoperimetric regions proven here with a continuity argument involving the isoperimetric profile (inspired by ideas from [10]). This continuity argument, while more complicated than the argument from [11, sec. 2], could be useful in other settings.

and

$$2\sqrt{1-2m_{\mathrm{ADM}}/r} \le rH \le 2.$$

From this, we see

$$H - 2/r = O(r^{-2}); \tag{2.2}$$

cf. [10, p. 425]. In conjunction with the results stated in Appendix G, with $\tau = 1$, we obtain

$$\begin{aligned}\int_\Sigma |\overset{\circ}{\bar{h}}|^2_{\bar{g}}\, d\bar{\mu} &= O\Big(\int_\Sigma |\overset{\circ}{h}|^2_g\, d\mu + \int_\Sigma |h|^2_g |x|^{-2}\, d\mu + \int_\Sigma |x|^{-4}\, d\mu\Big)\\ &= O\big(r^{-1} + \rho_0^{-2}\big).\end{aligned}$$

We next recall a consequence of J. Simons' identity for the trace-free part of the second fundamental form.

LEMMA 2.2 (cf. [17, cor. 5.3]). *There is a constant $c > 0$ with the following property. Consider in a Riemannian manifold a two-sided hypersurface with constant mean curvature H and trace-free second fundamental form $\overset{\circ}{h}$. Then*

$$2|\overset{\circ}{h}|^3 + \Delta|\overset{\circ}{h}| \ge -c(H|\overset{\circ}{h}|^2 + H|\mathrm{Rm}| + |\overset{\circ}{h}||\mathrm{Rm}| + |\nabla \mathrm{Rm}|) \tag{2.3}$$

holds weakly, where Δ is the induced Laplace-Beltrami operator and where Rm, ∇ Rm *are the ambient Riemannian curvature tensor and its first covariant derivative, both restricted along the surface.*

PROPOSITION 2.3. *There is a constant $c > 0$ depending only on (M, g) such that*

$$|\overset{\circ}{h}(x)| \le c\, r^{-5/4}$$

for all $x \in \Sigma$ such that $2|x| \ge r^{3/4}$.

PROOF. Assume that the assertion fails with $c = k$ along a sequence of regions Ω_k with area radius $r_k \to \infty$ and at points $x_k \in \Sigma_k = \partial\Omega_k$ where $r_k^{3/4} \le 2|x_k|$. We work in the chart at infinity $\{x \in \mathbb{R}^3 : |x| > 1/2\}$. If we rescale by $r_k^{-3/4}$ and pass to a subsequence, curvature estimates as in [11, app. B] show that the rescaled regions converge in $C^{2,\alpha}_{\mathrm{loc}}$ to a half-space in $\mathbb{R}^3 \setminus \{0\}$. Upon further translation by the points $r_k^{-3/4} x_k$, we find surfaces $\widetilde{\Sigma}_k$ in $B_{1/4}(0)$ with $0 \in \widetilde{\Sigma}_k$ that are locally isoperimetric with respect to a metric $\widetilde{g}_k$ on $B_{1/4}(0)$ and such that

$$r_k |\overset{\circ}{\widetilde{h}}_k(0)|^2_{\widetilde{g}_k} \ge k^2 \quad \text{and} \quad r_k \int_{\widetilde{\Sigma}_k} |\overset{\circ}{\widetilde{h}}_k|^2_{\widetilde{g}_k}\, d\widetilde{\mu}_k \le 48\pi(1 + o(1)) m_{\mathrm{ADM}}. \tag{2.4}$$

The second estimate follows from (2.1), inclusion, and scaling invariance. The surfaces $\widetilde{\Sigma}_k$ converge in $C^{2,\alpha}$ to a plane through the origin in $B_{1/4}(0)$. The Riemannian metrics $\widetilde{g}_k$ converge to the Euclidean metric on $B_{1/4}(0)$ with

$$\widetilde{\mathrm{Rm}_k} = O\big(r_k^{-3/4}\big) \quad \text{and} \quad \widetilde{\nabla}_k \widetilde{\mathrm{Rm}_k} = O\big(r_k^{-3/4}\big).$$

For large k, (2.3) and (2.4) are incompatible with the standard De Giorgi–Nash–Moser $L^2 \Rightarrow L^\infty$ estimate for subsolutions; see theorem 8.17 in [20]. □

Using (2.2), (G.1), (G.2), and Proposition 2.3, we see that

$$\bar{H}(x) = 2/r + O(r^{-3/2}) \quad \text{and} \quad \mathring{\bar{h}}(x) = O(r^{-5/4})$$

for all points $x \in \Sigma$ with $2|x| \geq r^{3/4}$. In particular, the Euclidean principle curvatures $\bar{\kappa}_i(x)$ of Σ satisfy

$$\bar{\kappa}_i(x) = 1/r + O(r^{-5/4}) \tag{2.5}$$

for $2|x| \geq r^{3/4}$, where $i = 1, 2$. Using the Gauss-Weingarten relations, we conclude that

$$\bar{\nabla}(\bar{\nu} - x/r) = O(r^{-5/4}) \tag{2.6}$$

on $\Sigma \setminus B_{r^{3/4}/2}$.

Let Σ' be a connected component of $\Sigma \setminus B_{r^{3/4}}$.

LEMMA 2.4. *There is $a \in \mathbb{R}^3$ with $|a| > r + r^{3/4}$ such that*

$$\bar{\nu}(x) - \frac{x - a}{r} = O(r^{-1/4}) \tag{2.7}$$

for all $x \in \Sigma'$.

PROOF. We first show that the diameter of Σ' is $O(r)$. We only need to consider the case where $\{x \in \Sigma : r^{3/4}/2 \leq |x| \leq r^{3/4}\}$ is nonempty. Using the co-area formula and the quadratic area growth[2] of stable constant mean curvature surfaces (cf. [11, lemma B.5]), we see that

$$\int_{r^{3/4}/2}^{r^{3/4}} \mathcal{H}^1_{\bar{g}}(\{x \in \Sigma : |x| = \sigma\}) d\sigma = O(r^{3/2}).$$

We can choose a regular value σ with $r^{3/4}/2 \leq \sigma \leq r^{3/4}$ such that the curve $\{x \in \Sigma : |x| = \sigma\}$ has length $O(r^{3/4})$. A standard variation of the argument leading to the Bonnet-Myers diameter estimate shows that any two points $p, q \in \Sigma'$ are connected by a curve in Σ' whose length is $O(r)$.

Integrating (2.6) along such curves, we see that there is $a \in \mathbb{R}^3$ so that (2.7) holds. Assume now that $|a| \leq r + r^{3/4}$. It follows that there is $x_0 \in \Sigma'$ with $|x_0| = r^{3/4}$. Using (2.7), it follows that

$$|a| \geq |a - x_0| - |x_0| \geq r - c r^{3/4}$$

where $c > 0$ is independent of Σ. Replacing a by

$$a' = (1 + (c + 2) r^{-1/4}) a$$

completes the proof. □

[2] Such an estimate is simple to prove for isoperimetric regions; cf. Lemma F.1.

Now by (2.5), there is an open subset $\Gamma \subset \mathbb{S}^2$ and $u \in C^{2,\alpha}(\Gamma)$ with

$$\Sigma' = \{a + u(\theta)\theta : \theta \in \Gamma\}.$$

Let us assume for definiteness that $a = |a|(0,0,1)$. We have the estimate

$$\begin{aligned} \mathbb{S}^2 \setminus \Gamma \subset \big\{&(\sin\phi\cos\vartheta, \sin\phi\sin\vartheta, \cos\phi) : \\ &\vartheta \in [0, 2\pi] \text{ and } \phi \in (\pi - 2r^{-1/4}, \pi]\big\}. \end{aligned} \tag{2.8}$$

We also remark that $\Gamma = \mathbb{S}^2$ when $|a| > 2r$.

Let $\bar{\nu}$ be the outward pointing unit normal and $\bar{p}$ the induced metric of Σ', both computed with respect to the Euclidean background metric. Then

$$\bar{\nu}(\theta) = \frac{u(\theta)\theta - (\nabla u)(\theta)}{\sqrt{u(\theta)^2 + |(\nabla u)(\theta)|^2}}, \quad \bar{p}_{ij}(\theta) = u(\theta)^2\omega_{ij} + (\partial_i u)(\theta)(\partial_j u)(\theta),$$

where both the gradient and its length are computed with respect to the standard metric ω_{ij} on $\mathbb{S}^2 \subset \mathbb{R}^3$ and where i, j are with respect to local coordinates on $\mathbb{S}^2$. It follows from (2.7) that

$$u = r + O(r^{3/4}) \quad \text{and} \quad \nabla u = O(r^{3/4}). \tag{2.9}$$

Note that it also follows that $\Sigma \setminus B_{r^{3/4}}$ is connected (so $\Sigma' = \Sigma \setminus B_{r^{3/4}}$).

In the remainder of this section, integrals will be with respect to the measure induced by the Euclidean background unless otherwise indicated.

LEMMA 2.5. *We have*

$$\int_{\{a+r\theta : \theta \in \mathbb{S}^2 \setminus \Gamma\}} \frac{1}{|x|} = o(r). \tag{2.10}$$

PROOF. We may assume that $r + r^{3/4} < |a| < 2r$. Using (2.8), we have that

$$\int_{\{a+r\theta : \theta \in \mathbb{S}^2 \setminus \Gamma\}} \frac{1}{|x|} = O\int_0^{2\pi}\int_{\pi - 2r^{-1/4}}^{\pi} \frac{r^2 \sin\phi \, d\phi \, d\vartheta}{\sqrt{|a|^2 + r^2 - 2|a|r\cos\phi}} = o(r)$$

as claimed. □

PROPOSITION 2.6. *We have*

$$\begin{aligned} \operatorname{area}(\Sigma) - \operatorname{area}_{\bar{g}}(\Sigma) &= \operatorname{area}(S_r(a)) - 4\pi r^2 + o(r) \\ \operatorname{vol}(\Omega) - \operatorname{vol}_{\bar{g}}(\Omega) &= \operatorname{vol}(B_r(a)) - \frac{4\pi r^3}{3} + o(r^2). \end{aligned}$$

PROOF. Set $\Sigma'' = \Sigma - \Sigma' = \Sigma \cap B_{r^{3/4}}$. Then

$$\begin{aligned} \operatorname{area}(\Sigma) &= \operatorname{area}_{\bar{g}}(\Sigma) + \frac{1}{2}\int_\Sigma (\delta^{ij} - \bar{\nu}^i\bar{\nu}^j)\sigma_{ij} + O\int_\Sigma \frac{1}{|x|^2} \\ &= \operatorname{area}_{\bar{g}}(\Sigma) + \frac{1}{2}\int_{\Sigma'} (\delta^{ij} - \bar{\nu}^i\bar{\nu}^j)\sigma_{ij} + O\int_{\Sigma''} \frac{1}{|x|} + O\int_\Sigma \frac{1}{|x|^2}. \end{aligned}$$

It follows from the quadratic area growth of Σ and Lemma F.2 that

$$\int_{\Sigma'} \frac{1}{|x|} = O(r), \qquad \int_{\Sigma''} \frac{1}{|x|} = O(r^{3/4}), \qquad \int_{\Sigma} \frac{1}{|x|^2} = O(\log r),$$

so that

$$\operatorname{area}(\Sigma) = \operatorname{area}_{\overline{g}}(\Sigma) + \frac{1}{2}\int_{\Sigma'} (\delta^{ij} - \overline{\nu}^i \overline{\nu}^j)\sigma_{ij} + o(r). \tag{2.11}$$

Now

$$\begin{aligned}
&\int_{x\in\Sigma'} \big((\delta^{ij} - \overline{\nu}^i\overline{\nu}^j)\sigma_{ij}\big)(x) \\
&\quad = \int_{\theta\in\Gamma} \big((\delta^{ij} - \overline{\nu}^i\overline{\nu}^j)\sigma_{ij}\big)(a + u(\theta)\theta)\, \overbrace{u(\theta)\sqrt{u(\theta)^2 + |(\nabla u)(\theta)|^2}}^{\text{area element}} \\
&\quad = r^2 \int_{\theta\in\Gamma} \big((\delta^{ij} - \overline{\nu}^i\overline{\nu}^j)\sigma_{ij}\big)(a + u(\theta)\theta)(1 + O(r^{-1/4})) \\
&\quad = O(r^{3/4}) + r^2 \int_{\theta\in\Gamma} \big((\delta^{ij} - \overline{\nu}^i\overline{\nu}^j)\sigma_{ij}\big)(a + u(\theta)\theta)
\end{aligned} \tag{2.12}$$

where we have used (2.9) in the second equality. Conversely,

$$\int_{\theta\in\Gamma} \big((\delta^{ij} - \overline{\nu}^i\overline{\nu}^j)\sigma_{ij}\big)(a + u(\theta)\theta) = o(r) + \int_{\theta\in\Gamma} (\delta^{ij} - \theta^i\theta^j)\sigma_{ij}(a + r\theta)$$

since

$$\begin{aligned}
&\int_{\theta\in\Gamma} (\delta^{ij} - \overline{\nu}^i\overline{\nu}^j)\sigma_{ij}(a + u(\theta)\theta) - \int_{\theta\in\Gamma} (\delta^{ij} - \theta^i\theta^j)\sigma_{ij}(a + r\theta) \\
&\quad = r^{-1/4}\, O\int_{\theta\in\Gamma} \frac{1}{|a + r\theta|} + r^{3/4}\, O\int_{\theta\in\Gamma} \frac{1}{|a + r\theta|^2} \\
&\quad = O(r^{3/4}\log r) = o(r).
\end{aligned} \tag{2.13}$$

Substituting (2.12) and (2.13) into (2.11) gives

$$\operatorname{area}(\Sigma) = \operatorname{area}_{\overline{g}}(\Sigma) + \frac{r^2}{2}\int_{\theta\in\Gamma} (\delta^{ij} - \theta^i\theta^j)\sigma_{ij}(a + r\theta) + o(r). \tag{2.14}$$

A direct computation shows that

$$\begin{aligned}
&\operatorname{area}(S_r(a)) \\
&\quad = 4\pi r^2 + \frac{1}{2}\int_{S_r(a)} (\delta^{ij} - \theta^i\theta^j)\sigma_{ij} + O\int_{S_r(a)} \frac{1}{|x|^2} \\
&\quad = 4\pi r^2 + \frac{r^2}{2}\int_{\Gamma} (\delta^{ij} - \theta^i\theta^j)\sigma_{ij}(a + r\theta) \\
&\qquad + O\int_{S_r(a)-\Gamma} \frac{1}{|x|} + o(r). \\
&\quad = 4\pi r^2 + \frac{r^2}{2}\int_{\Gamma} (\delta^{ij} - \theta^i\theta^j)\sigma_{ij}(a + r\theta) + o(r)
\end{aligned} \tag{2.15}$$

where we have used (2.10) in the last equality. Combining (2.14) and (2.15) yields

$$\operatorname{area}(\Sigma) - \operatorname{area}_{\overline{g}}(\Sigma) = \operatorname{area}(S_r(a)) - 4\pi r^2 + o(r).$$

Finally, we have

$$\begin{aligned}
&\operatorname{vol}(\Omega) - \operatorname{vol}(B_r(a)) \\
&\quad = \int_\Omega \sqrt{\det(g_{ij})} - \int_{B_r(a)} \sqrt{\det(g_{ij})} \\
&\quad = \operatorname{vol}_{\overline{g}}(\Omega) - \operatorname{vol}_{\overline{g}}(B_r(a)) \\
&\qquad + O\int_{\{x : r - cr^{3/4} \le |x-a| \le r + cr^{3/4}\}} \frac{1}{|x|} + O\int_{\{x \in \Omega : |x| < r^{3/4}\}} \frac{1}{|x|} \\
&\quad = \operatorname{vol}_{\overline{g}}(\Omega) - \operatorname{vol}_{\overline{g}}(B_r(a)) + o(r^2)
\end{aligned}$$

where we have used (2.9) in the third inequality. This completes the proof. □

We arrive at the main results of this section, asserting that the isoperimetric deficit of large outlying stable constant mean curvature spheres is *very close* to Euclidean. The strategy of the approximation argument we have used here is illustrated in Figure 2.1.

COROLLARY 2.7. *We have*

$$\frac{2}{\operatorname{area}(\Sigma)}\left(\operatorname{vol}(\Omega) - \frac{\operatorname{area}(\Sigma)^{3/2}}{6\sqrt{\pi}}\right) \le o(1).$$

PROOF. We abbreviate

$$z = \int_{S_r(a)} (\delta^{ij} - \overline{\nu}^i \overline{\nu}^j)\sigma_{ij}.$$

Note that $z = O(r)$ and that $\operatorname{area}_{\overline{g}}(\Sigma) = 4\pi r^2 + o(r^2)$. Using Propositions I.2 and 2.6 and also the Euclidean isoperimetric inequality in the last step, we obtain that

$$\begin{aligned}
&\operatorname{vol}(\Omega) - \frac{\operatorname{area}(\Sigma)^{3/2}}{6\sqrt{\pi}} \\
&\quad = \operatorname{vol}_{\overline{g}}(\Omega) + \frac{zr}{4} \\
&\qquad - \frac{\operatorname{area}_{\overline{g}}(\Sigma)^{3/2}}{6\sqrt{\pi}}\left(1 + \frac{z\,r}{2\operatorname{area}_{\overline{g}}(\Sigma)} + o\left(\frac{r}{\operatorname{area}_{\overline{g}}(\Sigma)}\right)\right)^{3/2} + o(r^2) \\
&\quad = \operatorname{vol}_{\overline{g}}(\Omega) - \frac{\operatorname{area}_{\overline{g}}(\Sigma)^{3/2}}{6\sqrt{\pi}} + o(r^2) \\
&\quad \le o(r^2).
\end{aligned}$$

□

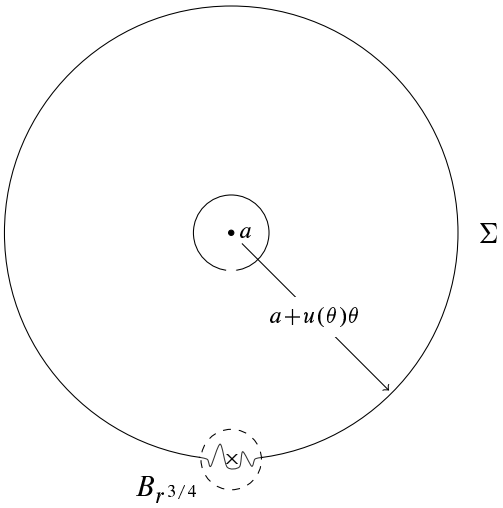

FIGURE 2.1. We show that the boundary Σ of the large component of an isoperimetric region, is *very* close to a sphere $S_r(a)$ outside of $B_{r^{3/4}}$; note that "×" represents the origin here. This allows us to approximate the isoperimetric deficit of Σ by that of $S_r(a)$. We show that in the scenario depicted here, the isoperimetric deficit of $S_r(a)$ (and thus of Σ) is too close to Euclidean.

3 Proof of Theorem 1.1 Assuming (3.1) and (3.2)

Throughout this section, we assume that (M, g) is a complete (nonflat) asymptotically flat Riemannian 3-manifold with nonnegative scalar curvature that satisfies the decay assumptions

$$\partial_I \sigma_{ij} = O(|x|^{-1-|I|}) \quad \text{as } |x| \to \infty \tag{3.1}$$

for all multi-indices I of order $|I| \leq 4$ (so $\tau = 1$ and decay of two additional derivatives) and

$$R(x) = O(|x|^{-3-\gamma}) \quad \text{as } |x| \to \infty \tag{3.2}$$

for some $\gamma > 0$ in the chart at infinity (1.1). S. Ma has shown in [30] that under these assumptions there is a compact subset $C \subset M$ so that each leaf Σ^H of the canonical foliation is the only stable constant mean curvature sphere of mean curvature $H \in (0, H_0)$ enclosing C. As observed by the first- and second-named authors, this statement remains true for surfaces of any genus; cf. [11, sec. 2].

Consider a sequence of large isoperimetric regions $\Omega_{V_k} \subset M$ with $V_k \to \infty$. Consider the two alternatives (a) and (b) in Lemma B.2. In (a), Ω_{V_k} eventually contains any compact set C. In particular, by the work of S. Ma [30] mentioned above, we find that $\partial\Omega_{V_k}$ is an element of the canonical foliation; cf. the discussion at the end of Appendix D.

On the other hand, if (b) from Lemma B.2 applies, then $\Omega_{V_k} = \Omega_k^{\text{res}} \cup \Omega_k$ where Ω_k^{res} is contained in a small neighborhood of the horizon and for k sufficiently large, $\Omega_k \cap B_{\rho_0} = \varnothing$ for any $\rho_0 > 1$.

In this case, the analysis from the previous section applies. In particular, Corollary 2.7 shows that the isoperimetric deficit of Ω_k, and thus Ω_{V_k}, is very close to Euclidean. By comparing, via Lemma C.1, with *centered* coordinate balls, this contradicts the isoperimetric property of Ω_{V_k}.

4 Mean Curvature Flow of Large Isoperimetric Regions

Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat at rate $\tau > 1/2$ and has nonnegative scalar curvature and positive mass $m_{\text{ADM}} > 0$. The analysis in this and the subsequent sections will establish the proof of Theorem 1.1 in full generality.

Let Ω_{V_k} be isoperimetric regions of volumes $V_k \to \infty$. Let Ω_k be the unique large component of Ω_{V_k}. We recall from Appendix B that Ω_k is connected with connected outer boundary $\partial\Omega_k \setminus \partial M$, and that Ω_k is outer area-minimizing in (M, g).

Let $\{\Omega_k(t)\}_{t\geq 0}$ denote the level set flow with initial condition Ω_k; see [27, §10.3]. Then $\Omega_k(t)$ is mean-convex in the sense of [42, p. 670] by theorem 3.1 in [42]. Moreover, by theorem 5.1 in [42], the 2-rectifiable Radon measures

$$\mu_k(t) = \mathcal{H}^2 \lfloor \partial^* \Omega_k(t) \tag{4.1}$$

define an integral Brakke flow $\{\mu_k(t)\}_{t\geq 0}$ in (M, g). For almost every time $t \geq 0$, $\mu_k(t)$ is the Radon measure associated to an integer rectifiable varifold $V_{\mu_k(t)}$.

LEMMA 4.1. *There is a constant $c > 0$ depending only on (M, g) so that*

$$\text{area}(B_\rho \cap \partial^* \Omega_k(t)) \leq c\rho^2$$

for all $\rho \geq 1$, $k \geq 1$, and $t \geq 0$.

PROOF. By [42, theorem 3.5], $\Omega_k(t)$ is outward area-minimizing in Ω_k. Combined with the fact that Ω_k is outward area-minimizing, we see that $\Omega_k(t)$ is outward area-minimizing in (M, g). The claim follows from comparison with coordinate spheres. □

Define the volume radius $\rho_k > 0$ by the expression

$$\text{vol}(\Omega_k) = \frac{4\pi}{3}\rho_k^3.$$

We may view $\mu_k(t) \lfloor (M \setminus K)$ as a measure on $\{x \in \mathbb{R}^3 : |x| > 1/2\}$ using the chart at infinity (1.1). In fact, consider the map

$$\eta_k : \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{given by} \quad x \mapsto x/\rho_k$$

and the rescaled measures

$$\widetilde{\mu}_k(t) = \eta_{k*}\big(\mu_k(\rho_k^2 t) \lfloor (M \setminus K)\big)$$

on $\{x \in \mathbb{R}^3 : \rho_k|x| > 1/2\}$. Then $\{\widetilde{\mu}_k(t)\}_{t\geq 0}$ is a Brakke motion on $\{x \in \mathbb{R}^3 : \rho_k|x| > 1/2\}$ with respect to the metric

$$\widetilde{g}_k(x) = \sum_{i,j=1}^{3} g_{ij}(\rho_k x) dx^i \otimes dx^j$$

on $\{x \in \mathbb{R}^3 : \rho_k|x| > 1/2\}$. As $k \to \infty$, $\widetilde{g}_k$ converges to the standard Euclidean inner product in $C^\infty_{\mathrm{loc}}(\mathbb{R}^3\setminus\{0\})$. We let $\widetilde{\Omega}_k$ be the subset of $\{x \in \mathbb{R}^3 : \rho_k\,|x| > 1/2\}$ such that

$$\Omega_k \setminus K \cong \{\rho_k x : x \in \widetilde{\Omega}_k\}.$$

We also let $\widetilde{\Omega}_k(t)$ be the subset of $\{x \in \mathbb{R}^3 : \rho_k|x| > 1/2\}$ such that

$$\Omega_k(t) \setminus K \cong \{\rho_k x : x \in \widetilde{\Omega}_k(t)\}.$$

By the remarks following Lemma B.2, there is $\xi \in \mathbb{R}^3$ such that, upon passing to a subsequence,

$$\widetilde{\Omega}_k \to B_1(\xi) \quad \text{in } C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$$

as $k \to \infty$. Our goal will be to show that $\xi = 0$.

PROPOSITION 4.2. *There is an integral Brakke flow $\{\widetilde{\mu}(t)\}_{t\geq 0}$ on $\mathbb{R}^3 \setminus \{0\}$ with the following three properties.*

(1) *There is a subsequence $\{\ell(k)\}_{k=1}^\infty$ such that, for all $t \geq 0$,*

$$\widetilde{\mu}_{\ell(k)}(t) \rightharpoonup \widetilde{\mu}(t)$$

as Radon measures on $\mathbb{R}^3 \setminus \{0\}$.

(2) *For almost every $t \geq 0$, there is a further subsequence $\{\ell(k,t)\}_{k=1}^\infty$ of $\{\ell(k)\}_{k=1}^\infty$ such that*

$$V_{\widetilde{\mu}_{\ell(k,t)}(t)} \rightharpoonup V_{\widetilde{\mu}(t)}$$

as varifolds on $\mathbb{R}^3 \setminus \{0\}$.

(3) *There is a constant $c > 0$ so that*

$$\widetilde{\mu}(t)(B_\rho(0)) \leq c\rho^2$$

for all $\rho > 0$ and $t \geq 0$.

PROOF. The first two claims follow from T. Ilmanen's compactness theorem for integral Brakke flows, theorem 7.1 in [27]. This result is only stated for sequences of Brakke flows with respect to a fixed complete Riemannian metric in [27]. However, the same proof as in [27] applies in the present setting. The quadratic area bounds carry over from Lemma 4.1. □

In view of Proposition J.5, it is now clear that $\{\mu(t)\}_{t\geq 0}$ extends to an integral Brakke flow in $\mathbb{R}^3$ with initial condition

$$\widetilde{\mu}(0) = \mathcal{H}^2 \lfloor S_1(\xi).$$

Proposition J.6 shows that such a Brakke motion follows classical mean curvature flow except possibly for sudden extinction:

$$\widetilde{\mu}(t) = \mathcal{H}^2 \lfloor S_{\sqrt{1-4t}}(\xi)$$

for all $t \in [0, T]$ where $T \in [0, \frac{1}{4})$. The *particular* flow at hand is constructed as the limit of level set flows. We use spherical barriers to show that the limiting flow cannot disappear suddenly; i.e., we will show that $T = \frac{1}{4}$.

LEMMA 4.3. *We have that* $\widetilde{\mu}(t) = \mathcal{H}^2 \lfloor S_{\sqrt{1-4t}}(\xi)$ *for all* $t \in [0, \frac{1}{4})$.

PROOF. If not, then there is $T \in [0, \frac{1}{4})$ so that $\widetilde{\mu}(t) = \mathcal{H}^2 \lfloor S_{\sqrt{1-4t}}(\xi)$ for $t \in [0, T]$ and $\widetilde{\mu}(t) = 0$ for $t > T$. We will prove the result for $|\xi| \geq 1$ and leave the straightforward modification to the case of $|\xi| < 1$ to the reader.

Assume that $T = 0$. Let $\varepsilon > 0$ be small. Upper semicontinuity of density for surfaces with bounded mean curvature (see [41, cor. 17.8]) implies that $B_{\sqrt{1-4\varepsilon}}(\xi) \subset \widetilde{\Omega}_k(0) = \widetilde{\Omega}_k$ for all sufficiently large k. Using that $\widetilde{g}_k$ converges to the standard Euclidean inner product in $C^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ and the avoidance principle for the level set flow, we see that $B_{\sqrt{1-9\varepsilon}}(\xi) \subset \widetilde{\Omega}_k(\varepsilon)$ provided that k is sufficiently large. Recall that $\widetilde{\mu}_k(t) = \mathcal{H}^2 \lfloor \partial^* \widetilde{\Omega}_k(t)$. We obtain a contradiction with the assumption that $\widetilde{\mu}_k(\varepsilon) \rightharpoonup \widetilde{\mu}(\varepsilon) = 0$.

Assume now that $T \in (0, \frac{1}{4})$. Let $0 < \varepsilon < (1-4T)/100$. Upper semi-continuity of Gaussian density for integral Brakke flows implies that

$$B_{\sqrt{1-4T-4\varepsilon}}(\xi) \subset \widetilde{\Omega}_k(T)$$

for all k sufficiently large. Arguing as in the previous case, we see that

$$B_{\sqrt{1-4T-9\varepsilon}}(\xi) \subset \widetilde{\Omega}_k(T+\varepsilon).$$

This is a contradiction for the same reason as before. □

Using B. White's version [43] of K. Brakke's regularity theorem [4] for mean curvature flow, we obtain the following consequence.

COROLLARY 4.4. *Let* $(x,t) \in (\mathbb{R}^3 \setminus \{0\}) \times [0,\infty)$ *with* $(x,t) \neq (\xi, \frac{1}{4})$. *There is a neighborhood of* (x,t) *in* $\mathbb{R}^3 \times \mathbb{R}$ *where* $\{\widetilde{\Omega}_k(t)\}_{t \geq 0}$ *defines a classical mean curvature flow with respect to the Riemannian metric* $\widetilde{g}_k$ *provided that* k *is sufficiently large. These flows converge to the shrinking sphere* $S_{\sqrt{1-4t}}(\xi)$ *as* $k \to \infty$ *locally smoothly away from the spacetime set* $(\{0\} \times [0,\infty)) \cup \{(\xi, \frac{1}{4})\}$.

We define the *disconnecting time* for the rescaled flow by

$$\widetilde{T}(|\xi|) = \begin{cases} \frac{1-|\xi|^2}{4}, & |\xi| < 1, \\ 0, & |\xi| \geq 1. \end{cases}$$

Note that the bulk of $\Omega_k(t)$ is disjoint from the center of (M, g) after time $t = \rho_k^2 \widetilde{T}(|\xi|)(1 + o(1))$.

Assume now that $\xi \neq 0$. Choose $\varepsilon > 0$ such that

$$100\varepsilon < 1 - 4\widetilde{T}(|\xi|). \tag{4.2}$$

We can make this choice such that

$$t_k = \rho_k^2(\varepsilon + \widetilde{T}(|\xi|)) \quad \text{and} \quad T_k = \rho_k^2\left(\frac{1}{4} - \varepsilon\right).$$

are smooth times for all the level set flows $\{\Omega_k(t)\}_{t\geq 0}$. Indeed, by the work of B. White [42], almost every time is a smooth time for the individual flows. For every $t \in [t_k, T_k]$ there is a unique large component $\Gamma_k(t)$ of $\Omega_k(t)$ by Corollary 4.4. The boundary $\Sigma_k(t)$ of $\Gamma_k(t)$ is smooth and close to a Euclidean sphere with radius $(\rho_k^2 - 4t)^{1/2}$ and center $\rho_k\xi$ in the chart at infinity (1.1). Moreover, as $k \to \infty$,

$$\begin{aligned} \operatorname{area}((\partial^*\Omega_k(t)) \setminus \Sigma_k(t)) &= o\big(\rho_k^2\big), \\ \operatorname{vol}(\Omega_k(t) \setminus \Gamma_k(t)) &= o(\rho_k^3). \end{aligned}$$

Recall that the Hawking mass of a closed, two-sided surface $\Sigma \subset M$ is defined as

$$m_H(\Sigma) = \sqrt{\frac{\operatorname{area}(\Sigma)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2\, d\mu\right).$$

Let

$$m_k = \sup_{t\in[t_k,T_k]} m_H(\Sigma_k(t)).$$

COROLLARY 4.5. *We have that*

$$\lim_{k\to\infty} m_k = 0.$$

PROOF. The surface $\Sigma_k(t)$ is geometrically close to the coordinate sphere

$$S_{\sqrt{\rho_k^2-4t}}(\rho_k\xi)$$

in the chart at infinity (1.1) by Corollary 4.4. The assertion thus follows from Appendix H. □

We denote by

$$A_m : (0, \infty) \to (0, \infty)$$

the isoperimetric profile of Schwarzschild with mass $m > 0$. Thus, given $V > 0$,

$$A_m(V) = \left(1 + \frac{m}{2r}\right)^4 4\pi r^2$$

where $r = r(V) > m/2$ is such that

$$V = 4\pi\int_{\frac{m}{2}}^r \left(1 + \frac{m}{2r}\right)^6 r^2\, dr.$$

We denote by

$$V_m : (0, \infty) \to (0, \infty)$$

the inverse of this function. We recall the following expansion. It follows from a straightforward computation in view of H. Bray's characterization of isoperimetric surfaces in Schwarzschild as centered coordinate spheres [5, theorem 8]. The claim about the error term is proven in lemma 10 of [29].

LEMMA 4.6. *We have that*

$$V_m(A) = \frac{1}{6\sqrt{\pi}} A^{3/2} + \frac{m}{2} A + O(A^{1/2})$$

as $A \to \infty$. *The error is uniform with respect to the parameter* m *in a given range* $0 < m \leq m_0$.

G. Huisken has shown [23, 24] that the quantity

$$t \mapsto -\operatorname{vol}(\Omega_t) + V_m(\operatorname{area}(\Sigma_t)) \tag{4.3}$$

is nonincreasing along a classical mean curvature flow of boundaries

$$\{\Sigma_t = \partial\Omega_t\}_{t \in (a,b)}$$

provided that $m_H(\Sigma_t) \leq m$ and $|\Sigma_t| > 16\pi m^2$ for all $t \in (a, b)$.

J. Jauregui and D. Lee [29] have introduced a modification of the level set flow starting from a mean convex region along which G. Huisken's monotonicity holds. Their result applies beautifully to our setting.

Let Ω be the (unique) large component of a large isoperimetric region in (M, g). We consider the *modified level set flow* $\{\widehat{\Omega}(t)\}_{t \geq 0}$ with $\widehat{\Omega}(0) = \Omega$ defined by J. Jauregui and D. Lee in definition 24 and definition 27 of [29]. The modified flow agrees with the original level set flow $\{\Omega(t)\}_{t \geq 0}$ except that components of the original flow are frozen when their perimeter drops below $36\pi(m_{\mathrm{ADM}})^2$. J. Jauregui and D. Lee show that G. Huisken's monotonicity holds along their modified level set flow. In the statement of their result below, $T \geq 0$ as in lemma 29 of [29] is the time when the flow freezes up completely.

PROPOSITION 4.7 ([29, prop. 30]). *The quantity*

$$t \mapsto -\operatorname{vol}(\widehat{\Omega}(t)) + V_{m_{\mathrm{ADM}}}(\operatorname{area}(\partial^* \widehat{\Omega}(t)))$$

is nonincreasing on $[0, T]$.

We return to our previous setting, where each Ω_k is the large component of a large isoperimetric region and where the rescaled regions $\widetilde{\Omega}_k$ converge to $B_1(\xi)$ for some $\xi \neq 0$. We have already seen that the original level set flow $\{\Omega_k(t)\}_{t \geq 0}$ with initial condition $\Omega_k(0) = \Omega_k$ has the property that, for $t \in [t_k, T_k]$, there is a *unique* large component $\Gamma_k(t)$ of $\Omega_k(t)$. The boundary $\Sigma_k(t) = \partial\Gamma_k(t)$ of this component is smooth. We recall that

$$t_k = \rho_k^2(\varepsilon + \widetilde{T}(|\xi|)) \quad \text{and} \quad T_k = \rho_k^2\left(\frac{1}{4} - \varepsilon\right)$$

have been chosen as smooth times for the level set flow $\{\Omega_k(t)\}_{t\geq 0}$. The surface $\Sigma_k(t)$ is close to a Euclidean sphere of radius $(\rho_k^2 - 4t)^{1/2}$ with center at $\rho_k\xi$ in the chart at infinity (1.1). Consider the modified flow $\{\widehat{\Omega}_k(t)\}_{t\geq 0}$ of J. Jauregui and D. Lee described above. By what we have just said,

$$\operatorname{area}(\Sigma_k(t_k)) \geq 36\pi\,(m_{\mathrm{ADM}})^2$$

provided that k is sufficiently large. We see that the large components $\Gamma_k(t_k)$ are not affected by the freezing that defines the passing from the original to the modified level set flow—their perimeter is too large. Thus $\widehat{\Omega}_k(t_k)$ is the disjoint union $E_k(t_k) \cup \Gamma_k(t_k)$ where

$$\operatorname{vol}(E_k(t_k)) = o(\rho_k^3) \quad \text{and} \quad \operatorname{area}(\partial E_k(t_k)) = o\big(\rho_k^2\big). \tag{4.4}$$

5 Proof of Theorem 1.1 When $\tau > 1/2$

We continue with the notation of Section 4. The strategy of the proof is illustrated in Figure 5.1.

PROPOSITION 5.1. $\xi = 0$.

PROOF. Assume that $\xi \neq 0$. We continue with the notation set forth above. Note that

$$\{\Sigma_k(t)\}_{t\in[t_k,T_k]}$$

is a smooth mean curvature flow where $\Sigma_k(t) = \partial\Gamma_k(t)$. In Corollary 4.5 we have seen that the Hawking masses of the surfaces along this flow are bounded by $m_k = o(1)$ as $k \to \infty$. By G. Huisken's monotonicity (4.3) for $\Sigma_k(t)$ applied with the Hawking mass bound $m = m_k = o(1)$, we have that

$$\begin{aligned} &-\operatorname{vol}(\Gamma_k(t_k)) + \frac{1}{6\sqrt{\pi}}\operatorname{area}(\Sigma_k(t_k))^{3/2} + o\big(\rho_k^2\big) \\ &\quad\geq -\operatorname{vol}(\Gamma_k(T_k)) + \frac{1}{6\sqrt{\pi}}\operatorname{area}(\Sigma_k(T_k))^{3/2} + o\big(\rho_k^2\big) \end{aligned}$$

where we have also used that

$$\operatorname{area}(\Sigma_k(T_k)) = 4\varepsilon\rho_k^2 + o\big(\rho_k^2\big) \geq 36\pi\,(m_k)^2$$

as $k \to \infty$. Conversely, by the sharp isoperimetric inequality (C.2) for (M, g),

$$\begin{aligned} -\operatorname{vol}(\Gamma_k(T_k)) + \frac{1}{6\sqrt{\pi}}\operatorname{area}(\Sigma_k(T_k))^{3/2} &\geq -m_{\mathrm{ADM}}\operatorname{area}(\Sigma_k(T_k)) \\ &= -16\pi\varepsilon m_{\mathrm{ADM}}\rho_k^2 + o\big(\rho_k^2\big). \end{aligned}$$

Combining these two estimates, we obtain

$$-\operatorname{vol}(\Gamma_k(t_k)) + \frac{1}{6\sqrt{\pi}}\operatorname{area}(\Sigma_k(t_k))^{3/2} \geq -16\pi\varepsilon m_{\mathrm{ADM}}\rho_k^2 + o\big(\rho_k^2\big). \tag{5.1}$$

We now apply Proposition 4.7 to the modified weak flow $\{\widehat{\Omega}_k(t)\}_{t\geq 0}$ between the (smooth) times $t = 0$ and $t = t_k$. In the first line below we use that Ω_k—as the

substantial component of a large isoperimetric region—almost saturates the sharp isoperimetric inequality (C.2) on (M, g).

$$\begin{aligned}
0 &= -\operatorname{vol}(\Omega_k) + \frac{1}{6\sqrt{\pi}}\operatorname{area}(\partial\Omega_k)^{3/2} + \frac{m_{\mathrm{ADM}}}{2}\operatorname{area}(\partial\Omega_k) + o(\rho_k^2)\\
&\geq -\operatorname{vol}(\widehat{\Omega}_k(t_k)) + \frac{1}{6\sqrt{\pi}}\operatorname{area}(\partial\widehat{\Omega}_k(t_k))^{3/2} + \frac{m_{\mathrm{ADM}}}{2}\operatorname{area}(\partial\widehat{\Omega}_k(t_k)) + o\big(\rho_k^2\big)\\
&= -\operatorname{vol}(\Gamma_k(t_k)) + \frac{1}{6\sqrt{\pi}}\operatorname{area}(\Sigma_k(t_k))^{3/2} - \operatorname{vol}(E_k(t_k))\\
&\quad + \frac{1}{6\sqrt{\pi}}\big((\operatorname{area}(\Sigma_k(t_k)) + \operatorname{area}(\partial E_k(t_k)))^{3/2} - \operatorname{area}(\Sigma_k(t_k))^{3/2}\big)\\
&\quad + \frac{m_{\mathrm{ADM}}}{2}\operatorname{area}(\Sigma_k(t_k)) + o\big(\rho_k^2\big)\\
&\geq -\operatorname{vol}(E_k(t_k))\\
&\quad + \frac{1}{6\sqrt{\pi}}\big((\operatorname{area}(\Sigma_k(t_k)) + \operatorname{area}(\partial E_k(t_k)))^{3/2} - \operatorname{area}(\Sigma_k(t_k))^{3/2}\big)\\
&\quad + \frac{m_{\mathrm{ADM}}}{2}\operatorname{area}(\Sigma_k(t_k)) - 16\pi\varepsilon m_{\mathrm{ADM}}\rho_k^2 + o\big(\rho_k^2\big).
\end{aligned}$$

The final inequality follows from (4.4) and (5.1).

Assume first that $\operatorname{area}(\partial E_k(t_k)) = O(1)$ as $k \to \infty$. Then $\operatorname{vol}(E_k(t_k)) = O(1)$ as well, and

$$\begin{aligned}
&-\operatorname{vol}(E_k(t_k)) + \frac{1}{6\sqrt{\pi}}\big(\big(\operatorname{area}(\Sigma_k(t_k)) + \operatorname{area}(\partial E_k(t_k))\big)^{3/2} - \operatorname{area}(\Sigma_k(t_k))^{3/2}\big)\\
&\quad \geq -O(1)
\end{aligned}$$

as $k \to \infty$. Thus

$$\operatorname{area}(\Sigma_k(t_k)) \leq (8\varepsilon + o(1))4\pi\rho_k^2. \tag{5.2}$$

This contradicts the choice $\varepsilon > 0$ in (4.2), because

$$\operatorname{area}(\Sigma_k(t_k)) = (1 - 4\varepsilon - 4\widetilde{T}(|\xi|) + o(1))4\pi\rho_k^2.$$

Assume now that $\operatorname{area}(\partial E_k(t_k)) \to \infty$ as $k \to \infty$. Then

$$\operatorname{vol}(E_k(t_k)) \leq \frac{1}{6\sqrt{\pi}}\operatorname{area}(\partial E_k(t_k))^{3/2} + (m_{\mathrm{ADM}} - o(1))\operatorname{area}(\partial E_k(t_k))$$

by the sharp isoperimetric inequality (C.2). Combining this with the above and (4.4), we have

$$\begin{aligned}
0 \geq \frac{1}{6\sqrt{\pi}}&\big((\operatorname{area}(\Sigma_k(t_k)) + \operatorname{area}(\partial E_k(t_k)))^{3/2}\\
&\qquad - \operatorname{area}(\Sigma_k(t_k))^{3/2} - \operatorname{area}(\partial E_k(t_k))^{3/2}\big)\\
&+ \frac{m_{\mathrm{ADM}}}{2}\operatorname{area}(\Sigma_k(t_k)) - 16\pi\varepsilon m_{\mathrm{ADM}}\rho_k^2 + o\big(\rho_k^2\big).
\end{aligned}$$

Using that

$$x^{3/2} + y^{3/2} \leq (x+y)^{3/2}$$

for all $x, y \geq 0$, we arrive again at the contradictory estimate (5.2). □

PROOF OF THEOREM 1.1. Combining Lemma B.2 and Proposition 5.1, we see that every sufficiently large isoperimetric region is connected and close to the centered coordinate ball $B_1(0)$ when put to scale of its volume in the chart at infinity (1.1). By the uniqueness of large stable constant mean curvature spheres described in Appendix D, the outer boundary of such an isoperimetric region is a leaf of the canonical foliation. □

Appendix A General Properties of the Isoperimetric Profile

Let (M, g) be an asymptotically flat Riemannian 3-manifold as defined in Section 1. We recall below several properties of the *isoperimetric profile* $A : (0, \infty) \to (0, \infty)$ of (M, g) as defined by (1.3) that we use throughout this paper. The results on the regularity of the isoperimetric profile are given in or follow easily from, e.g., [3, 5, 19, 38].

Locally, the isoperimetric profile can be written as the sum of a concave and a smooth function. In particular, the isoperimetric profile is absolutely continuous. The left derivate $A^-(V)$ and the right derivative $A^+(V)$ exist at every $V > 0$. They agree at all but possibly countably many $V > 0$. Moreover,

$$\lim_{W \searrow V} A'^+(W) \leq A'^+(V) \leq A'^-(V) \leq \lim_{W \nearrow V} A'^-(W).$$

Assume that, for some $V > 0$, there is $\Omega_V \in \mathcal{R}_V$ with

$$A(V) = \text{area}(\partial\Omega_V) - \text{area}(\partial M).$$

The proof of theorem 1.2 in [17] shows that such *isoperimetric regions* exist for every sufficiently large volume $V > 0$ when the mass of (M, g) is positive. They exist for every volume $V > 0$ when the scalar curvature of (M, g) is nonnegative by proposition K.1 in [7]. The outer boundary $\partial\Omega_V \setminus \partial M$ is a stable constant mean curvature surface. Its mean curvature H is positive when computed with respect to the outward unit normal. Moreover,

$$A'^+(V) \leq H \leq A'^-(V).$$

In particular, the isoperimetric profile is a strictly increasing function. Moreover, at volumes $V > 0$ where the isoperimetric profile is differentiable, the outer boundaries of all isoperimetric regions of volume V have the same constant mean curvature.

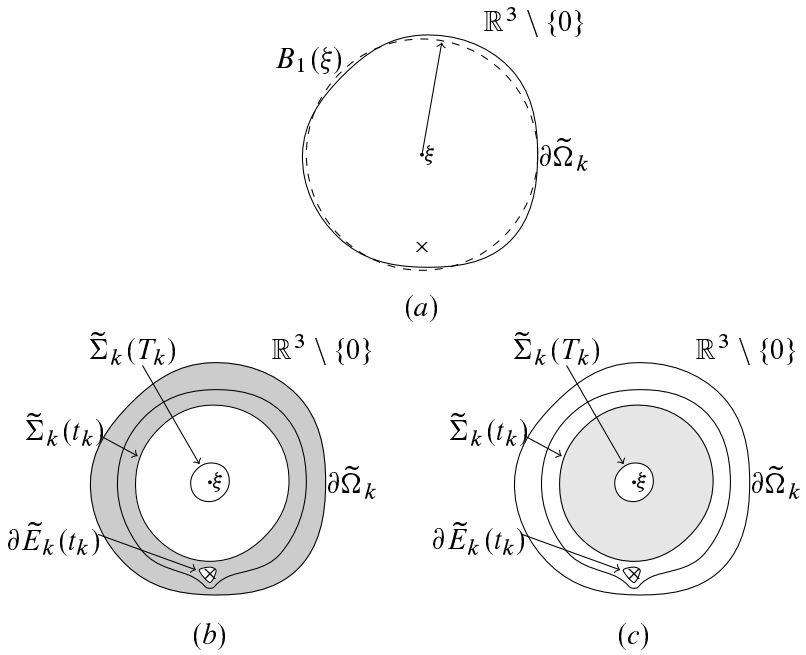

FIGURE 5.1. We depict here the case where $0 < |\xi| < 1$. In (a), a sequence of large isoperimetric regions Ω_k is assumed to limit to $B_1(\xi)$ after rescaling. The convergence is smooth on compact subsets of $\mathbb{R}^3 \setminus \{0\}$. Here, the origin is denoted by "$\times$."

In (b) and (c), we depict boundaries of the (modified) level set flows. We show that the large component of the level set flow "disconnects." The large disconnected component is labeled $\widetilde{\Sigma}_k(t_k)$. It is possible that there are additional components $\widetilde{E}_k(t_k)$ of the modified flow.

In (b), the change of the isoperimetric deficit as the flow sweeps out the shaded region is estimated by the Hawking mass bound of m_{ADM}. On the other hand, in (c), the lightly shaded region is swept out by surfaces with Hawking mass bounded by $o(1)$ as $k \to \infty$. This leads to improved estimates for the deficit, showing that the original region Ω_k cannot have been isoperimetric.

When $|\xi| = 1$, a similar situation occurs, except the flow disconnects from the origin after a short time (in the rescaled picture). We must wait this short time before arguing as in (c), so there will be a thin region as in (b) in this case. If $|\xi| > 1$, the flow is completely disconnected, so we do not need to consider the shaded region as in (b).

Appendix B Divergent Sequences of Isoperimetric Regions

The following result of the first- and the second-named authors is included as corollary 1.13 in [7]. It is a consequence of the solution of the following conjecture by R. Schoen: The only asymptotically flat Riemannian 3-manifold with nonnegative scalar curvature that admits a noncompact area-minimizing boundary is flat Euclidean space.

LEMMA B.1. *Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat with nonnegative scalar curvature and positive mass. Let $U \subset M$ be a bounded open subset that contains the boundary of M. There is $V_0 > 0$ so that for every isoperimetric region Ω_V of volume $V \geq V_0$, either $U \subset \Omega_V$ or $U \cap \Omega_V$ is a thin smooth region that is bounded by ∂M and a nearby stable constant mean curvature surface.*

The conclusion of the lemma clearly fails in Euclidean space. Under the additional assumption that the scalar curvature of (M, g) is everywhere positive, this result was observed by the second-named author and J. Metzger as corollary 6.2 in [16]. Together with elementary observations on the number of components of large isoperimetric regions as in Section 5 of [15] and the proof of theorem 1.12 in [7], we obtain the following dichotomy for sequences of isoperimetric regions with divergent volumes.

LEMMA B.2. *Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat with nonnegative scalar curvature and positive mass. Let Ω_{V_k} be an isoperimetric region of volume V_k where $V_k \to \infty$. After passing to a subsequence, exactly one of the following alternatives occurs*:

(a) *Each Ω_{V_k} is connected, $\partial\Omega_{V_k} \setminus \partial M$ is connected, and the sequence is increasing to M.*
(b) *Each Ω_{V_k} splits into a union $\Omega^{\text{res}}_{V_k}$ and $\Omega^{\infty}_{V_k}$ where the $\Omega^{\infty}_{V_k}$ are connected with connected boundary and divergent in M as $k \to \infty$, and where each $\Omega^{\text{res}}_{V_k}$ is contained in an ε_k-neighborhood of ∂M where $\varepsilon_k \to 0$ as $k \to \infty$.*

In particular, every isoperimetric region Ω_V in (M, g) of sufficiently large volume $V > 0$ has *exactly one* large connected component—either Ω_V in alternative (a) or Ω^{∞}_V in alternative (b). Note that Theorem 1.1 implies that alternative (b) in Lemma B.2 doesn't occur.

We include several additional observations—extracted from the proofs of Theorem 1.2 in [17] and Theorem 1.12 in [7]—about the sequences in Lemma B.2. Let

$$\widetilde{\Omega}_{V_k} \subset \{x \in \mathbb{R}^3 : \lambda_k |x| > 1/2\}$$

be such that

$$\Omega_{V_k} \setminus K \cong \{\lambda_k x : x \in \widetilde{\Omega}_{V_k}\}$$

where

$$\lambda_k = \sqrt[3]{(3V_k)/(4\pi)}.$$

Then, possibly after passing to a further subsequence,

$$\widetilde{\Omega}_{V_k} \to B_1(\xi)$$

in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ for some $\xi \in \mathbb{R}^3$. In particular,

$$\begin{aligned} \operatorname{area}(\Sigma_{V_k}) &= 4\pi\lambda_k^2(1+o(1)), \\ H_{\Sigma_{V_k}} &= 2(1+o(1))/\lambda_k, \end{aligned}$$

as $k \to \infty$ where $\Sigma_{V_k} = \partial\Omega_{V_k} \setminus \partial M$.

Appendix C Sharp Isoperimetric Inequality

The characterization of the ADM-mass through the isoperimetric deficit of large centered coordinate spheres in Lemma C.1 below was proposed by G. Huisken [23] and proved by X.-Q. Fan, P. Miao, L.-F. Tam, and the third-named author as corollary 2.3 in [18].

LEMMA C.1. *Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat of rate $\tau > 1/2$. Then*

$$m_{\mathrm{ADM}}(M,g) = \lim_{\rho\to\infty} \frac{2}{\operatorname{area}(S_\rho)}\left(\operatorname{vol}(B_\rho) - \frac{\operatorname{area}(S_\rho)^{3/2}}{6\sqrt{\pi}}\right).$$

The next result was also proposed by G. Huisken [23, 24]. A detailed proof, following the ideas of G. Huisken, was given by J. Jauregui and D. Lee as theorem 3 in [29].

THEOREM C.2. *Let (M, g) be a complete Riemannian 3-manifold with nonnegative scalar curvature that is asymptotically flat of rate $\tau > 1/2$. Then*

$$m_{\mathrm{ADM}}(M,g) = \lim_{V\to\infty} \frac{2}{A(V)}\left(V - \frac{A(V)^{3/2}}{6\sqrt{\pi}}\right).$$

We recall from [29] that the inequality

$$m_{\mathrm{ADM}}(M,g) \le \limsup_{V\to\infty} \frac{2}{A(V)}\left(V - \frac{A(V)^{3/2}}{6\sqrt{\pi}}\right)$$

follows from Lemma C.1 and the elementary observation that, for every $V > 0$, the function

$$x \mapsto \frac{2}{x}\left(V - \frac{x^{3/2}}{6\sqrt{\pi}}\right)$$

is decreasing on $(0, \infty)$. We include a short new proof of the reverse inequality

$$m_{\mathrm{ADM}}(M,g) \ge \lim_{V\to\infty} \frac{2}{A(V)}\left(V - \frac{A(V)^{3/2}}{6\sqrt{\pi}}\right) \tag{C.1}$$

below. We also refer to the recording of the Marston Morse lecture given by G. Huisken [24] for the original argument.

ALTERNATIVE PROOF OF (C.1). Assume first that $\partial M = \varnothing$.

Let $V > 0$ large be such that $A'(V)$ exists. An isoperimetric region Ω of volume V is connected with connected, outward area-minimizing boundary Σ of constant mean curvature $A'(V) = H > 0$. Using the work of G. Huisken and T. Ilmanen [25] as stated in Lemma E.1, we see that

$$\begin{aligned}&\sqrt{\frac{A(V)}{16\pi}}\left(1 - \frac{1}{16\pi}A'(V)^2 A(V)\right)\\ &\quad = \sqrt{\frac{\operatorname{area}(\Sigma)}{16\pi}}\left(1 - \frac{1}{16\pi}H^2 \operatorname{area}(\Sigma)\right) \leq m_{\mathrm{ADM}}.\end{aligned}$$

From this, we compute that

$$\begin{aligned}\left(V - \frac{A(V)^{3/2}}{6\sqrt{\pi}}\right)' &= 1 - \frac{1}{4\sqrt{\pi}}A'(V)A(V)^{1/2}\\ &= \frac{1 - \frac{1}{16\pi}A'(V)^2 A(V)}{1 + \frac{1}{4\sqrt{\pi}}A'(V)A(V)^{1/2}} \leq \frac{4\sqrt{\pi}A(V)^{-1/2}}{1 + \frac{1}{4\sqrt{\pi}}A'(V)A(V)^{1/2}} m_{\mathrm{ADM}}.\end{aligned}$$

Using the remarks following Lemma B.2, we see that $A'(V)\sqrt{A(V)}$ approaches $4\sqrt{\pi}$ as $V \to \infty$. It follows that the above expression is bounded above by

$$\frac{1}{2}A'(V)m_{\mathrm{ADM}}(1 + o(1))$$

as $V \to \infty$. Using that the isoperimetric profile is absolutely continuous, it follows that

$$\limsup_{V\to\infty} \frac{2}{A(V)}\left(V - \frac{A(V)^{3/2}}{6\sqrt{\pi}}\right) \leq m_{\mathrm{ADM}}.$$

In the general case, where $\partial M \neq \varnothing$, we work with the (unique) large component of a large isoperimetric region instead. We omit the necessary but purely formal modifications of the proof. □

COROLLARY C.3 (Sharp isoperimetric inequality). *Let (M, g) be an asymptotically flat Riemannian 3-manifold with nonnegative scalar curvature. Let $\Omega \subset M$ be a compact region. Then*

$$\operatorname{vol}(\Omega) \leq \frac{\operatorname{area}(\partial\Omega)^{3/2}}{6\sqrt{\pi}} + \frac{m_{\mathrm{ADM}}}{2}\operatorname{area}(\partial\Omega) + o(1)\operatorname{area}(\partial\Omega) \tag{C.2}$$

as $\operatorname{vol}(\Omega) \to \infty$.

Appendix D Canonical Foliation

Here we collect results on the existence and uniqueness of a canonical foliation through stable constant mean curvature spheres of the end of an asymptotically flat Riemannian 3-manifold (M, g) with positive mass. The asymptotic assumptions

in the discussion are tailored to our applications. We refer to [33] and [34, sec. 5] for more general results.

The results discussed below depart from the pioneering work of G. Huisken and S.-T. Yau [26] and of J. Qing and G. Tian [36] for initial data *asymptotic to Schwarzschild*. We also mention the crucial intermediate results by L.-H. Huang [21] for *asymptotically even data*. We refer to the recent articles [9, 22, 30, 33] for an overview of the literature on this subject.

The following uniqueness and existence results are, in the stated generality, due to C. Nerz [34, sec. 5]. Let (M, g) be a Riemannian 3-manifold that is asymptotically flat at rate $\tau > 1/2$ and has nonnegative scalar curvature and $m_{\mathrm{ADM}} > 0$. There are a number $H_0 > 0$, a compact subset $C \subset M$ with $B_1 \subset C$, and a diffeomorphism

$$\Phi : (0, H_0) \times \mathbb{S}^2 \to M \setminus C$$

such that

$$\Sigma^H = \Phi(\{H\} \times \mathbb{S}^2)$$

is a constant mean curvature sphere with mean curvature $H > 0$ for every $H \in (0, H_0)$. In the chart at infinity (1.1),

$$(H/2)\, \Sigma^H \to S_1(0) = \{x \in \mathbb{R}^3 : |x| = 1\}$$

smoothly as $H \searrow 0$. We have, by the remark preceding Proposition A.1 in [33], that

$$m_{\mathrm{ADM}} = \lim_{H \searrow 0} \sqrt{\frac{\mathrm{area}(\Sigma^H)}{16\pi}} \left(1 - \frac{H^2\, \mathrm{area}(\Sigma^H)}{16\pi}\right). \tag{D.1}$$

Moreover, Σ^H is the unique stable constant mean curvature sphere of mean curvature H that is geometrically close to the coordinate sphere $S_{2/H}(0)$ in the chart at infinity (1.1).

S. Ma has shown in [30] that under the stronger decay assumptions on the metric stated here as (3.1) and (3.2), the compact subset $C \subset M$ above can be chosen so that each leaf Σ^H of the canonical foliation is the only stable constant mean curvature sphere of mean curvature $H \in (0, H_0)$ that encloses C.

Appendix E A Priori Estimates for the Hawking Mass

The next lemma due to G. Huisken and T. Ilmanen is extracted from section 6 in [25]. Recall from section 4 in [25] that M is diffeomorphic to the complement in $\mathbb{R}^3$ of a finite union of open balls with disjoint closures. Fix a complete Riemannian manifold $(\widehat{M}, \widehat{g})$ with $\widehat{M} \cong \mathbb{R}^3$ that contains (M, g) isometrically. We think of M as being included in $\widehat{M}$ below.

LEMMA E.1 ([25]). *Assume (M, g) to be a complete Riemannian 3-manifold that is asymptotically flat with nonnegative scalar-curvature. Let $\Sigma \subset M$ be a connected closed surface that is outward area-minimizing in $(\widehat{M}, \widehat{g})$. Then*

$$\sqrt{\frac{\text{area}(\Sigma)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2\, d\mu\right) \leq m_{\text{ADM}}.$$

COROLLARY E.2. *Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat with nonnegative scalar curvature. The outer boundary $\Sigma = \partial\Omega \setminus \partial M$ of the unique large component Ω of a large isoperimetric region Ω_V in (M, g) is connected and outward area-minimizing in $(\widehat{M}, \widehat{g})$. In particular,*

$$\sqrt{\frac{\text{area}(\Sigma)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2\, d\mu\right) \leq m_{\text{ADM}}.$$

PROOF. We have already seen in Appendix B that Σ is connected. Let $\widehat{\Omega} \subset \widehat{M}$ be the least area enclosure of Ω in $(\widehat{M}, \widehat{g})$. Recall from e.g. Theorem 1.3 in [25] that the boundary $\widehat{\Sigma}$ of $\widehat{\Omega}$ is $C^{1,1}$ and smooth away from the coincidence set $\widehat{\Sigma} \cap \Sigma$. Assume that $\widehat{\Omega} \neq \Omega$. It follows that the volume of $(M \cap \widehat{\Omega}) \cup \Omega^{\text{res}}$ is strictly larger than that of the isoperimetric region $\Omega \cup \Omega^{\text{res}}$ so that by the monotonicity of the isoperimetric profile of (M, g) its boundary area is less. A cut-and-paste argument using that the area of $\widehat{\Sigma}$ is less than that of Σ shows otherwise—a contradiction. □

COROLLARY E.3. *Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat with nonnegative scalar curvature. Consider $\Sigma \subset M$ a large connected stable constant mean curvature surface. Then,*

$$\sqrt{\frac{\text{area}(\Sigma)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2\, d\mu\right) \leq (1 + o(1))m_{\text{ADM}}$$

as $\text{area}(\Sigma) \to \infty$.

PROOF. This follows exactly as before when Σ is outer-minimizing. To handle the case when Σ is not outer-minimizing, we can apply the previous argument to the outer-minimizing hull of Σ and then use the fact that after rescaling to unit size, Σ is close to a round sphere (and thus the difference in area between Σ and its outer-minimizing hull is small). The details are given in [11, prop. D.1]. □

D. Christodoulou and S.-T. Yau [13] have proven (E.2) below. Estimate (E.1) follows from a variation of their argument as in the proof of theorem 6 in [37].

LEMMA E.4 (cf. [13,37]). *Let $\Sigma \subset M$ be a connected closed stable constant mean curvature surface in a Riemannian 3-manifold (M, g). Then*

$$H^2\,\text{area}(\Sigma) + \frac{2}{3}\int_\Sigma (R + |\mathring{h}|^2) d\mu \leq \frac{64\pi}{3}. \tag{E.1}$$

When Σ is a sphere, then

$$H^2 \operatorname{area}(\Sigma) + \frac{2}{3}\int_\Sigma (R + |\mathring{h}|^2) d\mu \leq 16\pi. \tag{E.2}$$

Here, R denotes the ambient scalar curvature and H and $\mathring{h}$ denote, respectively, the constant scalar mean curvature and the trace-free part of the second fundamental form of Σ with respect to a choice of unit normal, and $d\mu$ is the area element of Σ with respect to the induced metric.

Appendix F Elementary Growth Estimates

The elementary and well-known fact stated in the lemma below follows from an explicit "cut and paste" argument by comparison with balls B_ρ for $\rho > 1$ large.

LEMMA F.1. *Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat. There is a constant $c > 0$ depending only on (M, g) such that, for every isoperimetric region $\Omega \subset M$,*

$$\operatorname{area}(B_\rho \cap \partial\Omega) \leq c\rho^2$$

for all $\rho > 1$.

The following lemma is a standard consequence of the layer-cake representation of a function. We frequently apply this result in conjunction with the previous lemma to surfaces $\Sigma = \partial\Omega$ where $\Omega \subset M$ is an isoperimetric region.

LEMMA F.2. *Let (M, g) be an asymptotically flat 3-manifold. Let $\Sigma \subset M$ be a surface such that, for some $c > 0$,*

$$\operatorname{area}(B_\rho \cap \Sigma) \leq c\rho^2$$

for all $\rho > 1$. Then, for $\alpha > 0$ and $1 < \sigma \leq \rho$,

$$\int_{\Sigma\cap(B_\rho\setminus B_\sigma)} |x|^{-\alpha} d\mu \leq \frac{\operatorname{area}(\Sigma \cap (B_\rho \setminus B_\sigma))}{\rho^\alpha} + c\alpha \int_\sigma^\rho t^{1-\alpha}\, dt.$$

Appendix G Geometry in the Asymptotically Flat End

Consider a Riemannian metric

$$g = g_{ij}\, dx^i \otimes dx^j \quad \text{where } g_{ij} = \delta_{ij} + \sigma_{ij}$$

on $\mathbb{R}^3$ such that

$$|x||\sigma_{ij}| + |x|^2|\partial_k \sigma_{ij}| = O(|x|^{-\tau}) \quad \text{as } |x| \to \infty$$

for some $\tau > 1/2$. We denote the Euclidean background metric by

$$\overline{g} = \delta_{ij}\, dx^i \otimes dx^j.$$

Let $\Sigma \subset \mathbb{R}^3$ be a two-sided surface. The unit normal, the second fundamental form, the trace-free second fundamental form, the mean curvature, and the induced surface measure of Σ are denoted by ν, h, $\mathring{h}$, H, and μ, respectively. These

geometric quantities can also be computed with respect to the standard Euclidean metric $\overline{g}$. To distinguish these Euclidean quantities from those with respect to the curved metric, we denote them using an additional bar: $\overline{\nu}$, $\overline{h}$, $\overline{\mathring{h}}$, $\overline{H}$, and $\overline{\mu}$. A standard computation as in, e.g., [25, p. 418] shows how to compare the respective quantities:

$$\begin{aligned}
\nu(x) - \overline{\nu}(x) &= O(|x|^{-\tau}),\\
h(x) - \overline{h}(x) &= O(|h(x)||x|^{-\tau}) + O(|x|^{-1-\tau}),\\
H - \overline{H}(x) &= O(|h(x)||x|^{-\tau}) + O(|x|^{-1-\tau}), \qquad \text{(G.1)}\\
\mathring{h}(x) - \overline{\mathring{h}}(x) &= O(|h(x)||x|^{-\tau}) + O(|x|^{-1-\tau}), \qquad \text{(G.2)}\\
d\mu - d\overline{\mu} &= O(|x|^{-1-\tau})\, d\overline{\mu}.
\end{aligned}$$

Appendix H The Hawking Mass of Outlying Spheres

We continue with the notation of Appendix G. Let $\delta > 0$. Let $\Sigma \subset \mathbb{R}^3$ be geometrically close to a coordinate sphere $S_\rho(a)$ with $|a| > (1+\delta)\rho$ and $\rho > 1$ large. More precisely, we ask that the rescaled surface

$$\rho^{-1}\,\Sigma = \{\rho^{-1}\, x : x \in \Sigma\}$$

be C^2-close to a coordinate sphere of radius 1 in $\mathbb{R}^3$. We claim that

$$m_H(\Sigma) = o(1)$$

as $\rho \to \infty$. To see this, we follow the strategy of G. Huisken and T. Ilmanen in their proof of the asymptotic comparison lemma 7.4 in [25]. We use the positivity of a term dropped in [25] in conjunction with estimates of C. De Lellis and S. Müller [14] to handle an additional technical difficulty brought about by our weaker decay assumptions $\tau > 1/2$. All integrals below are with respect to the Euclidean background metric unless explicitly noted otherwise.

Let $r > 0$ so that

$$\operatorname{area}_{\overline{g}}(\Sigma) = 4\pi r^2.$$

Note that r and ρ are comparable. Following G. Huisken and T. Ilmanen [25, (7.11)], we compute

$$\begin{aligned}
&16\pi - \int_\Sigma H^2\, d\mu\\
&\quad = 16\pi - \int_\Sigma \overline{H}^2 + \int_\Sigma \Big(-\frac{1}{2} H^2 \operatorname{tr}_\Sigma \sigma + 2Hg(\sigma, h) - H^2 \sigma(\nu,\nu)\\
&\qquad\qquad + 2H \operatorname{tr}_\Sigma (\nabla_\cdot \sigma)(\nu, \cdot) - H \operatorname{tr}_\Sigma \nabla_\nu \sigma \Big) d\mu\\
&\quad + O\int_\Sigma |\sigma|^2 |h|^2 + O\int_\Sigma |\partial \sigma|^2.
\end{aligned}$$

The error terms are both $O(r^{-2\tau})$, since

$$\int_\Sigma |h|^2\, d\mu = O(1).$$

By the Gauss equation and the Gauss-Bonnet formula,

$$16\pi - \int_\Sigma \bar{H}^2 = -2\int_\Sigma |\mathring{\bar{h}}|^2_{\bar{g}}.$$

Using this in the above equation and computing as in [25, p. 420], we arrive at

$$\begin{aligned}
&16\pi - \int_\Sigma H^2\, d\mu \\
&\quad = -2\int_\Sigma |\mathring{\bar{h}}|^2_{\bar{g}} \\
&\qquad + \frac{1}{r}\int_\Sigma (H\,\mathrm{tr}_\Sigma\,\sigma - 2H\sigma(\nu,\nu) + 4\,\mathrm{tr}_\Sigma(\nabla_\cdot\sigma)(\nu,\cdot) - 2\,\mathrm{tr}_\Sigma\,\nabla_\nu\sigma)d\mu \\
&\qquad + O\int_\Sigma |H - 2/r|(H|\sigma| + |\partial\sigma|) + O\int_\Sigma H|\mathring{h}||\sigma| + O(r^{-2\tau}).
\end{aligned}$$

Finally, integrating by parts as in Huisken-Ilmanen (7.15), we find

$$\int_\Sigma 2\,\mathrm{tr}_\Sigma(\nabla_\cdot\sigma)(\nu,\cdot)d\mu = \int_\Sigma (2H\sigma(\nu,\nu) - H\,\mathrm{tr}_\Sigma\,\sigma)d\mu + O\int_\Sigma |\mathring{h}||\sigma|$$

so that

$$\begin{aligned}
16\pi - \int_\Sigma H^2 d\mu = -2\int_\Sigma |\mathring{\bar{h}}|^2_{\bar{g}} &+ O\int_\Sigma |H - 2/r|(H|\sigma| + |\partial\sigma|) \\
&+ O\int_\Sigma H|\mathring{h}||\sigma| + O(r^{-2\tau}) \\
&+ \frac{2}{r}\int_\Sigma (\mathrm{tr}(\nabla_\cdot\sigma)(\nu,\cdot) - \mathrm{tr}\,\nabla_\nu\sigma)d\mu.
\end{aligned}$$

Using that the scalar curvature is integrable and that Σ is outlying and divergent as $r \to \infty$, we see that the "mass integral" on the second line is $o(r^{-1})$. Using (G.1) and the trivial estimate $|h(x)| = O(r^{-1})$, we may rewrite the above expression as

$$\begin{aligned}
&16\pi - \int_\Sigma H^2 d\mu \\
&\quad = -2\int_\Sigma |\mathring{\bar{h}}|^2_{\bar{g}} + O\int_\Sigma |\bar{H} - 2/r|(|\sigma|/r + |\partial\sigma|) \\
&\qquad + O\int_\Sigma H|\mathring{h}||\sigma| + O\int_\Sigma (|\sigma|/r + |\partial\sigma|)^2 + o(r^{-1}).
\end{aligned}$$

It is clear that this additional error term is $o(r^{-1})$. Simplifying, we find

$$16\pi - \int_\Sigma H^2 \, d\mu = -2\int_\Sigma |\mathring{\bar{h}}|^2_{\bar{g}} + O\left(r^{-1-\tau}\int_\Sigma |\bar{H} - 2/r|\right) + O\left(r^{-1-\tau}\int_\Sigma |\mathring{h}|\right) + o(r^{-1}).$$

Using now that $|\mathring{h}(x)| = |\mathring{\bar{h}}(x)|_{\bar{g}} + O(r^{-1-\tau})$ by (G.2), we obtain

$$16\pi - \int_\Sigma H^2 d\mu = -2\int_\Sigma |\mathring{\bar{h}}|^2_{\bar{g}} + O\left(r^{-1-\tau}\int_\Sigma |\bar{H} - 2/r|\right) + O\left(r^{-1-\tau}\int_\Sigma |\mathring{\bar{h}}|\right) + o(r^{-1}).$$

Using Hölder's inequality, we find

$$16\pi - \int_\Sigma H^2 \, d\mu \leq -\int_\Sigma |\mathring{\bar{h}}|^2_{\bar{g}} + O\left(r^{-1-\tau}\int_\Sigma |\bar{H} - 2/r|\right) + o(r^{-1}).$$

Using now the estimate

$$\int_\Sigma (\bar{H} - 2/r)^2 \leq c\int_\Sigma |\mathring{\bar{h}}|^2_{\bar{g}} \tag{H.1}$$

due to C. De Lellis and S. Müller [14] where $c > 0$ is a universal constant, it follows that

$$O\left(r^{-1-\tau}\int_\Sigma |\bar{H} - 2/r|\right) \leq \int_\Sigma |\mathring{\bar{h}}|^2_{\bar{g}} + O(r^{-2\tau}).$$

Thus

$$16\pi - \int_\Sigma H^2 \, d\mu \leq o(r^{-1}) \quad \text{or, equivalently,} \quad m_H(\Sigma) = o(1)$$

as $r \to \infty$.

Remark H.1. Note that Σ is convex since it is geometrically close to $S_\rho(a)$. We mention that there are two alternative proofs of (H.1) in this case. One is due to G. Huisken and uses the inverse mean curvature flow of mean-convex, star-shaped regions in $\mathbb{R}^3$—see theorem 3.3 in [35]. A second proof is due to D. Perez [35, theorem 3.1], who proves (H.1) for convex hypersurfaces in $\mathbb{R}^{n+1}$ and proceeds via integration by parts with an appropriately chosen solution to the Poisson equation.

Appendix I Area and Volume of Large, Outlying Coordinate Spheres

The computations below follow closely the ideas leading to Corollary 2.3 in [18], which we have stated here as Lemma C.1.

Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat at rate $\tau = 1$.

Given $a \in \mathbb{R}^3$ and $\rho > 0$, we let

$$S_\rho(a) = \{x \in \mathbb{R}^3 : |x - a| = \rho\} \quad \text{and} \quad B_\rho(a) = \{x \in \mathbb{R}^3 : |x - a| < \rho\}.$$

We also abbreviate

$$n^i(x) = \frac{x^i - a^i}{|x-a|}.$$

Integration is with respect to the Euclidean metric in the chart at infinity (1.1) unless we indicate otherwise.

LEMMA I.1. *Let $a \in \mathbb{R}^3$ and $\rho > 0$. Then, as $|a| - \rho \to \infty$,*

$$\sum_{i,j=1}^{3} \int_{S_\rho(a)} (\partial_i \sigma_{ij} - \partial_j \sigma_{ii}) n^j = o(1).$$

PROOF. We compute

$$\begin{aligned} o(1) = \int_{B_\rho(a)} R &= \sum_{i,j=1}^{3} \int_{B_\rho(a)} (\partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii}) + O \int_{B_\rho(a)} \frac{1}{|x|^4} \\ &= \sum_{i,j=1}^{3} \int_{S_\rho(a)} (\partial_i \sigma_{ij} - \partial_j \sigma_{ii}) n^j + O\left(\frac{1}{|a|-\rho}\right) \end{aligned}$$

where we use the decay assumptions on the metric and the integrability of the scalar curvature. □

PROPOSITION I.2. *We have, as $\rho \to \infty$ and $|a| - \rho \to \infty$,*

$$\text{area}(S_\rho(a)) = 4\pi\rho^2 + \frac{1}{2} \int_{S_\rho(a)} (\delta^{ij} - n^i n^j) \sigma_{ij} + o(\rho), \tag{I.1}$$

$$\text{vol}(B_\rho(a)) = \frac{4\pi\rho^3}{3} + \frac{\rho}{4} \int_{S_\rho(a)} (\delta^{ij} - n^i n^j) \sigma_{ij} + o(\rho^2). \tag{I.2}$$

PROOF. Let $t \in [1, \rho]$. Note that

$$\text{area}(S_t(a)) = 4\pi t^2 + \frac{1}{2} \int_{S_t(a)} (\delta^{ij} - n^i n^j) \sigma_{ij} + O \int_{S_t(a)} \frac{1}{|x|^2}. \tag{I.3}$$

Now,

$$\begin{aligned} \int_{S_t(a)} \frac{1}{|x|^2} &= \int_0^{2\pi} \int_0^{\pi} \frac{t^2 \sin\phi}{|a|^2 + t^2 - 2|a| t \cos\phi} \, d\phi \, d\theta \\ &= \frac{\pi t}{|a|} \log\left(\frac{|a|+t}{|a|-t}\right) = o(t), \end{aligned} \tag{I.4}$$

which gives (I.1). Differentiating (I.3), we obtain

$$\begin{aligned} &\partial_t \, \text{area}(S_t(a)) \\ &\quad = 8\pi t + \frac{1}{2} \int_{S_t(a)} n^k \partial_k \big((\delta^{ij} - n^i n^j) \sigma_{ij} \big) + \frac{1}{t} \int_{S_t(a)} (\delta^{ij} - n^i n^j) \\ &\qquad + O \int_{S_t(a)} |x|^{-3} + \frac{1}{t} O \int_{S_t(a)} |x|^{-2}. \end{aligned}$$

Using that $n^k \partial_k n^i = 0$ for all $i = 1, 2, 3$, it follows that

$$\begin{aligned}\partial_t \operatorname{area}(S_t(a)) &= 8\pi t + \frac{1}{2}\int_{S_t(a)} n^k(\delta^{ij} - n^i n^j)\partial_k \sigma_{ij} + \frac{1}{t}\int_{S_t(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} \\ &\quad + O\int_{S_t(a)} \frac{1}{|x|^3} + \frac{1}{t} O\int_{S_t(a)} \frac{1}{|x|^2}.\end{aligned}$$

Observe that

$$\begin{aligned}&\int_{S_t(a)} n^i n^j n^k \partial_k \sigma_{ij} \\ &= \int_{S_t(a)} n^i n^k \partial_k \big(n^j \sigma_{ij}\big) \\ &= -\int_{S_t(a)} (\delta^{ik} - n^i n^k)\partial_k \big(n^j \sigma_{ij}\big) + \int_{S_t(a)} \delta^{ik}\partial_k \big(n^j \sigma_{ij}\big) \\ &= -\frac{2}{t}\int_{S_t(a)} n^i n^j \sigma_{ij} + \int_{S_t(a)} \delta^{ik} n^j \partial_k \sigma_{ij} + \frac{1}{t}\int_{S_t(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} \\ &= -\frac{2}{t}\int_{S_t(a)} n^i n^j \sigma_{ij} + \int_{S_t(a)} \delta^{ik} n^j \partial_k \sigma_{ij} + \frac{1}{t}\int_{S_t(a)} (\delta^{ij} - n^i n^j)\sigma_{ij},\end{aligned}$$

where we have used the first variation formula in the second equality. Thus

$$\begin{aligned}\partial_t \operatorname{area}(S_t(a)) &= 8\pi t + \frac{1}{2}\int_{S_t(a)} \delta^{ik} n^j (\partial_j \sigma_{ik} - \partial_i \sigma_{kj}) + \frac{1}{t}\int_{S_t(a)} n^i n^j \sigma_{ij} \\ &\quad + \frac{1}{2t}\int_{S_t(a)} (\delta^{ij} - n^i n^j)\sigma_{ij} + \frac{1}{t} O\int_{S_t(a)} \frac{1}{|x|^2} + O\int_{S_t(a)} \frac{1}{|x|^3}.\end{aligned} \tag{I.5}$$

Substituting (I.3) into (I.5) and applying Lemma I.1 leads to

$$\begin{aligned}\partial_t \operatorname{area}(S_t(a)) &= \frac{\operatorname{area}(S_t(a))}{t} + 4\pi t + \frac{1}{t}\int_{S_t(a)} n^i n^j \sigma_{ij} + O\int_{S_t(a)} \frac{1}{|x|^3} \\ &\quad + \frac{1}{t} O\int_{S_t(a)} \frac{1}{|x|^2} + o(1).\end{aligned} \tag{I.6}$$

Next, we give an estimate of $\operatorname{vol}(B_t(a))$. By the co-area formula,

$$\begin{aligned}\partial_t \operatorname{vol}(B_t(a)) &= \int_{S_t(a)} \frac{t\, d\mu}{\sqrt{g_{ij}(x^i - a^i)(x^j - a^j)}} \\ &= \operatorname{area}(S_t(a)) + \frac{1}{2}\int_{S_t(a)} n^i n^j \sigma_{ij} + O\int_{S_t(a)} \frac{1}{|x|^2}\end{aligned} \tag{I.7}$$

which in conjunction with (I.6) yields

$$\begin{aligned}&\partial_t \operatorname{area}(S_t(a))\\ &\quad= \frac{\operatorname{area}(S_t(a))}{t} + 4\pi t + \frac{1}{t}(2\partial_t \operatorname{vol}(B_t(a)) - 2\operatorname{area}(S_t(a)))\\ &\qquad + O\int_{S_t(a)} \frac{1}{|x|^3} + \frac{1}{t}\, O\int_{S_t(a)} \frac{1}{|x|^2} + o(1).\end{aligned} \tag{I.8}$$

It follows that

$$\begin{aligned}\partial_t(t\operatorname{area}(S_t(a))) &= 4\pi t^2 + 2\partial_t \operatorname{vol}(B_t(a))\\ &\quad + tO\int_{S_t(a)} \frac{1}{|x|^3} + O\int_{S_t(a)} \frac{1}{|x|^2} + o(t).\end{aligned}$$

Integrating from 1 to ρ yields

$$\begin{aligned}&\rho\operatorname{area}(S_\rho(a))\\ &\quad= \frac{4\pi\rho^3}{3} + 2\operatorname{vol}(B_\rho(a)) + O\int_1^\rho\int_{S_t(a)} \frac{t}{|x|^3}\,dt\\ &\qquad + O\int_1^\rho\int_{S_t(a)} \frac{1}{|x|^2}\,dt + o(\rho^2).\end{aligned} \tag{I.9}$$

A direct computation shows

$$\int_{S_t(a)} \frac{1}{|x|^3} = \frac{2\pi t}{|a|}\left(\frac{1}{|a|-t} - \frac{1}{|a|+t}\right)$$

so that

$$\int_1^\rho\int_{S_t(a)} \frac{t}{|x|^3}\,dt = o(\rho^2). \tag{I.10}$$

Similarly,

$$\int_1^\rho\int_{S_t(a)} \frac{1}{|x|^2}\,dt = o(\rho^2). \tag{I.11}$$

Substituting (I.10) and (I.11) into (I.9) gives (I.2). □

Appendix J Extension of Brakke Flow Across a Point

In this appendix, we follow the notation, conventions, and some of the ideas in T. Ilmanen's article [27].

Let (M, g) be an $(n+1)$-dimensional Riemannian manifold M. Consider a Radon measure μ on M and a nonnegative test function $\phi \in C_c^2(M)$. We first recall definition 6.2 in [27] of the quantity $\mathcal{B}(\mu, \phi)$: In all of the four cases,

(1) $\mu \lfloor \{\phi > 0\}$ is not a n-rectifiable Radon measure,
(2) $|\delta V| \lfloor \{\phi > 0\}$ is not a Radon measure on $\{\phi > 0\}$ where V is the varifold on $\{\phi > 0\}$ associated to μ,

(3) $|\delta V| \llcorner \{\phi > 0\}$ is singular with respect to $\mu \llcorner \{\phi > 0\}$, or
(4) $\int \phi |\mathbf{H}|^2 \, d\mu = \infty$ where $\mathbf{H} = \frac{d(\delta V)}{d\mu} \llcorner \{\phi > 0\}$,

we let $\mathcal{B}(\mu, \phi) = -\infty$. Else, we set

$$\mathcal{B}(\mu, \phi) = \int \big(-\phi |\mathbf{H}|^2 + \nabla_{T_x \mu} \phi \cdot \mathbf{H}\big) d\mu.$$

Recall from definition 6.3 in [27] that a family $\{\mu_t\}_{t \geq 0}$ of Radon measures on M is a *Brakke flow* if

$$\bar{D}_t \mu_t(\phi) \leq \mathcal{B}(\mu_t, \phi)$$

for all $t \geq 0$ and all nonnegative $\phi \in C^2_c(M)$.

We now discuss the extension of Brakke flows across a point. First, we define an injective map of Radon measures

$$\widehat{\mathcal{M}}(\mathbb{R}^{n+1} \setminus \{0\}) := \{\mu \in \mathcal{M}(\mathbb{R}^{n+1} \setminus \{0\}) : \mu(B_1(0) \setminus \{0\}) < \infty\} \to \mathcal{M}(\mathbb{R}^{n+1})$$

that extends $\mu \in \widehat{\mathcal{M}}(\mathbb{R}^{n+1} \setminus \{0\})$ to a Radon measure $\widehat{\mu} \in \mathcal{M}(\mathbb{R}^{n+1})$ such that

$$\widehat{\mu}(\{0\}) = 0.$$

This map restricts to an injection of integer n-rectifiable Radon measures

$$\widehat{\mathcal{M}}(\mathbb{R}^{n+1} \setminus \{0\}) \cap \mathcal{IM}_n(\mathbb{R}^{n+1} \setminus \{0\}) \to \mathcal{IM}_n(\mathbb{R}^{n+1}),$$

which in turn lifts to an injection of integer n-rectifiable varifolds

$$\{V \in \mathbf{IV}_n(\mathbb{R}^{n+1}) : \mu_V(B_1(0) \setminus \{0\}) < \infty\} \to \mathbf{IV}_n(\mathbb{R}^{n+1}),$$

which we denote by

$$V \mapsto \widehat{V}.$$

The extension of a stationary varifold across a point is not necessarily again stationary as shown by the following well-known example.

Example J.1. Let $\theta_1, \ldots, \theta_m \in \mathbb{R}$. Consider the rays $\ell_k = [0, \infty) \, e^{i\theta_k} \subset \mathbb{R}^2$. The varifold $V = \bigcup_{k=1}^m |\ell_k|$ is stationary as an element of $\mathcal{IM}_1(\mathbb{R}^2 \setminus \{0\})$. It is stationary as an element of $\mathcal{IM}_1(\mathbb{R}^2)$ if and only if $e^{i\theta_1} + \cdots + e^{i\theta_m} = 0$.

However, the phenomenon in the previous example is particular to dimension $n = 1$.

LEMMA J.2. *Let $n \geq 2$. There are radial functions $\chi_k \in C^\infty_c(B_1(0))$ with $0 \leq \chi_k \leq 1$ such that $\chi_k(x) = 1$ when $|x| < 1/(2k^2)$ and $\chi_k(x) = 0$ when $|x| > 1/k$ and constants $c_k \searrow 0$ with the following property: Let μ be a measure on $B_1(0) \setminus \{0\}$ such that, for some $c > 0$,*

$$\mu(B_\rho(0) \setminus \{0\}) \leq c\rho^n \tag{J.1}$$

for all $0 < \rho \leq 1$. Then

$$\frac{1}{c} \int |\nabla \chi_k|^2 \, d\mu \leq c_k.$$

Below we will often work with the functions

$$\varphi_k = 1 - \chi_k \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\}). \tag{J.2}$$

Note that $0 \le \varphi_k \to 1$ locally uniformly on $\mathbb{R}^{n+1} \setminus \{0\}$ and that, under the assumptions of the previous lemma,

$$\lim_{k\to\infty} \int |\nabla \varphi_k|^2 \, d\mu = 0.$$

We include a proof of the following, well-known result as preparation for Proposition J.5.

LEMMA J.3 (Extending stationary varifolds across a point). *Let $n \ge 2$. Let V be a stationary n-rectifiable varifold on $\mathbb{R}^{n+1} \setminus \{0\}$ such that $\mu_V(B_1(0) \setminus \{0\}) < \infty$. The extension $\widehat{V}$ of V across the origin is stationary as an n-rectifiable varifold on $\mathbb{R}^{n+1}$.*

PROOF. Let $\varphi_k \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ be cutoff functions as in (J.2). Note that (J.1) holds by the monotonicity formula for stationary varifolds as stated in (17.5) of [41]. Let $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$. Then

$$0 = \int_\Sigma \operatorname{div}_\Sigma(\varphi_k X) d\mu_V = \int_\Sigma \varphi_k \operatorname{div}_\Sigma X \, d\mu_V + \int_\Sigma X \cdot \operatorname{proj}_{T\Sigma} \nabla \varphi_k \, d\mu_V$$

because $V = \underline{v}(\Sigma, \theta)$ is stationary in $\mathbb{R}^n \setminus \{0\}$. As $k \to \infty$, the first term on the right tends to $(\delta \widehat{V})(X)$, while the second term tends to 0. □

A similar argument gives the following result.

LEMMA J.4. *Let $n \ge 2$. Let V be an n-rectifiable varifold on $\mathbb{R}^{n+1} \setminus \{0\}$ such that, for some $c > 0$,*

$$\mu_V(B_\rho(0) \setminus \{0\}) \le c\rho^n$$

for all $0 < \rho \le 1$. Let $\phi \in C^\infty_c(\mathbb{R}^{n+1})$ be a nonnegative function such that V is n-rectifiable on $\{x \in \mathbb{R}^{n+1} \setminus \{0\} : \phi(x) > 0\}$ with absolutely continuous first variation such that $\int \phi |\mathbf{H}|^2 \, d\mu_V < \infty$. The first variation of the extension $\widehat{V}$ of V across the origin is absolutely continuous on $\{x \in \mathbb{R}^{n+1} : \phi(x) > 0\}$.

PROOF. Let $X \in C^1_c(\{x \in \mathbb{R}^{n+1} : \phi(x) > 0\}, \mathbb{R}^{n+1})$. Let $\varphi_k \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ be cutoff functions as in (J.2). We compute that

$$\begin{aligned}(\delta V)(\varphi_k \sqrt{\phi} X) &= \int \varphi_k \sqrt{\phi}\, \mathbf{H} \cdot X \, d\mu_V \le \left(\int \phi |\mathbf{H}|^2 \, d\mu_V \right)^{\frac{1}{2}} \left(\int |X|^2 \, d\mu_V \right)^{\frac{1}{2}} \\ &\le C \, \| X \|_{L^2(\mu_V)}\end{aligned}$$

and

$$\begin{aligned}(\delta V)(\varphi_k \sqrt{\phi} X) = &\int \varphi_k \sqrt{\phi}, \operatorname{div}_\Sigma X \, d\mu_V + \int \sqrt{\phi} (\operatorname{proj}_{T\Sigma} \nabla \varphi_k) \cdot X \, d\mu_V \\ &+ \int \varphi_k (\operatorname{proj}_{T\Sigma} \nabla \sqrt{\phi}) \cdot X \, d\mu_V\end{aligned}$$

where $V = \underline{v}(\Sigma, \theta)$. In the last expression, the second term tends to 0 by Hölder's inequality and the construction of φ_k, while the first term tends to $(\delta \widehat{V})(X)$.

Finally, by Hölder's inequality, we may bound the third term by

$$\| |\nabla \phi|/\phi \|_{L^2(\mu_V)} \| X \|_{L^2(\mu_V)}.$$

The first quantity here can be bounded using the estimate in lemma 6.6 of [27]. Putting these facts together, we find that

$$|(\delta \widehat{V})(\sqrt{\phi} X)| \leq C \| X \|_{L^2(\mu_V)} = C \| X \|_{L^2(\mu_{\widehat{V}})}.$$

This completes the proof. □

We now turn to the situation for Brakke flows.

PROPOSITION J.5 (Extending Brakke flows across a point). *Let $n \geq 2$. Let $\{\mu_t\}_{t \geq 0}$ be a codimension 1 integral Brakke flow on $\mathbb{R}^{n+1} \setminus \{0\}$ such that, for some constant $c > 0$,*

$$\mu_t(B_\rho(0) \setminus \{0\}) \leq c\rho^n$$

for all $t \geq 0$ and $0 < \rho \leq 1$. Then $\{\widehat{\mu}_t\}_{t \geq 0}$ is a codimension 1 integral Brakke flow on $\mathbb{R}^{n+1}$.

PROOF. We use the cutoff functions $\varphi_k \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ from (J.2). Let $0 \leq \phi \in C_c^2(\mathbb{R}^{n+1})$.

Recall from lemma 6.6 in [27] that, on $\{x \in \mathbb{R}^{n+1} : \phi(x) > 0\}$,

$$\frac{|\nabla \phi|^2}{\phi} \leq 2 \max |\nabla^2 \phi|. \tag{J.3}$$

In a first step, we verify that, for all $t \geq 0$,

$$\lim_{k \to \infty} \mathcal{B}\big(\mu_t, \varphi_k^2 \phi\big) = \mathcal{B}(\widehat{\mu}_t, \phi). \tag{J.4}$$

Assume first that $\mathcal{B}(\widehat{\mu}_t, \phi) > -\infty$. Then

$$\begin{aligned} -\infty < \mathcal{B}\big(\widehat{\mu}_t, \varphi_k^2 \phi\big) &= \mathcal{B}\big(\mu_t, \varphi_k^2 \phi\big) \\ &= \int_{\{\varphi_k^2 \phi > 0\}} \big(-\varphi_k^2 \phi |\mathbf{H}|^2 + \varphi_k^2 (\mathrm{proj}_{T^\perp \Sigma_t} \nabla \phi) \cdot \mathbf{H} \\ &\qquad + 2\varphi_k \phi (\mathrm{proj}_{T^\perp \Sigma_t} \nabla \varphi_k) \cdot \mathbf{H} \big) d\mu_t \end{aligned} \tag{J.5}$$

for all k. The sum of the first two terms in (J.5) tends to $\mathcal{B}(\widehat{\mu}_t, \phi)$ as $k \to \infty$ by dominated convergence. Using the Hölder inequality and the properties of the functions φ_k, we see that the third term tends to 0. This establishes (J.4) when $\mathcal{B}(\widehat{\mu}_t, \phi) > -\infty$.

Assume now that $\liminf_{j \to \infty} \mathcal{B}(\mu_t, \varphi_k^2 \phi) > -\infty$. From (J.5), we see that

$$\int_{\{x \in \mathbb{R}^{n+1} \setminus \{0\} : \phi(x) > 0\}} \phi |\mathbf{H}|^2 \, d\mu_t = \limsup_{k \to \infty} \int_{\{\varphi_k^2 \phi > 0\}} \varphi_k^2 \phi |\mathbf{H}|^2 \, d\mu_t < \infty.$$

Lemma J.4 shows that $\mathcal{B}(\widehat{\mu}_t, \phi) > -\infty$. The claim follows from our earlier computation.

Finally, it is easy to see that $\lim_{k\to\infty} \mathcal{B}(\mu_t, \varphi_k^2\phi) = -\infty$ when $\mathcal{B}(\widehat{\mu}_t, \phi) = -\infty$.

Estimating (J.5) as in §6.7 of [27], we see that

$$\mathcal{B}\big(\mu_t, \varphi_k^2\phi\big) \le \int_{\{\varphi_k^2\phi>0\}} \bigg(-\frac{1}{4}\varphi_k^2\phi|\mathbf{H}|^2 + \frac{1}{2}\varphi_k^2\frac{|\nabla\phi|^2}{\phi} + 4\phi|\nabla\varphi_k|^2 \bigg) d\mu_t.$$

In combination with (J.3) and the uniform mass bounds, we obtain

$$\sup_k \sup_{t\ge 0} \mathcal{B}(\mu_t, \varphi_k^2\phi) = C(\phi) < \infty.$$

As in [27, §7.2(i)], in conjunction with the Brakke property for $\{\mu_t\}_{t\ge 0}$, this estimate implies that

$$t \mapsto \mu_t\big(\varphi_k^2\phi\big) - C(\phi)t$$

is nonincreasing. Passing to the limit as $k \to \infty$ and using the uniform mass bounds, it follows that

$$t \mapsto \widehat{\mu}_t(\phi) - C(\phi)t$$

is nonincreasing.

We now verify that

$$\bar{D}_t\widehat{\mu}_t(\phi) \le \mathcal{B}(\widehat{\mu}_t, \phi)$$

for all $t \ge 0$. The argument follows a step of the proof of theorem 7.1 on pp. 40–41 in [27] closely.

Fix $t \ge 0$. We may assume that $-\infty < \bar{D}_t\widehat{\mu}_t(\phi)$. Suppose that there are times $t_k \nearrow t$ with

$$\bar{D}_t\widehat{\mu}_t(\phi) \le \frac{\widehat{\mu}_{t_k}(\phi) - \widehat{\mu}_t(\phi)}{t_k - t} + o(1)$$

as $k \to \infty$. (The case where $t_k \searrow t$ is analogous.) By choosing the indices $\ell(k)$ to tend to infinity sufficiently fast, we arrange that

$$-\infty < \bar{D}_t\widehat{\mu}_t(\phi) \le \frac{\mu_t\big(\varphi_{\ell(k)}^2\phi\big) - \mu_{t_k}\big(\varphi_{\ell(k)}^2\phi\big)}{t - t_k} + o(1)$$

as $k \to \infty$. Arguing as on p. 40 in [27], we see that there are $s_k \in [t_k, t]$ with

$$-\infty < \bar{D}_t\widehat{\mu}_t(\phi) \le \mathcal{B}\big(\mu_{s_k}, \varphi_{\ell(k)}^2\phi\big) + o(1) \tag{J.6}$$

as $k \to \infty$. In particular,

$$\limsup_{k\to\infty} \int \varphi_{\ell(k)}^2\phi|\mathbf{H}|^2\, d\mu_{s_k} < \infty.$$

The measures $\mu_{s_k} \lfloor \{x \in \mathbb{R}^{n+1} \setminus \{0\} : \phi(x) > 0\}$ converge to $\mu_t \lfloor \{x \in \mathbb{R}^{n+1} \setminus \{0\} : \phi(x) > 0\}$ as $k \to \infty$ by the same argument as on page 41 of [27]. In fact,

the associated varifolds converge. It follows (cf. theorem 7.3 in [27]) that

$$\limsup_{k\to\infty} \mathcal{B}\big(\mu_{s_k}, \varphi^2_{\ell(k)}\phi\big) \leq \mathcal{B}(\widehat{\mu}_t, \phi).$$

Together with (J.6) this finishes the proof. □

Example J.1 shows that there is no analogue of Proposition J.5 when $n = 1$. Indeed, stationary varifolds are (constant) Brakke flows.

The following result and its proof should be compared with the constancy theorem for stationary varifolds [41, §41].

PROPOSITION J.6 (Constancy theorem). *Let $\{\mu_t\}_{t\geq 0}$ be an integral Brakke flow in $\mathbb{R}^3$ such that $\mu(0) = \mathcal{H}^2 \lfloor S_1(\xi)$. There is $T \in [0, \frac{1}{4}]$ such that*

$$\mu_t = \mathcal{H}^2 \lfloor S_{\sqrt{1-4t}}(\xi) \quad \textit{for all } t \in [0, T] \textit{ and } \mu_t = 0 \textit{ for all } t > T.$$

PROOF. The avoidance principle for Brakke flows as stated in §10.7 of [27] shows that

$$\operatorname{supp} \mu_t \subset S_{\sqrt{1-4t}}(\xi)$$

for all $t \in [0, \frac{1}{4}]$. The entropy of $S_1(\xi)$ is less than $3/2$. The entropy decreases along the Brakke flow by lemma 7 of [28]. Using that $\{\mu_t\}_{t\geq 0}$ is an integral Brakke flow, we see that for almost every $t \geq 0$ the measure μ_t has an approximate tangent plane with multiplicity 1 at x for μ_t–almost every x. Thus, for almost every $t \geq 0$, there is a measurable subset $\Sigma_t \subset S_{\sqrt{1-4t}}(\xi)$ with $\mu_t = \mathcal{H}^2 \lfloor \Sigma_t$.

We claim that Σ_t as a varifold with multiplicity 1 has absolutely continuous first variation for almost every $t \geq 0$. Indeed, by §7.2(ii) in [27], given $\phi \in C^\infty_c(\mathbb{R}^3)$ we have that $-\infty < \bar{D}_t \mu_t(\phi)$ for almost every $t \geq 0$. Let ϕ such that $\phi(x) = 1$ for all $x \in B_2(\xi)$. Using that $\bar{D}_t \mu_t(\phi) \leq \mathcal{B}(\mu_t, \phi)$ for a Brakke motion, the claim follows.

For such $t \geq 0$ and every $X \in C^1_c(\mathbb{R}^3, \mathbb{R}^3)$, we have that

$$\begin{aligned}(\delta\mu_t)(X) &= \int_{S_{\sqrt{1-4t}}(\xi)} \chi_{\Sigma_t} \operatorname{div}_{S_{\sqrt{1-4t}}(\xi)} X \\ &\leq c_1 \bigg(\int_{S_{\sqrt{1-4t}}(\xi)} |X|^2 \bigg)^{1/2} \leq c_2 \sup_{S_{\sqrt{1-4t}}(\xi)} |X|.\end{aligned}$$

It follows that the perimeter of Σ_t as a subset of $S_{\sqrt{1-4t}(\xi)}$ vanishes. The Poincaré inequality (as in lemma 6.4 of [41]) shows that either Σ_t or its complement in $S_{\sqrt{1-4t}}(\xi)$ is a set of 2–dimensional measure zero. We have thus shown that for almost every $t \geq 0$, either $\mu_t = \mathcal{H}^2 \lfloor S_{\sqrt{1-4t}}(\xi)$ or $\mu(t) = 0$.

By §7.2 (ii) of [27], we have that

$$\lim_{t\nearrow s} \mu_s(\phi) \geq \mu_s(\phi) \geq \lim_{t\searrow s} \mu_t(\phi)$$

for all $\phi \in C^\infty_c(\mathbb{R}^3)$ and all $s \geq 0$. This finishes the proof. □

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Bibliography

[1] Arnowitt, R.; Deser, S.; Misner, C. Coordinate invariance and energy expressions in general relativity. *Phys. Rev. (2)* **122** (1961), no. 3, 997–1006. doi:10.1103/PhysRev.122.997

[2] Bartnik, R. The mass of an asymptotically flat manifold. *Comm. Pure Appl. Math.* **39** (1986), no. 5, 661–693. doi:10.1002/cpa.3160390505

[3] Bavard, C.; Pansu, P. Sur le volume minimal de $\mathbf{R}^2$. *Ann. Sci. École Norm. Sup. (4)* **19** (1986), no. 4, 479–490.

[4] Brakke, K. A. *The motion of a surface by its mean curvature*. Mathematical Notes, 20. Princeton University Press, Princeton, N.J., 1978.

[5] Bray, H. L. The Penrose inequality in general relativity and volume comparison theorems involving scalar curvature. Ph.D. thesis, Stanford University, 1997.

[6] Brendle, S.; Eichmair, M. Large outlying stable constant mean curvature spheres in initial data sets. *Invent. Math.* **197** (2014), no. 3, 663–682. doi:10.1007/s00222-013-0494-8

[7] Carlotto, A.; Chodosh, O.; Eichmair, M. Effective versions of the positive mass theorem. *Invent. Math.* **206** (2016), no. 3, 975–1016. doi:10.1007/s00222-016-0667-3

[8] Carlotto, A.; Schoen, R. Localizing solutions of the Einstein constraint equations. *Invent. Math.* **205** (2016), no. 3, 559–615. doi:10.1007/s00222-015-0642-4

[9] Cederbaum, C.; Nerz, C. Explicit Riemannian manifolds with unexpectedly behaving center of mass. *Ann. Henri Poincaré* **16** (2015), no. 7, 1609–1631. doi:10.1007/s00023-014-0346-0

[10] Chodosh, O. Large isoperimetric regions in asymptotically hyperbolic manifolds. *Comm. Math. Phys.* **343** (2016), no. 2, 393–443. Available at: `http://dx.doi.org/10.1007/s00220-015-2457-y`

[11] Chodosh, O.; Eichmair, M. Global uniqueness of large stable CMC surfaces in asymptotically flat 3-manifolds. Preprint, 2017. arXiv:1703.02494 [math.DG]

[12] Chodosh, O.; Eichmair, M. On far-outlying constant mean curvature spheres in asymptotically flat Riemannian 3-manifolds. *J. Reine Angew. Math.* **767** (2020), 161–191. doi:10.1515/crelle-2019-0034

[13] Christodoulou, D.; Yau, S.-T. Some remarks on the quasi-local mass. *Mathematics and general relativity (Santa Cruz, CA, 1986)*, 9–14. Contemporary Mathematics, 71. American Mathematical Society, Providence, R.I., 1988. doi:10.1090/conm/071/954405

[14] De Lellis, C.; Müller, S. Optimal rigidity estimates for nearly umbilical surfaces. *J. Differential Geom.* **69** (2005), no. 1, 75–110. doi:10.4310/jdg/1121540340

[15] Eichmair, M.; Metzger, J. On large volume preserving stable CMC surfaces in initial data sets. *J. Differential Geom.* **91** (2012), no. 1, 81–102.

[16] Eichmair, M.; Metzger, J. Large isoperimetric surfaces in initial data sets. *J. Differential Geom.* **94** (2013), no. 1, 159–186.
[17] Eichmair, M.; Metzger, J. Unique isoperimetric foliations of asymptotically flat manifolds in all dimensions. *Invent. Math.* **194** (2013), no. 3, 591–630. doi:10.1007/s00222-013-0452-5
[18] Fan, X.-Q.; Shi, Y.; Tam, L.-F. Large-sphere and small-sphere limits of the Brown-York mass. *Comm. Anal. Geom.* **17** (2009), no. 1, 37–72. doi:10.4310/CAG.2009.v17.n1.a3
[19] Flores, A. M.; Nardulli, S. Continuity and differentiability properties of the isoperimetric profile in complete noncompact Riemannian manifolds with bounded geometry. Preprint, 2014. arXiv:1404.3245 [math.MG]
[20] Gilbarg, D.; Trudinger, N. S. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer, Berlin, 2001.
[21] Huang, L.-H. Foliations by stable spheres with constant mean curvature for isolated systems with general asymptotics. *Comm. Math. Phys.* **300** (2010), no. 2, 331–373. doi:10.1007/s00220-010-1100-1
[22] Huang, L.-H. On the center of mass in general relativity. *Fifth International Congress of Chinese Mathematicians. Part 1, 2*, 575–591. AMS/IP Studies in Advanced Mathematics, 51, pt. 1, vol. 2. American Mathematical Society, Providence, RI, 2012.
[23] Huisken, G. An isoperimetric concept for mass and quasilocal mass. *Oberwolfach Rep.*, vol. 3, 87–88, 2006. doi:10.4171/OWR/2006/02
[24] Huisken, G. "An Isoperimetric Concept for the Mass in General Relativity". Marston Morse Lecture, 2009. https://www.youtube.com/watch?v=4aG5L49p428
[25] Huisken, G.; Ilmanen, T. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differential Geom.* **59** (2001), no. 3, 353–437.
[26] Huisken, G.; Yau, S.-T. Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature. *Invent. Math.* **124** (1996), no. 1-3, 281–311. doi:10.1007/s002220050054
[27] Ilmanen, T. Elliptic regularization and partial regularity for motion by mean curvature. *Mem. Amer. Math. Soc.* **108** (1994), no. 520, x+90. doi:10.1090/memo/0520
[28] Ilmanen, T. Singularities of mean curvature flow of surfaces. Preprint, 1995.
[29] Jauregui, J. L.; Lee, D. A. Lower semicontinuity of mass under C^0 convergence and Huisken's isoperimetric mass. *J. Reine Angew. Math.* **756** (2019), 227–257. doi:10.1515/crelle-2017-0007
[30] Ma, S. On the radius pinching estimate and uniqueness of the CMC foliation in asymptotically flat 3-manifolds. *Adv. Math.* **288** (2016), 942–984. doi:10.1016/j.aim.2015.11.009
[31] Metzger, J. Foliations of asymptotically flat 3-manifolds by 2-surfaces of prescribed mean curvature. *J. Differential Geom.* **77** (2007), no. 2, 201–236.
[32] Nardulli, S. The isoperimetric profile of a smooth Riemannian manifold for small volumes. *Ann. Global Anal. Geom.* **36** (2009), no. 2, 111–131. doi:10.1007/s10455-008-9152-6
[33] Nerz, C. Foliations by stable spheres with constant mean curvature for isolated systems without asymptotic symmetry. *Calc. Var. Partial Differential Equations* **54** (2015), no. 2, 1911–1946. doi:10.1007/s00526-015-0849-7
[34] Nerz, C. Foliations by stable spheres with constant mean curvature for isolated systems without asymptotic symmetry. Preprint, 2016. arXiv:1408.0752 [math.AP]
[35] Perez, D. On nearly umbilical hypersurfaces. Ph.D. thesis, University of Zurich, 2011.
[36] Qing, J.; Tian, G. On the uniqueness of the foliation of spheres of constant mean curvature in asymptotically flat 3-manifolds. *J. Amer. Math. Soc.* **20** (2007), no. 4, 1091–1110. doi:10.1090/S0894-0347-07-00560-7
[37] Ritoré, M.; Ros, A. Stable constant mean curvature tori and the isoperimetric problem in three space forms. *Comment. Math. Helv.* **67** (1992), no. 2, 293–305.
[38] Ros, A. The isoperimetric problem. *Global theory of minimal surfaces*, 175–209. Clay Mathematics Proceedings, 2. American Mathematical Society, Providence, R.I., 2005.

[39] Schoen, R.; Yau, S.-T. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.* **65** (1979), no. 1, 45–76.

[40] Shi, Y. The isoperimetric inequality on asymptotically flat manifolds with nonnegative scalar curvature. *Int. Math. Res. Not. IMRN* (2016), no. 22, 7038–7050. doi:10.1093/imrn/rnv395

[41] Simon, L. *Lectures on geometric measure theory*. Proceedings of the Centre for Mathematical Analysis, Australian National University, 3. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[42] White, B. The size of the singular set in mean curvature flow of mean-convex sets. *J. Amer. Math. Soc.* **13** (2000), no. 3, 665–695. doi:10.1090/S0894-0347-00-00338-6

[43] White, B. A local regularity theorem for mean curvature flow. *Ann. of Math. (2)* **161** (2005), no. 3, 1487–1519. doi:10.4007/annals.2005.161.1487

[44] Witten, E. A new proof of the positive energy theorem. *Comm. Math. Phys.* **80** (1981), no. 3, 381–402.

[45] Yu, H. Isoperimetry for asymptotically flat 3-manifolds with positive ADM mass. Preprint, 2020. arXiv:2008.13307 [math.DG]

OTIS CHODOSH
Department of Mathematics
Building 380
Stanford University
Stanford, CA 94305
USA
E-mail: ochodosh@stanford.edu

YUGUANG SHI
Key Laboratory of Pure
and Applied Mathematics
School of Mathematical Sciences
Peking University
Beijing 100871
CHINA
E-mail: ygshi@math.pku.edu.cn

MICHAEL EICHMAIR
Faculty of Mathematics
University of Vienna
Oskar-Morgenstern-Platz 1
1090 Vienna
AUSTRIA
E-mail: michael.eichmair@univie.ac.at

HAOBIN YU
Department of Mathematics
Hangzhou Normal University
Hangzhou, 311121
CHINA
E-mail: yhbmath@hznu.edu.cn

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