

On classification of super-modular categories of rank 8

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We develop categorical and number-theoretical tools for the classification of super-modular categories. We apply these tools to obtain a partial classification of super-modular categories of rank 8. In particular we find three distinct families of prime categories in rank 8 in contrast to the lower rank cases for which there is only one such family.

Keywords: Super-modular categories; Galois action; classification by rank.

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1. Introduction

The classification of braided fusion categories (BFCs) stands as a formidable, yet enticing problem. There are many approaches to this problem with varying levels of preciseness and corresponding degrees of difficulty. As examples, one might try to classify by categorical dimension [11, 12, 14, 27, 39, 49], by Witt class [19, 20], by dimension of a generating object [1, 23, 24], or by rank [43, 44]. Each of these approaches has a different motivation and has seen some measure of success. For example, classifying by categorical dimension is related to the problem of classifying

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groups by their orders, while classifying by the dimension of a generating object is related to the classification of subfactors of finite index and depth. Classification by rank can be motivated physically: for condensed matter systems (e.g. topological phases of matter) modeled by braided fusion categories, the rank of the category corresponds to the number of distinguishable indecomposable particle species [40]. In this paper, we will be interested in classification of *unitary* BFCs by (low) rank as motivated by this physical interpretation.

Interestingly, the classification of low-rank fusion categories has not progressed very far; it is an open question whether there are finitely many fusion categories of each rank whereas with the braiding assumption rank-finiteness is known [15, 32]. The classification of pivotal fusion categories is complete up to rank 3 [42]. Adding the braiding assumption allows one to go a bit further. For example, there is a complete classification of pre-modular categories of rank at most 5 [9, 17]. One reason is as follows, which also serves to motivate this paper more specifically: It is well known [22] that if $\mathcal{B}$ is a braided fusion category and $\text{Rep}(G) \cong \mathcal{B}'_{\text{Tan}} \subset \mathcal{B}$ is the maximal Tannakian subcategory of the Müger center $\mathcal{B}'$ of $\mathcal{B}$, then the G -de-equivariantization $\mathcal{B}_G$ of $\mathcal{B}$ is either non-degenerate (has trivial Müger center) or slightly degenerate (has Müger center equivalent to sVec). For unitary BFCs this produces either a unitary modular tensor category (in the non-degenerate case) or a super-modular category (in the slightly degenerate case). Thus, if one is interested in unitary braided fusion categories “modulo finite group representations” one is led to study modular or super-modular categories.

Techniques for classifying modular categories are well-established ([16, 44]), and the classification up to rank 6 is nearly complete [18, 31]. Those methods cannot always be applied to general braided fusion categories. For example, a key approach in [16] is to use the representation theory of the modular group $\text{SL}(2, \mathbb{Z})$ to constrain the (modular) S - and (twist) T -matrices, whereas a super-modular category does not provide such representations as the S -matrix has determinant 0. On the other hand, there is an important conjecture known as the *minimal modular extension* (MME) conjecture [10, 19] that predicts that any super-modular category $\mathcal{B}$ can be embedded in a modular category $\mathcal{C}$ with $\dim(\mathcal{C}) = 2 \dim(\mathcal{B})$. Necessarily such a $\mathcal{C}$ will be a *spin modular* category, i.e. a modular category with a distinguished fermion f , and $\mathcal{B} = \langle f \rangle'$ is the Müger centralizer of the category generated by f .

Some techniques for classifying super-modular categories have been developed recently [10, 13], which lead to a complete classification up to rank 6. There are only 2 such categories: modulo trivial Deligne product constructions and up to fusion rules there are only two examples with rank ≤ 6 , and both of them belong to the a family of super-modular categories arising from quantum groups. A particularly useful technique is to formally condense the fermion at the level of fusion rules and modular data to obtain a fermionic quotient, which has naive fusion rules. These can be studied using the concept of a sVec -enriched fusion category [35, 46], but we will not pursue that here. In this paper, we make progress towards the classification of rank 8 super-modular categories using a stratification by Galois group and some

new techniques. We find many nontrivial examples in contrast to lower ranks, and we were unable to give a definitively complete classification — that is, we expect our list to be complete, but do not have an unconstrained proof.

For the following the (standard) notation is explained in Appendix A.

Theorem 1.1. (1) *The following are constructions of prime rank 8 super-modular categories as centralizers of a distinguished fermion in spin modular categories:*

- (a) $\text{PSU}(2)_{14} = \langle f \rangle' \subset \text{SU}(2)_{14}$ where f is the unique fermion corresponding to highest weight 7ϖ .
 - (b) $[\text{PSU}(2)_6 \boxtimes \text{PSU}(2)_6]_{\mathbb{Z}_2} = \langle \overline{(f, \mathbf{1})} \rangle' \subset ([\text{SU}(2)_6 \boxtimes \text{SU}(2)_6]_{\mathbb{Z}_2})_0$ where the $\mathbb{Z}_2$ -de-equivariantization in both cases is with respect to the boson (f, f) where f has highest weight 3ϖ , and $\overline{(f, \mathbf{1})}$ is the image of $(f, \mathbf{1})$ under de-equivariantization.
 - (c) $\langle f \rangle' \subset \text{SO}(12)_2$, where f is either of the fermions labeled by $2\varpi_5$ or $2\varpi_6$.
- (2) *Moreover, if we assume that the naive fusion rules $\{\hat{N}_{ij}^k = N_{ij}^k + N_{ij}^{fk}\}_{i,j,k}$ and the simple objects' dimensions d_i are each bounded by 14, then any prime super-modular category of rank 8 has the same fusion rules as one of the above.*

A more precise classification with less stringent bounds can be found in Sec. [3](#).

While we cannot claim this is a complete classification as we have placed bounds in some cases on naive fusion rule multiplicities or dimensions, it is possible that we have listed all possibilities. A counterexample would have large naive fusion multiplicities/dimensions compared to the known examples: the largest naive fusion multiplicity we find among fermionic quotients is 4 while the largest dimension of a simple object is $3 + 2\sqrt{2} \approx 5.8$. There is some precedent for these types of constraints: [\[30\]](#) gives a classification of low rank modular categories with bounded fusion multiplicities and [\[48\]](#) uses numerical techniques to study low rank modular categories with constrained categorical dimension. Although our result is not complete, we provide some new powerful methods for classifying super-modular categories, and illustrate the utility of the existing techniques.

In this paper we assume that the reader is familiar with the notions and basic properties of fusion, braided and modular tensor categories. For details, we refer to [\[2, 26\]](#). We provide the most relevant details and derive some general results in Sec. [2](#). In Sec. [3](#), we state our main results in detail and complete the first step of our classification, which determine the naive fusion rules. In Sec. [4](#), we lift the naive fusion rules to those of super-modular categories. In Appendix A, we explain some of the notations and give S - and T -matrices for a realization of each prime super-modular category of rank 8 that appear in our constrained classification.

2. Preliminaries

In this section, we first introduce the notion of super-modular categories and some of its properties. Most of the results can be found in ([\[10, 13\]](#)) and the references therein. Then we discuss the Galois symmetry for super-modular categories.

2.1. Centralizers

One may always define an S -matrix for any ribbon fusion category $\mathcal{B}$; however, it may be degenerate. The failure of modularity is encoded in the **Müger center** $\mathcal{B}'$, which is the subcategory of transparent objects. Here, an object X is called **transparent** if all the double braidings with X are trivial: $c_{Y,X}c_{X,Y} = \text{Id}_{X \otimes Y}$ for all $Y \in \mathcal{B}$. In general, we have the following notion of the centralizer of the braiding.

Definition 2.1. The **Müger centralizer** of a subcategory $\mathcal{D}$ of a pre-modular category $\mathcal{B}$ is the full fusion subcategory

$$\mathcal{D}' = C_{\mathcal{B}}(\mathcal{D}) = \{X \in \mathcal{B} \mid c_{Y,X}c_{X,Y} = \text{Id}_{X \otimes Y}, \forall Y \in \mathcal{D}\}.$$

The **Müger center** of $\mathcal{B}$ is the centralizer $\mathcal{B}'$ of $\mathcal{B}$ itself, that is, $\mathcal{B}' = C_{\mathcal{B}}(\mathcal{B})$.

While the notation $\mathcal{D}'$ is slightly ambiguous as it is relative to an ambient category, the context will always make it clear.

By a theorem of Bruguières [8], the simple objects in $\mathcal{B}'$ are those X with $\tilde{S}_{X,Y} = d_X d_Y$ for all simple Y , where $d_Y = \dim(Y) = \tilde{S}_{1,Y}$ is the categorical dimension of the object Y . The Müger center is **symmetric**, that is, $c_{Y,X}c_{X,Y} = \text{Id}_{X \otimes Y}$ for all $X, Y \in \mathcal{B}'$. Symmetric fusion categories have been classified by Deligne in terms of representations of supergroups [21]. In the case that $\mathcal{B}' \cong \text{Rep}(G)$ (i.e. $\mathcal{B}'$ is **Tannakian**), the de-equivariantization procedure of Bruguières [8] and Müger [36] yields a modular category $\mathcal{B}_G$ of dimension $\dim(\mathcal{B})/|G|$. Otherwise, by taking a maximal Tannakian subcategory $\text{Rep}(G) \subset \mathcal{B}'$, the de-equivariantization $\mathcal{B}_G$ has Müger center $(\mathcal{B}_G)' \cong \text{sVec}$, the symmetric fusion category of super-vector spaces. Generally, a braided fusion category $\mathcal{B}$ with $\mathcal{B}' \cong \text{sVec}$ as symmetric fusion category is called **slightly degenerate** [22], while if $\mathcal{B}' \cong \text{Vec}$, $\mathcal{B}$ is **non-degenerate**.

The symmetric fusion category sVec has a unique spherical structure compatible with unitarity and has S - and T -matrices: $S_{\text{sVec}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $T_{\text{sVec}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

From this point on we will assume that all our categories are unitary, so that sVec is a unitary spherical symmetric fusion category and all categorical dimensions are equal to the largest eigenvalue of the corresponding fusion matrix, i.e. the Frobenius–Perron dimension. In particular, for any simple object X , $d_X \geq 1$.

2.2. Definition of a super-modular category

Definition 2.2. A unitary pre-modular category $\mathcal{B}$ is called **super-modular** if $\mathcal{B}' \simeq \text{sVec}$.

Remark 1. In other terminology, we say $\mathcal{B}$ is super-modular if its Müger center is generated by a **fermion**, that is, an object f with $f^{\otimes 2} \cong 1$ and $\theta_f = -1$. We restrict to unitary categories both for mathematical convenience and for their physical significance. On the other hand, there is a non-unitary version sVec^- of sVec : the underlying (non-Tannakian) symmetric fusion category is the same, but with the other possible spherical structure, which leads to negative categorical dimensions.

We could define super-modular categories more generally as pre-modular categories $\mathcal{B}$ with Müger center equivalent to either of sVec or sVec^- . However, we do not know of any examples $\mathcal{B}$ with $\mathcal{B}' \cong \text{sVec}^-$ that are not simply of the form $\mathcal{C} \boxtimes \text{sVec}^-$ for some modular category $\mathcal{C}$.

Super-modular categories (or slight variations) have been studied from several perspectives, see [5, 7, 10, 13, 19, 20, 33, 49] for a few examples. An algebraic motivation for studying these categories is the following: any unitary braided fusion category is the equivariantization [22] of either a modular or super-modular category (see [45, Theorem 2]). Physically, super-modular categories provide a framework for studying fermionic topological phases of matter [10]. Topological motivations include the study of spin 3-manifold invariants ([3, 4, 45]) and (3+1)-TQFTs ([47]).

A braided fusion category is called **prime** if it contains no non-trivial non-degenerate braided fusion subcategories. Indeed, if $\mathcal{D} \subset \mathcal{B}$ with $\mathcal{D}$ non-degenerate and $\mathcal{B}$ a braided fusion category then $\mathcal{B} \cong \mathcal{D} \boxtimes \mathcal{D}'$ as braided fusion categories [22, Theorem 3.13] (see also [38]). As a special case of non-prime categories we say a super-modular category $\mathcal{C}$ is **split** if $\mathcal{C} \simeq \text{sVec} \boxtimes \mathcal{D}$ for some modular subcategory $\mathcal{D} \subset \mathcal{C}$, and otherwise $\mathcal{C}$ is **non-split**.

2.3. Spin modular categories

A **spin modular category** $\mathcal{C}$ is a modular category with a distinguished fermion. Let $\mathcal{C}$ be a spin modular category, with fermion f , (unnormalized) S -matrix $\tilde{S}$ and T -matrix T . Proposition II.3 of [10] provides a number of useful symmetries of $\tilde{S}$ and T :

- (1) $\tilde{S}_{f,\alpha} = \epsilon_\alpha d_\alpha$, where $\epsilon_\alpha = \pm 1$ and $\epsilon_f = 1$,
- (2) $\theta_{f\alpha} = -\epsilon_\alpha \theta_\alpha$,
- (3) $\tilde{S}_{f\alpha,\beta} = \epsilon_\beta \tilde{S}_{\alpha,\beta}$.

Remark 2. We have a canonical $\mathbb{Z}_2$ -grading $\mathcal{C}_0 \oplus \mathcal{C}_1$ with simple objects $X \in \mathcal{C}_0$ if $\epsilon_X = 1$ and $X \in \mathcal{C}_1$ when $\epsilon_X = -1$. The trivial component $\mathcal{C}_0$ is a super-modular category, since $\mathcal{C}'_0 = \langle f \rangle \cong \text{sVec}$.

Definition 2.3. Let $\mathcal{B}$ be a ribbon fusion category. A **minimal modular extension** MME of $\mathcal{B}$ is a modular category $\mathcal{C}$ such that $\mathcal{B} \subset \mathcal{C}$ and $\text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{B}') \text{FPdim}(\mathcal{B})$.

It is known that not every ribbon fusion category has a minimal modular extension [29]. Notice that if $\mathcal{B}$ is super-modular, a minimal modular extension of $\mathcal{B}$ is a spin modular category $(\mathcal{C}, f)$, where the fermion f is transparent in $\mathcal{B}$. It is conjectured (see [10, 19]) that *every* super-modular category has an MME, and it is known [10, 33] that if one exists there are precisely 16 inequivalent such extensions. A complete classification of rank ≤ 8 super-modular categories would include

a classification of $\text{rank} \leq 14$ spin modular categories, whereas if the MME conjecture is true a classification of spin modular categories of $\text{rank} \leq 16$ would imply a classification of super-modular categories of $\text{rank} \leq 8$.

2.4. Fermionic quotient

One interesting feature of super-modular categories $\mathcal{B}$ is that their S and T matrices have tensor decompositions:

Theorem 2.1. [10, Theorem 3.5] *Let $\mathcal{B}$ be a super-modular category, then $\tilde{S} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \hat{S}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{T}$, with $\hat{S}$ a symmetric invertible matrix and $\hat{T}$ a diagonal matrix.*

Recall that for the category sVec , we have $\tilde{S}_{\text{sVec}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $T_{\text{sVec}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Definition 2.4. $\hat{S}$ and $\hat{T}$ are called the S - and T -matrix of the fermionic quotient.

By the following proposition, pointed super-modular categories always split.

Proposition 2.1 ([22, Corollary A.19.]). *Let $\mathcal{B}$ be a pointed super-modular category, then $\mathcal{B} \simeq \mathcal{C} \boxtimes \text{sVec}$, where $\mathcal{C}$ is a pointed modular category.*

Let f be the transparent fermion in a super-modular category $\mathcal{B}$ with label set $\Pi_{\mathcal{B}}$. By the following lemma, we know that $f \otimes -$ is fixed-point-free on $\Pi_{\mathcal{B}}$. We will omit the $\otimes$ symbol and denote $f \otimes X$ simply as fX .

Lemma 2.1 ([37, Lemma 5.4]). *Let $\mathcal{B}$ be a super-modular category with transparent fermion f . Then $fX \not\cong X$ for any $X \in \Pi_{\mathcal{B}}$.*

As a direct consequence of the previous lemma, we have that super-modular categories have even rank.

Lemma 2.2. *Let $\mathcal{B}$ be a super-modular category with transparent fermion f . Then $fX \not\cong X^*$ for any $X \in \mathcal{B}$.*

Proof. By the balancing equation (given in by the third equality), we have that

$$\begin{aligned} -\theta_X d_X &= \theta_X \theta_f d_f d_X \\ &= \theta_X \theta_f S_{f,X} = \sum_Y N_{f,X}^Y d_Y \theta_Y \\ &= d_{fX} \theta_{fX} = d_X \theta_{fX}. \end{aligned}$$

Therefore, $\theta_{fX} = -\theta_X$. But since $\theta_{X^*} = \theta_X$, it follows that $fX \not\cong X^*$. $\square$

Thus there is a non-canonical partition of the label set $\Pi_{\mathcal{B}} = \Pi_0 \sqcup f\Pi_0$. We can arrange this partition such that $0 \in \Pi_0$ and such that $X^* \in \Pi_0$ if $X \in \Pi_0$. For a

rank $2r$ super-modular $\mathcal{B}$, we have $0, \dots, r-1 \in \Pi_0$ and $f = f0, \dots, f(r-1) \in f\Pi_0$, where fi is the label for fX_i , $i = 0, \dots, r-1$.

For $i, j, k \in \Pi_0$, we define the **naive fusion rule**

$$\hat{N}_{ij}^k = \dim \operatorname{Hom}(X_i \otimes X_j, X_k) + \dim \operatorname{Hom}(X_i \otimes X_j, f \otimes X_k) = N_{ij}^k + N_{ij}^{fk},$$

and corresponding **naive fusion matrices** $(\hat{N}_i)_{k,j} := \hat{N}_{i,j}^k$. The semisimple commutative algebra they generate will be denoted $\hat{\mathcal{U}}_{\mathcal{B}}$.

Proposition 2.2 ([13, Proposition 2.7]). *Let $\mathcal{B}$ be a super-modular category, then*

- (a) $\hat{S}$ is symmetric and $\hat{S}\bar{\hat{S}} = \frac{D^2}{2}I$.
- (b) $\hat{N}_i\hat{N}_j = \hat{N}_j\hat{N}_i$ for any $i, j \in \Pi_0$.
- (c) Let $\{x_i | i \in \Pi_0\}$ denote the basis of the algebra $\hat{\mathcal{U}}_{\mathcal{B}}$. Then the functions $\phi_i(x_j) := \hat{S}_{ij}/\hat{S}_{0i}$ for $0 \leq i \leq r-1$ form a set of orthogonal characters of $\hat{\mathcal{U}}_{\mathcal{B}}$. Thus $\hat{S}$ simultaneously diagonalizes the matrices $\hat{N}_i$.
- (d) We have a Verlinde type formula in this context given by $\hat{N}_{ij}^k = \frac{2}{D^2} \sum_{m \in \Pi_0} \frac{\hat{S}_{im}\hat{S}_{jm}\bar{\hat{S}}_{km}}{d_m}$.

Corollary 2.1. *Let $\mathcal{B}$ be a super-modular category and $\hat{N}_{ij}^k$ be its naive fusion rule, where $i, j, k \in \Pi_0$. We have the following symmetries*

$$\hat{N}_{ij}^k = \hat{N}_{ji}^k = \hat{N}_{ik^*}^{j^*} = \hat{N}_{i^*j^*}^{k^*}, \quad \hat{N}_{ij}^0 = \delta_{ij^*}$$

Proof. The first equation is a direct consequence of Proposition 2.2 (d). The second equation can be derived by combining (a) and (d) of Proposition 2.2. $\square$

Remark 3. One can combine Corollary 2.1 and [2, Eq. (2.4.3)] to get more relations for the fusion coefficients. For example, we have $N_{ij}^{fk} = N_{ik^*}^{fj^*}$. In fact, the result follows from $\hat{N}_{ij}^k = N_{ij}^k + N_{ij}^{fk} = N_{ik^*}^{j^*} + N_{ij}^{fk} = N_{ik^*}^{j^*} + N_{ik^*}^{fj^*} = \hat{N}_{ik^*}^{j^*}$.

Mimicking the proof for modular categories (see, e.g. [25, Lemma 1.2]), one can derive the following property of the dimensions for super-modular categories.

Corollary 2.2 ([49, Corollary 3.4]). *Let $\mathcal{B}$ be a super-modular category, then $d_i^2 | \frac{D^2}{2}$.*

Proof. By Proposition 2.2, we know that $\hat{S}\bar{\hat{S}} = \frac{D^2}{2}I$, hence, we have

$$\frac{D^2}{2} = \sum_{j \in \Pi_0} \hat{S}_{ij}\bar{\hat{S}}_{jk} = \sum_{j \in \Pi_0} \hat{S}_{ij}\hat{S}_{jk^*}.$$

The second equation comes from the fact that for pre-modular categories, we have $\bar{\hat{S}}_{ij} = \hat{S}_{ij^*}$ since we can embed them into their Drinfeld center. Therefore, we have

$\sum_{j \in \Pi_0} \frac{\hat{S}_{ij}}{d_j} \frac{\hat{S}_{jk^*}}{d_j} = \frac{D^2/2}{d_i^2}$. The result follows since the left-hand side is an algebraic integer. $\square$

The following property of the second Frobenius–Schur indicator for self-dual objects is useful in Sec. [3.2](#).

Lemma 2.3 ([\[13\]](#), [Lemma 2.8.](#)). *Let $\mathcal{B}$ be a super-modular category and X_i a simple object such that $X_i \cong X_i^*$ (i.e., X_i is self-dual), then*

$$\pm 1 = \nu_2(X_i) = \frac{2}{D^2} \sum_{j,k \in \Pi_0} \hat{N}_{j,k}^i d_j d_k \left(\frac{\theta_j}{\theta_k} \right)^2.$$

Corollary 2.3 (Balancing equation for super-modular categories). *For a super-modular category of rank $2r$, we have*

$$\theta_i \theta_j \hat{S}_{ij} = \sum_{k=0}^{r-1} (N_{i^*j}^k - N_{i^*j}^{fk}) \theta_k d_k.$$

Proof. The balancing equation [\[2\]](#) for a pre-modular category gives us

$$\begin{aligned} \theta_i \theta_j \hat{S}_{ij} &= \sum_{k=0}^{2r-1} N_{i^*j}^k \theta_k d_k \\ &= \sum_{k=0}^{r-1} N_{i^*j}^k \theta_k d_k + \sum_{k=r}^{2r-1} N_{i^*j}^k \theta_k d_k \\ &= \sum_{k=0}^{r-1} N_{i^*j}^k \theta_k d_k + \sum_{k=r}^{2r-1} N_{i^*j}^{fk} \theta_{fk} d_{fk} \\ &= \sum_{k=0}^{r-1} (N_{i^*j}^k - N_{i^*j}^{fk}) \theta_k d_k. \end{aligned}$$

□

2.5. Galois symmetries for super-modular categories

In this section, we discuss the Galois symmetry in the fermionic quotient of a super-modular category, which is parallel to the modular setting. We extend results that are well known for modular categories to this setting.

Let $\mathcal{B}$ be a super-modular category and $\hat{S}$, $\hat{T}$ and $\hat{N}_i$ defined as above. We have the following relation for the entries of $\hat{S}$ and $\hat{N}_i$ [\[13\]](#), Eq. (2.3):

$$\frac{\hat{S}_{ij} \hat{S}_{ik}}{\hat{S}_{0,i}} = \sum_{m \in \Pi_0} \hat{N}_{jk}^m \hat{S}_{im}. \quad (1)$$

This means that $\hat{\lambda}_{ij} := \frac{\hat{S}_{ij}}{\hat{S}_{0j}}$ are eigenvalues of the matrices $\hat{N}_j$ with eigenvectors $(\hat{S}_{im})_{m \in \Pi_0}$. Defining the diagonal matrix $(\hat{\Lambda}_i)_{jk} = \delta_{jk} \frac{\hat{S}_{ij}}{\hat{S}_{0j}}$, then Eq. [\(1\)](#) can be written as $\hat{N}_i \hat{S} = \hat{S} \hat{\Lambda}_i$ for all $i \in \Pi_0$.

Remark 4. Let $\mathbb{Q}(\hat{S})$ be the smallest field containing all elements of the S -matrix. Similarly to the modular setting, $\mathbb{Q}(\hat{S})$ is Galois over $\mathbb{Q}$. Define $\text{Gal}(\mathcal{B}) =$

$\text{Gal}(\mathbb{Q}(\hat{S})/\mathbb{Q})$. Then $\text{Gal}(\mathcal{B})$ is an abelian subgroup of $\mathfrak{S}_r$, where $2r$ is the rank of the corresponding super modular category and $\mathfrak{S}_r$ is the symmetric group on r letters. We will use σ for both the element of the Galois group $\text{Gal}(\mathcal{B})$ and its associated element in $\mathfrak{S}_r$. Indeed, since $\sigma(\frac{\hat{S}_{ik}}{\hat{S}_{0k}})$ is a character of $\hat{\mathcal{U}}_{\mathcal{B}}$ (see Proposition 2.2), the following defines $\sigma(k)$ for $k \in \Pi_0$:

$$\sigma\left(\frac{\hat{S}_{ik}}{\hat{S}_{0k}}\right) = \frac{\hat{S}_{i\sigma(k)}}{\hat{S}_{0\sigma(k)}} = \frac{\hat{S}_{i\sigma(k)}}{d_{\sigma(k)}}. \quad (2)$$

Lemma 2.4. *Let $\hat{S}$ be as above for a super-modular category $\mathcal{B}$.*

- (i) *Let $\sigma \in \text{Gal}(\mathcal{B})$. Then $\sigma(k)^* = \sigma(k^*)$ for all $k \in \Pi_0$.*
- (ii) *The algebraic integers $\hat{S}_{k,\sigma(0)}$ are real numbers.*
- (iii) *We have $\left|\frac{\hat{S}_{k,\sigma(0)}}{d_{\sigma(k)}}\right|^2 = 1$ for all k, σ .*

Proof. Let $\tau \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be complex conjugation. Now, since $\overline{\hat{S}_{ij}} = \hat{S}_{ij}^*$ we have

$$\begin{aligned} \frac{S_{j,k^*}}{d_{k^*}} &= \overline{\left(\frac{\hat{S}_{j,k}}{d_k}\right)} \\ &= \tau(\hat{S}_{j,k}/d_k) = \hat{S}_{j,\tau(k)}/d_{\tau(k)}. \end{aligned}$$

Thus τ sends the normalized k th column to the $\tau(k)$ th column which is also the k^* th column. Since $\text{Gal}(\mathcal{B})$ is abelian, we have $\sigma(k)^* = \tau\sigma(k) = \sigma\tau(k) = \sigma(k^*)$.

The second result now follows from the following computation:

$$\overline{\hat{S}_{k,\sigma(0)}} = \hat{S}_{k,\sigma(0)^*} = \hat{S}_{k,\sigma(0^*)} = \hat{S}_{k,(0)}.$$

For the third result, we compute

$$\begin{aligned} \sigma(D^2) &= 2 \sum_{j \in \Pi_0} \sigma(d_j)^2 = 2 \sum_{j \in \Pi_0} \sigma(d_j)\sigma(d_j^*) \\ &= 2 \sum_{j \in \Pi_0} \frac{\hat{S}_{j,\sigma(0)}}{d_{\sigma(0)}} \frac{\hat{S}_{j^*,\sigma(0)}}{d_{\sigma(0)}} \\ &= \frac{2}{d_{\sigma(0)}^2} \sum_{j \in \Pi_0} \hat{S}_{j,\sigma(0)}(\hat{S}_{j,\sigma(0)})^* = \frac{D^2}{d_{\sigma(0)}^2}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sigma(D^2) &= 2 \sum_{j \in \Pi_0} \sigma(\hat{S}_{j,k}\hat{S}_{j,k^*}) = 2 \sum_{j \in \Pi_0} \sigma(\hat{S}_{j,k})\sigma(\hat{S}_{j,k^*}) \\ &= 2 \sum_{j \in \Pi_0} \left(\frac{\hat{S}_{j,\sigma(k)}\hat{S}_{k,\sigma(0)}}{d_{\sigma(0)}d_{\sigma(k)}} \right) \left(\frac{S_{j,\sigma(k^*)}S_{k^*,\sigma(0)}}{d_{\sigma(0)}d_{\sigma(k^*)}} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\hat{S}_{k,\sigma(0)}\hat{S}_{k^*,\sigma(0)}}{d_{\sigma(0)}^2 d_{\sigma(k)} d_{\sigma(k^*)}} 2 \sum_{j \in \Pi_0} \hat{S}_{j,\sigma(k^*)} \hat{S}_{j,\sigma(k)} \\
&= \frac{\hat{S}_{k,\sigma(0)}\hat{S}_{k^*,\sigma(0)}}{d_{\sigma(0)}^2 d_{\sigma(k)} d_{\sigma(k^*)}} D^2.
\end{aligned}$$

Since $d_{\sigma(k^*)} = d_{\sigma(k)^*} = d_{\sigma(k)}$ and $\hat{S}_{k^*,\sigma(0)} = \overline{\hat{S}_{k,\sigma(0)}} = \hat{S}_{k,\sigma(0)}$, the result follows because $D^2/d_{\sigma(0)}^2$ is nonzero. $\square$

We can also derive a result parallel to [16, Eq. (2.12)] for the S -matrix of the fermionic quotient.

Corollary 2.4. *Let $\sigma \in \text{Gal}(\mathcal{B})$ and j, k the indices of simple objects in Π_0 . Then*

$$\sigma(\hat{S}_{j,k}) = \pm \frac{\hat{S}_{j,\sigma(k)}}{d_{\sigma(0)}}.$$

Moreover, we have the following symmetries:

$$\hat{S}_{j,k} = \pm \hat{S}_{\sigma(j),\sigma^{-1}(k)}. \quad (3)$$

Proof. By Eq. (2), we have

$$\begin{aligned}
\sigma(\hat{S}_{j,k}) &= \hat{S}_{j,\sigma(k)} \sigma(d_k) / d_{\sigma(k)}, \\
\sigma(d_k) &= \hat{S}_{k,\sigma(0)} / d_{\sigma(0)}.
\end{aligned}$$

In particular,

$$\sigma(\hat{S}_{j,k}) = \frac{\hat{S}_{j,\sigma(k)} \hat{S}_{k,\sigma(0)}}{d_{\sigma(0)} d_{\sigma(k)}}.$$

So it suffices to show that $\frac{\hat{S}_{k,\sigma(0)}}{d_{\sigma(k)}} = \pm 1$ which follows from Lemma 2.4. For Eq. (3), we use the symmetry of the $\hat{S}$ -matrix and apply $\sigma \circ \sigma^{-1}$ to the first equation. $\square$

Let $(\mathcal{C}, f)$ be a spin modular category, recall that the fermion f gives a grading $\mathcal{C}_0 \oplus \mathcal{C}_1$.

Lemma 2.5. *Let $(\mathcal{C}, f)$ be spin modular with (unnormalized) S -matrix S , and $\hat{S}$ the S -matrix for the fermionic quotient. Then $[\mathbb{Q}(S) : \mathbb{Q}(\hat{S})] = 2^n$, for some n .*

Proof. Denote by $S^{(0,0)}, S^{(0,1)} = [S^{(1,0)}]^T$ and $S^{(1,1)}$ the 2×2 blocks of the S -matrix S relative to the grading $\mathcal{C}_0 \oplus \mathcal{C}_1$. Suppose that $X_a, X_b \in \mathcal{C}_1$ so that $S_{b,a}$ is an entry in $S^{(1,1)}$. Then, since the normalized i th column $S_{i,a}/d_a$ is a character of the Grothendieck ring $K_0(\mathcal{C})$ for each i , we see that $(S_{b,a})^2 = d_a^2 \sum_j N_{b,a}^j S_{j,a}/d_a$. Since $N_{b,a}^j = 0$ if $X_j \in \mathcal{C}_1$ we find that $(S_{b,a})^2$ lies in the field generated by the entries of $S^{(0,1)}$. In particular, $[\mathbb{Q}(S^{(1,1)}) : \mathbb{Q}(S^{(0,1)})] = 2^k$ for some k , since every entry of $S^{(1,1)}$ satisfies a polynomial equation of degree ≤ 2 over $S^{(0,1)}$.

Now, let $S_{b,c}$ be an entry of $S^{(0,1)} = [S^{(1,0)}]^T$, i.e. $X_b \in \mathcal{C}_1$ and $X_c \in \mathcal{C}_0$. A similar argument shows that $(S_{b,c})^2$ lies in the field generated by $S^{(0,0)}$, so that $[\mathbb{Q}(S^{(0,1)}) : \mathbb{Q}(S^{(0,0)})] = 2^\ell$. Since $\mathbb{Q}(\hat{S}) = \mathbb{Q}(S^{(0,0)})$, the result follows. $\square$

Example 2.1. Consider the Ising modular category with label set $\{1, \sigma, \psi\}$. It is a spin modular category with fermion ψ . Its S -matrix is

$$\frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}.$$

The subcategory generated by 1 and ψ is sVec , and we have $[\mathbb{Q}(S) : \mathbb{Q}(S_{\text{sVec}})] = 2$.

Question 1. Is there a relationship between the Galois group of the S -matrix of a braided fusion category $\mathcal{B}$ and that of its Drinfeld center $\mathcal{Z}(\mathcal{B})$?

The following lemma can probably be generalized to non-self-dual categories, but we will only use it in the self-dual case:

Lemma 2.6. *Suppose that $\mathcal{B}$ is a self-dual super-modular category and z is a label in the fermionic quotient such that $d_z = 1$ and $\hat{S}_{z,z} \neq 1$. Then $\mathcal{B}$ contains a modular pointed subcategory equivalent to $\mathcal{C}(\mathbb{Z}_2, Q)$ (i.e. Sem or $\overline{\text{Sem}}$).*

Proof. The hypothesis immediately implies that $\mathcal{B}$ contains an invertible, self-dual simple object Z . Since $S_{Z,Z} = \hat{S}_{z,z} \neq 1$, the object Z is not self-centralizing, hence generates a modular subcategory of dimension 2. $\square$

Question 2. Can we drop the self-duality condition in the above, with the same conclusion?

2.6. Rank finiteness

The rank-finiteness property can be extended to categories that do not necessarily admit a spherical structure. It was recently proved that rank-finiteness holds for G -crossed braided fusion categories.

Theorem 2.2 ([32, Corollary 4.7.]). *There are finitely many equivalence classes of G -crossed braided fusion categories of any given rank.*

This motivates us to pursue a classification of low-rank super-modular categories parallel to [16, 44]. A classification of super-modular categories of rank ≤ 6 is given in [13]. It is shown, for example, that the fusion rules of any non-split super-modular category of rank ≤ 6 are the same as $\text{PSU}(2)_{4k+2}$ for $k = 0, 1$ and 2.

3. Classification of Super-modular Categories by Rank

3.1. Main results

Similarly to modular categories, the Galois group $\text{Gal}(\mathcal{B})$ of a super-modular category $\mathcal{B}$ defined in Sec. 2.5 is an abelian subgroup of the symmetric group $\mathfrak{S}_r$, where $2r$ is the rank of $\mathcal{B}$ (see Remark 4).

In this section, we consider the problem of classifying rank $2r = 8$ super-modular categories. If $\mathcal{B}$ is non-self dual, we can denote the four simple objects in Π_0 as $1, Y, X, X^*$. The naive fusion rules satisfy the relations in Corollary 2.1 and the argument in [44, Appendix A.2] works for this case. Therefore, we sometimes assume the super-modular categories are self-dual, in which case $\hat{S}$ has real entries.

The abelian subgroups (up to relabeling, but with 0 distinguished) G of $\mathfrak{S}_4$ are listed in the following table.

In this section, we determine the possible $\hat{S}$ -matrices for super-modular categories, and then derive the fusion rules in Sec. 4. We summarize our results into the following.

Theorem 3.1. *Suppose $\mathcal{B}$ is a rank 8 self-dual super-modular category and G is its Galois group as in Table 1 then:*

- If $G = \langle (23) \rangle, \langle (01), (23) \rangle$ or $\langle (123) \rangle$, then $\mathcal{B}$ does not exist.
- If $G = \langle (0) \rangle$, then $\mathcal{B}$ is **pointed**, i.e. of the form $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2, Q) \boxtimes \text{sVec}$.
- If $G = \langle (01) \rangle$, then $\mathcal{B}$ is **prime** and **weakly integral** with the same fusion rules as the centralizer of either fermion in $\text{SO}(12)_2$.
- If $G = \langle (01)(23), (02)(13) \rangle$, then $\mathcal{B}$ has the same fusion as $\text{Fib} \boxtimes \text{PSU}(2)_6$.
- If $G = \langle (0123) \rangle$ and $\hat{N}_{ij}^k < 14$, then $\mathcal{B}$ is **prime** and has the same fusion rules as $\text{PSU}(2)_{14}$.
- If $G = \langle (012) \rangle$ and $\hat{N}_{ij}^k < 21$, then $\mathcal{B}$ has the same fusion rules as $\text{PSU}(2)_7 \boxtimes \text{sVec}$.
- If $G = \langle (01)(23) \rangle$ and $d_i \leq 14$ for all i , then the fusion rules of $\mathcal{B}$ are the same as $[\text{PSU}(2)_6 \boxtimes \text{PSU}(2)_6]_{\mathbb{Z}_2}$ and is **prime**, $\text{Fib} \boxtimes \text{Fib} \boxtimes \text{sVec}$, $\text{Sem} \boxtimes \text{Fib} \boxtimes \text{sVec}$ or $\text{Sem} \boxtimes \text{PSU}(2)_6$.

In several cases the proofs in [44] for the classification of rank 4 modular use techniques and results that apply to super-modular categories as well, so we do not repeat the proof here. For many computations the Gröbner basis software in Maple is useful — we used Maple 2018 for our calculations.

Table 1. Abelian subgroups of $\mathfrak{S}_4$.

$\langle 1 \rangle$	$\langle (0) \rangle$
$\mathbb{Z}_2$	$\langle (01) \rangle, \langle (23) \rangle, \langle (01)(23) \rangle$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\langle (01)(23), (02)(13) \rangle, \langle (01), (23) \rangle$
$\mathbb{Z}_3$	$\langle (012) \rangle, \langle (123) \rangle$
$\mathbb{Z}_4$	$\langle (0123) \rangle$

3.2. $\hat{S}$ -matrices for rank 8

The naive fusion coefficients $\hat{N}_{ij}^k$ can be computed by the entries of $\hat{S}$ via the Verlinde formula (see Proposition 2.2 (d)). More precisely, to get the $\hat{N}_{ij}^k$'s, it suffices to determine the $\hat{S}$ -matrix.

Remark 5. We denote by ϕ_n the positive real root of the equation $x^2 - nx - 1 = 0$, where n is an integer, i.e. $\phi_n = \frac{n + \sqrt{n^2 + 4}}{2}$. If an algebraic number ϕ has conjugate $-\frac{1}{\phi}$, then ϕ must be of the form ϕ_n for some $n \in \mathbb{Z}$.

Theorem 3.2. *If $\mathcal{B}$ is a rank 8 non-self dual super-modular category, then the corresponding $\hat{S}$ -matrix, up to relabeling the simple objects, has the following form:*

$$\hat{S} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \pm i & \mp i \\ 1 & -1 & \mp i & \pm i \end{pmatrix}.$$

Proof. The proof in [44, Appendix A.2] carries through, *mutatis mutandis*. $\square$

Remark 6. Having dispensed with the non-self-dual case, we assume for the rest of this section that all categories are self-dual. In particular the naive fusion coefficients are cyclically symmetric (see Corollary 2.1), so we will denote $\hat{N}_{ij}^k$ by $n_{i,j,k}$.

Theorem 3.3. *There are no rank 8 self-dual super-modular categories with Galois group $G = \langle (23) \rangle$, $\langle (01), (23) \rangle$ or $\langle (123) \rangle$.*

Proof. (1) If $G = \langle (23) \rangle$, applying Eq. (3) with $\sigma = \langle (23) \rangle$, we have the following form for the $\hat{S}$ -matrix:

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_2 \\ d_1 & s_{11} & s_{12} & \epsilon_1 s_{12} \\ d_2 & s_{12} & s_{22} & s_{23} \\ d_2 & \epsilon_1 s_{12} & s_{23} & \epsilon_2 s_{22} \end{pmatrix}.$$

As 0 and 1 are fixed by G , by Eq. (2), we know that $d_1, d_2, \frac{s_{11}}{d_1}, \frac{s_{12}}{d_1}, \frac{s_{12}^2}{d_2^2}$ and $\frac{s_{22}s_{23}}{d_2^2}$ are rationals as they are fixed by the Galois group. Since they are also algebraic integers (see [26, Proposition 8.13.11]), we know these are integers. Consequently, $s_{11}, s_{12}, s_{22}s_{23}$ are also integers.

If $\epsilon_1 = -1$, the orthogonality of the columns of $\hat{S}$ gives

$$d_1(1 + s_{11}) = 0,$$

$$d_1 d_2 + s_{11} s_{12} + s_{12} s_{22} - s_{12} s_{23} = 0,$$

$$d_1 d_2 - s_{11} s_{12} + s_{12} s_{23} - \epsilon_2 s_{12} s_{22} = 0.$$

So we have $s_{11} = -1$. If $\epsilon_2 = 1$, then we have $d_1 d_2 = 0$, which is a contradiction. If $\epsilon_2 = -1$, we have $d_1 d_2 = -s_{12} s_{22}$. Plugging this into the second equation above,

we get $s_{12}(1 + s_{23}) = 0$. If $s_{12} = 0$, then $d_1 d_2 = 0$, which is impossible. If $s_{23} = -1$, then s_{22} is an integer. Then all the entries of $\hat{S}$ are integers, which contradicts the assumption that G is $\mathbb{Z}_2$.

If $\epsilon_1 = 1$, the orthogonality of the columns of $\hat{S}$ gives

$$d_2^2 + s_{12}^2 + s_{22}s_{23} + \epsilon_2 s_{22}s_{23} = 0.$$

If $\epsilon_2 = -1$, then $d_2^2 + s_{12}^2 = 0$, a contradiction. If $\epsilon_2 = 1$, by applying a Gröebner basis algorithm on Maple, we get $(2s_{22} + s_{11} + 1)(2d_1 d_2 + s_{11}s_{12} + 2s_{12}s_{22} - s_{12}) = 0$. One sees that if either factor is 0, we will have trivial G , a contradiction.

(2) Assume $G = \langle (01), (23) \rangle$. Using Eq. (B), we get

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \pm 1 & \pm d_2 & \pm d_3 \\ d_2 & \pm d_2 & s_{22} & s_{23} \\ d_3 & \pm d_3 & s_{23} & \pm s_{22} \end{pmatrix}.$$

It follows from $\hat{S}^2 = \frac{D^2}{2}I$ that $2d_2^2 + s_{22}^2 + s_{23}^2 = 2d_3^2 + s_{22}^2 + s_{23}^2$. Since d_i 's are positive, $d_2 = d_3$.

Let

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_2 \\ d_1 & \epsilon_1 & \epsilon_2 d_2 & \epsilon_3 d_2 \\ d_2 & \epsilon_2 d_2 & s_{22} & s_{23} \\ d_2 & \epsilon_3 d_2 & s_{23} & \epsilon_4 s_{22} \end{pmatrix}.$$

This case can be eliminated using orthogonality of the columns of $\hat{S}$. Applying a Gröebner basis algorithm to these equations we find that the only possible sign choice is given by $\epsilon_1 = \epsilon_4 = 1$ and $\epsilon_2 = \epsilon_3 = -1$. We can further deduce that $s_{23} = -1$, $s_{22} = d_1$ and $d_1 = d_2^2$. Therefore, we have

$$\hat{S} = \begin{pmatrix} 1 & d_2^2 & d_2 & d_2 \\ d_2^2 & 1 & -d_2 & -d_2 \\ d_2 & -d_2 & d_2^2 & -1 \\ d_2 & -d_2 & -1 & d_2^2 \end{pmatrix}.$$

Notice that $G = \text{Gal}(\mathbb{Q}(d_2)/\mathbb{Q})$. Computing the characteristic polynomial for $\hat{N}_2$, we have

$$p_2(x) = x^4 + \left(-2d_2 + \frac{2}{d_2}\right)x^3 + \left(d_2^2 + \frac{1}{d_2^2} - 4\right)x^2 + \left(2d_2 - \frac{2}{d_2}\right)x + 1$$

Therefore, $-2d_2 + \frac{2}{d_2}$ must be an integer. In particular, d_2 satisfies a quadratic equation over $\mathbb{Q}$. This means $\text{Gal}(\mathbb{Q}(d_2)/\mathbb{Q})$ is either trivial or $\mathbb{Z}_2$, which contradicts the fact that G is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

- (3) If $G = \langle (123) \rangle$, then G fixes 0. Therefore, $\hat{S}_{i,0} = d_i$ are rational numbers. Since the dimensions d_i 's are always algebraic integers, then they must be integers in this case. Moreover, $d_i = \hat{S}_{0,1} = \pm \hat{S}_{0,i+1} = \pm d_{i+1}$. So, by the positivity of the dimensions (i.e. unitarity assumption), we have

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_1 & d_1 \\ d_1 & s_{11} & \epsilon_1 s_{33} & \epsilon_2 s_{22} \\ d_1 & \epsilon_1 s_{33} & s_{22} & \epsilon_3 s_{11} \\ d_1 & \epsilon_2 s_{22} & \epsilon_3 s_{11} & s_{33} \end{pmatrix}.$$

From Corollary 2.2 we have $d_1^2 | (1 + 3d_1^2)$. We can deduce that $d_1 = 1$. Since d_1 is the largest (in magnitude) eigenvalue of the fusion matrices N_1, N_2 and N_3 , we see that the other eigenvalues (which are real numbers) satisfy $\pm \hat{S}_{ii}/d_1 = \pm \hat{S}_{i,i} = \pm 1$. This means the entries of $\hat{S}$ are ± 1 's which contradicts the assumption of G being nontrivial. $\square$

Theorem 3.4. *If $G = \langle (0) \rangle$, then the corresponding $\hat{S}$ -matrix, up to relabeling the simple objects, is one of the following:*

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Proof. If G is trivial, then the proof of [44, Theorem 4.1, Case 7] goes through *mutatis mutandis* showing that the corresponding super-modular category is pointed. Thus by Proposition 2.1 the super-modular category splits, so that $\hat{S}$ has the same form as the S -matrix of some rank 4 pointed modular category [44] as in the statement. $\square$

Theorem 3.5. *If $G = \langle (01) \rangle$, then the corresponding $\hat{S}$ is*

$$\begin{pmatrix} 1 & 1 & 2 & \sqrt{6} \\ 1 & 1 & 2 & -\sqrt{6} \\ 2 & 2 & -2 & 0 \\ \sqrt{6} & -\sqrt{6} & 0 & 0 \end{pmatrix}.$$

Proof. By Eq. (3), we have

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \epsilon_1 & \epsilon_2 d_2 & \epsilon_3 d_3 \\ d_2 & \epsilon_2 d_2 & s_{22} & s_{23} \\ d_3 & \epsilon_3 d_3 & s_{23} & s_{33} \end{pmatrix}.$$

We first assume that $\epsilon_1 = 1$. Then we can have $\epsilon_2 \epsilon_3 = -1$ or $\epsilon_2 = \epsilon_3 = -1$.

For the first case, we can assume $\epsilon_2 = 1$, $\epsilon_3 = -1$ and interchange N_2 and N_3 if necessary. Then the orthogonality of $\hat{S}$ gives us $s_{23}(s_{22} + s_{33}) = 0$ and $2d_1 + d_2^2 - d_3^2 = 0$. Assume that $s_{22} + s_{33} = 0$, then since the columns of $\hat{S}$ are of equal length $2d_2^2 + s_{22}^2 = 2d_3^2 + s_{33}^2$. This gives that $d_2 = d_3$, and that $d_1 = 0$, which is a contradiction. So, we must have $s_{23} = 0$. Then $\hat{S}$ becomes

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & 1 & d_2 & -d_3 \\ d_2 & d_2 & s_{22} & 0 \\ d_3 & -d_3 & 0 & s_{33} \end{pmatrix}.$$

Since $\sigma = (01)$ is the only nontrivial element of the Galois group, we conclude that $m = \frac{d_2(d_1+1)}{d_1}$, $n = \frac{d_3(d_1-1)}{d_1}$, $t = \frac{s_{22}}{d_2}$, $u = \frac{s_{33}}{d_3}$, $v = \frac{d_2^2}{d_1}$, $w = \frac{(d_1+1)}{d_1}$ and $x = \frac{d_3^2}{d_1}$ are integers as coefficients of the minimal polynomials of the $\hat{N}_i$. Note that m, v, w and x are strictly greater than 0 and $n \geq 0$. Since $d_2 + d_1d_2 + d_2s_{22} = 0$, we have $s_{22} < 0$ so $t < 0$. Moreover, we have $t^2 - u^2 \neq 0$. In fact, if $t^2 - u^2 = 0$, then $u^2 + 2 = \frac{s_{33}^2 + 2d_3^2}{d_3^2} = \frac{s_{22}^2 + 2d_2^2}{d_3^2} = \frac{d_2^2}{d_3^2} \left(\frac{s_{22}^2 + 2d_2^2}{d_2^2} \right) = \frac{d_2^2}{d_3^2} (t^2 + 2)$. This implies that $d_2 = d_3$. Using $2d_1 + d_2^2 - d_3^2 = 0$, we have $d_1 = 0$, a contradiction. Thus $t^2 - u^2 \neq 0$ and we have

$$m = -\frac{2t(u^2 + 2)}{t^2 - u^2}, \quad n = \frac{2u(t^2 + 2)}{t^2 - u^2}, \quad v = \frac{2(u^2 + 2)}{t^2 - u^2},$$

$$w = \frac{2(t^2u^2 + t^2 + u^2)}{t^2 - u^2}, \quad x = \frac{2(t^2 + 2)}{t^2 - u^2}.$$

Since $x > 0$, we have $t^2 - u^2 > 0$. We have $n_{2,2,2} = \frac{t(t^2 - u^2 - 2)}{(t^2 - u^2)}$. In order to have $n_{2,2,2} \geq 0$, we must have $t^2 - u^2 - 2 \leq 0$. The only integer solution satisfying all the restrictions here is $t = -1$ and $u = 0$. Then $s_{33} = 0$ and $s_{22} = -d_2$. Thus, we have $d_1 = 1$. The orthogonality condition on the columns of $\hat{S}$ gives that $2d_2 - d_2^2 = 0$. This implies that $d_2 = 2$ and $d_3 = \sqrt{6}$.

If $\epsilon_2 = \epsilon_3 = -1$, we have

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & 1 & -d_2 & -d_3 \\ d_2 & -d_2 & s_{22} & s_{23} \\ d_3 & -d_3 & s_{23} & s_{33} \end{pmatrix}.$$

Similarly to the previous case, we have $m = \frac{d_3(d_1-1)}{d_2}$, $n = \frac{d_1^2+1}{d_1}$, $t = \frac{d_3^2}{d_1}$, $u = \frac{s_{22}}{d_2}$, $v = \frac{d_2^2}{d_1}$, $w = \frac{s_{33}}{d_3}$, $x = \frac{s_{23}}{d_2}$, $y = \frac{s_{23}}{d_3}$ and $z = \frac{d_2(d_1-1)}{d_1}$ are integers. Here, we have $nv - z^2 - 2v = 0$, $t + v - 2 = 0$ and $m^2 + z^2 - 2n + 4 = 0$. Note that $m^2 + n^2 \neq 0$ since $n \neq 0$. So, we have $n = \frac{m^2 + z^2}{2} + 2$, $t = \frac{2m^2}{m^2 + z^2}$, and $v = \frac{2z^2}{m^2 + z^2}$. Since t is an integer, we have $m^2 \geq z^2$. Similarly, we have $z^2 \geq m^2$. Thus $|m| = |z|$ so $t = v = 1$.

This means $d_2 = d_3 = \sqrt{d_1}$. Then $m = d_1 - 1$ and d_1 is an integer. From $|m| = |z|$, we get $d_1 - 1 = \frac{d_2(d_1-1)}{d_1}$. If $d_1 = 1$, then we have $d_2 = d_3 = 1$. This would force all the entries of $\hat{S}$ to be integers, which a contradiction to the assumption that the Galois group is $\mathbb{Z}_2$. If $d_1 > 1$, then we have $d_2 = d_1$. Recall that $d_2 = d_3 = \sqrt{d_1}$. This means either $d_2 = d_3 = d_1 = 0$ or $d_2 = d_3 = d_1 = 1$, again a contradiction.

If $\epsilon_1 = -1$, the orthogonality of the columns of $\hat{S}$ gives $\epsilon_2 d_2^2 + \epsilon_3 d_3^2 = 0$. Thus, we have $\epsilon_2 \epsilon_3 = -1$ and $d_2 = d_3$. But then we have $\sigma(d_2) = \frac{d_2}{d_1} = -\frac{d_2}{d_1}$ so $d_2 = 0$, a contradiction. $\square$

Theorem 3.6. *If $G = \langle (01)(23), (02)(13) \rangle$, then the corresponding $\hat{S}$ has the following form:*

$$\begin{pmatrix} 1 & \phi_1 \phi_2 & \phi_1 & \phi_2 \\ \phi_1 \phi_2 & 1 & -\phi_2 & -\phi_1 \\ \phi_1 & -\phi_2 & -1 & \phi_1 \phi_2 \\ \phi_2 & -\phi_1 & \phi_1 \phi_2 & -1 \end{pmatrix}.$$

Proof. By Eq. (3), we have the corresponding $\hat{S}$:

$$\begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \epsilon_1 & \epsilon_2 d_3 & \epsilon_3 d_2 \\ d_2 & \epsilon_2 d_3 & \epsilon_4 & \epsilon_5 d_1 \\ d_3 & \epsilon_3 d_2 & \epsilon_5 d_1 & \epsilon_6 \end{pmatrix}.$$

Using orthogonality of the columns of $\hat{S}$ and the fact that $d_i \geq 1$, there are only 2 possibilities for ϵ_i 's, namely,

- (1) $\epsilon_1 = 1, \epsilon_2 = -1, \epsilon_3 = -1, \epsilon_4 = 1, \epsilon_5 = -1, \epsilon_6 = 1$, or
- (2) $\epsilon_1 = 1, \epsilon_2 = -1, \epsilon_3 = -1, \epsilon_4 = -1, \epsilon_5 = 1, \epsilon_6 = -1$.

For the first case, the orthogonality of $\hat{S}$ gives $d_1 = d_2 d_3$, $d_2 = d_1 d_3$ and $d_3 = d_1 d_2$. So, we have $d_1 d_2 d_3 = (d_1 d_2 d_3)^2$, we have $d_1 d_2 d_3 = 1$. Since $d_i \geq 1$ for all i , this implies that $d_1 = d_2 = d_3 = 1$. This cannot happen since the corresponding Galois group should be trivial, which is a contradiction to our assumption.

Consider the second case. The orthogonality of $\hat{S}$ gives $d_1 = d_2 d_3$. So, we can write the corresponding matrix as

$$\hat{S} = \begin{pmatrix} 1 & d_2 d_3 & d_2 & d_3 \\ d_2 d_3 & 1 & -d_3 & -d_2 \\ d_2 & -d_3 & -1 & d_2 d_3 \\ d_3 & -d_2 & d_2 d_3 & -1 \end{pmatrix}.$$

Note that Eq. (2) indicates that d_2 and $-1/d_2$ are conjugates. By Remark 5, we know that $d_2 = \phi_m$ for some $m \in \mathbb{Z}$. Similarly, $d_3 = \phi_n$ for some integer n .

Thus, we have

$$\hat{S} = \begin{pmatrix} 1 & \phi_m \phi_n & \phi_m & \phi_n \\ \phi_m \phi_n & 1 & -\phi_n & -\phi_m \\ \phi_m & -\phi_n & -1 & \phi_m \phi_n \\ \phi_n & -\phi_m & \phi_m \phi_n & -1 \end{pmatrix}.$$

The corresponding $\hat{N}_i$ matrices have integer entries in terms of m and n . More precisely, we have

$$\hat{N}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & mn & m & n \\ 0 & m & 0 & 1 \\ 0 & n & 1 & 0 \end{pmatrix}, \quad \hat{N}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & m & 0 & 1 \\ 1 & 0 & m & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \hat{N}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & n & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & n \end{pmatrix}.$$

Using the formula given in Lemma 2.3, we calculate the 2nd Frobenius–Schur indicator for the simple object X_2 :

$$\begin{aligned} \nu_2(X_2) &= \pm 1 \\ &= \frac{2}{D^2} \left(d_2 \left(\frac{1}{\theta_2} \right)^2 + m d_1^2 + d_1 d_3 \left(\frac{\theta_1}{\theta_3} \right)^2 + m d_2^2 + d_1 \theta_2^2 + d_1 d_3 \left(\frac{\theta_3}{\theta_1} \right)^2 \right) \end{aligned}$$

from this, we obtain

$$\begin{aligned} \pm \frac{D^2}{2} &= m (d_1^2 + d_2^2) + d_2 (\theta_2^2 + \theta_2^{-2}) + d_1 d_3 \left(\left(\frac{\theta_1}{\theta_3} \right)^2 + \left(\frac{\theta_1}{\theta_3} \right)^{-2} \right) \\ &= m (d_2^2 d_3^2 + d_2^2) + 2d_2 \operatorname{Re} (\theta_2^2) + 2d_2 d_3^2 \operatorname{Re} \left(\frac{\theta_1}{\theta_3} \right)^2 \\ &\leq \frac{D^2}{2} = 1 + d_2^2 d_3^2 + d_2^2 + d_3^2 \\ &\Rightarrow 0 \geq m d_2^2 (d_3^2 + 1) + 2d_2 \operatorname{Re} (\theta_2^2) + 2d_2 d_3^2 \operatorname{Re} \left(\frac{\theta_1}{\theta_3} \right)^2 - 1 - d_2^2 d_3^2 - d_2^2 - d_3^2 \\ &= m d_2^2 (d_3^2 + 1) - 2d_2 (d_3^2 + 1) - d_2^2 (d_3^2 + 1) - (d_3^2 + 1) \\ &= (m d_2^2 - 2d_2 - d_2^2 - 1)(d_3^2 + 1) \\ &\Rightarrow 0 \geq (m d_2^2 - 2d_2 - d_2^2 - 1) \end{aligned}$$

$$\begin{aligned}
 &= d_2^2(m-1) - 2d_2 - 1 \\
 &= \phi_m^2(m-1) - 2\phi_m - 1 \\
 &= (m\phi_m + 1)(m-1) - 2\phi_m - 1 \\
 &= (m-2)(\phi_m(m+1) + 1).
 \end{aligned}$$

Thus m must be 0, 1, or 2.

Similarly, we calculate the 2nd Frobenius–Schur indicator for X_3 :

$$\begin{aligned}
 \nu_2(X_3) &= \pm 1 = \frac{2}{D^2} \left(d_3\theta_3^{-2} + nd_1^2 + d_1d_2 \left(\frac{\theta_1}{\theta_2} \right)^2 + d_1d_2 \left(\frac{\theta_2}{\theta_1} \right)^2 + nd_3^2 + d_3\theta_3^2 \right) \\
 &\Rightarrow \pm \frac{D^2}{2} = 2d_3 \operatorname{Re}(\theta_3^2) + n(d_2^2d_3^2 + d_3^2) + 2d_1^2d_3 \operatorname{Re} \left(\frac{\theta_1}{\theta_2} \right)^2 \\
 &\leq \frac{D^2}{2} = 1 + d_2^2d_3^2 + d_2^2 + d_3^2 \\
 &\Rightarrow 0 \geq 2d_3 \operatorname{Re}(\theta_2^2) + nd_3^2(d_2^2 + 1) + 2d_3d_2^2 \operatorname{Re} \left(\frac{\theta_1}{\theta_2} \right)^2 - 1 - d_2^2d_3^2 - d_2^2 - d_3^2 \\
 &\geq -2d_3 + nd_3^2(d_2^2 + 1) - 2d_3d_2^2 - d_3^2(d_2^2 + 1) - (1 + d_2^2) \\
 &= (nd_3^2 - 2d_3 - d_3^2 - 1)(d_2^2 + 1) \\
 &\Rightarrow 0 \geq (nd_3^2 - 2d_3 - d_3^2 - 1) \\
 &= d_3^2(n-1) - 2d_3 - 1 \\
 &= \phi_n^2(n-1) - 2\phi_n - 1 \\
 &= (n\phi_n + 1)(n-1) - 2\phi_n - 1 \\
 &= (n-2)(\phi_n(n+1) + 1).
 \end{aligned}$$

So n must be 0, 1, or 2.

Up to symmetry, we can exclude the cases $(m, n) = (0, 0), (1, 1), (1, 0), (2, 2)$ since the corresponding Galois groups are not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The possible value for this case, up to symmetry, is $(m, n) = (1, 2)$. Note that $\phi_1 = \frac{1+\sqrt{5}}{2}$ and $\phi_2 = 1 + \sqrt{2}$. $\square$

In the last few cases we were unable to complete the classification in general — instead we placed bounds on the $\hat{N}_{ij}^k$'s. Since $N_{ij}^k \leq 2\|N_i\|_{\max}$, this could also be done in terms of bounds on the N_i 's. Sometimes it is easier to work in terms of a bound on the dimensions d_i . Indeed, the proof of [15, Lemma 3.14] goes through with no change, from which we conclude: $\hat{N}_{ij}^k \leq d_i \leq 4\|\hat{N}_i\|_{\max}$.

Theorem 3.7. *If $G = \langle(0123)\rangle$ and $\hat{N}_{ij}^k < 14$, the corresponding $\hat{S}$ is*

$$\begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & -d_2 & d_3 & 1 \\ d_2 & d_3 & -1 & -d_1 \\ d_3 & 1 & -d_1 & d_2 \end{pmatrix},$$

where $d_1 = 1 + \sqrt{2} + \sqrt{2 + \sqrt{2}}$, $d_2 = 1 + \sqrt{2} + \sqrt{2(2 + \sqrt{2})}$, and $d_3 = 1 + \sqrt{2 + \sqrt{2}}$.

Proof. Applying Eq. (3) with $\sigma = \langle(0123)\rangle$, we have the following form of $\hat{S}$ matrix:

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \epsilon_1 d_2 & \epsilon_2 d_3 & \epsilon_3 \\ d_2 & \epsilon_2 d_3 & \epsilon_4 & \epsilon_5 d_1 \\ d_3 & \epsilon_3 & \epsilon_5 d_1 & \epsilon_6 d_2 \end{pmatrix}.$$

Using a Maple's Gröbner basis algorithm, we deduce that $\epsilon_1 = \epsilon_4 = \epsilon_5 = -1$ and $\epsilon_2 = \epsilon_3 = \epsilon_6 = 1$.

So

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & -d_2 & d_3 & 1 \\ d_2 & d_3 & -1 & -d_1 \\ d_3 & 1 & -d_1 & d_2 \end{pmatrix}.$$

Let $p_1(x) = x^4 - c_1 x^3 + c_2 x^2 + c_3 x - 1$ be the characteristic polynomial of $\hat{N}_1$. Then $p_3(x) = x^4 - c_3 x^3 - c_2 x^2 + c_1 x - 1$, where $c_i \in \mathbb{Z}$ for $i = 1, 2$ and 3 . Note that $c_1 = \text{Trace}(\hat{N}_1) \geq 0$ and $c_3 = \text{Trace}(\hat{N}_3) \geq 0$ as the $\hat{N}_i$'s are matrices with nonnegative integer entries. Let $p_2(x) = x^4 - b_1 x^3 + b_2 x^2 + b_3 x + 1$ be the characteristic polynomial of $\hat{N}_2$, where $b_1 = b_3 = d_2 + \frac{d_3}{d_1} - \frac{1}{d_2} - \frac{d_1}{d_3}$ and $b_2 = -2 + \frac{d_1}{d_2 d_3} - \frac{d_3}{d_1 d_2} - \frac{d_2 d_1}{d_3} + \frac{d_2 d_3}{d_1}$.

The orthogonality of the rows of $\hat{S}$ gives $d_1 = d_1 d_2 - d_2 d_3 - d_3$, $d_3 = -d_1 + d_1 d_2 - d_2 d_3$, $d_1 d_2 = d_3 + d_1 + d_2 d_3$ and $d_2 d_3 = -d_1 + d_1 d_2 - d_3$. So, we have $b_2 = -6$ and $b_3 = -b_1$. Thus $p_2(x) = x^4 - b_1 x^3 - 6x^2 + b_1 x + 1$, where $b_1 = \text{Trace}(\hat{N}_2) \geq 0$.

Note that $c_1 + c_3 = 2\frac{(d_2+1)d_3}{d_2} + 4\frac{d_2}{(d_2+1)d_3}$. This gives $c_1 + c_3 \geq 4\sqrt{2}$. Since c_1 and c_3 are integers, we have $c_1 + c_3 \geq 6$. Moreover, we have $4b_1 - c_1^2 + 8c_2 + c_3^2 = 0$.

Let $\Delta = c_1 - c_3$ and $\Sigma = c_1 + c_3$, then $c_2 = \frac{1}{16}[3\Delta\Sigma \pm \sqrt{(32 + \Delta^2)(-32 + \Sigma^2)}]$ and $b_1 = \frac{1}{8}[-\Delta\Sigma \mp \sqrt{(32 + \Delta^2)(-32 + \Sigma^2)}]$. Let $P = \frac{16c_2 - 3\Delta\Sigma}{\Delta^2 + 32} = \pm \sqrt{\frac{\Sigma^2 - 32}{\Delta^2 + 32}}$.

We compute the $n_{i,j,k}$'s and we get the following relations:

$$n_{1,1,1} = \frac{5c_1 - 3c_3}{8} - \frac{(c_1 - c_3)P}{8},$$

$$n_{1,1,2} = 1 - P = 1 + n_{1,2,3} = 2 + n_{2,3,3},$$

$$\begin{aligned}
 n_{1,1,3} &= \frac{c_1 + c_3}{8} - \frac{(c_1 - c_3)P}{8} = n_{1,3,3} = \frac{1}{2}(n_{1,1,1} + n_{3,3,3}), \\
 n_{1,2,2} &= \frac{c_1 + c_3}{4} + \frac{(c_1 - c_3)P}{4} = n_{2,2,3}, \\
 n_{2,2,2} &= \frac{c_1^2 - c_3^2}{4} - 2c_2 + 2P = b_1 + 2P,
 \end{aligned}$$

Recall that the fusion coefficients are integral. In particular, since $n_{2,2,2}$ is an integer, we know that c_1 and c_3 are both even. Thus Δ and Σ are divisible by 2. Via a computer search for integer solutions using the above equations, we found there is only one solution when $n_{i,j,k} < 14$, with $c_1 = c_3 = 4$ and $c_2 = 2P = -2$. The corresponding $\hat{S}$ matrix for this case is the one in the statement (and is the same as that of $\text{PSU}(2)_{14}$). $\square$

We can make further progress using more sophisticated number theoretical arguments:

Lemma 3.1. *If Σ and Δ are divisible by 4, the corresponding super-modular categories have $c_1 = c_3 = \sqrt{2}(\zeta^{2i-1} - \bar{\zeta}^{2i-1})$, $c_2 = -(\zeta^{2i-1} + \bar{\zeta}^{2i-1})$ and $P = -\frac{1}{2}(\zeta^{2i-1} + \bar{\zeta}^{2i-1})$, where $\zeta = 1 + \sqrt{2}$, $\bar{\zeta} = 1 - \sqrt{2}$ and $i \geq 1$ is an integer.*

Proof. Assume that Σ and Δ in the proof above are also divisible by 4. Denote $a = \frac{\Sigma}{4}$, $b = \frac{\Delta}{4}$ and $c = P$. Then we have the following Diophantine equation

$$a^2 - (b^2 + 2)c^2 = 2.$$

Lemma 3.2 shows that $b = 0$. Consequently, we have $c_1 = c_3$, and the Diophantine equation becomes $a^2 - 2c^2 = 2$. Since $a = \frac{c_1}{2} \geq 0$ and $c = P = \frac{c_2}{2} \leq -1$ the resulting solutions are

$$a(i) := \frac{1}{\sqrt{2}}(\zeta^{2i-1} - \bar{\zeta}^{2i-1}), \quad c(i) = -\frac{1}{2}(\zeta^{2i-1} + \bar{\zeta}^{2i-1}),$$

where $1 \leq i$ and $\zeta = 1 + \sqrt{2}$ and $\bar{\zeta} = 1 - \sqrt{2}$. This determines all possible fusion rules under these assumptions. The first few are $(a, c) \in \{(2, -1), (10, -7), (58, -41), (338, -239), \dots\}$. $\square$

Some cases can be ruled out if we assume the MME conjecture using Lemma 2.5 as follows.

Example 3.1. In the case $(a, c) = (58, -41)$, we find that d_1 is a root of the irreducible polynomial $x^4 - 2 \cdot 58x^3 - 82x^2 + 2 \cdot 58x - 1$. The smallest cyclotomic field in which d_1 resides has degree $464 = 2^4 \cdot 29$ (i.e., the conductor of $\mathbb{Q}(d_1)$ is 464). Now suppose that the corresponding super-modular category $\mathcal{B}$ has a MME $(\mathcal{C}, f)$. Then the order of the T matrix of $\mathcal{C}$ is divisible by 29, so that $7|\varphi(29)|[\mathbb{Q}(T) : \mathbb{Q}]$. But Lemma 2.5 and the results of [41] imply that $[\mathbb{Q}(T) : \mathbb{Q}] = 2^m$ for some m (since $[\mathbb{Q}(T) : \mathbb{Q}(S)] = 2^t$). Thus no such category can exist.

Remark 7. The $(a, c) = (10, -7)$ case cannot be dealt with in this way since the corresponding conductor is 80.

Lemma 3.2. Assume a, b and c are integers and $a^2 - (b^2 + 2)c^2 = 2$, then $b = 0$.

Proof. Reducing modulo 8 both sides of the equation, there are three cases to consider since a square modulo 8 is 0, 1, or 4.

- If $b^2 \equiv 1 \pmod{8}$, then we have $a^2 - 2 \equiv 3c^2 \pmod{8}$. This gives no solutions.
- If $b^2 \equiv 0 \pmod{8}$, then we have $c \equiv 1 \pmod{8}$ and $a \equiv 4 \pmod{8}$.
- If $b^2 \equiv 4 \pmod{8}$, then we have $c \equiv 1 \pmod{8}$ and $a \equiv 0 \pmod{8}$.

Therefore, we must have that a and b are even and c is odd. Moreover, if $4|b$, then $4 \nmid a$ and vice versa.

Now we consider both sides of $a^2 - (b^2 + 2)c^2 = 2$ modulo 4. This gives us $b^2 + 2 \equiv 2 \pmod{4}$. Let $B = b^2 + 2$, and then we need to solve the following Pell-like equation

$$a^2 - Bc^2 = 2$$

As b is even, B is not divisible by 4. So we write $B = m^2d$, where d is square-free and even and m is odd.

Claim: $d = 2$. Assume otherwise, then we can prove that $a^2 - Bc^2 = 2$ has no solutions by looking at the class group of $\mathbb{Z}[\sqrt{d}]$ via genus theory. In fact, assume $d \neq 2$ and even. Then the equation $a^2 - d(mc)^2 = 2$ can be written as

$$a^2 - dy^2 = 2.$$

If the above equation has no integer solution, then $a^2 - Bc^2 = 2$ has no solution. Now, we consider the quadratic number field $K = \mathbb{Q}(\sqrt{d})$. We denote the class group of K by C_K (see [28] p. 45), which is a finite abelian group. Let $V = (\mathbb{Z}/2\mathbb{Z})^g$, where g is the number of distinct prime dividing d . Let $e_i = (0, \dots, 1, \dots, 0)$ be the basis of V , where $i = 1, \dots, g$ and 1 is on the i^{th} position. Let $C_{K,2}$ be the subgroup of C_K consisting of the elements of order 2. For primes $p_1, \dots, p_g \in \mathbb{Z}$, denote the corresponding prime ideals as $\mathfrak{p}_1, \dots, \mathfrak{p}_g \in \mathbb{Z}[\sqrt{d}]$. Define the map

$$\begin{aligned} \phi : V &\rightarrow C_{K,2} \\ e_i &\mapsto [\mathfrak{p}_i]. \end{aligned}$$

This assignment gives a group homomorphism. By Corollary 1 in [28, Chap. 5], we know that ϕ is surjective and $\ker(\phi) = \{0, (1, 1, \dots, 1)\}$. Consequently, $C_{K,2} \simeq (\mathbb{Z}/\mathbb{Z}_2)^{g-1}$. In particular, if $g \geq 2$, then for any prime $p|d$, $\mathfrak{p} = (p, \sqrt{d})$ is not principal.

Now, we return to our equation $a^2 - dy^2 = 2$, where $d \neq 2$ and even. Consider the ideal $(a + y\sqrt{d}) \subseteq \mathbb{Z}[\sqrt{d}]$, which has norm 2. We have $(a + y\sqrt{d})(a - y\sqrt{d}) = (2)$. Moreover, we have $(2, \sqrt{d})^2 = (2)$. By the unique factorization, we have $(2, \sqrt{d}) = (a + y\sqrt{d})$. However, if $g \geq 2$, $(2, \sqrt{d})$ is not principal. Consequently, there is no integer solutions for a and y when $d \neq 2$.

Thus, we have

$$a^2 - 2m^2c^2 = 2, \quad b^2 - 2m^2 = -2.$$

One can further deduce that $4|b$. Let $b = 4\beta$, the second equation gives us $m^2 - 8\beta^2 = 1$. This is a Pell-equation. Note that $(m, \beta) = (3, 1)$ is the smallest nontrivial solution. Let $z = 3 + 2\sqrt{2}$ and denote its conjugate as $\bar{z}$. The solutions (m, β) of the equation are given by

$$m_n = \frac{z^n + \bar{z}^n}{2}, \quad \beta_n = \frac{z^n - \bar{z}^n}{4\sqrt{2}},$$

where n is a positive integer. We also have $a^2 - 2y^2 = 2$, which is a Pell-type equation. Notice that $(a, y) = (2, 1)$ is a solution. Let $s = 2 + \sqrt{2}$. By the theorem of Mahler [34], the solutions are given by

$$a_k = \frac{s^k + \bar{s}^k}{2\sqrt{2}^{(k-1)}}, \quad y_k = \frac{s^k - \bar{s}^k}{2\sqrt{2}^k},$$

where k is an odd positive integer. By modifying the indices, we know the solutions of the pair (m_n, y_n) are given by

$$y_n = \frac{(z+1)^{2n+1} - (7-z)^{2n+1}}{2^{3n+2}\sqrt{2}}, \quad m_n = \frac{z^n + (6-z)^n}{2},$$

where $n \in \mathbb{N}$. Recall that the values of m and y are related by $y = mc$, where m and c are both odd. In particular, $y \geq m$. Now, we consider the function given by $f(x) = \frac{y_x}{m_x}$. Using standard calculus, we know that f is a monotonic increasing function and $\lim_{x \rightarrow \infty} f(x) = 1 + \sqrt{2}$. Therefore, the only possible solution here is $m = 1$. Consequently, we have $b = 0$. $\square$

Remark 8. If $n_{i,j,k} < 115$, by a computer search for positive integer values, we find two more solutions with $(\Sigma, \Delta) = (40, 0)$ and $(232, 0)$, which are correspond to $i = 2, 3$ in Lemma 3.1. The first possible solution with $\Sigma \equiv 2 \pmod{4}$ has $(\Sigma, \Delta) = (434, 18)$ and $n_{1,1,1} = 115$.

Theorem 3.8. If $G = \langle (012) \rangle$ and $\hat{N}_{ij}^k < 21$, then $\hat{S}$ is

$$\begin{pmatrix} 1 & d & 1+d & d^2-1 \\ d & -(1+d) & -1 & d^2-1 \\ 1+d & -1 & d & -(d^2-1) \\ d^2-1 & d^2-1 & -(d^2-1) & 0 \end{pmatrix},$$

where d is the largest real root of the polynomial $x^3 - 3x - 1 = 0$.

Proof. Applying Eq. (B) to $\sigma = (012)$, we get

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \epsilon_1 d_2 & \epsilon_2 & \epsilon_3 d_3 \\ d_2 & \epsilon_2 & \epsilon_4 d_1 & \epsilon_5 d_3 \\ d_3 & \epsilon_3 d_3 & \epsilon_5 d_3 & s_{33} \end{pmatrix}.$$

A computation using $\hat{S}^2 = \frac{D^2}{2}I$ and $d_i \geq 1$ reduces the sign choices to the following three cases:

- (1) $\epsilon_3 = \epsilon_4 = -1, \epsilon_1 = \epsilon_5 = 1, \epsilon_2 = -1$,
- (2) $\epsilon_3 = \epsilon_4 = 1, \epsilon_1 = \epsilon_5 = -1, \epsilon_2 = -1$, or
- (3) $\epsilon_3 = \epsilon_4 = -1, \epsilon_1 = \epsilon_5 = -1, \epsilon_2 = 1$.

In case (3), we find that $d_3^2 + d_1 d_2 - (d_1 + d_2) = 0$. However, since $d_i \geq 1$, we have $d_3^2 + d_1 d_2 \geq 2$ and $-(d_1 + d_2) \leq -2$. So, the equality holds if and only if $d_1 + d_2 = 2 = d_3^2 + d_1 d_2$, which forces $d_1 = d_2 = d_3 = 1$. This is impossible since the Galois group is nontrivial by hypothesis.

Case (1) is equivalent to case (2) by permuting columns/rows 2 and 3 and relabeling $d_1 \leftrightarrow d_2$. So, without loss of generality, we may assume we are in case (2). Let $g(x) = x^3 - c_1 x^2 + c_2 x - c_3$ be an irreducible polynomial for which d_3 is a root. Note that $c_1 = \frac{d_3}{d_1 d_2}(d_1 d_2 + d_2 - d_1)$, $c_2 = \frac{d_3^2}{d_1 d_2}(d_2 - d_1 - 1)$, and $c_3 = -\frac{d_3^3}{d_1 d_2}$. The orthogonality of the rows of $\hat{S}$ shows that $c_1 = -c_3$. Moreover, $\frac{c_2}{c_3} = -\lambda_{33} \in \mathbb{Z}$. Let $n = \lambda_{33}$ and $c = -c_3 = c_1$, so we have $g(x) = x^3 - cx^2 + ncx + c$. Since the Galois group is $\mathbb{Z}_3$, we have that $\frac{\text{dis}(g)}{c^2} = c^2(n^2 + 4) - 2nc(9 + 2n^2) - 27$ is a square.

Take t to be the positive root of this, that is, $t = \frac{(d_1-1)(d_1+d_2)(1+d_2)}{d_1 d_2}$.

Note that $c = \frac{d_3^3}{d_1 d_2} > 0$. Moreover $t > 0$. Computing the fusion rules, we get

$$\begin{aligned} n_{1,1,1} &= \frac{(t - nc - 1)}{2} - \frac{t}{n^2 + 3}, & n_{1,1,2} &= n_{1,3,3} = \frac{-cn + 2n^2 + t - 3}{2(n^2 + 3)}, \\ n_{1,1,3} &= \frac{cn^2 + 2c - nt + 3n}{2(n^2 + 3)}, & n_{1,2,2} &= n_{2,3,3} = \frac{cn - 2n^2 + t + 3}{2(n^2 + 3)}, \\ n_{1,2,3} &= \frac{c - 3n}{n^2 + 3}, & n_{2,2,2} &= \frac{1 + nc + t}{2} - \frac{t}{3 + n^2}, \\ n_{2,2,3} &= \frac{2c + 3n + cn^2 + nt}{2(3 + n^2)}, & n_{3,3,3} &= \frac{c + n^3}{n^2 + 3}. \end{aligned}$$

If we restrict $n_{i,j,k} < 21$, the only integer values of n, t and c that satisfy $t^2 = c^2(n^2 + 4) - 2nc(9 + 2n^2) - 27$ and yield $n_{i,j,k} \in \mathbb{Z}$ is $(n, t, c) = (0, 3, 3)$. The corresponding $\hat{S}$ -matrix is the one given in the statement and is the same as that of $\text{PSU}(2)_7$ (see [44]). $\square$

Remark 9. Here is an alternative approach that is less computationally intensive, but assumes the minimal modular extension conjecture holds. First note that c is a divisor of $\dim(\mathcal{C})$, so that if we assume the MME conjecture holds then, by the Cauchy theorem [15], any prime divisor p of c must divide the order N of the T -matrix of any minimal modular extension of the corresponding super-modular category. Now, by Lemma 2.5 we have $\varphi(N) = [\mathbb{Q}(T) : \mathbb{Q}] = 3 \cdot 2^k$ since $|G| = 3$. Thus if $p \mid c$, we also have $\varphi(p) = 2^a 3^b$ where $b \in \{0, 1\}$ and at most one prime divisor p can have $3 \mid \varphi(p)$. Thus the prime divisors of c are somewhat uncommon (for example Fermat primes).

For $n = 0$, the discriminant equation above yields the Diophantine equation $(2c)^2 - 27 = t^2$, which has finitely many solutions. The only values of $c > 0$ that correspond to a solution are: 3 and 7. Since $n_{3,3,3} \in \mathbb{Z}$, when $n = 0$ we have $3 \mid c$. So $c = 3$ which, in turn, implies $t = 3$, giving the same solution as above. So in this case we do not need to assume the MME conjecture.

For $n = 1$ the Diophantine discriminant equation $5c^2 - 22c - 27 = t^2$ has infinitely many solutions, with the smallest few c values:

$$c \in \{7, 31, 199, 1351, 9247, 63367, 434311, 2976799, 20403271\}.$$

The method above eliminates all of these values of c except for 7 (note that $9 \mid \varphi(1351) = 2^7 3^2$). In the case that $c = 7$, we find that $t = 8$ which implies $n_{1,1,1} = -2$, so this cannot occur.

Theorem 3.9. *If $G = \langle (01)(23) \rangle$ and $d_i < 14$ for all i , then the corresponding $\hat{S}$ is one of the following:*

$$\begin{pmatrix} 1 & \phi_1^2 & \phi_1 & \phi_1 \\ \phi_1^2 & 1 & -\phi_1 & -\phi_1 \\ \phi_1 & -\phi_1 & -1 & \phi_1^2 \\ \phi_1 & -\phi_1 & \phi_1^2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \phi_2^2 & \phi_2 & \phi_2 \\ \phi_2^2 & 1 & -\phi_2 & -\phi_2 \\ \phi_2 & -\phi_2 & -1 & \phi_2^2 \\ \phi_2 & -\phi_2 & \phi_2^2 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \phi_1 & 1 & \phi_1 \\ \phi_1 & -1 & \phi_1 & -1 \\ 1 & \phi_1 & -1 & -\phi_1 \\ \phi_1 & -1 & -\phi_1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \phi_2 & 1 & \phi_2 \\ \phi_2 & -1 & \phi_2 & -1 \\ 1 & \phi_2 & -1 & -\phi_2 \\ \phi_2 & -1 & -\phi_2 & 1 \end{pmatrix}.$$

Proof. Similar as previous cases, we have

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \epsilon_1 & \epsilon_2 d_3 & \epsilon_3 d_2 \\ d_2 & \epsilon_2 d_3 & s_{22} & s_{23} \\ d_3 & \epsilon_3 d_2 & s_{23} & s_{33} \end{pmatrix}.$$

Case (1). $\epsilon_1 = 1$. Using Maple's Gröbner basis algorithm, we deduced that

$$(s_{33} + 1)(s_{23} - 1)(s_{23} + 1) = 0.$$

First, we assume $s_{33} + 1 = 0$, then we have $s_{33} = s_{22} = -1$, $\epsilon_2 = \epsilon_3 = -1$, $\epsilon_1 = 1$ and $s_{23} = d_1 = d_2 d_3$. Therefore the corresponding $\hat{S}$ is given by

$$\hat{S} = \begin{pmatrix} 1 & d_2 d_3 & d_2 & d_3 \\ d_2 d_3 & 1 & -d_3 & -d_2 \\ d_2 & -d_3 & -1 & d_2 d_3 \\ d_3 & -d_2 & d_2 d_3 & -1 \end{pmatrix}.$$

Note that this is exactly the same matrix we derived in Theorem 3.6. But here we do not get a contradiction since the Galois group is $\mathbb{Z}_2$. Thus the same argument using the 2nd Frobenius-Schur indicator works here. Since the Galois group is $\mathbb{Z}_2$, we have solutions for S -matrix when $(m, n) = (1, 1), (1, 0), (2, 0)$ and $(2, 2)$, i.e. $(d_2, d_3) = (\phi_i, \phi_i)$ or $(\phi_i, 1)$ for $i = 1, 2$. The cases $(1, 1)$ and $(2, 2)$ yield the first two $\hat{S}$ -matrices above, while for $(2, 0)$ and $(1, 0)$ the Galois group $G \neq \langle (01)(23) \rangle$, a contradiction. However, see Case 2 where these solutions do occur.

If $s_{23} - 1 = 0$, one can show that the corresponding Galois group is trivial.

Now, we assume $s_{23} + 1 = 0$, then the matrix $\hat{S}$ has the form

$$\hat{S} = \begin{pmatrix} 1 & d_3^2 & d_3 & d_3 \\ d_3^2 & 1 & -d_3 & -d_3 \\ d_3 & -d_3 & d_3^2 & -1 \\ d_3 & -d_3 & -1 & d_3^2 \end{pmatrix}.$$

Note this is the same matrix as the previous one if $d_2 = d_3$ and permuting the matrices $\hat{N}_2$ and $\hat{N}_3$.

Case (2). $\epsilon_1 = -1$. In this case, the $\hat{S}$ is of the form

$$\hat{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & -1 & d_3 & -d_2 \\ d_2 & d_3 & s_{22} & s_{23} \\ d_3 & -d_2 & s_{23} & -s_{22} \end{pmatrix}.$$

Note that the conjugate of d_1 is $-\frac{1}{d_1}$. Moreover, we know that if $d_1 = 1$, then the corresponding Galois group is trivial. Thus the field $\mathbb{Q}(\hat{S}) = \mathbb{Q}(d_1)$, where $d_1 = \phi_n = \frac{n + \sqrt{n^2 + 4}}{2}$ for some n . Now, we assume $k\sqrt{P} = \sqrt{n^2 + 4}$, where k is an integer and P is a square-free integer. Then $d_1 = \frac{n + k\xi}{2}$, where $\xi = \sqrt{P}$. Then $\mathbb{Q}(\hat{S}) = \mathbb{Q}(\xi)$. As all the entries of $\hat{S}$ are algebraic integers, we can assume $d_2 = a + b\xi$, $d_3 = c + d\xi$, $s_{22} = e + f\xi$, $s_{23} = g + h\xi$, where a, b, c, d, e, f, g and h are either half integers or integers. Then using Maple's Gröbner basis algorithm to eliminate non-rational variables we obtain 21 Diophantine equations (over $\frac{1}{2}\mathbb{Z}$).

Note that $\hat{N}_{12}^3 = -1$ if $d = 0$ or $2h - k = 0$. One Diophantine equation we derive is

$$2b^2h - b^2k + 2d^2h + d^2k = 0,$$

Which can be written as $\frac{b^2}{d^2} = -\frac{2h+k}{2h-k}$. So we have $(2h - k)(2h + k) \leq 0$, and since $k > 0$, we see that $h \in (-\frac{k}{2}, \frac{k}{2})$. The condition $d_1 < 14$ implies $n \leq 13$ and $k \leq \sqrt{n^2 + 4}$, and k is determined by n , so we do a brute force search for solutions using parameters (n, h, k) . There are 13 cases which pass the non-negative and integral condition of the naive fusion coefficients $\hat{N}_{ij}^k$, which are the cases when $n = 1, \dots, 13$ and $h = -\frac{k}{2}$, for each k corresponding to n . In fact, for these cases, the corresponding $\hat{S}$ matrix has the following form:

$$\begin{pmatrix} 1 & \phi_n & 1 & \phi_n \\ \phi_n & -1 & \phi_n & -1 \\ 1 & \phi_n & -1 & -\phi_n \\ \phi_n & -1 & -\phi_n & 1 \end{pmatrix}.$$

All the cases can be ruled out by Lemma 2.6 except when $(n, k, h) = (1, 1, -\frac{1}{2})$ and $(n, k, h) = (2, 2, -1)$. For the first case, we have $a = 2d, b = 0, c = d, e = -1, f = 0$, and $g = -\frac{1}{2}$. Then $n_{3,3,3} = 2d - \frac{1}{2d}$, which is non-negative and integral. Thus $d = -\frac{1}{2}$ or $\frac{1}{2}$. Note that $d_2 = -1$ if $d = -\frac{1}{2}$, which is a contradiction. If $d = \frac{1}{2}$, the corresponding $\hat{S}$ -matrix has a modular realization as $\text{Fib} \boxtimes \text{Sem}$. For the second case, we have $n_{2,2,2} = d - \frac{1}{d}$. Thus $d = 1$ and the corresponding S -matrix has a modular realization as $\text{PSU}(2)_6 \boxtimes \text{Sem}$. These are the second pair of $\hat{S}$ -matrices. $\square$

4. Fusion Rules

Recall that the naive fusion coefficients are defined as $\hat{N}_{ij}^k = N_{ij}^k + N_{ij}^{fk}$, where $i, j, k \in \Pi_0$. To get the fusion coefficients N_{ij}^k for the corresponding super-modular categories, we need to determine how these $\hat{N}_{ij}^k$ split. Note that for the pointed cases, such as the ones in Theorems 3.2 and 3.4, the corresponding super-modular categories split by Proposition 2.1. Moreover, the $\hat{S}$ matrices in Theorem 3.4 give the same naive fusion coefficients. From this discussion, we have the following results:

Lemma 4.1. *If $\mathcal{B}$ is non-self dual super-modular category of rank 8, then $\mathcal{B}$ has the same fusion rules as $\mathcal{C}(\mathbb{Z}_4, Q) \boxtimes \text{sVec}$ where $\mathcal{C}(\mathbb{Z}_4, Q)$ is a pointed modular category with $\mathbb{Z}_4$ fusion rules.*

Lemma 4.2. *If $\mathcal{B}$ is a self-dual super-modular category with Galois group $G = \langle (0) \rangle$, then $\mathcal{B}$ has the same fusion rules as $\mathcal{D} \boxtimes \text{sVec}$, where $\mathcal{D}$ is a Toric code modular category.*

Lemma 4.3. *Let $\mathcal{B}$ be a self-dual super-modular category with $\hat{S}$ of the following form:*

$$\begin{pmatrix} 1 & 1 & 2 & \sqrt{6} \\ 1 & 1 & 2 & -\sqrt{6} \\ 2 & 2 & -2 & 0 \\ \sqrt{6} & -\sqrt{6} & 0 & 0 \end{pmatrix}.$$

Then $\mathcal{B}$ has the same fusion rules as the centralizer $\langle f \rangle'$ of either fermion f in the modular category $\text{SO}(12)_2$ (see the Appendix A).

Proof. $\hat{N}_{11}^1 = \hat{N}_{11}^2 = \hat{N}_{12}^3 = \hat{N}_{22}^3 = \hat{N}_{33}^3 = 0$, $\hat{N}_{12}^2 = \hat{N}_{13}^3 = \hat{N}_{22}^2 = 1$ and $\hat{N}_{23}^3 = 2$.

We can assume that $N_{22}^2 = 1$ and $N_{22}^{f2} = 0$ by interchanging X_2 and fX_2 if necessary. Similarly, we assume $N_{13}^3 = 1$ and $N_{13}^{f3} = 0$ by interchanging X_3 and fX_3 and X_1 and fX_1 simultaneously, if needed. Using the modified balancing equation on $\hat{S}_{23}$, we get $0 = (N_{23}^3 - N_{23}^{f3})\theta_3\sqrt{6}$. So, we have $N_{23}^3 = N_{23}^{f3} = 1$. Now we have

- (1) $f^{\otimes 2} = \mathbf{1}$,
- (2) $X_1^{\otimes 2} = \mathbf{1}$,
- (3) $X_2^{\otimes 2} = \mathbf{1} \oplus aX_1 \oplus bfX_1 \oplus X_2$,
- (4) $X_3^{\otimes 2} = \mathbf{1} \oplus X_1 \oplus X_2 \oplus fX_2$,
- (5) $X_1 \otimes X_2 = aX_2 \oplus bfX_2$,
- (6) $X_1 \otimes X_3 = X_3$,
- (7) $X_2 \otimes X_3 = X_3 \oplus fX_3$.

Computing $X_2 \otimes X_2 \otimes X_3$ in two ways gives us: $(2+a)X_3 \oplus (b+1)fX_3 = 2X_3 \oplus 2fX_3$. So we have $a = 0$ and $b = 1$. $\square$

Lemma 4.4. *Let $\mathcal{B}$ be a self-dual super-modular category with*

$$\hat{S} = \begin{pmatrix} 1 & \phi_1\phi_2 & \phi_1 & \phi_2 \\ \phi_1\phi_2 & 1 & -\phi_2 & -\phi_1 \\ \phi_1 & -\phi_2 & -1 & \phi_1\phi_2 \\ \phi_2 & -\phi_1 & \phi_1\phi_2 & -1 \end{pmatrix}.$$

Then $\mathcal{B}$ has the same fusion rules as $\text{Fib} \boxtimes \text{PSU}(2)_6$.

Proof. The naive fusion coefficients are: $\hat{N}_{11}^1 = \hat{N}_{33}^3 = 2$, $\hat{N}_{11}^2 = \hat{N}_{12}^3 = \hat{N}_{22}^2 = 1$, $\hat{N}_{12}^2 = \hat{N}_{13}^3 = \hat{N}_{22}^3 = \hat{N}_{23}^3 = 0$. As $\hat{N}_{22}^2 = N_{22}^2 + N_{22}^{f2} = 1$, we assume $N_{22}^2 = 1$ and $N_{22}^{f2} = 0$ by interchanging X_2 and fX_2 if necessary. Then we have $X_2^{\otimes 2} = \mathbf{1} \oplus X_2$, so X_2 generates a subcategory $\mathcal{F}$ with fusion rules like those of Fib , which is necessarily modular. Therefore $\mathcal{B} \cong \mathcal{F} \boxtimes \mathcal{D}$ where $\mathcal{D}$ is a super-modular category of rank 4 [22, Theorem 3.13]. The classification results in [13] imply that $\mathcal{B}$ has the same fusion rules as $\text{Fib} \boxtimes \text{PSU}(2)_6$. $\square$

Lemma 4.5. *Let $\mathcal{B}$ be a self-dual super-modular category with $\hat{S}$ of the following form*

$$\begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & -d_2 & d_3 & 1 \\ d_2 & d_3 & -1 & -d_1 \\ d_3 & 1 & -d_1 & d_2 \end{pmatrix},$$

where $d_1 = 1 + \sqrt{2} + \sqrt{2 + \sqrt{2}}$, $d_2 = 1 + \sqrt{2} + \sqrt{2(2 + \sqrt{2})}$ and $d_3 = 1 + \sqrt{2 + \sqrt{2}}$. Then $\mathcal{B}$ has the same fusion rules as $\text{PSU}(2)_{14}$.

Proof. The corresponding naive fusion coefficients are: $\hat{N}_{11}^1 = \hat{N}_{11}^3 = \hat{N}_{12}^3 = \hat{N}_{13}^3 = \hat{N}_{33}^3 = 1$, $\hat{N}_{11}^2 = \hat{N}_{12}^2 = \hat{N}_{22}^2 = \hat{N}_{23}^2 = 2$ and $\hat{N}_{23}^3 = 0$. Since $\hat{N}_{11}^1 = N_{11}^1 + N_{11}^{f1} = 1$, we can assume $N_{11}^1 = 1$ and $N_{11}^{f1} = 0$ by interchanging X_1 and fX_1 if necessary. Similarly, since $\hat{N}_{33}^3 = N_{33}^3 + N_{33}^{f3} = 1$, we can assume $N_{33}^3 = 1$ and $N_{33}^{f3} = 0$. Finally, we may use the X_2 versus fX_2 labeling ambiguity to assume that $N_{13}^2 = 1$. We have

- (1) $f^{\otimes 2} = \mathbf{1}$,
- (2) $X_1^{\otimes 2} = \mathbf{1} \oplus X_1 \oplus aX_2 \oplus bfX_2 \oplus cX_3 \oplus dfX_3$, where $a + b = 2$, $c + d = 1$,
- (3) $X_2^{\otimes 2} = \mathbf{1} \oplus gX_1 \oplus hfX_1 \oplus lX_2 \oplus mfX_2 \oplus pX_3 \oplus qfX_3$, where $g + h = 2$, $l + m = 2$ and $p + q = 2$,
- (4) $X_3^{\otimes 2} = \mathbf{1} \oplus rX_1 \oplus sfX_1 \oplus X_3$, where $r + s = 1$,
- (5) $X_1 \otimes X_2 = aX_1 \oplus bfX_1 \oplus gX_2 \oplus hfX_2 \oplus X_3$,
- (6) $X_1 \otimes X_3 = cX_1 \oplus dfX_1 \oplus X_2 \oplus rX_3 \oplus sfX_3$,
- (7) $X_2 \otimes X_3 = X_1 \oplus pX_2 \oplus qfX_2$.

Computing $X_1 \otimes X_3 \otimes X_3$ in two ways and comparing the coefficients of X_1 , fX_1 , X_2 and fX_2 , we have $c + r = 2$, $d + s = 0$, $ar + bs + 1 = c + p$ and $br + as = d + q$. Thus, we have $c = r = 1$, $d = s = 0$, $a = p$ and $b = q$. Applying Corollary 2.3 to $\hat{S}_{23}$, we have $|d_1| = |d_1\theta_1 + (p - q)d_2\theta_2| \geq |(p - q)d_2\theta_2| - d_1$. If $|p - q| = 2$, then $4.26 \approx d_1 \geq |2d_2 - d_1| \approx 5.79$, which is impossible. So we have $p = q = 1$. Therefore $a = b = 1$. Computing $X_2 \otimes X_3 \otimes X_3$ in two different ways and comparing the coefficients of X_2 and fX_2 , we have $g = h = 1$. Tensoring $X_2 \otimes X_2 \otimes X_3$ in two ways and comparing the coefficients of X_1 and fX_1 , we have $l = 1$ and $m = 1$. $\square$

Lemma 4.6. *Let $\mathcal{B}$ be a self-dual super-modular category with*

$$\hat{S} = \begin{pmatrix} 1 & d & 1 + d & d^2 - 1 \\ d & -(1 + d) & -1 & d^2 - 1 \\ 1 + d & -1 & d & -(d^2 - 1) \\ d^2 - 1 & d^2 - 1 & -(d^2 - 1) & 0 \end{pmatrix},$$

where d is the largest real root of $x^3 - 3x - 1 = 0$. Then $\mathcal{B}$ has the same fusion rules as $\text{PSU}(2)_7 \boxtimes \text{sVec}$.

Proof. We have $\hat{N}_{11}^1 = \hat{N}_{11}^2 = \hat{N}_{13}^3 = 0$ and $\hat{N}_{11}^3 = \hat{N}_{12}^2 = \hat{N}_{12}^3 = \hat{N}_{22}^2 = \hat{N}_{22}^3 = \hat{N}_{23}^3 = \hat{N}_{33}^3 = 1$.

Note that since $\hat{N}_{22}^2 = N_{22}^2 + N_{22}^{f2} = 1$, we can assume $N_{22}^2 = 1$ and $N_{22}^{f2} = 0$ by interchanging X_2 and fX_2 if necessary. Similarly, we can assume $N_{33}^3 = 1$, $N_{33}^{f3} = 0$, $\hat{N}_{22}^1 = 1$ and $\hat{N}_{22}^{f1} = 0$. We have

- (1) $f^{\otimes 2} = \mathbf{1}$,
- (2) $X_1^{\otimes 2} = \mathbf{1} \oplus aX_3 \oplus bfX_3$, where $a + b = 1$,
- (3) $X_2^{\otimes 2} = \mathbf{1} \oplus X_1 \oplus X_2 \oplus gX_3 \oplus hfX_3$, where $g + h = 1$,
- (4) $X_3^{\otimes 2} = \mathbf{1} \oplus lX_2 \oplus mfX_2 \oplus X_3$, where $l + m = 1$,
- (5) $X_1 \otimes X_2 = X_2 \oplus pX_3 \oplus qfX_3$, where $p + q = 1$,
- (6) $X_1 \otimes X_3 = aX_1 \oplus bfX_1 \oplus pX_2 \oplus qfX_2$,
- (7) $X_2 \otimes X_3 = pX_1 \oplus qfX_1 \oplus gX_2 \oplus hfX_2 \oplus lX_3 \oplus mfX_3$.

Computing $X_1 \otimes X_1 \otimes X_2$ in two different ways and comparing the coefficients of X_2 and fX_2 , we have $ag + bh = 1$, $bg + ah = 0$. Thus, we have $a = g$ and $b = h$. Similarly, comparing the coefficients of X_3 and fX_3 in $X_1 \otimes X_1 \otimes X_3$ gives us $a = 1$ and $b = 0$. Computing $X_2 \otimes X_2 \otimes X_3$ and comparing the coefficients of X_3 and fX_3 , we have $l = 1$ and $m = 0$. Computing $X_1 \otimes X_3 \otimes X_3$ in two different ways and comparing the coefficients for X_2 and fX_2 , we have $p = 1$ and $q = 0$. Observing that the simple objects $\mathbf{1}, X_1, X_2$ and X_3 generate a fusion subcategory with the same fusion rules as $\text{PSU}(2)_7$ we obtain the stated result. $\square$

Lemma 4.7. *Let $\mathcal{B}$ be a self-dual super-modular category. Suppose that the corresponding $\hat{S}$ has one of the following forms:*

$$\begin{pmatrix} 1 & \phi_1^2 & \phi_1 & \phi_1 \\ \phi_1^2 & 1 & -\phi_1 & -\phi_1 \\ \phi_1 & -\phi_1 & -1 & \phi_1^2 \\ \phi_1 & -\phi_1 & \phi_1^2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \phi_2^2 & \phi_2 & \phi_2 \\ \phi_2^2 & 1 & -\phi_2 & -\phi_2 \\ \phi_2 & -\phi_2 & -1 & \phi_2^2 \\ \phi_2 & -\phi_2 & \phi_2^2 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \phi_1 & 1 & \phi_1 \\ \phi_1 & -1 & \phi_1 & -1 \\ 1 & \phi_1 & -1 & -\phi_1 \\ \phi_1 & -1 & -\phi_1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \phi_2 & 1 & \phi_2 \\ \phi_2 & -1 & \phi_2 & -1 \\ 1 & \phi_2 & -1 & -\phi_2 \\ \phi_2 & -1 & -\phi_2 & 1 \end{pmatrix},$$

then $\mathcal{B}$ has the same fusion rules as $\text{Fib} \boxtimes \text{Fib} \boxtimes \text{sVec}$, $[\text{PSU}(2)_6 \boxtimes \text{PSU}(2)_6]_{\mathbb{Z}_2}$, $\text{Sem} \boxtimes \text{PSU}(2)_6 \boxtimes \text{sVec}$, or $\text{Sem} \boxtimes \text{Fib} \boxtimes \text{sVec}$, respectively.

Proof. Consider the first $\hat{S}$ -matrix. We have $\hat{N}_{11}^1 = \hat{N}_{11}^2 = \hat{N}_{11}^3 = \hat{N}_{12}^3 = \hat{N}_{22}^2 = \hat{N}_{33}^3 = 1$ and $\hat{N}_{12}^2 = \hat{N}_{13}^3 = \hat{N}_{22}^3 = \hat{N}_{23}^3 = 0$. Without loss of generality, we may assume $N_{22}^2 = 1$, $N_{22}^{f2} = 0$ by interchanging X_2 and fX_2 if necessary. Thus $X_2^{\otimes 2} = \mathbf{1} \oplus X_2$, so X_2 generates a subcategory $\mathcal{F}$ with fusion rules like those of Fib , which is necessarily modular. In particular $\mathcal{B} \cong \mathcal{F} \boxtimes \mathcal{D}$, where $\mathcal{D}$ is a super-modular category

of rank 4. The classification results of [13] now imply that $\mathcal{B}$ has the same fusion rules as $\text{Fib} \boxtimes \text{Fib} \boxtimes \text{sVec}$.

For the second $\hat{S}$ -matrix, we have that the associated naive fusion coefficients are $\hat{N}_{11}^1 = 4$, $\hat{N}_{11}^2 = \hat{N}_{11}^3 = \hat{N}_{22}^2 = \hat{N}_{33}^3 = 2$, $\hat{N}_{12}^3 = 1$, $\hat{N}_{12}^2 = \hat{N}_{13}^3 = \hat{N}_{22}^3 = \hat{N}_{23}^3 = 0$. We may assume $N_{12}^3 = 1$ and $N_{12}^{f3} = 0$ by interchanging X_3 and fX_3 if necessary. Using Corollary 2.3 on $\hat{S}_{12}$ gives

$$-\theta_1\theta_2\phi_2 = (N_{12}^1 - N_{12}^{f1})\phi_2^2\theta_1 + \phi_2\theta_3.$$

Dividing by ϕ_2 , we have

$$-\theta_1\theta_2 = (N_{12}^1 - N_{12}^{f1})\phi_2\theta_1 + \theta_3.$$

Taking absolute value on both sides, we get

$$1 = |(N_{12}^1 - N_{12}^{f1})\phi_2\theta_1 + \theta_3| \geq |(N_{12}^1 - N_{12}^{f1})\phi_2| - 1.$$

So we must have $N_{12}^1 = N_{12}^{f1} = 1$. Similarly, applying Corollary 2.3 to $\hat{S}_{33}$ and $\hat{S}_{13}$ gives

$$-\theta_3^2 = 1 + (N_{33}^3 - N_{33}^{f3})\phi_2\theta_3, \quad -\theta_1\theta_3\phi_2 = (N_{13}^1 - N_{13}^{f1})\phi_2^2 + \phi_2\theta_2$$

and we get $N_{33}^3 = N_{33}^{f3} = 1$ and $N_{11}^3 = N_{11}^{f3} = 1$. A parallel calculation for $\hat{S}_{22}$ yields $N_{22}^2 = N_{22}^{f2} = 1$. By using Corollary 2.3 again for $\hat{S}_{11}$, we get

$$\theta_1^2 = (N_{11}^1 - N_{11}^{f1})\phi_2^2\theta_1 + 1.$$

The potential choices of (N_{11}^1, N_{11}^{f1}) are $(2, 2)$, $(4, 0)$, $(0, 4)$, $(1, 3)$ and $(3, 1)$, but since $\phi_2^2 > 2$ the only possibility is $(2, 2)$. This category has the same fusion rules as $[\text{PSU}(2)_6 \boxtimes \text{PSU}(2)_6]_{\mathbb{Z}_2}$, see the Appendix A.

In the last two cases, observe that $\mathcal{B}$ must contain a modular subcategory of the form $\mathcal{C}(\mathbb{Z}_2, Q)$ by Lemma 2.6. Then $\mathcal{B} \cong \mathcal{C}(\mathbb{Z}_2, Q) \boxtimes \mathcal{D}$, where $\mathcal{D}$ is a rank 4 super-modular category. The result now follows from the classification in [13]. $\square$

Appendix A.

Here, we record the data for some of the realizations of the super-modular categories that appear in this paper, both modular and super-modular, as well as the families of categories in which they reside. We write the T -matrix as an n -tuple with the understanding that these are the diagonal entries.

Appendix A.1. Pointed modular categories

Pointed braided fusion categories are classified, see [22]. They correspond to pairs (A, Q) , where Q is a symmetric quadratic form on A (with values in $U(1)$). The fusion rules of $\mathcal{C}(A, Q)$ are the same as the multiplication in A , and the S - and T -matrices are determined by Q as follows: $S_{a,b} = \frac{Q(a+b)}{Q(a)Q(b)}$ and $\theta_a = Q(a)$. If the symmetric bilinear form given by $S_{a,b}$ is non-degenerate then $\mathcal{C}(A, Q)$ is modular.

For example the semion theory $\text{Sem} = \mathcal{C}(\mathbb{Z}_2, Q)$ that appears in our classification has the following modular data: $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and $T = (1, i)$.

Appendix A.2. $\text{PSU}(2)_k$

The rank $k + 1$ modular category $\text{SU}(2)_k$ obtained from $U_q \mathfrak{sl}_2$ at $q = e^{\pi i/(2+k)}$ contains the subcategory $\text{PSU}(2)_k$ whose simple objects have even labels (“integer spin” in the physics literature). Denote by ϖ the fundamental weight of type A_1 , so that X_ϖ tensor generates $\text{SU}(2)_k$. The object labeled by $\frac{k}{2}\varpi$ is always invertible. When $k \equiv 2 \pmod{4}$ the category $\text{PSU}(2)_k$ is super-modular with $f = X_{\frac{k}{2}\varpi}$, when $4 \mid k$, there is a boson $b = X_{\frac{k}{2}\varpi}$ in $\text{PSU}(2)_k$, and when k is odd, $\text{PSU}(2)_k$ is modular, with $X_{\frac{k}{2}\varpi}$ a semion (not in $\text{PSU}(2)_k$).

The (modular) Fibonacci theory $\text{Fib} = \text{PSU}(2)_3^{\text{ev}}$ as well as $\text{PSU}(2)_7$ appear in our classification, and the data can be found in [44].

Some low rank super-modular categories that appear in this paper are

- $\text{PSU}(2)_6$ with data:

$$S = \begin{pmatrix} 1 & 1 + \sqrt{2} \\ 1 + \sqrt{2} & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad T = (1, i) \otimes (1, -1).$$

- $\text{PSU}(2)_{10}$ with data:

$$S = \begin{pmatrix} 1 & 2 + \sqrt{3} & 1 + \sqrt{3} \\ 2 + \sqrt{3} & 1 & -1 - \sqrt{3} \\ 1 + \sqrt{3} & -1 - \sqrt{3} & 1 + \sqrt{3} \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \\ T = (1, -1, e^{\pi i/3}) \otimes (1, -1).$$

- $\text{PSU}(2)_{14}$ with data:

$$S = \begin{pmatrix} 1 & 1 + x & 1 + \sqrt{2} + x & 1 + \sqrt{2} + \sqrt{2}x \\ 1 + x & 1 + \sqrt{2} + \sqrt{2}x & 1 & -1 - \sqrt{2} - x \\ 1 + \sqrt{2} + x & 1 & -1 - \sqrt{2} - \sqrt{2}x & 1 + x \\ 1 + \sqrt{2} + \sqrt{2}x & -1 - \sqrt{2} - x & 1 + x & -1 \end{pmatrix} \\ \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{where} \\ x = \sqrt{2 + \sqrt{2}} \quad \text{and} \quad T = (1, e^{i\pi/4}, e^{3i\pi/4}, -i) \otimes (1, -1).$$

The full sequence of super-modular categories $\text{PSU}(2)_{4m+2}$ was studied in [6, 10], where the modular data can be found. If we order the simple objects $[1, X_1, \dots, X_{r-1}, fX_{r-1}, \dots, fX_1, f] = [Y_0, \dots, Y_{2(r-1)}]$ the fusion rules are completely determined by the rule $Y_1 \otimes Y_k \cong Y_{k-1} \oplus Y_k \oplus Y_{k+1}$ for $0 < k < 2(r-1)$ and the obvious rules involving $Y_{2(r-1)} = f$ and $Y_0 = 1$.

Appendix A.3. Other examples

The following are spin modular categories coming from quantum groups with fermion f so that the subcategory $\langle f \rangle'$ is super-modular, where $r, m \in \mathbb{N}$:

- $SU(4k+2)_{4m+2}$,
- $SO(2k+1)_{2m+1}$,
- $Sp(2r)_m$ with $rm = 2 \pmod{4}$,
- $SO(2r)_m$ with $r = 2 \pmod{4}$ and $m = 2 \pmod{4}$,
- $(E_7)_{4m+2}$.

The pointed sub-category of the rank 13 modular category $SO(12)_2$ is $\text{sVec} \boxtimes \text{sVec}$ and hence contains two fermions labeled by $2\varpi_5$ and $2\varpi_6$, where ϖ_i are the fundamental weights of type D_6 . The centralizer of either of these fermions is super-modular and has modular data:

$$S := \begin{pmatrix} 1 & 1 & 2 & \sqrt{6} \\ 1 & 1 & 2 & -\sqrt{6} \\ 2 & 2 & -2 & 0 \\ \sqrt{6} & -\sqrt{6} & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and}$$

$$T = (1, 1, e^{2\pi i/3}, e^{3\pi i/8}) \otimes (1, -1).$$

If we label the simple objects of dimension $\sqrt{6}$ by X_3 and fX_3 then the fusion rules are determined by $X_3^{\otimes 2} \cong \mathbf{1} \oplus X_1 \oplus X_2 \oplus fX_2$, $X_1^{\otimes 2} \cong \mathbf{1}$, $X_2^{\otimes 2} \cong \mathbf{1} \oplus fX_1 \oplus X_2$ and $X_2 \otimes X_3 \cong X_3 \oplus fX_3$.

Finally, we observe that if $(\mathcal{C}_1, f_1)$ and $(\mathcal{C}_2, f_2)$ are spin modular categories, then $(f_1, f_2) \in \mathcal{C}_1 \boxtimes \mathcal{C}_2$ is a boson and hence can be condensed to obtain a new spin modular category $([\mathcal{C}_1 \boxtimes \mathcal{C}_2]_{\mathbb{Z}_2})_0$, where we de-equivariantize by $\text{Rep}(\mathbb{Z}_2) \cong \langle (f_1, f_2) \rangle$ and then take the trivial component of the corresponding $\mathbb{Z}_2$ -grading. For example applying this to $\text{PSU}(2)_6$ we obtain the prime rank 8 example $(\text{PSU}(2)_6 \boxtimes \text{PSU}(2)_6)_{\mathbb{Z}_2}$ with data:

$$S := \begin{pmatrix} 1 & 3+2\sqrt{2} & 1+\sqrt{2} & 1+\sqrt{2} \\ 3+2\sqrt{2} & 1 & -1-\sqrt{2} & -1-\sqrt{2} \\ 1+\sqrt{2} & -1-\sqrt{2} & -1 & 3+2\sqrt{2} \\ 1+\sqrt{2} & -1-\sqrt{2} & 3+2\sqrt{2} & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and}$$

$$T = (1, -1, i, i) \otimes (1, -1).$$

The fusion rules may be readily determined from those of $\text{PSU}(2)_6$ by condensing the boson $b := (f_1, f_1)$. Note that $b \otimes X \not\cong X$ for any simple X so that there is no ambiguity in labeling the objects in the de-equivariantization. Setting $f :=$

$[(f_1, \mathbf{1})] = [(\mathbf{1}, f_1)]$ we have

$$X_1^{\otimes 2} \cong \mathbf{1} \oplus 2(X_1 \oplus fX_1) \oplus X_2 \oplus fX_2 \oplus X_3 \oplus fX_3,$$

$$X_1 \otimes X_2 \cong X_3 \oplus X_1 \oplus fX_1$$

$$X_1 \otimes X_3 \cong X_2 \oplus X_3 \oplus fX_3, \quad X_2 \otimes X_3 \cong X_1, \quad \text{and} \quad X_2^{\otimes 2} \cong \mathbf{1} \oplus X_2 \oplus fX_2$$

from which all fusion rules can be recovered.

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