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Stochastic PDE limit of the dynamic ASEP

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Abstract: We study a stochastic PDE limit of the height function of the dynamic asymmetric simple exclusion process (dynamic ASEP). Introduced in Borodin (Symmetric elliptic functions, IRF models, and dynamic exclusion processes, 2017), the dynamic ASEP has a jump parameter $q \in (0, 1)$ and a dynamical parameter $\alpha > 0$. It degenerates to the standard ASEP height function when α goes to 0 or ∞. We consider a *very weakly asymmetric regime*, i.e. for ε tending to zero we set $q = e^{-\varepsilon}$. We show that under the parabolic scaling the height function of the dynamic ASEP converges to the solution of the space-time Ornstein–Uhlenbeck (OU) process. We also introduce the dynamic ASEP on a ring with generalized rate functions. Under the very weakly asymmetric scaling, we show that the dynamic ASEP (with generalized jump rates) on a ring also converges to the solution of the space-time OU process on [0, 1] with periodic boundary conditions.

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1. Introduction

The dynamic ASEP, introduced in [Bor17], is a continuous time Markov process defined in terms of a temporally evolving height function $s_t(x) \in \mathbb{Z}$ with time $t \in \mathbb{R}_{\geq 0}$ and

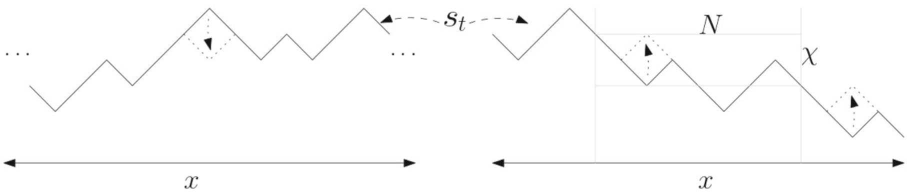

Fig. 1. Dynamic ASEP height function s_t on the full line (left) and periodic domain (right). In the periodic case, the N represents the periodicity length and χ represents the change in the height function over that length

space $x \in \mathbb{Z}$. (We may also extend the space variable to $x \in \mathbb{R}$ by linear interpolation between values at $x \in \mathbb{Z}$.) The height function satisfies the *solid-on-solid condition* that $s_t(x) - s_t(x+1) \in \{1, -1\}$ for all t and x. The Markovian update rule depends on two parameters: the asymmetry $q \in (0, 1)$ and the dynamic parameter $\alpha \in (0, \infty)$. The values of the height function at each $x \in \mathbb{Z}$ are updated according to independent exponential clocks with the following rates (assuming that the change does not violate the solid-on-solid condition, Fig. 1):

$$s_t(x) \mapsto s_t(x) - 2 \text{ at rate } \frac{q(1+\alpha q^{-s_t(x)})}{1+\alpha q^{-s_t(x)+1}}, \quad s_t(x) \mapsto s_t(x) + 2 \text{ at rate } \frac{1+\alpha q^{-s_t(x)}}{1+\alpha q^{-s_t(x)-1}}. \tag{1.1}$$

The term *dynamic* alludes to the fact that the above rates depend on the height function. On taking α to 0 or ∞, the rate parameters converge to q and 1, or 1 and q, thus recovering the standard ASEP height function rates. The dynamic ASEP with $\alpha \in (0, \infty)$ has a preferred height of roughly $\log_q(\alpha)$—above this height there is a tendency for the height function to decrease (i.e. the rate for decreasing exceeds that for increasing) and below this height the opposite happens. In light of this markedly different behavior compared to the standard ASEP, it is natural to explore what becomes of the various asymptotic phenomena enjoyed by the standard ASEP. Notice that via a height shift $s \mapsto s - \log_q \alpha$, the rates above reduce to the $\alpha = 1$ rates. (The resulting shifted height function now lives on a shift of the lattice $\mathbb{Z}$.) Owing to this observation, we may, without loss of generality assume that $\alpha = 1$ throughout the rest of this work. We will still use α in stating our main results, but in the proof we will set $\alpha = 1$ to simplify notation.

In this paper we prove a stochastic PDE (SPDE) limit for the dynamic ASEP under *very weakly asymmetric scaling*. This is the first SPDE limit result shown for this type of system. The limiting SPDE is a space-time Ornstein–Uhlenbeck (OU) process. For the standard ASEP under the same scaling, the limiting SPDE is the additive stochastic heat equation (or Edward–Wilkinson equation). See [DMPS89,DG91] for reference. The difference between these two equations is the presence of a linear drift in the OU which introduces a preferred height which the process drifts towards. Thus, the effect of the dynamic parameter survives in our limit.

Before going into greater depth about our present contribution, let us recall the previous work on this process.

Besides introducing the model, [Bor17] developed a generalization of the method introduced by [BP18,BP16] (in studying non-dynamic higher spin vertex models) in order to compute contour integral formulas for expectations of a class of observables for certain initial conditions (in particular for the wedge, where $s_0(x) = |x|$). Taking the limit $q \to 1$ leads to a dynamic version of the SSEP where the corresponding jump rates are

$$s_t(x) \mapsto s_t(x) - 2 \text{ at rate } \frac{s_t(x) - \lambda}{s_t(x) - 1 - \lambda}, \quad s_t(x) \mapsto s_t(x) + 2 \text{ at rate } \frac{s_t(x) - \lambda}{s_t(x) + 1 - \lambda},$$

for some $\lambda < 0$. In that limit, the observables and formulas simplify sufficiently so that [Bor17] was able to probe some asymptotics (for wedge initial data).

In particular [Bor17, Theorem 11.2] explores the hydrodynamic scaling for the dynamic SSEP. There is some freedom in how to scale the dynamic parameter. For one choice, the hydrodynamic limit is the same as for the standard SSEP (i.e. governed by the heat equation). Surprisingly, for other choices of scaling for the dynamic parameter, there is no deterministic hydrodynamic limit—the height function remains random under scaling.

The next development regarding dynamic ASEP was the Markov duality derived by [BC17] between it and the standard ASEP (Section 2 herein). The duality function which intertwines these two processes is the same observable which had arisen in the earlier work of [Bor17]. [BC17] also derived a translation invariant, stationary measure for the dynamic ASEP (see Definition 1.2).

There are a few other works related to the dynamic ASEP (though we will not make use of them herein). [Agg18] introduced dynamic analogs of the higher spin vertex models from [CP16], and [BM18] and [ABB18] introduced other types of dynamic analogs of growth models and vertex models (i.e. the growth rates depend on the height through an additional dynamic parameter). For the dynamic stochastic six vertex model, [BG18] used the formulas from [Bor17] to derive a law of large numbers and Gaussian central limit theorem under a particular choice of scaling.

1.1. Main results. We consider the scaling limit of the dynamic ASEP defined on the full line $\mathbb{Z}$ as well a more general version of it defined on a finite interval of $\mathbb{Z}$ with periodic boundary conditions. In each of these two settings we provide a different method of proof. In the full line case we employ a remarkable generalization of the microscopic Hopf–Cole (or Gärtner) transform [Gär88] along with some estimates similar to [BG97]. In the periodic setting we employ a variant of "da Prato–Debussche trick" from [DPD03]. This second approach can be used to prove a scaling limit of a generalization of the dynamic ASEP, defined in Section 1.1.2, to which the discrete Hopf–Cole transform cannot be applied. We consider this generalized model in a periodic setting, because this allows us to avoid some challenging technical points regarding the growth of stochastic processes at infinity that would arise in applying the method of [DPD03] on the line. Besides technicalities, we believe that both methods present in this paper have value and may find further applications in studying other scaling limits of this or related systems.

1.1.1. Full line results We define the following linear stochastic heat equation with additive space-time white noise (with $A < 0$ this is sometimes called a *space-time Ornstein–Uhlenbeck process*)

$$\partial_t \mathcal{Z}_t(x) = \partial_x^2 \mathcal{Z}_t(x) + A\mathcal{Z}_t(x) + B_t\, \xi(t, x) \tag{1.2}$$

on $[0, +\infty) \times \mathbb{R}$, with the initial state $\mathcal{Z}_0 : \mathbb{R} \to \mathbb{R}$ at time $t = 0$, where ξ is the space-time white noise on $\mathbb{R}^2$ on some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, $A \le 0$ is a constant and $B : \mathbb{R} \to \mathbb{R}$ is a differentiable, locally bounded function. The mild formulation of (1.2) is

$$\mathcal{Z}_t(x) = (e^{t\partial_x^2}\mathcal{Z}_0)(x) + A\int_0^t (e^{(t-s)\partial_x^2}\mathcal{Z}_s)(x)ds + \int_0^t B_s(e^{(t-s)\partial_x^2}\xi_s)(x)ds, \qquad (1.3)$$

where $(e^{t\partial_x^2} f)(x)$ denotes the action of the heat semigroup $\{e^{t\partial_x^2}\}_{t\geq 0}$ on a function $f : \mathbb{R} \to \mathbb{R}$ via

$$(e^{t\partial_x^2} f)(x) = \int_{\mathbb{R}} p_t(x-y) f(y)dy,$$

where $p_t(\bullet)$ denotes the Green's function of the operator $\partial_t - \partial_x^2$, i.e. p_t is the heat kernel, solving the PDE

$$\partial_t p_t(x) = \partial_x^2 p_t(x), \qquad p_0(x) = \delta_x,$$

where δ_x is the Dirac delta-function. The last term on the r.h.s. of (1.3) is defined as a Wiener integral, see [DPZ14] or [Wal86]. For the existence, uniqueness and the continuity of the mild solution (1.3), see [Wal86, Thm. 3.5].

We denote the space of all continuous functions on $\mathbb{R}$ by $\mathcal{C}(\mathbb{R})$, and endow it with the topology of uniform convergence on the compact sets of $\mathbb{R}$. Then $D([0,\infty), \mathcal{C}(\mathbb{R}))$ denotes the space of all $\mathcal{C}(\mathbb{R})$-valued càdlàg functions on $[0,\infty)$, equipped with the Skorokhod topology [Bil99]. In what follows, we use the notation "$\Rightarrow$" to denote weak convergence in the respective topology, and $f^\varepsilon \Rightarrow f$ means that the random function f^ε weakly converges to f as $\varepsilon \to 0$. The L^n-norm with respect to the underlying probability measure will be denoted $\| \bullet \|_n := \mathbb{E}[| \bullet |^n]^{1/n}$.

We are now ready to state our main theorem for the dynamic ASEP on the full line.

Theorem 1.1. *Consider the dynamic ASEP with ε-dependent asymmetry parameter $q = e^{-\varepsilon}$ and fixed dynamic parameter $\alpha \in (0,\infty)$. Define the ε-rescaled height function as*

$$\hat{s}_t^\varepsilon(x) := \varepsilon^{\frac{1}{2}}\big(s_{\varepsilon^{-2}t}(\varepsilon^{-1}x) - \log_q \alpha\big). \qquad (1.4)$$

Assume that $\hat{s}_0^\varepsilon(\bullet)$ is a sequence of near stationary initial condition, i.e. for some $u > 0$, $\beta \in (0, \frac{1}{4})$ and all $k \in \mathbb{N}$, there exists $C_0 = C_0(u, \beta, k)$ such that the following bounds hold

$$\|e^{|\hat{s}_0^\varepsilon(x)|}\|_{2k} \leq C_0 e^{u|x|}, \qquad (1.5a)$$

$$\|\hat{s}_0^\varepsilon(x) - \hat{s}_0^\varepsilon(x')\|_{2k} \leq C_0 |x - x'|^{2\beta} e^{u(|x|+|x'|)}, \qquad (1.5b)$$

for all $x, x' \in \mathbb{R}$. Assume that $\hat{s}_0^\varepsilon(\bullet) \Rightarrow \mathcal{Z}_0(\bullet)$ in $\mathcal{C}(\mathbb{R})$. Then $\hat{s}_\varepsilon$ converges weakly as $\varepsilon \to 0$ in $D([0,\infty), \mathcal{C}(\mathbb{R}))$ to the unique solution of (1.2) with $A = -\frac{1}{4}$ and $B \equiv \sqrt{2}$, and initial data $\mathcal{Z}_0$.

The proof of Theorem 1.1 relies on the remarkable fact that dynamic ASEP admits a microscopic Hopf–Cole (or Gärtner) transform. Namely, as a function of $t \geq 0$ and $x \in \mathbb{Z}$,

$$e^{(1-\sqrt{q})^2 t}\big(\alpha^{\frac{1}{2}} q^{-\frac{s_t(x)}{2}} - \alpha^{-\frac{1}{2}} q^{\frac{s_t(x)}{2}}\big) \qquad (1.6)$$

solves a microscopic stochastic heat equation (see Proposition 2.1 for a precise statement). This result is closely related to the one-particle version of the Markov duality proved in [BC17]. We note that under any minute perturbation of the rates, the duality relation would be lost and consequently the Hopf–Cole transform would fail to satisfy a semi-discrete stochastic heat equation as in Proposition 2.1 below. Theorem 2.5 shows that the process in (1.6) converges to a certain space-time OU process. Since the Hopf–Cole transform of (1.6) is a bijective continuously differentiable transformation of the height function, the desired convergence claimed in Theorem 1.1 follows from Theorem 2.5 via continuous mapping theorem. Hence, the challenge boils down to proving Theorem 2.5. This is done via convergence of martingale problems. The proof of the convergence of a linear martingale problem as well as tightness are standard. Identifying the noise (via the quadratic martingale problem) relies upon a 'key estimate' (Lemma 4.4) which is similar to that used in [BG97] (and developed further in subsequent generalizations such as [CST18]) when proving the KPZ equation limit of the standard ASEP.

It was not immediately clear to us which scalings would produce a meaningful limiting SPDE for dynamic ASEP. In our investigation, we were informed by the analysis of the stationary measure for dynamic ASEP. As we now explain, the stationary measure, defined below, converges to a spatial OU process when $q = e^{-\varepsilon}$, height fluctuations are scaled like ε^{-1} and space is scaled like ε^{-2}. This suggested that we should apply the same scalings to the dynamic ASEP, though it does not tell us how to scale time. Any candidate SPDE limit for dynamic ASEP should have the spatial OU as its stationary measure. In fact, the space-time OU process has a spatial OU process as its stationary measure (this can be proved by writing the exact formula for the space-time OU process and sending the time variable to infinity, see [Hai09, Ex. 2.2]).

Definition 1.2 (Stationary measure). We say that $\{s_0(x)\}_{x\in\mathbb{Z}}$ is distributed according to the stationary initial data if it equals in law the trajectory of a Markov process in x with the following transition probability from $s_0(x)$ to $s_0(x-1)$:

$$s_0(x-1) = s_0(x) + 1 \quad \text{with probability } \frac{q^{s_0(x)}}{\alpha + q^{s_0(x)}},$$
$$s_0(x-1) = s_0(x) - 1 \quad \text{with probability } \frac{\alpha}{\alpha + q^{s_0(x)}},$$

and with the one-point marginal at any $x \in 2\mathbb{Z}$ given by

$$\mathbb{P}\big(s_0(x) = 2n\big) = \frac{\alpha^{-2n} q^{n(2n-1)}(1+\alpha^{-1}q^{2n})}{(-\alpha^{-1}, -q\alpha, q; q)_\infty}, \quad n \in \mathbb{Z}, \tag{1.7}$$

and at any $x \in 2\mathbb{Z}+1$ given by

$$\mathbb{P}\big(s_0(x) = 2n+1\big) = \frac{\alpha^{-1} q^{n(2n+1)}(1+\alpha^{-1}q^{2n+1})}{(-q\alpha^{-1}, -\alpha, q; q)_\infty}, \quad n \in \mathbb{Z}.$$

Here, $(a_1, a_2, a_3; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty (a_3; q)_\infty$, where $(a; q)_\infty := \prod_{i=0}^{\infty}(1 - q^i a)$ is the q-Pochhammer symbol, defined for the values a and q, for which the infinite product converges. Theorem 2.15 of [BC17] shows that this initial data is stationary for the dynamic ASEP, meaning that for any fixed $t > 0$, $\{s_t(x)\}_{x\in\mathbb{Z}}$ has the same distributions as $\{s_0(x)\}_{x\in\mathbb{Z}}$.

Note that the distribution of $\{s_0(x)\}_{x\in\mathbb{Z}}$ is spatially stationary up to parity, i.e. one point distributions $s_0(0)$ (resp. $s_0(1)$) is same as $s_0(2k)$ (resp. $s_0(2k+1)$) for all $k \in \mathbb{Z}$. It is natural, therefore, to look for scaling limits in which the stationary measure has a non-trivial limit, and then to use that non-trivial limit to help guess what the limit of the entire space-time process could be. The following lemma and corollary are proved in Appendix D.

Lemma 1.3. *Let $\{s_0(x)\}_{x\in\mathbb{Z}}$ be distributed according to the stationary initial data (Definition 1.2), and let $\hat{s}_0^\varepsilon(x) = \varepsilon^{\frac{1}{2}}(s_0(\varepsilon^{-1}x) - \log_q \alpha)$ be extended piece-wise linearly to $x \in \mathbb{R}$. Then*

(1) *$\hat{s}_0^\varepsilon$ satisfies (1.5);*
(2) *$\hat{s}_0^\varepsilon \Rightarrow \mathcal{Z}_0$ in $\mathcal{C}(\mathbb{R})$, as $\varepsilon \to 0$, where $\mathcal{Z}_0$ is a stationary solution of the SDE*

$$d\mathcal{Z}_0(x) = -\frac{1}{2}\mathcal{Z}_0(x)dx + \frac{1}{2}d\mathcal{W}(x), \tag{1.8}$$

where $x \in \mathbb{R}$ plays a role of a time variable, and $\mathcal{W}$ is a two sided Brownian motion.

In particular, $\mathcal{Z}_0$ is a stationary Ornstein–Uhlenbeck process, and $\mathcal{Z}_0(0)$ is a standard Gaussian random variable.

Applying Theorem 1.1, we may now show that the stationary dynamic ASEP converges to the stationary solution to the space-time OU process.

Corollary 1.4. *Using the notation of Theorem 1.1, consider dynamic ASEP started from the stationary initial data (Definition 1.2). Then the scaled height function $\hat{s}^\varepsilon$ converges to the solution of (1.2) with $A = -\frac{1}{4}$, $B \equiv \sqrt{2}$ and started from the initial data $\mathcal{Z}_0$, given by the stationary solution of (1.8).*

1.1.2. Periodic results The proof of Theorem 1.1 relies heavily on the microscopic Hopf–Cole transform for the dynamic ASEP. However, this transformation is intimately connected with how the rate functions (1.1) are defined. In particular, it may fail to hold after a slight change in the definition of the rate functions. We will define such a more general model now.

For $\chi \in \mathbb{Z}$ and $N \in \mathbb{N}$ with $\chi \equiv N \bmod 2$ we consider the space Ω_χ^N of functions $s : \mathbb{Z} \to \mathbb{R}$ such that $s(x) - s(x+1) \in \{-1, 1\}$ and $s(x + mN) = s(x) + \chi m$ for any integer m. For a function $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$, we define a generalization of the dynamic ASEP on a finite interval $x \in [0, N) \cap \mathbb{Z}$ with periodic boundary conditions and with the following update rates (assuming that they do not violate the condition on the ± 1 slopes):

$$\begin{aligned} s_t(x) &\mapsto s_t(x) - 2 \;\text{ at rate }\; \frac{q(1 + q^{-\mathfrak{f}(s_t(x)-\chi x/N)})}{1 + q^{-\mathfrak{f}(s_t(x)-\chi x/N)+1}}, \\ s_t(x) &\mapsto s_t(x) + 2 \;\text{ at rate }\; \frac{1 + q^{-\mathfrak{f}(s_t(x)-\chi x/N)}}{1 + q^{-\mathfrak{f}(s_t(x)-\chi x/N)-1}}. \end{aligned} \tag{1.9}$$

Each change of the height function is extended to all $x \in \mathbb{Z}$, so that $s_t(\bullet) \in \Omega_\chi^N$. We assume the function $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$ to satisfy the following assumption.

Assumption 1.5. There exist $\mathfrak{a} \geq 0$, $\gamma \in [0, \frac{1}{2})$ and $c \geq 0$ such that $|\mathfrak{f}(z) - \mathfrak{f}(0) - \mathfrak{a}z| \leq c|z|^\gamma$ for all $z \in \mathbb{R}$.

The bound $\gamma < \frac{1}{2}$ guarantees quick convergence of $\mathfrak{f}(z)$ to $\mathfrak{a}z$ at infinity. In particular, this assumption is crucial in the estimate in Lemma 5.8. Note that if $\mathfrak{f}(z) = z$ and $\chi = 0$, then we get the rates (1.1). It is unknown if a Hopf–Cole transform exists for this general dynamic ASEP. However, using the method of [DPD03], we can show the rescaled height function of the periodic dynamic ASEP with the rate functions (1.9) converges to the solution of (1.2) with suitably chosen constants. In the statement of the following theorem, we use the standard Hölder spaces $\mathcal{C}^\eta$ on $\mathbb{R}$, for $\eta > 0$, with the norm $\|\cdot\|_{\mathcal{C}^\eta}$.

Theorem 1.6. *Let $s_t(x)$ be the height function of the periodic generalized dynamic ASEP with the rates (1.9), with a function $\mathfrak{f}$ satisfying Assumption 1.5, with period $N \in \mathbb{N}$ and with $\chi \in \mathbb{Z}$ such that $\chi \equiv N \bmod 2$, and let $s_0(\bullet) \in \Omega_\chi^N$. Let us denote $\varepsilon = \frac{1}{N}$, and let $q = e^{-\varepsilon}$ and*

$$\hat{s}_t^\varepsilon(x) := \varepsilon^{\frac{1}{2}}(s_{\varepsilon^{-2}t}(\varepsilon^{-1}x) - \chi x)$$

be the rescaled height function, extended piece-wise linearly to $x \in \mathbb{R}$. Note that $\hat{s}_t^\varepsilon(x)$ is 1-periodic in the variable x.

For some $\eta \in (\frac{1}{3}, \frac{1}{2})$ assume that there exists a 1-periodic η-Hölder continuous function $\mathcal{Z}_0$ (defined on the same probability space as the initial data $\hat{s}_0^\varepsilon$), such that

$$\limsup_{\varepsilon \to 0} \mathbb{E}\|\hat{s}_0^\varepsilon\|_{\mathcal{C}^\eta} < \infty, \qquad \lim_{\varepsilon \to 0} \mathbb{E}\|\hat{s}_0^\varepsilon - \mathcal{Z}_0\|_{\mathcal{C}^\eta} = 0.$$

Then, for every $T > 0$ the following bound holds

$$\limsup_{\varepsilon \to 0} \mathbb{E}\Big[\sup_{t \in [0,T]} \|\hat{s}_t^\varepsilon\|_{\mathcal{C}^\eta}\Big] < \infty,$$

and $\hat{s}^\varepsilon$ converges weakly in $D([0,\infty), \mathcal{C}(\mathbb{R}))$ to the 1-periodic solution of (1.2) with $A = -\frac{\mathfrak{a}}{4}$, $B \equiv \sqrt{2}$, the initial data $\mathcal{Z}_0$, and a spatially 1-periodic driving white noise ξ.

It is unknown if the dynamic ASEP with the general rates (1.9) has any "integrable" structure, in contrast to the model considered in Theorem 1.1. In particular, invariant measures for this model are unknown and not used in our approach. Also since there is no apparent microscopic Hopf–Cole transform, we have to work 'directly' with the stochastic PDEs and their approximations, in the spirit of [DPD03]. We provide heuristics for our argument in Section 5.1. Note that we have restricted ourselves to the periodic model, in order to avoid significant difficulties with growth of processes at infinity. There are a few instances where this sort of difficulty (in the context of regularity structures [HL18] and paracontrolled distributions [PR18]) has been surmounted by use of suitable weighted function spaces, though it still requires case-by-case analysis. Although we expect Theorem 1.6 to hold also on the whole line, it is not clear whether one can use such weighted spaces to prove it. Our full-line analysis used to prove Theorem 1.1 does not require such methods since we have the microscopic Hopf–Cole transform at our disposal.

The assumption $\eta < \frac{1}{2}$ in Theorem 1.6 is natural, since this is the spatial regularity of the solution to the linear stochastic PDE (1.2), see for example [DPZ14]. However, the restriction $\eta > \frac{1}{3}$ is a consequence of the method we are using to analyze the discrete stochastic PDE, governing the evolution of $\hat{s}^\varepsilon$. More precisely, for regularities below $\frac{1}{3}$ we lose control on the non-linearity in this stochastic PDE (see Lemma 5.8).

1.2. Further directions. For the standard ASEP, there are several PDE and SPDE results that may have generalizations to the dynamic setting. For instance, the standard ASEP enjoys a hydrodynamic limit (i.e. a law of large numbers for its height function) which is determined by the integrated inviscid Burgers equation (or Hamilton–Jacobi equation) with quadratic flux [AV87,FKS91,BGRS02]. Under very weak asymmetry scaling, the limit inviscid equation is replaced by its viscous analog [DMPS89,Gär88,DG91]. When the scaling of the asymmetry decreases from being weak and very weak, the limiting density field of ASEP does not feel the strength of the asymmetry and solves an Ornstein–Uhlenbeck equation with a linear drift [GJ12]. At the weakly asymmetric scaling, the limiting density field of ASEP solves the KPZ equation [BG97,GJ12]. We expect that one can show similar results for the dynamic ASEP by making necessary changes in the model and suitable modifications of our arguments. It is presently unclear how the dynamic parameter (and preferred height) influences these hydrodynamic limit results. However, we expect that, at least under very weak asymmetry scaling, it should be possible to use the methods from this paper to answer this question.

The KPZ equation arises as a scaling limit of the height function fluctuations of the weak asymmetry standard ASEP [BG97] (weak asymmetry means $q = e^{-\sqrt{\varepsilon}}$ versus our very weak asymmetry where $q = e^{-\varepsilon}$). It is presently unclear what becomes of the dynamic ASEP under this weak asymmetry. Part of the challenge is that the stationary initial data simply converges to 0. This may suggest that there should be an SPDE limit of the weak asymmetry ($q = e^{-\sqrt{\varepsilon}}$) dynamic ASEP, its solution will tend over time to 0. The exact form of this limiting SPDE is not yet clear. Another possibility is to tune the value of α in a time-dependent manner. This may enable us to access a KPZ equation-type limit. We leave this for future work.

1.3. Notation. We define here several objects which are used throughout the article. We use the standard notation $\vee$ and $\wedge$ for the maximum and minimum functions respectively. As before, the L^n-norm with respect to the underlying probability measure will be denoted by $\| \bullet \|_n := \mathbb{E}[| \bullet |^n]^{1/n}$.

For $\eta \in (0, 1)$, we denote by $\mathcal{C}^\eta$ the standard space of η-Hölder functions on $\mathbb{R}$, equipped with the norm $\| \bullet \|_{\mathcal{C}^\eta}$. The Hölder space $\mathcal{C}^\eta$ of non-integer regularity $\eta \geq 1$ consists of $\lfloor \eta \rfloor$ times continuously differentiable functions whose $\lfloor \eta \rfloor$-th derivative is $(\eta - \lfloor \eta \rfloor)$-Hölder continuous. When we apply all these norms to functions or distributions on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, we identify them with their periodic extensions. The ε-discretization of $\mathbb{R}$ will be denoted by $\varepsilon\mathbb{Z}$, and we denote by $\langle\langle \bullet, \bullet \rangle\rangle$ and $\langle\langle \bullet, \bullet \rangle\rangle_\varepsilon$ the standard and discretized pairings:

$$\langle\langle \zeta, \varphi \rangle\rangle := \int_{\mathbb{R}} \zeta(x)\varphi(x)dx, \qquad \langle\langle \zeta, \varphi \rangle\rangle_\varepsilon := \varepsilon \sum_{x \in \varepsilon\mathbb{Z}} \zeta(x)\varphi(x).$$

We prefer to use this non-standard notation for the pairing, to avoid confusions with the bracket processes of martingale.

For a test function $\varphi : \mathbb{R} \to \mathbb{R}$, we define its λ-scaled and x-centered version $\varphi_x^\lambda(y) := \lambda^{-1}\varphi(\lambda^{-1}(y - x))$, where $\lambda \in (0, 1]$ and $x \in \mathbb{R}$. Then for $\eta < 0$ and we define the space $\mathcal{C}^\eta$, which is the Besov space $\mathcal{B}^\eta_{\infty,\infty}$ of distributions ζ (see [BCD11] for a definition), characterized by the bound

$$|\langle\langle \zeta, \varphi_x^\lambda \rangle\rangle| \leq C\lambda^\eta, \tag{1.10}$$

uniformly in $\lambda \in (0, 1]$, $x \in \mathbb{R}$ and $\lceil -\eta \rceil$-Hölder continuous functions φ, with the $\mathcal{C}^{\lceil -\eta \rceil}$ norm bounded by 1 and compactly supported in the unit ball, centered at the origin. For a discrete function $\zeta^\varepsilon : \varepsilon\mathbb{Z} \to \mathbb{R}$, we define the norm $\|\zeta^\varepsilon\|_{\mathcal{C}^\eta_\varepsilon}$ as the smallest constant $C \geq 0$, independent of ε, such that the bound

$$|\langle\!\langle \zeta^\varepsilon, \varphi^\lambda_x \rangle\!\rangle_\varepsilon| \leq C(\lambda \vee \varepsilon)^\eta, \tag{1.11}$$

with the same quantities as in (1.10). Obviously, a piece-wise linearly extended function ζ^ε satisfying $\|\zeta^\varepsilon\|_{\mathcal{C}^\eta_\varepsilon} \leq C$ also belongs to $\mathcal{C}^\eta$.

By $\| \bullet \|_{V \to W}$ we denote the operator norm of a linear map acting from the space V to W. For time-dependent functions or distributions ζ_t we define the following norms

$$\|\zeta\|_{\mathcal{C}^\eta_{T,\varepsilon}} := \sup_{t \in [0,T]} \|\zeta_t\|_{\mathcal{C}^\eta_\varepsilon}, \qquad \|\zeta\|_{L^\infty_T} := \sup_{t \in [0,T]} \|\zeta_t\|_{L^\infty}.$$

To make our notation lighter, sometimes we will write "$\lesssim$" for a bound "$\leq$" up to a multiplier, independent of relevant quantities. We will also use $\mathcal{O}(\varepsilon)$ to denote a function which is bounded in absolute value by $C\varepsilon$ as $\varepsilon \to 0$ for some constant C which does not depend on ε or any other varying parameters. The function $\mathcal{O}(\varepsilon)$ can be random, in which case the bound holds almost surely for a non-random constant C.

We define the discrete derivatives in the usual way: $\nabla^\pm f(x) := \pm(f(x \pm 1) - f(x))$, and we write for brevity $\nabla := \nabla^+$. The discrete Laplacian is defined as $\Delta := \nabla^+ - \nabla^-$. The rescaled versions of these operators act on functions $\varphi : \mathbb{R} \to \mathbb{R}$ as $\nabla^\pm_\varepsilon \varphi(x) := \pm(\varphi(x \pm \varepsilon) - \varphi(x))/\varepsilon$ and $\Delta_\varepsilon := (\nabla^+_\varepsilon - \nabla^-_\varepsilon)/\varepsilon$. As before we sometimes write ∇_ε in place of ∇^+_ε.

Since we are going to work with càdlàg martingales, we will use the two bracket processes associated to them. More precisely, the predictable quadratic covariation $\langle M, N \rangle_t$ of two martingales $(M_t)_{t \geq 0}$ and $(N_t)_{t \geq 0}$ is the unique adapted process with bounded total variation, such that $M_t N_t - \langle M, N \rangle_t$ is a martingale. Furthermore, the quadratic covariation $[M, N]_t$ is defined by

$$[M, N]_t := M_t N_t - M_0 N_0 - \int_0^t M_{s-} dN_s - \int_0^t N_{s-} dM_s,$$

where $M_{s-} := \lim_{r \uparrow s} M_r$ is the left limit of M at time s. We refer to [JS03, Ch. I.4] for properties of these two bracket processes. In particular the difference $[M, N]_t - \langle M, N \rangle_t$ is always a martingale.

Outline

The rest of paper is organized as follows. In Section 2, we show that a modified Hopf–Cole transform of the height function of the dynamic ASEP on the full line satisfies a microscopic stochastic heat equation (SHE). The main result of Section 2 is Theorem 2.5. The section also contains the proof of Theorem 1.1 which follows fairly easily from Theorem 2.5. Sections 3 and 4 are devoted to the proof of Theorem 2.5: Section 3 provides moment bounds on the microscopic SHE which imply tightness of the space-time process, Section 4 uses martingale problems to identify all limit solutions as the unique one from Theorem 2.5. In Section 5, we prove Theorem 1.6. Appendix A contains some of the important properties of various heat kernels. In Appendix B and Appendix C, we include few other important inputs which are mainly needed for the proof of Theorem 1.6. Finally, Appendix D contains a proof of Lemma 1.3 and Corollary 1.4.

2. Microscopic SHE and Proof of Theorem 1.1

The ultimate goal of this section is to prove Theorem 1.1 (this comes at the end of the section). In Proposition 2.1 we derive a microscopic stochastic heat equation (SHE) satisfied by a modified Hopf–Cole transform of the dynamic ASEP and compute the quadratic variation of the associated martingale. Under very weakly asymmetric scaling and some assumption on the initial data (Definition 2.4), we take a continuum limit of the discrete SHE in Theorem 2.5.

Dynamic ASEP depends on two parameters q and α, however, up to a shift in the height function by $\log_q \alpha$, the parameter α can always be set to 1. Under this shift, the height function will live on a shift of the integer lattice. We will not labor this point further since in our scaling limits, such a shift is inconsequential. The state space of the dynamic ASEP is

$$\mathcal{S} := \Big\{s = \{s(x)\}_{x\in\mathbb{Z}} : |s(x+1) - s(x)| = 1, \forall x \in \mathbb{Z}\Big\}.$$

Each site $x \in \mathbb{Z}$ has independent "up" and "down" step exponential clocks with rates given below. When the clock at x rings, the height $s(x)$ is updated according to the following rule, assuming that the update does not result in s exiting $\mathcal{S}$:

$$\begin{aligned} s(x) &\mapsto s(x) - 2 \text{ at rate } a^{\downarrow}(s(x)) := \frac{q(1+q^{-s(x)})}{1+q^{-s(x)+1}}, \\ s(x) &\mapsto s(x) + 2 \text{ at rate } a^{\uparrow}(s(x)) := \frac{1+q^{-s(x)}}{1+q^{-s(x)-1}}. \end{aligned} \tag{2.1}$$

The infinitesimal generator of the dynamic ASEP is denoted by $\mathcal{L}$ and acts on functions $f : \mathcal{S} \to \mathbb{R}$ as

$$(\mathcal{L}f)(s) = \sum_{x\in\mathbb{Z}} \Big[\eta_x^{\downarrow}(s)\, a^{\downarrow}(s(x))\big(f(s^{x,-2}) - f(s)\big) + \eta_x^{\uparrow}(s)\, a^{\uparrow}(s(x))\big(f(s^{x,+2}) - f(s)\big)\Big], \tag{2.2}$$

where $\eta_x^{\downarrow}(s) := \mathbb{1}_{\{s(x)>s(x-1)\vee s(x+1)\}}$ and $\eta_x^{\uparrow}(s) := \mathbb{1}_{\{s(x)<s(x-1)\wedge s(x+1)\}}$, and where the height functions $s^{x,\pm 2}$ are obtained from s by replacing $s(x)$ with $s(x) \pm 2$.

The following proposition associates a microscopic SHE to the dynamic ASEP.

Proposition 2.1. *Consider the dynamic ASEP process $s_t(x)$ with $q = e^{-\varepsilon}$ (as explained above, we have taken $\alpha = 1$ without lost of generality). Let us define the constant $\theta := (1 - \sqrt{q})^2$ and the function*

$$Z_t(x) := e^{\theta t}\big(q^{-\frac{s_t(x)}{2}} - q^{\frac{s_t(x)}{2}}\big) = 2e^{\theta t} \sinh\Big(\frac{\varepsilon s_t(x)}{2}\Big), \tag{2.3}$$

for $t \in \mathbb{R}_{\geq 0}$ and $x \in \mathbb{Z}$. Let furthermore

$$M_t(x) := Z_t(x) - \int_0^t \big(\mathcal{L}Z_r(x) + \theta Z_r(x)\big)dr, \tag{2.4}$$

where the generator $\mathcal{L}$, defined in (2.2), acts on $s_r(x)$ in $Z_r(x)$.

Then, for each x, the process $t \mapsto M_t(x)$ is a martingale with respect to the natural filtration of $\{s_t\}_{t\geq 0}$ and Z satisfies the following microscopic SHE

$$dZ_t(x) = \sqrt{q}\Delta Z_t(x)dt + dM_t(x), \tag{2.5}$$

where the nearest-neighbour discrete Laplacian Δ, defined in Section 1.3, acts on the variable x. Moreover, the martingales $M_t(x)$ have the following properties:

(1) *the predictable quadratic covariation of the martingale* (2.4) *is given by*

$$\frac{d}{dt}\big\langle M(y), M(x)\big\rangle_t = \mathbb{1}_{\{x=y\}} \frac{\varepsilon^2 e^{2\theta t}}{2}\big(1 + q^{-s_t(x)}\big)\big(1 + q^{\frac{s_t(x+1)+s_t(x-1)}{2}}\big) \\ \times \Big(1 - \nabla^+ s_t(x)\nabla^- s_t(x) + \mathcal{E}_t^\varepsilon(x)\Big), \tag{2.6}$$

where $\nabla^\pm$ *are the discrete derivatives, defined in Section* 1.3*, and the function* $\mathcal{E}^\varepsilon$ *satisfies* $|\mathcal{E}_t^\varepsilon(x)| \leq C\varepsilon$ *uniformly in t and x, for a non-random constant C;*

(2) *for some non-random constant* $C > 0$*, the following bound holds*

$$\frac{d}{dt}\langle M(x), M(x)\rangle_t \leq C\varepsilon^2\big(Z_t(x)^2 + e^{2t\theta}\big). \tag{2.7}$$

Remark 2.2. Taking expectation in (2.5) kills the martingale and shows that $\mathbb{E}[Z_t(x)]$ solves a semi-discrete heat equation. This is essentially the one-particle case of the duality for dynamic ASEP, proved in [BC17, Thm. 2.3].

Proof of Proposition 2.1. We will first demonstrate (2.5), and explicitly construct the martingale therein. In this proof it will be convenient for us to overload the notation for Z and write $Z_t(x; s_t)$ instead of $Z_t(x)$. This mean the same thing, but explicitly emphasizes the dependence of Z on s_t. Owing to the definition of the dynamic ASEP (2.1), we have

$$dZ_t(x; s_t) = \eta_x^\downarrow(s_t)(Z_t(x; s_t^{x,-2}) - Z_t(x; s_t))a^\downarrow(s_t(x))dP_t^\downarrow(x) \\ + \eta_x^\uparrow(s_t)(Z_t(x; s_t^{x,+2}) - Z_t(x; s_t))a^\uparrow(s_t(x))dP_t^\uparrow(x) + \theta Z_t(x; s_t)dt, \tag{2.8}$$

where $\{P_t^\downarrow(x)\}_{x\in\mathbb{Z}}$ and $\{P_t^\uparrow(x)\}_{x\in\mathbb{Z}}$ are independent Poisson processes of intensities 1, and the functions $\eta^\downarrow$ and $\eta^\uparrow$ are defined below (2.2). We introduce the compensated Poisson processes, which are martingales, by setting

$$M_t^\downarrow(x) := \int_0^t \eta_x^\downarrow(s_r)a^\downarrow(s_r(x))d(P_r^\downarrow(x) - r), \tag{2.9}$$

and likewise with $\uparrow$ in place of $\downarrow$. Furthermore, we define the martingales $\{M_t(x)\}_{x\in\mathbb{Z}}$ via

$$M_t(x) := \int_0^t \big(Z_r(x; s_r^{x,-2}) - Z_r(x; s_r)\big)dM_r^\downarrow(x) + \int_0^t \big(Z_r(x; s_r^{x,+2}) - Z_r(x; s_r)\big)dM_r^\uparrow(x). \tag{2.10}$$

With these martingales at hand, we can rewrite (2.8) as

$$dZ_t(x; s_t) = \mathcal{L}Z_t(x; s_t)dt + \theta Z_t(x; s_t)dt + dM_t(x),$$

where the generator $\mathcal{L}$ is defined in (2.2). To complete the proof of (2.5), it remains to show

$$\mathcal{L}Z_t(x; s_t) + \theta Z_t(x; s_t) = \sqrt{q}\Delta Z_t(x; s_t),$$

where Δ acts on the variable x in $Z_t(x; s_t)$. This follows from the one-particle duality in [BC17, Thm. 2.3], though it can also be easily deduced (in the spirit of the analogous transform for ASEP in [BG97]) by expressing both sides in terms of $Z_t(x; s_t)$, $\eta_x^\downarrow(s_t)$

and $\eta_x^{\uparrow}(s_t)$, and then checking that they match over all possible values of the pair $\eta_x^{\downarrow}(s_t)$ and $\eta_x^{\uparrow}(s_t)$.

Now, we will prove properties (1) and (2) of the bracket processes of the martingales M. Owing to the independence of the exponential clocks, $\frac{d}{dt}\langle M(y), M(x)\rangle_t = 0$ for $y \neq x \in \mathbb{Z}$. When $y = x$, (2.10) yields

$$\frac{d}{dt}\langle M(x), M(x)\rangle_t = A_t(x) + B_t(x), \tag{2.11}$$

where the two terms are

$$\begin{aligned} A_t(x) &:= \big(Z_t(x; s_t^{x,-2}) - Z_t(x; s_t)\big)^2 \frac{d}{dt}\langle M^{\downarrow}(x), M^{\downarrow}(x)\rangle_t, \\ B_t(x) &:= \big(Z_t(x; s_t^{x,+2}) - Z_t(x; s_t)\big)^2 \frac{d}{dt}\langle M^{\uparrow}(x), M^{\uparrow}(x)\rangle_t. \end{aligned} \tag{2.12}$$

From their definitions as compensated Poisson processes, $\frac{d}{dt}\langle M^{\downarrow}(x), M^{\downarrow}(x)\rangle_t = \eta_x^{\downarrow}(s_t)a^{\downarrow}(s_t(x))$ and likewise for $\uparrow$. Moreover, we have the following readily checked identities:

$$\begin{aligned} \eta_x^{\downarrow}(s_t) &= \mathbb{1}_{\{s_t(x) > s_t(x-1) = s_t(x+1)\}} = \frac{1}{(q-1)^2}\big(q^{\frac{1-\nabla^+ s_t(x)}{2}} - 1\big)\big(q^{\frac{1+\nabla^- s_t(x)}{2}} - 1\big), \\ \eta_x^{\uparrow}(s_t) &= \mathbb{1}_{\{s_t(x) < s_t(x-1) = s_t(x+1)\}} = \frac{1}{(q-1)^2}\big(q^{\frac{1+\nabla^+ s_t(x)}{2}} - 1\big)\big(q^{\frac{1-\nabla^- s_t(x)}{2}} - 1\big), \end{aligned}$$

where $\nabla^{\pm}$ are the discrete derivatives, defined in Section 1.3. Substituting this into (2.12) and using (2.3) and (2.1) yields

$$\begin{aligned} A_t(x) &= e^{2\theta t}\big(q^{\frac{1-\nabla^+ s_t(x)}{2}} - 1\big)\big(q^{\frac{1+\nabla^- s_t(x)}{2}} - 1\big)\big(1 + q^{s_t(x)-1}\big)\big(1 + q^{-s_t(x)}\big), \\ B_t(x) &= q^{-1}e^{2\theta t}\big(q^{\frac{1+\nabla^+ s_t(x)}{2}} - 1\big)\big(q^{\frac{1-\nabla^- s_t(x)}{2}} - 1\big)\big(1 + q^{s_t(x)+1}\big)\big(1 + q^{-s_t(x)}\big). \end{aligned} \tag{2.13}$$

We seek to establish that the sum of these two terms can be written as (2.6).

First, we observer that the values of s_t, contributing to $A_t(x)$, satisfy $s_t(x) > s_t(x-1) \vee s_t(x+1)$, and we can write $1+q^{s_t(x)-1} = 1+q^{(s_t(x+1)+s_t(x-1))/2}$. Similarly, for $B_t(x)$ we have $s_t(x) < s_t(x-1) \wedge s_t(x+1)$ and we can write $1+q^{s_t(x)+1} = 1+q^{(s_t(x+1)+s_t(x-1))/2}$.

Second, we will rewrite the terms in (2.13) which involve gradients. For this, we recall $q = e^{-\varepsilon} \in (0, 1]$ and observe that $(1 \pm \nabla^{\mp} s_t(x))/2 \in \{0, 1\}$. Then, using the bound $|e^a - 1 - a| \le a^2 e/2$, which holds for $|a| \le 1$, we obtain

$$\begin{aligned} \big(q^{\frac{1-\nabla^+ s_t(x)}{2}} - 1\big)\big(q^{\frac{1+\nabla^- s_t(x)}{2}} - 1\big) &= \frac{\varepsilon^2}{4}\big(1 - \nabla^+ s_t(x)\big)\big(1 + \nabla^- s_t(x)\big) + \mathcal{E}_t^{\varepsilon,1}(x) \\ &= \frac{\varepsilon^2}{4}\big(1 - \Delta s_t(x) - \nabla^+ s_t(x)\nabla^- s_t(x)\big) + \mathcal{E}_t^{\varepsilon,1}(x), \end{aligned}$$

where $|\mathcal{E}_t^{\varepsilon,1}(x)| \le C\varepsilon^3$, uniformly in t and x, and for some constant $C > 0$. Similarly, we can write

$$\begin{aligned}\left(q^{\frac{1+\nabla^+ s_t(x)}{2}}-1\right)\left(q^{\frac{1-\nabla^- s_t(x)}{2}}-1\right) &= \frac{\varepsilon^2}{4}\left(1+\nabla^+ s_t(x)\right)\left(1-\nabla^- s_t(x)\right)+\mathcal{E}^{\varepsilon,2}_t(x)\\ &= \frac{\varepsilon^2}{4}\left(1+\Delta s_t(x)-\nabla^+ s_t(x)\nabla^- s_t(x)\right)+\mathcal{E}^{\varepsilon,2}_t(x),\end{aligned}$$

where the function $\mathcal{E}^{\varepsilon,2}$ is bounded as $\mathcal{E}^{\varepsilon,1}$.

Substituting these identities into A and B, we arrive at

$$A_t(x)+B_t(x)=e^{2\theta t}\left(1+q^{-s_t(x)}\right)\left(1+q^{\frac{s_t(x+1)+s_t(x-1)}{2}}\right)D_t(x),$$

where

$$\begin{aligned}D_t(x) :=& \frac{\varepsilon^2}{4}(1+q^{-1})\left(1-\nabla^+ s_t(x)\nabla^- s_t(x)\right)-\frac{\varepsilon^2}{4}(1-q^{-1})\Delta s_t(x)\\ &+\mathcal{E}^{\varepsilon,1}_t(x)+q^{-1}\mathcal{E}^{\varepsilon,2}_t(x).\end{aligned}$$

Using $|\nabla^\pm s_t(x)|=1$ and $q^{-1}=1+\mathcal{O}(\varepsilon)$, this gives the required identity (2.6).

Now, we turn to the bound (2.7), for which we need to bound the r.h.s. of (2.11). Bounding the indicator functions $\eta^{\downarrow\uparrow}$ and the rates $a^{\downarrow\uparrow}$ in (2.9) by 1, and using the definition (2.3), we get

$$\begin{aligned}\frac{d}{dt}\langle M(x),M(x)\rangle_t &\le \left(Z_t(x;s^{x,-2}_t)-Z_t(x;s_t)\right)^2+\left(Z_t(x;s^{x,+2}_t)-Z_t(x;s_t)\right)^2\\ &= 4e^{2\theta t}\left(\sinh\left(\varepsilon s_t(x)/2-\varepsilon\right)-\sinh\left(\varepsilon s_t(x)/2\right)\right)^2\\ &\quad+4e^{2\theta t}\left(\sinh\left(\varepsilon s_t(x)/2+\varepsilon\right)-\sinh\left(\varepsilon s_t(x)/2\right)\right)^2. \qquad (2.14)\end{aligned}$$

From differentiability of $\sinh(x)$ one can derive $|\sinh(x+y)-\sinh(x)|^2\le cy^2(\sinh(x)^2+1)$, which holds uniformly in x and $|y|\le 1$, where the constant c is independent of x and y. Applying this bound to (2.14), we obtain (2.7). □

The following property of the process Z will be used in Section 4.

Proposition 2.3. *For Z defined in Proposition 2.1, we have the identity*

$$\varepsilon^{-2}e^{-2t\theta}\nabla^+ Z_t(x)\nabla^- Z_t(x)=\nabla^+ s_t(x)\nabla^- s_t(x)+\mathfrak{B}^\varepsilon_t(x), \qquad (2.15)$$

for a function $\mathfrak{B}^\varepsilon_t(x)$, satisfying $|\mathfrak{B}^\varepsilon_t(x)|\le C(e^{-2\theta t}Z_t(x)^2+\varepsilon)$ uniformly in t and x.

Proof. Using the definition of Z from (2.3) and the discrete derivative ∇^-, we can write

$$e^{-t\theta}\nabla^- Z_t(x)=q^{-\frac{s_t(x)}{2}}\left(1-q^{\frac{\nabla^- s_t(x)}{2}}\right)-q^{\frac{s_t(x)}{2}}\left(1-q^{-\frac{\nabla^- s_t(x)}{2}}\right).$$

Furthermore, recalling that $q=e^{-\varepsilon}$ and using the bound $|e^a-1-a|\le a^2e/2$, for $|a|\le 1$, the last expression equals

$$\frac{\varepsilon}{2}\left(q^{-\frac{s_t(x)}{2}}+q^{\frac{s_t(x)}{2}}\right)\left(\nabla^- s_t(x)+\mathfrak{B}^{\varepsilon,-}_t(x)\right)=\varepsilon\cosh\left(\frac{s_t(x)}{2}\right)\left(\nabla^- s_t(x)+\mathfrak{B}^{\varepsilon,-}_t(x)\right),$$

where $|\mathfrak{B}^{\varepsilon,-}_t(x)|\le C\varepsilon$, uniformly in t and x. Recalling the formula (2.3) and using $\cosh^2(x)-\sinh^2(x)=1$, we obtain

$$e^{-t\theta}\nabla^- Z_t(x)=\varepsilon\left(1+\frac{1}{4}e^{-2\theta t}Z_t(x)^2\right)^{1/2}\left(\nabla^- s_t(x)+\mathfrak{B}^{\varepsilon,-}_t(x)\right). \qquad (2.16)$$

Similarly, we can write

$$e^{-t\theta}\nabla^+ Z_t(x) = \varepsilon\Big(1+\frac{1}{4}e^{-2\theta t}Z_t(x)^2\Big)^{1/2}\big(\nabla^+ s_t(x)+\mathfrak{B}^{\varepsilon,+}_t(x)\big), \tag{2.17}$$

where $\mathfrak{B}^{\varepsilon,+}$ is bounded in the same way as $\mathfrak{B}^{\varepsilon,-}$. Multiplying (2.16) and (2.17), and using $|\nabla^\pm s(x)| = 1$, we arrive at (2.15). □

We aim now to state a convergence result for Z. This requires certain assumptions on the initial states.

Definition 2.4. Consider a sequence of random functions $\mathcal{Z}^\varepsilon_0 : \mathbb{R} \to \mathbb{R}$ indexed by ε. We call $\{\mathcal{Z}^\varepsilon_0\}_{\varepsilon>0}$ *near stationary initial data* with parameters $u > 0$ and $\beta \in \big(0, \frac{1}{4}\big)$, if for all $k \in \mathbb{N}$, there exist $C = C(u, \beta, k) > 0$, such that for all $x, x_1, x_2 \in \mathbb{R}$ one has the bounds

$$\|\mathcal{Z}^\varepsilon_0(x)\|_{2k} \le Ce^{u|x|}, \tag{2.18a}$$

$$\|\mathcal{Z}^\varepsilon_0(x_1) - \mathcal{Z}^\varepsilon_0(x_2)\|_{2k} \le C|x_1 - x_2|^{2\beta}e^{u(|x_1|+|x_2|)}. \tag{2.18b}$$

With this definition we are ready to state our convergence result for Z.

Theorem 2.5. *Consider the dynamic ASEP $s_t(x)$ with $q = e^{-\varepsilon}$, where without loss of generality we have taken $\alpha = 1$ (see the explanation at the beginning of this section). Extend $Z_t(x)$, defined in (2.3), to non-integer x by linear interpolation and define $\mathcal{Z}^\varepsilon_t(x) := \varepsilon^{-\frac{1}{2}}Z_{\varepsilon^{-2}t}(\varepsilon^{-1}x)$. Assume that $\mathcal{Z}^\varepsilon$ starts from an ε-dependent sequence of* near stationary initial data $\mathcal{Z}^\varepsilon_0$ *(see Definition 2.4). If $\mathcal{Z}^\varepsilon_0 \Rightarrow \mathcal{Z}_0$ in $\mathcal{C}(\mathbb{R})$ as $\varepsilon \to 0$, then $\mathcal{Z}^\varepsilon \Rightarrow \mathcal{Z}$ in $D([0,\infty), \mathcal{C}(\mathbb{R}))$, as $\varepsilon \to 0$, where $\mathcal{Z}$ is the unique solution of (1.2), with $A = 0$ and $B_t = \sqrt{2}e^{t/4}$, started from the initial state $\mathcal{Z}_0$.*

Proof. The moment bounds in Proposition 3.1 and an argument similar to [BG97, Thm. 3.3] (see also [DT16, Prop. 1.4]), readily yield tightness of $\{\mathcal{Z}^\varepsilon\}_{\varepsilon>0}$ with respect to the weak topology of $D([0,\infty), \mathcal{C}(\mathbb{R}))$. Identification of the limit with the solution of (1.2) is proved in Proposition 4.1. □

With this result at hand, we close this section by proving Theorem 1.1. For this, it will be convenient to write a formula for $\mathcal{Z}^\varepsilon$, which follows from (2.3):

$$\mathcal{Z}^\varepsilon_t(x) = 2\varepsilon^{-\frac{1}{2}}e^{\varepsilon^{-2}\theta t}\sinh\Big(\frac{\sqrt{\varepsilon}\hat{s}^\varepsilon_t(x)}{2}\Big), \tag{2.19}$$

where $\hat{s}^\varepsilon$ is defined in (1.4).

Proof of Theorem 1.1. Owing to (2.19), we write $\hat{s}^\varepsilon_0(x) = 2\varepsilon^{-\frac{1}{2}}\sinh^{-1}\big(\sqrt{\varepsilon}\mathcal{Z}^\varepsilon_0(x)/2\big)$. By using the fact that $\sinh^{-1}$ is a bijective continuously differentiable function, we prove Theorem 1.1 when $\{\mathcal{Z}^\varepsilon_t(\cdot)\}_{t\ge0}$ weakly converges as ε goes to 0. For showing the weak convergence of $\{\mathcal{Z}^\varepsilon_t(\cdot)\}_{t\ge0}$, we apply Theorem 2.5. To this end, we first show that $\mathcal{Z}^\varepsilon_0(\cdot)$ satisfies (2.18) under (1.5).

Using (2.19) and the bound $|\sinh(x)| \le |x|e^{|x|}$, we obtain

$$|\mathcal{Z}^\varepsilon_0(x)| \le |\hat{s}^\varepsilon_0(x)|e^{\sqrt{\varepsilon}|\hat{s}^\varepsilon_0(x)|/2}.$$

Taking the L^{2k}-norm, applying the Cauchy–Schwarz inequality and (1.5a), we obtain

$$\|\mathcal{Z}^\varepsilon_0(x)\|_{2k} \le \|\hat{s}^\varepsilon_0(x)\|_{4k}\|e^{\sqrt{\varepsilon}|\hat{s}^\varepsilon_0(x)|/2}\|_{4k} \le Ce^{u(1+\sqrt{\varepsilon}/2)|x|} \le Ce^{3u|x|/2},$$

for a constant C, depending on k. The second inequality follows by combining

$$|\hat{s}_0^\varepsilon(x)| \le e^{u|\hat{s}_0^\varepsilon(x)|}, \quad \|e^{\sqrt{\varepsilon}|\hat{s}_0^\varepsilon(x)|/2}\|_{4k} \le \|e^{|\hat{s}_0^\varepsilon(x)|}\|_{4k}^{\sqrt{\epsilon}/2}, \quad \forall x \in \mathbb{Z} \tag{2.20}$$

with an upper bound on $\|e^{|\hat{s}_0^\varepsilon(x)|}\|$ from (1.5a) for ε small and x tends ∞. The first bound in (2.20) holds since $|x| \le e^{|x|}$ for all $x \in \mathbb{Z}$ and the second bound is obtained via Hölder's inequality. We use $|\sinh(x) - \sinh(y)| \le |x - y|e^{|x|+|y|}$ to get

$$|\mathcal{Z}_0^\varepsilon(x_1) - \mathcal{Z}_0^\varepsilon(x_2)| \le |\hat{s}_0^\varepsilon(x_1) - \hat{s}_0^\varepsilon(x_2)|e^{\sqrt{\varepsilon}(|\hat{s}_0^\varepsilon(x_1)|+|\hat{s}_0^\varepsilon(x_2)|)/2}.$$

Combining this with (1.5), we obtain (2.18b). Then Theorem 2.5 implies that $\mathcal{Z}^\varepsilon$ converges weakly to the unique solution of (1.2) with $A = 0$ and $B_t = \sqrt{2}e^{t/4}$ in $D([0,\infty), \mathcal{C}(\mathbb{R}))$ as $\varepsilon \to 0$.

To show the weak convergence of $\hat{s}^\varepsilon$, we use the formula (2.19). The weak convergence $\lim_{\varepsilon\to 0} \mathcal{Z}_t^\varepsilon(x) = \mathcal{Z}_t(x)$ in $D([0,\infty), \mathcal{C}(\mathbb{R}))$ and $\lim_{\varepsilon\to 0} \varepsilon^{-2}t\theta = \frac{t}{4}$ imply that

$$\varepsilon^{-\frac{1}{2}} \sinh\Big(\frac{\sqrt{\varepsilon}\hat{s}_t^\varepsilon(x)}{2}\Big) = \frac{1}{2}e^{-\varepsilon^{-2}\theta t}\mathcal{Z}_t^\varepsilon(x)$$

weakly converges to $\frac{1}{2}e^{-t/4}\mathcal{Z}_t(x)$ in the same topology. Hence, continuous differentiability of the function $\sinh^{-1}(x)$ yields the weak convergence in $D([0,\infty), \mathcal{C}(\mathbb{R}))$:

$$\begin{aligned}\lim_{\varepsilon\to 0} \hat{s}_t^\varepsilon(x) &= \lim_{\varepsilon\to 0} 2\varepsilon^{-\frac{1}{2}} \sinh^{-1}\Big(\frac{\sqrt{\varepsilon}}{2}e^{-\varepsilon^{-2}\theta t}\mathcal{Z}_t^\varepsilon(x)\Big) \\ &= \lim_{\varepsilon\to 0} 2\varepsilon^{-\frac{1}{2}} \sinh^{-1}\Big(\frac{\sqrt{\varepsilon}}{2}e^{-\frac{t}{4}}\mathcal{Z}_t(x)\Big) = e^{-\frac{t}{4}}\mathcal{Z}_t(x),\end{aligned}$$

where in the last identity we made use of $(\sinh^{-1})'(0) = 1$. Theorem 2.5 and the chain rule imply that $e^{-t/4}\mathcal{Z}$ solves (1.2) with $A = -\frac{1}{4}$ and $B \equiv \sqrt{2}$, which completes the proof. □

3. Moment Bounds for Solutions of the Microscopic SHE

The following moment bounds are the main result of this section.

Proposition 3.1. *Consider the space time process $\mathcal{Z}^\varepsilon$, defined in Theorem 2.5, starting from a sequence of near stationary initial data $\mathcal{Z}_0^\varepsilon$ (see Definition 2.4) with parameters $u \in \mathbb{R}_{>0}$ and $\beta \in (0, \frac{1}{4})$. Then, for any $t > 0$ and $k \in \mathbb{N}$, there exists $C = C(u, \beta, k, t)$, such that*

$$\begin{aligned}\|\mathcal{Z}_t^\varepsilon(x)\|_{2k} &\le Ce^{u|x|}, \\ \|\mathcal{Z}_t^\varepsilon(x_1) - \mathcal{Z}_t^\varepsilon(x_2)\|_{2k} &\le C|x_1 - x_2|^{2\beta}e^{u(|x_1|+|x_2|)}, \\ \|\mathcal{Z}_{t_1}^\varepsilon(x) - \mathcal{Z}_{t_2}^\varepsilon(x)\|_{2k} &\le C(\varepsilon^2 \vee |t_1 - t_2|)^\beta e^{u|x|},\end{aligned}$$

for all $x, x_1, x_2 \in \mathbb{R}$ and $t, t_1, t_2 \in [0, t]$.

In our proof we find it easier to work with microscopic variables. For this, we define the process

$$\widetilde{\mathcal{Z}}^{\varepsilon}_t(x) = \varepsilon^{-\frac{1}{2}} Z_t(x), \tag{3.1}$$

so that $\mathcal{Z}^{\varepsilon}_T(X) = \widetilde{\mathcal{Z}}^{\varepsilon}_{\varepsilon^{-2}T}(\varepsilon^{-1}X)$, where $\mathcal{Z}^{\varepsilon}$ is defined in Theorem 2.5. Owing to (2.5), $\widetilde{\mathcal{Z}}^{\varepsilon}_t(x)$ satisfies a microscopic SHE:

$$d\widetilde{\mathcal{Z}}^{\varepsilon}_t(x) = \sqrt{q}\Delta\widetilde{\mathcal{Z}}^{\varepsilon}_t(x)dt + d\widetilde{\mathcal{M}}^{\varepsilon}_t(x), \tag{3.2}$$

where $\widetilde{\mathcal{M}}^{\varepsilon} := \varepsilon^{-\frac{1}{2}} M$. The mild solution to (3.2) is given by

$$\widetilde{\mathcal{Z}}^{\varepsilon}_t(x) = \sum_{y\in\mathbb{Z}} \mathfrak{p}^{\varepsilon}_t(x-y)\widetilde{\mathcal{Z}}^{\varepsilon}_0(y) + \sum_{y\in\mathbb{Z}} \int_0^t \mathfrak{p}^{\varepsilon}_{t-r}(x-y)d\widetilde{\mathcal{M}}^{\varepsilon}_r(y) \tag{3.3}$$

(in all stochastic integrals we always integrate with respect to the time variable), where $\mathfrak{p}^{\varepsilon}_t(x)$ solves the following semi-discrete PDE:

$$\partial_t \mathfrak{p}^{\varepsilon}_t(x) = \sqrt{q}\Delta\mathfrak{p}^{\varepsilon}_t(x), \quad \mathfrak{p}^{\varepsilon}_0(x) = \mathbb{1}_{\{x=0\}}. \tag{3.4}$$

Note that $\mathfrak{p}^{\varepsilon}_t$ is the transition kernel for a continuous time random walk starting from $x = 0$ which jumps symmetrically ± 1 with rate $\sqrt{q}$.

In terms of that, we can rewrite the bounds in Proposition 3.1 as follows:

$$\|\widetilde{\mathcal{Z}}^{\varepsilon}_t(x)\|_{2k} \le Ce^{u\varepsilon|x|}, \tag{3.5a}$$

$$\|\widetilde{\mathcal{Z}}^{\varepsilon}_t(x_1) - \widetilde{\mathcal{Z}}^{\varepsilon}_t(x_2)\|_{2k} \le C(\varepsilon|x_1 - x_2|)^{2\beta} e^{u\varepsilon(|x_1|+|x_2|)}, \tag{3.5b}$$

$$\|\widetilde{\mathcal{Z}}^{\varepsilon}_{t_1}(x) - \widetilde{\mathcal{Z}}^{\varepsilon}_{t_2}(x)\|_{2k} \le C\varepsilon^{2\beta}(1 \vee |t_1 - t_2|)^{\beta} e^{u\varepsilon|x|}, \tag{3.5c}$$

for all $x, x_1, x_2 \in \mathbb{Z}$ and $t, t_1, t_2 \in [0, \varepsilon^{-2}T]$. Proving these bounds immediately implies that analogous bounds in Proposition 3.1.

To prove (3.5), we first focus on bounding the second component of (3.3). For fixed $0 \le t_1 < t_2$, and for any $t \in [t_1, t_2]$, let us define the processes

$$\widetilde{M}^{\varepsilon;t_1,t_2}_t(x) := \sum_{y\in\mathbb{Z}} \int_{t_1}^t \mathfrak{p}^{\varepsilon}_{t_2-r}(x-y)d\widetilde{\mathcal{M}}^{\varepsilon}_r(y),$$

$$\widetilde{M}^{\varepsilon;\nabla,t_1,t_2}_t(x_1, x_2) := \sum_{y\in\mathbb{Z}} \int_{t_1}^t \big(\mathfrak{p}^{\varepsilon}_{t_2-r}(x_1-y) - \mathfrak{p}^{\varepsilon}_{t_2-r}(x_2-y)\big)d\widetilde{\mathcal{M}}^{\varepsilon}_r(y).$$

Furthermore, for any $r \in \mathbb{R}_{>0}$ and $y, x_1, x_2 \in \mathbb{R}$, we define the modified kernels

$$\bar{\mathfrak{p}}^{\varepsilon}_r(y) := (1 \wedge r^{-\frac{1}{2}})\mathfrak{p}^{\varepsilon}_r(y), \qquad \bar{\mathfrak{p}}^{\varepsilon;\nabla}_r(x_1, x_2; y) := (1 \wedge r^{-\frac{1}{2}})\mathfrak{p}^{\varepsilon;\nabla}_r(x_1, x_2; y),$$

where we have set $\mathfrak{p}^{\varepsilon;\nabla}_r(x_1, x_2; y) := \mathfrak{p}^{\varepsilon}_r(x_1 - y) - \mathfrak{p}^{\varepsilon}_r(x_2 - y)$.

In the following lemma we will derive necessary estimates on the norms of the processes $\widetilde{M}^{\varepsilon;t_1,t_2}$ and $\widetilde{M}^{\varepsilon;\nabla,t_1,t_2}$ which we will use in Section 3.1.

Lemma 3.2. *For any $\beta \in (0, \frac{1}{4})$ and any $k \in \mathbb{N}$, there exist positive constants $C_1 = C_1(k)$ and $C_2 = C_2(k, \beta)$, such that*

$$\|\widetilde{M}_t^{\varepsilon;t_1,t_2}(x)\|_{2k}^2 \leq C_1 \sum_{y\in\mathbb{Z}} \int_{t_1}^{t} \bar{\mathfrak{p}}_{t_2-r}^{\varepsilon}(x-y)\widetilde{\mathcal{A}}_r^{\varepsilon;k}(y)dr, \tag{3.6a}$$

$$\|\widetilde{M}_t^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2)\|_{2k}^2 \leq C_2(\varepsilon|x_1-x_2|)^{4\beta} \sum_{y\in\mathbb{Z}} \int_{t_1}^{t} \left|\bar{\mathfrak{p}}_{t_2-r}^{\varepsilon;\nabla}(x_1,x_2;y)\right| \widetilde{\mathcal{A}}_r^{\varepsilon;k}(y)dr, \tag{3.6b}$$

uniformly over $x, x_1, x_2 \in \mathbb{Z}$ and $t \in [t_1, t_2]$, where

$$\widetilde{\mathcal{A}}_r^{\varepsilon;k}(y) := \varepsilon^2 \|\widetilde{\mathcal{Z}}_r^{\varepsilon}(y)\|_{2k}^2 + \varepsilon e^{2r\theta}. \tag{3.7}$$

Proof. Using Burkholder–Davis–Gundy's inequality (Lemma B.1), we can bound

$$\|\widetilde{M}_t^{\varepsilon;t_1,t_2}(x)\|_{2k}^2 \leq C\Big(\mathbb{E}\Big[\big[\widetilde{M}^{\varepsilon;t_1,t_2}(x), \widetilde{M}^{\varepsilon;t_1,t_2}(x)\big]_t^k\Big]\Big)^{\frac{1}{k}}, \tag{3.8a}$$

$$\|\widetilde{M}_t^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2)\|_{2k}^2 \leq C\Big(\mathbb{E}\Big[\big[\widetilde{M}^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2), \widetilde{M}^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2)\big]_t^k\Big]\Big)^{\frac{1}{k}}, \tag{3.8b}$$

where $[\bullet, \bullet]_t$ is the quadratic variation of the martingales with respect to time t (suppressed in the notation for the martingales) and $C > 0$ is a constant which depends only on k. We will bound the r.h.s.'s of (3.8a) and (3.8b) by the respective r.h.s.'s of (3.6).

Expanding the quadratic variations yields

$$\big[\widetilde{M}^{\varepsilon;t_1,t_2}(x), \widetilde{M}^{\varepsilon;t_1,t_2}(x)\big]_t = \sum_{y\in\mathbb{Z}} \int_{t_1}^{t} (\mathfrak{p}_{t_2-r}^{\varepsilon}(x-y))^2 d\big[\widetilde{\mathcal{M}}^{\varepsilon}(y), \widetilde{\mathcal{M}}^{\varepsilon}(y)\big]_r, \tag{3.9a}$$

$$\big[\widetilde{M}^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2), \widetilde{M}^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2)\big]_t = \sum_{y\in\mathbb{Z}} \int_{t_1}^{t} (\mathfrak{p}_{t_2-r}^{\varepsilon;\nabla}(x_1,x_2;y))^2 d\big[\widetilde{\mathcal{M}}^{\varepsilon}(y), \widetilde{\mathcal{M}}^{\varepsilon}(y)\big]_r. \tag{3.9b}$$

We will focus our analysis on (3.9a), and for (3.9b) we give only the key steps, since the bounds follow from similar arguments.

To start with, we separate out the predictable quadratic covariation by writing

$$\text{r.h.s. of (3.9a)} = \widetilde{\mathcal{R}}_r^{\varepsilon;t_1,t_2}(x) + \sum_{y\in\mathbb{Z}} \int_{t_1}^{t} (\mathfrak{p}_{t_2-r}^{\varepsilon}(x-y))^2 d\langle\widetilde{\mathcal{M}}^{\varepsilon}(y), \widetilde{\mathcal{M}}^{\varepsilon}(y)\rangle_r, \tag{3.10}$$

where

$$\widetilde{\mathcal{R}}_t^{\varepsilon;t_1,t_2}(x) := \sum_{y\in\mathbb{Z}} \int_{t_1}^{t} \big(\mathfrak{p}_{t_2-r}^{\varepsilon}(x-y)\big)^2 d\Big([\widetilde{\mathcal{M}}^{\varepsilon}(y), \widetilde{\mathcal{M}}^{\varepsilon}(y)]_r - \langle\widetilde{\mathcal{M}}^{\varepsilon}(y), \widetilde{\mathcal{M}}^{\varepsilon}(y)\rangle_r\Big). \tag{3.11}$$

For any $y \in \mathbb{Z}$, the process $r \mapsto [\widetilde{\mathcal{M}}^{\varepsilon}(y), \widetilde{\mathcal{M}}^{\varepsilon}(y)]_r - \langle\widetilde{\mathcal{M}}^{\varepsilon}(y), \widetilde{\mathcal{M}}^{\varepsilon}(y)\rangle_r$ is a martingale with respect to the natural filtration of $\{s_r\}_{r\geq 0}$ [JS03, Ch. I.4], and hence so is $t \mapsto \widetilde{\mathcal{R}}_t^{\varepsilon;t_1,t_2}(y)$. Combining (3.9a) and (3.10), and applying the triangle inequality for the

L^k-norm and the Burkholder–Davis–Gundy inequality (Lemma B.1) for the martingale $\widetilde{\mathcal{R}}^{\varepsilon;t_1,t_2}_t(y)$, we obtain

$$\big\|\big[\widetilde{M}^{\varepsilon;t_1,t_2}(x),\widetilde{M}^{\varepsilon;t_1,t_2}(x)\big]_t\big\|_k \le C\big\|\big[\widetilde{\mathcal{R}}^{\varepsilon;t_1,t_2}(x),\widetilde{\mathcal{R}}^{\varepsilon;t_1,t_2}(x)\big]_t^{\frac12}\big\|_k + \Big\|\sum_{y\in\mathbb{Z}}\int_{t_1}^t(\mathfrak{p}^\varepsilon_{t_2-r}(x-y))^2 d\langle\widetilde{\mathcal{M}}^\varepsilon(y),\widetilde{\mathcal{M}}^\varepsilon(y)\rangle_r\Big\|_k. \tag{3.12}$$

We first bound the second term in the r.h.s. of (3.12), and then later the first one.

From (2.7), we can deduce an upper bound on the derivative of the bracket process $\langle\widetilde{\mathcal{M}}^\varepsilon(y),\widetilde{\mathcal{M}}^\varepsilon(y)\rangle_r$ yielding

$$\sum_{y\in\mathbb{Z}}\int_{t_1}^t(\mathfrak{p}^\varepsilon_{t_2-r}(x-y))^2 d\langle\widetilde{\mathcal{M}}^\varepsilon(y),\widetilde{\mathcal{M}}^\varepsilon(y)\rangle_r \le C\sum_{y\in\mathbb{Z}}\int_{t_1}^t\bar{\mathfrak{p}}^\varepsilon_{t_2-r}(x-y)2\varepsilon(|\widetilde{\mathcal{Z}}^\varepsilon_r(y)|^2+2e^{2r\theta})dr,$$

for some absolute constant C. Here, we made use of the first estimate in (A.4a) for the kernel $\mathfrak{p}^\varepsilon$. Taking L^k-norm on both sides of the above inequality and using the triangle inequality for the L^k-norm yields

$$\Big\|\int_{t_1}^t\sum_{y\in\mathbb{Z}}\bar{\mathfrak{p}}^\varepsilon_{t_2-r}(x-y)d\langle\widetilde{\mathcal{M}}^\varepsilon(y),\widetilde{\mathcal{M}}^\varepsilon(y)\rangle_r\Big\|_k \le C\sum_{y\in\mathbb{Z}}\int_{t_1}^t\bar{\mathfrak{p}}^\varepsilon_{t_2-r}(x-y)\widetilde{\mathcal{A}}^{\varepsilon;k}_r(y)dr, \tag{3.13}$$

where the function $\widetilde{\mathcal{A}}^{\varepsilon;k}$ is defined in (3.7).

Turning to the first term in the r.h.s. of (3.12), we expand the quadratic variation

$$\big[\widetilde{\mathcal{R}}^{\varepsilon;t_1,t_2}(x),\widetilde{\mathcal{R}}^{\varepsilon;t_1,t_2}(x)\big]_t = \sum_{y\in\mathbb{Z}}\sum_{t_1\le\tau\le t}\big(\mathfrak{p}^\varepsilon_{t_2-\tau}(x-y)\big)^4\big(\widetilde{\mathcal{Z}}^\varepsilon_\tau(y)-\widetilde{\mathcal{Z}}^\varepsilon_{\tau-}(y)\big)^4, \tag{3.14}$$

where for each y, the inner sum is over all $\tau\in[t_1,t]$ which are the random times when transitions $s_\tau(y)\to s_\tau(y)-2$ or $s_\tau(y)\to s_\tau(y)+2$ occur, and the number of which is almost surely finite. In the notation above, $f(\tau-)$ refers to the limit of $f(r)$ as $r\uparrow\tau$. Denote the number of such transitions at site y during the time interval $(r_1,r_2]$ by $N_y(r_1,r_2)$. Divide $[t_1,t]$ into $\ell=\lceil t-t_1\rceil$ sub-intervals $\mathcal{I}_1=(r_0,r_1]$, through $\mathcal{I}_\ell=(r_{\ell-1},r_\ell]$ where $r_0=t_1, r_1=r_0+1,\ldots,r_{\ell-1}=r_0+\lfloor t-t_1\rfloor$ and $r_\ell=t$, so each interval has length at most one. Denote $N^{(i)}_y:=N_y(\mathcal{I}_i)$. By direct computations

$$\big(\widetilde{\mathcal{Z}}^\varepsilon_\tau(y)-\widetilde{\mathcal{Z}}^\varepsilon_{\tau-}(y)\big)^2\le 2\big(\varepsilon^2\widetilde{\mathcal{Z}}^\varepsilon_\tau(y)^2+2\varepsilon e^{2\theta\tau}\big). \tag{3.15}$$

Next, we show

$$\max_{r\in\mathcal{I}_i}\big|\widetilde{\mathcal{Z}}^\varepsilon_r(y)\big|\le 2\big(e^\theta\big|\widetilde{\mathcal{Z}}^\varepsilon_{r_{i-1}}(y)\big|+e^{\theta r_i}\big)\exp(2\sqrt{\varepsilon}N^{(i)}_y). \tag{3.16}$$

For any $r\in\mathcal{I}_i$ and $y\in\mathbb{Z}$, $|s_r(y)-s_{r_{i-1}}(y)|$ is bounded above by $2N^{(i)}_y$. Recall from (2.3) that $\widetilde{\mathcal{Z}}^\varepsilon_r(y)=2\varepsilon^{-\frac12}e^{\theta r}\sinh(\varepsilon s_r(y)/2)$, which can be bounded by $2\varepsilon^{-\frac12}e^{\theta r}\sinh(\varepsilon|s_{r_{i-1}}$

$(y)|/2 + \varepsilon N_y^{(i)})$. Hence, using the identity $\sinh(a+b) = \sinh(a)\cosh(b) + \sinh(a)\cosh(b)$, for $a = \varepsilon|s_{r_{i-1}}(y)|/2$ and $b = \varepsilon N_y^{(i)}$, we can bound

$$\begin{aligned}\big|\widetilde{\mathcal{Z}}^\varepsilon_r(y)\big| &\le 2\varepsilon^{-\frac{1}{2}} e^{\theta r}\big(\sinh(a)\cosh(b) + \cosh(a)\sinh(b)\big)\\ &\le 2\Big(e^{\theta(r-r_{i-1})}\big|\widetilde{\mathcal{Z}}^\varepsilon_{r_{i-1}}(y)\big|\cosh(b) + \varepsilon^{-\frac{1}{2}} e^{\theta r}\sinh(b)\Big).\end{aligned}$$

The last line follows from $|\sinh(x)| = \sinh(|x|)$ and the inequalities $\cosh(a) \le 1 + \sinh(a)$ and $\sinh(b) \le \cosh(b)$. Using furthermore $\cosh(b) \le \exp(2\sqrt{\varepsilon}N_y^{(i)})$, $\varepsilon^{-\frac{1}{2}}\sinh(b) \le \exp\big(2\sqrt{\varepsilon}N_y^{(i)}\big)$ and $|r - r_{i-1}| \le 1$ for all $r \in \mathcal{I}_i$, we arrive at (3.16).

Combining (3.15) and (3.16) we get

$$\max_{r\in\mathcal{I}_i}\big(\widetilde{\mathcal{Z}}^\varepsilon_r(y) - \widetilde{\mathcal{Z}}^\varepsilon_{r-}(y)\big)^2 \le C\big(\varepsilon^2\big|\widetilde{\mathcal{Z}}^\varepsilon_{r_{i-1}}(y)\big|^2 + \varepsilon e^{2\theta r_i}\big)\exp(4\sqrt{\varepsilon}N_y^{(i)}),$$

for a constant $C \ge 0$. Using this bound in (3.14) yields

$$\text{r.h.s. of (3.14)} \le \sum_{i=1}^{\ell}\sum_{y\in\mathbb{Z}}\max_{r\in\mathcal{I}_i}\big(\mathfrak{p}^\varepsilon_{t_2-r}(x-y)\big)^4\; N_y^{(i)}\exp\big(8\sqrt{\varepsilon}N_y^{(i)}\big)\widetilde{B}^\varepsilon_i(y)^2, \quad (3.17)$$

where $\widetilde{B}^\varepsilon_i(y) := C(\varepsilon^2\widetilde{\mathcal{Z}}^\varepsilon_{r_{i-1}}(y)^2 + \varepsilon e^{2\theta r_i})$. We may bound the term $(\mathfrak{p}^\varepsilon_{t_2-r})^4 \le C(\bar{\mathfrak{p}}^\varepsilon_{t_2-r})^2$ using the first inequality of (A.4a). Taking the square root of both sides of (3.17) and using Minkowski's inequality $(a+b)^{1/2} \le a^{1/2} + b^{1/2}$ for any $a, b \ge 0$, we arrive at

$$(\text{r.h.s. of (3.14)})^{\frac{1}{2}} \le \sum_{i=1}^{\ell}\sum_{y\in\mathbb{Z}}\max_{r\in\mathcal{I}_i}\bar{\mathfrak{p}}^\varepsilon_{t_2-r}(x-y)\,(N_y^{(i)})^{\frac{1}{2}}\exp\big(4\sqrt{\varepsilon}N_y^{(i)}\big)\widetilde{B}^\varepsilon_i(y). \quad (3.18)$$

Taking L^k-norm of the both sides of (3.18) and using the triangle and Hölder inequalities for that norm yields

$$\begin{aligned}\big\|\big[\widetilde{\mathcal{R}}^{\varepsilon;t_1,t_2}(x), \widetilde{\mathcal{R}}^{\varepsilon;t_1,t_2}(x)\big]_t^{\frac{1}{2}}\big\|_k &\le \sum_{i=1}^{\ell}\sum_{y\in\mathbb{Z}}\max_{r\in\mathcal{I}_i}\bar{\mathfrak{p}}^\varepsilon_{t_2-r}(x-y)\big\|(N_y^{(i)})^{\frac{1}{2}}\exp\big(4\sqrt{\varepsilon}N_y^{(i)}\big)\widetilde{B}^\varepsilon_i(y)\big\|_k\\ &\le \sum_{i=1}^{\ell}\sum_{y\in\mathbb{Z}}\max_{r\in\mathcal{I}_i}\bar{\mathfrak{p}}^\varepsilon_{t_2-r}(x-y)\big\|(N_y^{(i)})^{\frac{1}{2}}\exp\big(4\sqrt{\varepsilon}N_y^{(i)}\big)\big\|_{2k}\big\|\widetilde{B}^\varepsilon_i(y)\big\|_{2k}. \quad (3.19)\end{aligned}$$

Owing to the first inequality of (A.4c), we have

$$\max_{r\in\mathcal{I}_i}\bar{\mathfrak{p}}^\varepsilon_{t_2-r}(x-y) \le C\bar{\mathfrak{p}}^\varepsilon_{t_2-r_{i-1}}(x-y), \quad (3.20)$$

for all $x, y \in \mathbb{Z}$, $i = 1, \ldots, \ell$ and for some absolute constant $C > 0$, when ε is sufficiently small. Since $N_y^{(i)}$ is a Poisson random variable, whose mean is bounded uniformly in the variables i and y (more precisely, the mean is at most 2), we can bound

$$\big\|(N_y^{(i)})^{\frac{1}{2}}\exp\big(4\sqrt{\varepsilon}N_y^{(i)}\big)\big\|_{2k} \le C. \quad (3.21)$$

Using the triangle inequality for the L^{2k}-norm, we may also bound

$$\|\widetilde{B}^\varepsilon_i(y)\|_{2k} \le C\Big(\varepsilon^2\big\|\widetilde{\mathcal{Z}}^\varepsilon_{r_{i-1}}(y)\|^2_{2k} + \varepsilon e^{2\theta r_i}\Big). \quad (3.22)$$

Since $\widetilde{\mathcal{Z}}^{\varepsilon}_{r_{i-1}}(y)$ is measurable w.r.t. the σ-algebra generated by $\{s_r\}_{r\le r_{i-1}}$, $N^{(i)}_y$ and $\widetilde{\mathcal{Z}}^{\varepsilon}_{r_{i-1}}(y)$ are independent. Applying the bounds in (3.20), (3.21) and (3.22) to the r.h.s. of (3.19) yields

$$\text{r.h.s. of (3.19)} \le Ce^{2\theta} \sum_{i=1}^{\ell} \sum_{y\in\mathbb{Z}} \bar{\mathfrak{p}}^{\varepsilon}_{t_2-r_{i-1}}(x-y)\Big(\varepsilon^2 \big\| \widetilde{\mathcal{Z}}^{\varepsilon}_{r_{i-1}}(y)\big\|^2_{2k} + \varepsilon e^{2\theta r_{i-1}}\Big).$$

Approximating the above sum over the integer values of i by the corresponding integral, we arrive at

$$\big\| \big[\widetilde{\mathcal{R}}^{\varepsilon;t_1,t_2}(x), \widetilde{\mathcal{R}}^{\varepsilon;t_1,t_2}(x)\big]^{\frac{1}{2}}_t \big\|_k \le C \sum_{y\in\mathbb{Z}} \int_{t_1}^{t} \bar{\mathfrak{p}}^{\varepsilon}_{t_2-r}(x-y) \widetilde{\mathcal{A}}^{\varepsilon;k}_r(y)dr, \tag{3.23}$$

for a constant $C = C(k)$, where we use the function (3.7). Applying the bounds (3.13) and (3.23) to the r.h.s. of (3.12) finishes the proof of (3.6a).

Now, we show the main steps in the proof of (3.6b), though leave off the details which are similar to those described above in proving (3.6a). As in (3.10), we can decompose the r.h.s. of (3.9b) into two parts and bound then in a similar way as in (3.12):

$$\begin{aligned} &\Big\| \big[\widetilde{\mathcal{M}}^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2), \widetilde{\mathcal{M}}^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2)\big]_t \Big\|_k \\ &\quad\le C \Big\| \big[\widetilde{\mathcal{R}}^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2), \widetilde{\mathcal{R}}^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2)\big]^{\frac{1}{2}}_t \Big\|_k \\ &\qquad + \Big\| \sum_{y\in\mathbb{Z}} \int_{t_1}^{t} (\mathfrak{p}^{\varepsilon;\nabla}_{t_2-s}(x_1,x_2;y))^2 d\langle \widetilde{\mathcal{M}}^{\varepsilon}(y), \widetilde{\mathcal{M}}^{\varepsilon}(y)\rangle_r \Big\|_k, \end{aligned} \tag{3.24}$$

where the definition of $\widetilde{\mathcal{R}}^{\varepsilon;\nabla,t_1,t_2}$ is similar to that of $\widetilde{\mathcal{R}}^{\varepsilon;t_1,t_2}$ except that $\mathfrak{p}^{\varepsilon}_{t_2-r}(x-y)$ is replaced by $\mathfrak{p}^{\varepsilon;\nabla}_{t_2-r}(x_1,x_2;y)$ in (3.11). In order to obtain a similar bound to (3.13), we combine the Hölder-type estimate for the kernel $\mathfrak{p}^{\varepsilon}_{t_2-r}$ in the variable x (see the second inequality of (A.4b)) with (2.7) and the triangle inequality for the L^k-norm. This yields

$$\begin{aligned} &\Big\| \sum_{y\in\mathbb{Z}} \int_{t_1}^{t} (\mathfrak{p}^{\varepsilon;\nabla}_{t_2-s}(x_1,x_2;y))^2 d\langle \widetilde{\mathcal{M}}^{\varepsilon}(y), \widetilde{\mathcal{M}}^{\varepsilon}(y)\rangle_r \Big\|_k \\ &\qquad\le C(\varepsilon|x_1-x_2|)^{4\beta} \sum_{y\in\mathbb{Z}} \int_{t_1}^{t} |\bar{\mathfrak{p}}^{\varepsilon;\nabla}_{t_2-r}(x_1,x_2;y)| \widetilde{\mathcal{A}}^{\varepsilon;k}_r(y)dr. \end{aligned} \tag{3.25}$$

An argument, similar to the one which we used to prove (3.23), yields

$$\begin{aligned} &\Big\| \big[\widetilde{\mathcal{R}}^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2), \widetilde{\mathcal{R}}^{\varepsilon;\nabla,t_1,t_2}(x_1,x_2)\big]^{\frac{1}{2}}_t \Big\|_k \\ &\qquad\le C(\varepsilon|x_1-x_2|)^{4\beta} \sum_{y\in\mathbb{Z}} \int_{t_1}^{t} |\bar{\mathfrak{p}}^{\varepsilon;\nabla}_{t_2-r}(x_1,x_2;y)| \widetilde{\mathcal{A}}^{\varepsilon;k}_r(y)dr. \end{aligned} \tag{3.26}$$

Finally, (3.8b) follows by combining (3.24), (3.25) and (3.26). ◻

The following lemma develops a microscopic version of a chaos series for $\widetilde{\mathcal{Z}}^{\varepsilon}$.

Lemma 3.3. *For any $k \geq 1$ there exists a constant $C = C(k) > 0$, such that*

$$\|\widetilde{\mathcal{Z}}^{\varepsilon}_t(x)\|^2_{2k} \leq 2 \sum_{y \in \mathbb{Z}} \mathfrak{p}^{\varepsilon}_t(x-y) \|\widetilde{\mathcal{Z}}^{\varepsilon}_0(y)\|^2_{2k} + 2C \int_0^t \sum_{y \in \mathbb{Z}} \bar{\mathfrak{p}}^{\varepsilon}_{t-r}(x-y) \widetilde{\mathcal{A}}^{\varepsilon;k}_r(y) dr, \tag{3.27a}$$

$$\|\widetilde{\mathcal{Z}}^{\varepsilon}_t(x)\|^2_{2k} \leq 2 \sum_{y \in \mathbb{Z}} \mathfrak{p}^{\varepsilon}_t(x-y) \|\widetilde{\mathcal{Z}}^{\varepsilon}_0(y)\|^2_{2k} + 2 \sum_{\ell=1}^{\infty} (C\varepsilon^2)^{\ell} \int_{\boldsymbol{r} \in \Delta^{(\ell)}_t} \sum_{\boldsymbol{y} \in \mathbb{Z}^{\ell}} \mathcal{K}^{\varepsilon;\ell}_{\boldsymbol{r}}(\boldsymbol{y}) \mathcal{D}^{\varepsilon;k}_{r_1,0}(y_1) d\boldsymbol{r}, \tag{3.27b}$$

uniformly in $x \in \mathbb{Z}$ and $t \geq 0$, where $\Delta^{(\ell)}_t := \{(r_1, \ldots, r_\ell) \in \mathbb{R}^{\ell}_{\geq 0} : 0 \leq r_1 \leq \ldots \leq r_\ell \leq t\}$, the function $\widetilde{\mathcal{A}}^{\varepsilon;k}$ is defined in (3.7), and

$$\begin{aligned} \mathcal{K}^{\varepsilon;\ell}_{\boldsymbol{r}}(\boldsymbol{y}) &:= \bar{\mathfrak{p}}^{\varepsilon}_{t-r_\ell}(x-y_\ell) \prod_{i=1}^{\ell-1} \bar{\mathfrak{p}}^{\varepsilon}_{r_{i+1}-r_i}(y_{i+1}-y_i), \\ \mathcal{D}^{\varepsilon;k}_{r_1,r_0}(y_1) &:= \sum_{y_0 \in \mathbb{Z}} \mathfrak{p}^{\varepsilon}_{r_1-r_0}(y_1-y_0) \|\widetilde{\mathcal{Z}}^{\varepsilon}_{r_0}(y_0)\|^2_{2k} + \varepsilon^{-1} e^{2\theta t}, \end{aligned} \tag{3.28}$$

for any $\boldsymbol{y} \in \mathbb{Z}^{\ell}$, $\boldsymbol{r} \in \Delta^{(\ell)}_t$ and $r_0 \in \mathbb{R}$, such that $r_0 \leq r_1$.

Proof. Applying the L^{2k}-norm triangle inequality to the decomposition of $\widetilde{\mathcal{Z}}^{\varepsilon}$ in (3.3) yields

$$\|\widetilde{\mathcal{Z}}^{\varepsilon}_t(x)\|_{2k} \leq \sum_{y \in \mathbb{Z}} \mathfrak{p}^{\varepsilon}_t(x-y) \big\|\widetilde{\mathcal{Z}}^{\varepsilon}_0(y)\big\|_{2k} + \Big\| \int_0^t \sum_{y \in \mathbb{Z}} \mathfrak{p}^{\varepsilon}_{t-r}(x-y) d\widetilde{\mathcal{M}}^{\varepsilon}_r(y) \Big\|_{2k}. \tag{3.29}$$

Since $\mathfrak{p}^{\varepsilon}_t(x-y)$ is a probability measure in y, applying Cauchy–Schwarz inequality yields

$$\Big(\sum_{y \in \mathbb{Z}} \mathfrak{p}^{\varepsilon}_t(x-y) \big\|\widetilde{\mathcal{Z}}^{\varepsilon}_0(y)\big\|_{2k} \Big)^2 \leq \sum_{y \in \mathbb{Z}} \mathfrak{p}^{\varepsilon}_t(x-y) \|\widetilde{\mathcal{Z}}^{\varepsilon}_0(y)\|^2_{2k}, \tag{3.30}$$

which bounds the first term on the r.h.s. of (3.29). Applying (3.6a), we bound the second term

$$\Big\| \int_0^t \sum_{y \in \mathbb{Z}} \mathfrak{p}^{\varepsilon}_{t-r}(x-y) d\widetilde{\mathcal{M}}^{\varepsilon}_r(y) \Big\|^2_{2k} \leq C \int_0^t \sum_{y \in \mathbb{Z}} \bar{\mathfrak{p}}^{\varepsilon}_{t-r}(x-y) \widetilde{\mathcal{A}}^{\varepsilon;k}_r(y) dr, \tag{3.31}$$

where $\widetilde{\mathcal{A}}^{\varepsilon;k}$ is defined in (3.7). Bounding the square of the sum of the two terms on the r.h.s. of (3.29) by twice the sum of their squares and applying (3.30) and (3.31) gives (3.27a). Since $\widetilde{\mathcal{A}}^{\varepsilon;k}_r(y)$ involves $\|\widetilde{\mathcal{Z}}^{\varepsilon}_t(x)\|^2_{2k}$, the above equation establishes a recursion which produces the series (3.27b). To complete the proof we must control the tail of the series (3.27b). This follows from the same bounds used in the proof of (3.5a) below, so we do not reproduce it here. □

3.1. Proof of Proposition 3.1.

Proof of (3.5a). Starting with the first term on the r.h.s. of (3.27b), we claim that there exists $C = C(k, \beta, u) > 0$ (which may change values between lines below) such that

$$\sum_{y\in\mathbb{Z}} \mathfrak{p}_t^\varepsilon(x-y)\|\widetilde{\mathcal{Z}}_0^\varepsilon(y)\|_{2k}^2 \leq C\sum_{y\in\mathbb{Z}} \mathfrak{p}_t^\varepsilon(x-y)e^{2\varepsilon u(|x-y|+|x|)} \leq Ce^{2\varepsilon u|x|}. \tag{3.32}$$

The first inequality follows from the bound (2.18a) and the triangle inequality. The second bound follows from the second inequality of (A.4a), with $\alpha = 0$.

Now, we turn to bound the second term on the r.h.s. of (3.27b). Due to the semigroup property of $\mathfrak{p}_t^\varepsilon(x-\bullet)$, for any $\boldsymbol{r} \in \Delta_t^{(\ell)}$ we have

$$\sum_{(y_1,y_2,\dots,y_\ell)\in\mathbb{Z}^\ell} \mathfrak{p}_{t-r_\ell}^\varepsilon(x-y_\ell)\prod_{i=1}^{\ell}\mathfrak{p}_{r_{i+1}-r_i}^\varepsilon(y_{i+1}-y_i) = \mathfrak{p}_t^\varepsilon(x-y_0), \tag{3.33}$$

which shows (since $\bar{\mathfrak{p}}^\varepsilon(\cdot) = (1\wedge r^{-\frac{1}{2}})\mathfrak{p}^\varepsilon(\cdot)$)

$$\begin{aligned}\int_{\boldsymbol{r}\in\Delta_t^{(\ell)}} \sum_{(y_0,y_1,\dots,y_\ell)\in\mathbb{Z}^{\ell+1}} &\bar{\mathfrak{p}}_{t-r_\ell}^\varepsilon(x-y_\ell)\prod_{i=1}^{\ell-1}\bar{\mathfrak{p}}_{r_{i+1}-r_i}^\varepsilon(y_{i+1}-y_i)\cdot \mathfrak{p}_{r_1-r_0}^\varepsilon(y_1-y_0)\|\widetilde{\mathcal{Z}}_{r_0}^\varepsilon(y_0)\|_{2k}^2 \\ &\leq \sum_{y_0\in\mathbb{Z}} \mathfrak{p}_t^\varepsilon(x-y_0)\|\widetilde{\mathcal{Z}}_0^\varepsilon(y)\|^2 \int_{\boldsymbol{r}\in\Delta_t^{(\ell)}} \frac{1}{\sqrt{t-r_\ell}}\prod_{i=1}^{\ell-1}\frac{1}{\sqrt{r_{i+1}-r_i}}d\boldsymbol{r}.\end{aligned} \tag{3.34}$$

Using this inequality and the fact that $e^{2\theta r_1} \leq e^{2\theta t}$ for any $r_1 \in (0,t)$, we now show that

$$\int_{\boldsymbol{r}\in\Delta_t^{(\ell)}} \sum_{\boldsymbol{y}\in\mathbb{Z}^\ell} \mathcal{K}_{\boldsymbol{r}}^{\varepsilon;\ell}(\boldsymbol{y})\,\mathcal{D}_{r_1,0}^{\varepsilon;k}(y_1)d\boldsymbol{r} \leq \mathcal{D}_{t,0}^{\varepsilon;k}(x)\int_{\boldsymbol{r}\in\Delta_t^{(\ell)}} \frac{1}{\sqrt{t-r_\ell}}\prod_{i=1}^{\ell-1}\frac{1}{\sqrt{r_{i+1}-r_i}}d\boldsymbol{r}, \tag{3.35}$$

where we use the functions $\mathcal{K}_{\boldsymbol{r}}^{\varepsilon;\ell}(\boldsymbol{y})$ and $\mathcal{D}_{r_1,0}^{\varepsilon;k}(y_1)$, defined in (3.28). To prove (3.35), we first express the integral on the left hand side as the sum two terms by distributing the integrals and the sum (inside the integrals) over the two summands of $\mathcal{D}_{r_1,0}^{\varepsilon;k}(y_1)$, namely, $\sum_{y_0\in\mathbb{Z}} \mathfrak{p}_{r_1}^\varepsilon(x-y)\|\widetilde{Z}_0^\varepsilon(y)\|_{2k}^2$ and $\varepsilon^{-1}e^{2\theta r_1}$. After splitting, we note that the first term on the left side of (3.35) is same as the left hand side of (3.34). Moreover, we bound the first term using (3.34). By (3.33) and the inequality $e^{2\theta r_1} \leq e^{2\theta t}$, the second term is bounded above by $e^{2\theta t}$ times the integral on the right hand side of (3.35). This proves (3.35).

The integral on the r.h.s. of (3.35) equals $t^{\frac{\ell}{2}}\Gamma(\frac{1}{2})^\ell/\Gamma(\frac{\ell+1}{2})$. Substituting this value and applying (3.32) (to bound the term $\sum_{y_0\in\mathbb{Z}} \mathfrak{p}_t^\varepsilon(x-y_0)\|\widetilde{\mathcal{Z}}_0^\varepsilon(y_0)\|_{2k}^2$ inside $\mathcal{D}_{t,0}^{\varepsilon;k}(x)$) yields

$$\int_{\boldsymbol{r}\in\Delta_t^{(\ell)}} \sum_{\boldsymbol{y}\in\mathbb{Z}^\ell} \mathcal{K}_{\boldsymbol{r}}^{\varepsilon;\ell}(\boldsymbol{y})\,\mathcal{D}_{r_1,0}^{\varepsilon;k}(y_1)d\boldsymbol{r} \leq \left(Ce^{2\varepsilon u|x|} + \varepsilon^{-1}e^{2\theta t}\right)\frac{\Gamma(\frac{1}{2})^\ell}{\Gamma(\frac{\ell+1}{2})}t^{\frac{\ell}{2}}. \tag{3.36}$$

Combining this with bounds of the form $\sum_{\ell=1}^{\infty} x^{\ell}/\Gamma(\ell/2) \le e^{Cx}$, for sufficiently large C, yields

$$\sum_{\ell=1}^{\infty}(C\varepsilon^2)^{\ell} \int_{\boldsymbol{r}\in\Delta_t^{(\ell)}} \sum_{\boldsymbol{y}\in\mathbb{Z}^{\ell}} \mathcal{K}_{\boldsymbol{r}}^{\varepsilon;\ell}(\boldsymbol{y})\mathcal{D}_{r_1,0}^{\varepsilon;k}(y_1)d\boldsymbol{r} \le C\big(e^{2\varepsilon u|x|} + \sqrt{\varepsilon^2 t}e^{2\theta t}\big)e^{C\varepsilon^2\sqrt{t}}.$$

Recalling that $t \le \varepsilon^{-2}T$ and $\theta = (1-\sqrt{q})^2$ for $q = e^{-\varepsilon}$, we have $\sqrt{\varepsilon^2 t} \le \sqrt{T}$, $t\theta \le T/4$ and $\varepsilon^2\sqrt{t} \le \varepsilon\sqrt{T}$. Combining this bound with that on the first term in (3.32) readily yields (3.5a).

Proof of (3.5b). Recalling (3.3) and using the triangle inequality for the L^{2k}-norm, we write

$$\|\widetilde{\mathcal{Z}}_t^{\varepsilon}(x_1) - \widetilde{\mathcal{Z}}_t^{\varepsilon}(x_2)\|_{2k} \le \textbf{(I)} + \textbf{(II)}$$

where

$$\textbf{(I)} := \Big\| \sum_{y\in\mathbb{Z}} \big(\mathfrak{p}_t^{\varepsilon}(x_1-y) - \mathfrak{p}_t^{\varepsilon}(x_2-y)\big)\widetilde{\mathcal{Z}}_0^{\varepsilon}(y)\Big\|_{2k},$$

$$\textbf{(II)} := \Big\| \sum_{y\in\mathbb{Z}} \int_0^t (\mathfrak{p}_{t-r}^{\varepsilon}(x_1-y) - \mathfrak{p}_{t-r}^{\varepsilon}(x_2-y))d\widetilde{\mathcal{M}}_r^{\varepsilon}(y)\Big\|_{2k}.$$

We start with bounding (**I**). Due to a priori bound of $\|\widetilde{\mathcal{Z}}_0^{\varepsilon}(\bullet)\|_{2k}$ from (2.18a) and the heat kernel estimate of the second inequality in (A.4a), $\sum_{y\in\mathbb{Z}} \mathfrak{p}_t^{\varepsilon}(x-y)\widetilde{\mathcal{Z}}_0^{\varepsilon}(y)$ is absolutely convergent for all $x \in \mathbb{Z}$. Rearranging (as is justified by the absolute convergence) the sum yields

$$\sum_{y\in\mathbb{Z}} \big(\mathfrak{p}_t^{\varepsilon}(x_1-y)-\mathfrak{p}_t^{\varepsilon}(x_2-y)\big)\widetilde{\mathcal{Z}}_0^{\varepsilon}(y) = \sum_{y\in\mathbb{Z}} \mathfrak{p}_t^{\varepsilon}(x_1-y)\big(\widetilde{\mathcal{Z}}_0^{\varepsilon}(y)-\widetilde{\mathcal{Z}}_0(x_2-x_1+y)\big). \quad (3.37)$$

Taking L^{2k}-norm on both sides of (3.37) and using subadditivity, we find that

$$\textbf{(I)} \le \sum_{y\in\mathbb{Z}} \mathfrak{p}_t^{\varepsilon}(x_1-y)\big\|\widetilde{\mathcal{Z}}_0^{\varepsilon}(y) - \widetilde{\mathcal{Z}}_0^{\varepsilon}(x_2-x_1+y)\big\|_{2k}. \quad (3.38)$$

As $\mathcal{Z}_0^{\varepsilon}$ satisfies (2.18b), we have

$$\big\|\widetilde{\mathcal{Z}}_0^{\varepsilon}(y) - \widetilde{\mathcal{Z}}_0^{\varepsilon}(x_2-x_1+y)\big\|_{2k} \le (\varepsilon|x_1-x_2|)^{2\beta}e^{\varepsilon u(|x_1-x_2+y|+|y|)}. \quad (3.39)$$

Using (3.39) and the triangle inequality in (3.38) yields

$$\textbf{(I)} \le e^{\varepsilon u|x_1-x_2|}\sum_{y\in\mathbb{Z}} \mathfrak{p}_t^{\varepsilon}(x_1-y)e^{2\varepsilon u|y|}.$$

Bounding the sum on the right hand side using the second inequality of (A.4a) with $\alpha = 0$ shows the desired bound

$$\textbf{(I)} \le C(\varepsilon|x_1-x_2|)^{4\beta}e^{\varepsilon u|x_1-x_2|}. \quad (3.40)$$

Now, we turn to (**II**). Using (3.6b) yields

$$(\mathbf{II})^2 \leq C(\varepsilon|x_1 - x_2|)^{4\beta} \sum_{y \in \mathbb{Z}} \int_0^t |\bar{\mathfrak{p}}^{\varepsilon;\nabla}_{t-r}(x_1, x_2; y)| \big(\varepsilon^2 \|\widetilde{\mathcal{Z}}^\varepsilon_r(y)\|^2_{2k} + \varepsilon e^{2r\theta}\big) dr.$$

Applying (3.27a) recursively to the r.h.s. we may, in a similar way as in the proof of Lemma 3.3, develop an infinite series bound

$$(\mathbf{II})^2 \leq 2(\varepsilon|x_1 - x_2|)^{4\beta} \sum_{\ell=1}^{\infty} (C\varepsilon^2)^\ell \int_{\boldsymbol{r} \in \Delta^{(\ell)}} \sum_{\boldsymbol{y}^{(\ell)} \in \mathbb{Z}^\ell} \mathcal{K}^{\varepsilon;\ell;\nabla}_{\boldsymbol{r}}(\boldsymbol{y}) \mathcal{D}^{\varepsilon;k}_{r_1,0}(y_1) d\boldsymbol{r} \tag{3.41}$$

where $\mathcal{D}^{\varepsilon;k}_{r_1,0}(y_1)$ is defined in Lemma 3.3 and

$$\mathcal{K}^{\varepsilon;\ell;\nabla}_{\boldsymbol{r}}(\boldsymbol{y}) := |\bar{\mathfrak{p}}^{\varepsilon;\nabla}_{t-r_\ell}(x_1, x_2; y_\ell)| \prod_{i=1}^{\ell-1} \bar{\mathfrak{p}}^\varepsilon_{r_{i+1}-r_i}(y_{i+1} - y_i).$$

By use of the triangle inequality $|\bar{\mathfrak{p}}^{\varepsilon;\nabla}_{t-r_\ell}(x_1, x_2; y_\ell)| \leq \bar{\mathfrak{p}}^\varepsilon_{t-r_\ell}(x_1 - y_\ell) + \bar{\mathfrak{p}}^\varepsilon_{t-r_\ell}(x_2 - y_\ell)$ we can bound the series on the r.h.s. of (3.41) in the same manner as in the proof of (3.5b). This eventually produces the first inequality below (second inequality uses $\sqrt{\varepsilon^2 t} \leq \sqrt{T}$ and $\varepsilon^2\sqrt{t} \leq \varepsilon\sqrt{T}$)

$$(\mathbf{II})^2 \leq C\sqrt{\varepsilon^2 t}(\varepsilon|x_1 - x_2|)^{4\beta} e^{2u\varepsilon(|x_1|+|x_2|)} e^{C\varepsilon^2\sqrt{t}} \leq Ce^{C\varepsilon\sqrt{T}}(\mathbf{I})^2.$$

This and (3.40) imply (**I**) + (**II**) $\leq C(\varepsilon|x_1 - x_2|)^{2\beta} e^{u\varepsilon(|x_1|+|x_2|)}$, finishing the proof of (3.5b).

Proof of (3.5c). Without loss of generality, we can assume $t_2 > t_1$. Using (3.3) and L^{2k}-norm triangle inequality yields

$$\big\|\widetilde{\mathcal{Z}}^\varepsilon_{t_1}(x) - \widetilde{\mathcal{Z}}^\varepsilon_{t_2}(x)\big\|_{2k} \leq (\hat{\mathbf{I}}) + (\hat{\mathbf{II}}), \tag{3.42}$$

where

$$(\hat{\mathbf{I}}) := \Big\| \sum_{y \in \mathbb{Z}} \mathfrak{p}^\varepsilon_{t_2-t_1}(x - y)(\widetilde{\mathcal{Z}}^\varepsilon_{t_1}(y) - \widetilde{\mathcal{Z}}^\varepsilon_{t_1}(x)) \Big\|_{2k},$$

$$(\hat{\mathbf{II}}) := \Big\| \sum_{y \in \mathbb{Z}} \int_{t_1}^{t_2} \mathfrak{p}^\varepsilon_{t_2-r}(x - y) d\widetilde{\mathcal{M}}^\varepsilon_r(y) \Big\|_{2k}.$$

First, we bound the term $(\hat{\mathbf{I}})$. By the L^{2k}-norm triangle inequality

$$(\hat{\mathbf{I}}) \leq \sum_{y \in \mathbb{Z}} \mathfrak{p}^\varepsilon_{t_2-t_1}(x - y) \big\|\widetilde{\mathcal{Z}}^\varepsilon_{t_1}(y) - \widetilde{\mathcal{Z}}^\varepsilon_{t_1}(x)\big\|_{2k}.$$

Using (3.5b) and the second inequality of (A.4a), with $\alpha = 2\beta$, yields

$$(\mathbf{I}) \leq (1 \vee |t_2 - t_1|)^\beta \varepsilon^{2\beta} e^{2\varepsilon u|x|}. \tag{3.43}$$

Now, we turn to ($\hat{\mathbf{II}}$). Applying (3.6a) and then recursively applying (3.27a) to ($\hat{\mathbf{II}}$) yields (in the same way as in (3.41))

$$(\hat{\mathbf{II}})^2 \leq 2\sum_{\ell=1}^{\infty}(C\varepsilon^2)^\ell \int_{\boldsymbol{r}\in\Delta^{(\ell)}_{t_1,t_2}} \sum_{\boldsymbol{y}\in\mathbb{Z}^\ell} \mathcal{K}^{\varepsilon;\ell}_{\boldsymbol{r}}(\boldsymbol{y})\mathcal{D}^{\varepsilon;k}_{r_1,t_1}(y_1)d\boldsymbol{r} \tag{3.44}$$

where $\boldsymbol{r} \in \Delta^{(\ell)}_{t_1,t_2} := \{t_1 \leq r_1 \ldots \leq r_\ell \leq t_2\}$ and the functions are defined in Lemma 3.3. In a similar way as used to derive (3.36), and using (3.5a) to bound $\|\widetilde{\mathcal{Z}}^\varepsilon_{t_1}(y_0)\|^2_{2k}$, we arrive at

$$\int_{\boldsymbol{r}\in\Delta^{(\ell)}_{t_1,t_2}} \sum_{\boldsymbol{y}\in\mathbb{Z}^\ell} \mathcal{K}^{\varepsilon;\ell}_{\boldsymbol{r}}(\boldsymbol{y})\mathcal{D}^{\varepsilon;k}_{r_1,t_1}(y_1) \leq \big(Ce^{2\varepsilon u|x|} + \varepsilon^{-1}e^{2(t_2-t_1)\theta}\big)\frac{\Gamma(\frac{1}{2})^\ell}{\Gamma(\frac{\ell+1}{2})}(t_2-t_1)^{\frac{\ell}{2}}.$$

Substituting above inequality into the r.h.s. of (3.44) and summing over all $\ell \in \mathbb{Z}_{\geq 1}$ yields

$$(\hat{\mathbf{II}})^2 \leq C(1 \vee |t_2-t_1|)^{2\beta}\varepsilon^{4\beta}e^{\varepsilon^2|t_2-t_1|^{1/2}}e^{2u\varepsilon|x|}. \tag{3.45}$$

In deriving this we used the bound $\varepsilon(t_2-t_1)^{1/2} \leq C(1 \vee |t_2-t_1|)^{2\beta}\varepsilon^{4\beta}$, which is valid for $\beta \in (0, 1/4)$. Finally, substituting (3.45) and (3.43) into (3.42), we arrive at (3.5c).

4. Identification of the Limit for Solutions of the Microscopic SHE

The following proposition is the main result of this section

Proposition 4.1. *In the setting of Theorem 2.5, let $\mathcal{Z}^\varepsilon_0 \Rightarrow \mathcal{Z}_0$ in $\mathcal{C}(\mathbb{R})$ as $\varepsilon \to 0$. Then, every convergent subsequence of $\{\mathcal{Z}^\varepsilon\}_{\varepsilon>0}$ in $D([0,\infty), \mathcal{C}(\mathbb{R}))$ has the same limit, which is the unique solution of (1.2), with $A = 0$ and $B_t = \sqrt{2}e^{t/4}$, started from the initial state $\mathcal{Z}_0$.*

We will use the following martingale problem to uniquely identify the limiting SPDE (1.2).

Definition 4.2. Consider a stochastic process $\mathcal{Z}$ in $D([0,\infty), \mathcal{C}(\mathbb{R}))$ such that for any $t > 0$, $u > 0$ and $k \in \mathbb{N}$, there exists $C = C(t, u, k) > 0$ satisfying

$$\sup_{r\in[0,t]}\sup_{x\in\mathbb{R}} e^{-u|x|}\|\mathcal{Z}_r(x)\|_{2k} \leq C.$$

Let $C^\infty_b(\mathbb{R})$ be the set all infinitely differentiable bounded functions. Then, $\mathcal{Z}$ is the solution of the martingale problem for the SPDE (1.2) with $A = 0$ and $B_t = \sqrt{2}e^{ct}$ started from $\mathcal{Z}_0$, if for any $\varphi \in \mathcal{C}^\infty_b(\mathbb{R}) \cap L^2(\mathbb{R})$ the processes

$$\mathfrak{M}_t(\varphi) := \mathcal{Z}_t(\varphi)-\mathcal{Z}_0(\varphi)-2\int_0^t \mathcal{Z}_r(\varphi'')dr, \quad \mathfrak{N}_t(\varphi) := \big(\mathfrak{M}_t(\varphi)\big)^2-\frac{e^{2ct}-1}{c}\|\varphi\|^2_{L^2} \tag{4.1}$$

are local martingales, where $\mathcal{Z}_t(\varphi) := \int_{-\infty}^{\infty}\mathcal{Z}_t(y)\varphi(y)dy$.

The martingale problem uniquely identities the law of the solution to the SPDE (1.2). Therefore, in order to show Proposition 4.1 we will demonstrate a microscopic martingale problem and show that on convergent subsequences, limit laws satisfy the martingale problem in Definition 4.2. Fix $\varphi \in C_b^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ and a subsequence $\mathcal{Z}^{\varepsilon_n}$ which weakly converges to a limit $\mathcal{Z}$ in $D([0,\infty), \mathcal{C}(\mathbb{R}))$. For the convenience, in this section we will drop the subscript n from ε_n (though at this point we have not ruled out different limits along different subsequences) and always assume that $\mathcal{Z}^\varepsilon$ converges to a limit $\mathcal{Z}$. Recalling that $\mathcal{Z}^\varepsilon_t(x) = \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}t}(\varepsilon^{-1}x)$, let us denote $\mathcal{Z}^\varepsilon_t(\varphi) := \langle\langle \mathcal{Z}^\varepsilon_t, \varphi\rangle\rangle_\varepsilon$, where the pairing is defined in Section 1.3. Define furthermore

$$\mathfrak{M}^\varepsilon_t(\varphi) := \mathcal{Z}^\varepsilon_t(\varphi) - \mathcal{Z}^\varepsilon_0(\varphi) - \sqrt{q}\int_0^t (\Delta_\varepsilon \mathcal{Z}^\varepsilon)_r(\varphi) dr, \tag{4.2a}$$

$$\mathfrak{N}^\varepsilon_t(\varphi) := \big(\mathfrak{M}^\varepsilon_t(\varphi)\big)^2 - \int_0^{\varepsilon^{-2}t} \sum_{y\in\mathbb{Z}} (\varepsilon\varphi(\varepsilon y))^2 d\big\langle \widetilde{\mathcal{M}}^\varepsilon(y), \widetilde{\mathcal{M}}^\varepsilon(y)\big\rangle_r, \tag{4.2b}$$

where the discrete Laplacian Δ_ε acts on the variable x. We could have defined $\mathcal{M}^\varepsilon_t(x) = \widetilde{\mathcal{M}}^\varepsilon_{\varepsilon^{-2}t}(\varepsilon^{-1}x)$ in which case the second term in (4.2b) would take a slightly more appealing form. However, we find it simpler to work with this more microscopic expression below.

Owing to Lemma 2.1, it is straightforward to see that $\mathfrak{M}^\varepsilon_t(\varphi)$ is a local martingale with respect to the natural filtration of $\{s_{\varepsilon^{-2}t}\}_{t\geq 0}$. To see that $\mathfrak{N}^\varepsilon_t(\varphi)$ is also a local martingale, we note that the second term on the r.h.s. of (4.2b) is the bracket process $\langle \mathfrak{M}^\varepsilon(\varphi), \mathfrak{M}^\varepsilon(\varphi)\rangle_t$.

Since $\mathcal{Z}^\varepsilon_t(\varphi)$ converges weakly to $\mathcal{Z}_t(\varphi)$, by Skorokhod's representation theorem [Bil99, p.70] we can embed these processes onto a common probability space on which they converge almost surely. In the following lemma we provide convergence of various terms from (4.2) with respect to this common probability space.

Lemma 4.3. *For every fixed $t \geq 0$, we have the following limits*

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[\big|\mathcal{Z}^\varepsilon_t(\varphi) - \mathcal{Z}_t(\varphi)\big|\Big] = 0, \tag{4.3a}$$

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\left|2\sqrt{q}\int_0^t (\Delta_\varepsilon \mathcal{Z}^\varepsilon)_r(\varphi)dr - 4\int_0^t \mathcal{Z}_r(\varphi'')dr\right|\right] = 0, \tag{4.3b}$$

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\left|\int_0^{\varepsilon^{-2}t} \sum_{y\in\mathbb{Z}} \big(\varepsilon\varphi(\varepsilon y)\big)^2 d\big\langle \widetilde{\mathcal{M}}^\varepsilon(y), \widetilde{\mathcal{M}}^\varepsilon(y)\big\rangle_r - 4(e^{t/2}-1)\|\varphi\|^2_{L^2}\right|\right] = 0. \tag{4.3c}$$

Proof. By Fatou's lemma and the almost sure convergence of $\mathcal{Z}^\varepsilon_t(\varphi)$ to $\mathcal{Z}_t(\varphi)$ (by Skorokhod)

$$\mathbb{E}\big[|\mathcal{Z}_t(\varphi)|\big] \leq \liminf_{\varepsilon\to 0} \mathbb{E}\big[|\mathcal{Z}^\varepsilon_t(\varphi)|\big].$$

Owing to (3.5a) and the decay of $\varphi(x)$ as $x \to \infty$ (since $\varphi \in C_b^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$), the L^1-norm of $\{\mathcal{Z}^\varepsilon_t(\varphi)\}_{\varepsilon\geq 0}$ is uniformly bounded as $\varepsilon \to 0$, and thus $\mathbb{E}\big[|\mathcal{Z}_t(\varphi)|\big] < \infty$. Since $\{\mathcal{Z}^\varepsilon_t(\varphi) - \mathcal{Z}_t(\varphi)\}_\varepsilon$ converges almost surely to 0 and is uniformly bounded in L^1-norm as $\varepsilon \to 0$, we obtain, by dominated convergence, (4.3a).

Turning to (4.3b), observe that $\sqrt{q} \to 1$ as $\varepsilon \to 0$ and that, via summation by parts,

$$(\Delta_\varepsilon \mathcal{Z}^\varepsilon)_r(\varphi) = \varepsilon \sum_{x\in\mathbb{Z}} \mathcal{Z}_r^\varepsilon(\varepsilon x)\Delta_\varepsilon\varphi(\varepsilon x) = 2\mathcal{Z}_r^\varepsilon(\varphi'') + \varepsilon\langle\!\langle \mathcal{Z}_r^\varepsilon, \psi^\varepsilon\rangle\!\rangle_\varepsilon, \tag{4.4}$$

where $\psi^\varepsilon(x) := \Delta_\varepsilon\varphi(\varepsilon x) - \varphi''(x)$. Theorem 3.1 shows that $\mathbb{E}[\|\mathcal{Z}_r^\varepsilon\|_{2k}]$ is uniformly bounded in ε and $r \in [0, t]$. This along with the ε prefactor implies that the last term in (4.4) vanishes in distribution. Since $\mathcal{Z}^\varepsilon$ converges weakly to $\mathcal{Z}$ in $D([0, t), \mathcal{C}(\mathbb{R}))$, we get from (4.4) the convergence in distribution

$$2\sqrt{q}\int_0^t (\Delta_\varepsilon \mathcal{Z}^\varepsilon)_r(\varphi)dr \quad \Rightarrow \quad 4\int_0^t \mathcal{Z}_r(\varphi'')dr. \tag{4.5}$$

Due to (3.5a), rapid decay of φ and uniform bounds on the L^{2k}-norm of $\mathcal{Z}^\varepsilon$, the l.h.s. of (4.5) is uniformly bounded in L^1 as $\varepsilon \to 0$. Combining this with the Skorokhod almost sure convergence representation and the dominated convergence theorem, as in proving (4.3a), yields (4.3b).

The proof of (4.3c) is the most involved and ultimately relies on some self-averaging, whose idea goes back to [BG97]. Applying the change of variable $\varepsilon^2 r \mapsto r$ inside the integral of (4.3c) and then, computing the bracket process (inside that integral) by applying (2.6) and (2.15), we arrive at

$$\begin{aligned}&\int_0^{\varepsilon^{-2}t} \sum_{y\in\mathbb{Z}} (\varepsilon\varphi(\varepsilon y))^2 d\big\langle \widetilde{\mathcal{M}}^\varepsilon(y), \widetilde{\mathcal{M}}^\varepsilon(y)\big\rangle_r \\ &\quad = \int_0^t \varepsilon \sum_{y\in\mathbb{Z}} \varphi(\varepsilon y)^2 \Big(2e^{2\varepsilon^{-2}r\theta} - 2\varepsilon^{-1}\nabla^+\widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r}(y)\nabla^-\widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r}(y) + U^\varepsilon_{\varepsilon^{-2}r}(y)\Big)\,dr,\end{aligned} \tag{4.6}$$

where $\widetilde{\mathcal{Z}}^\varepsilon$ is defined in (3.1) and

$$\begin{aligned}U_r^\varepsilon(x) :=& \frac{1}{2}e^{2r\theta}\big((1+q^{-s_r(x)})(1+q^{\frac{s_r(x+1)+s_r(x-1)}{2}}) - 4\big) \\ &\times \big(1 - \nabla^+ s_r(x)\nabla^- s_r(x) + \mathcal{E}_r^\varepsilon(x)\big) + 2e^{2r\theta}\big(\mathcal{E}_r^\varepsilon(x) + \mathfrak{B}_r^\varepsilon(x)\big).\end{aligned} \tag{4.7}$$

Here, we use the functions $\mathcal{E}^\varepsilon$ and $\mathfrak{B}^\varepsilon$ which are defined in (2.6) and (2.15) respectively.

There are three terms inside the parenthesis on the r.h.s. of (4.6). We will address the third term, then the first and then the (much harder) second term. Starting with the third term U^ε, a direct computation shows that

$$q^{\mp s_r(x)/2} = \sqrt{1 + \frac{1}{4}\varepsilon e^{-2r\theta}(\widetilde{\mathcal{Z}}_r^\varepsilon(x))^2} \pm \frac{1}{2}\varepsilon^{\frac{1}{2}}e^{-r\theta}\widetilde{\mathcal{Z}}_r^\varepsilon(x).$$

Moreover, the simple bound $\sqrt{1+x^2} \le 1 + |x|$ yields

$$\Big|q^{\mp s_r(x)/2} - 1\Big| \le \varepsilon^{\frac{1}{2}}e^{-r\theta}\big|\widetilde{\mathcal{Z}}_r^\varepsilon(x)\big|.$$

Combining this with the fact that $|\nabla^\pm s_r(x)| = 1$ and with the upper bounds on $\mathcal{E}^\varepsilon$ and $\mathfrak{B}^\varepsilon$, provided in Propositions 2.1 and 2.3 respectively, yields

$$|U_r^\varepsilon(x)| \le \varepsilon^{\frac{1}{2}}F(r, x),$$

where $F(r,x)$ is a polynomial of third degree in $|\widetilde{\mathcal{Z}}^\varepsilon_r(x-1)|$, $|\widetilde{\mathcal{Z}}^\varepsilon_r(x)|$ and $|\widetilde{\mathcal{Z}}^\varepsilon_r(x+1)|$, whose coefficients are bounded uniformly in $r \in [0, \varepsilon^{-2}t]$ and $x \in \mathbb{Z}$. From this and the growth bound on $\widetilde{\mathcal{Z}}^\varepsilon$ in (3.5a), it follows that for any $k \geq 1$, the expression $\sup_{x\in\mathbb{Z}} e^{-2u\varepsilon|x|}\|U^\varepsilon_r(x)\|_{2k}$ is uniformly bounded over all $r \in [0, \varepsilon^{-2}t]$. By this uniform boundedness of U^ε, the fact that φ has bounded support and the dominated convergence theorem, $\int_0^t \varepsilon \sum_{y\in\mathbb{Z}} \varphi(\varepsilon y)^2 U^\varepsilon_{\varepsilon^{-2}r}(y)dr$ vanishes as $\varepsilon \to 0$.

Turning to the first term in (4.6), due to the continuity of φ, $\varepsilon \sum_{y\in\mathbb{Z}} \varphi(\varepsilon y)^2 \to \|\varphi\|^2_{L^2}$ as $\varepsilon \to 0$. Combining this with $2\varepsilon^{-2}r\theta \to r/2$ and the fact that $\int_0^t e^{r/2}dr = 2(e^{t/2}-1)$ yields

$$\int_0^t \varepsilon \sum_{y\in\mathbb{Z}} \varphi(\varepsilon y)^2 2e^{2\varepsilon^{-2}r\theta}dr \quad \to \quad 4(e^{t/2}-1)\|\varphi\|^2_{L^2},$$

as $\varepsilon \to 0$. This limiting term is precisely what is subtracted off in (4.3c). Therefore, to complete the proof of (4.3c) we must show $\lim_{\varepsilon\to 0}\mathbb{E}[(\mathcal{R}^\varepsilon)^2] = 0$, where

$$\mathcal{R}^\varepsilon := \Big|\int_0^t \sum_{y\in\mathbb{Z}} \varphi(\varepsilon y)^2 \nabla^+ \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r}(y)\nabla^- \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r}(y)dr\Big|.$$

The rest of this proof is devoted to showing this.

Expanding $(\mathcal{R}^\varepsilon)^2$ in terms of a double-integral in r and r' time variables, and introducing a conditional expectation with respect to the natural filtration $\mathcal{F}_{\varepsilon^{-2}r'}$ up to time $\varepsilon^{-2}r'$, yields

$$\begin{aligned}\mathbb{E}[(\mathcal{R}^\varepsilon)^2] = 2\mathbb{E}\Big[\int_0^t \sum_{y_1\in\mathbb{Z}} \varphi(\varepsilon y_1)^2 \nabla^+ \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r'}(y_1)\nabla^- \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r'}(y_1) \\ \times \int_{r'}^t \mathbb{E}\Big[\sum_{y_2\in\mathbb{Z}} \varphi(\varepsilon y_2)^2 \nabla^+ \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r}(y_2)\nabla^- \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r}(y_2)\Big|\mathcal{F}_{\varepsilon^{-2}r'}\Big]drdr'\Big].\end{aligned}$$

Owing to this expression along with the rapid decay of φ and uniform bounds on the norms of $\widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r'}$ for all $r' \in [0,t]$ as $\varepsilon \to 0$, we can establish the bound

$$\begin{aligned}\mathbb{E}[(\mathcal{R}^\varepsilon)^2] \leq 2 \sum_{y_1,y_2\in\mathbb{Z}} (\varphi(\varepsilon y_1)\varphi(\varepsilon y_2))^2 \int_0^t \Big\|\nabla^+ \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r'}(y_1)\nabla^- \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r'}(y_1)\Big\|_2 \\ \times \int_{r'}^t \Big\|\mathbb{E}\Big[\nabla^+ \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r}(y_2)\nabla^- \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r}(y_2)\Big|\mathcal{F}_{\varepsilon^{-2}r'}\Big]\Big\|_2 drdr', \qquad (4.8)\end{aligned}$$

by interchanging the summation and expectation, as well as the expectation and integral and then applying the Cauchy–Schwarz inequality. For the first integral in (4.8), from (3.5a), $q = e^{-\varepsilon}$ and $|\nabla^\pm s_{r'}(x)| = 1$, we obtain

$$\sup_{x\in\mathbb{Z}} e^{-2u\varepsilon|x|}\big\|\nabla^+ \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r'}(x)\nabla^- \widetilde{\mathcal{Z}}^\varepsilon_{\varepsilon^{-2}r'}(x)\big\|_2 \leq C\varepsilon, \qquad (4.9)$$

for all $r' \in [0, t]$ where $C = C(t, u)$. For the second integral in (4.8) we have

$$\begin{aligned}\int_{r'}^{t} &\Big\| \mathbb{E}\Big[\nabla^+ \widetilde{\mathcal{Z}}^{\varepsilon}_{\varepsilon^{-2}r}(x) \nabla^- \widetilde{\mathcal{Z}}^{\varepsilon}_{\varepsilon^{-2}r}(x) \Big| \mathcal{F}_{\varepsilon^{-2}r'}\Big]\Big\|_2 dr \\ &\le \int_{r'}^{r'+\sqrt{\varepsilon}} \Big\| \mathbb{E}\Big[\nabla^+ \widetilde{\mathcal{Z}}^{\varepsilon}_{\varepsilon^{-2}r}(x) \nabla^- \widetilde{\mathcal{Z}}^{\varepsilon}_{\varepsilon^{-2}r}(x) \Big| \mathcal{F}_{\varepsilon^{-2}r'}\Big]\Big\|_2 dr \\ &\quad + C \int_{r'+\sqrt{\varepsilon}}^{t} e^{2(u+1)\varepsilon|x|}\big(\varepsilon^2 (r-r')^{-\frac{3}{2}} + \varepsilon^{\frac{3}{2}}\big) dr \\ &\le C\varepsilon^{\frac{3}{2}} e^{2u\varepsilon|x|} + C\varepsilon^{\frac{5}{4}} e^{2(u+1)\varepsilon|x|}. \end{aligned} \tag{4.10}$$

The first inequality comes from splitting the integral into two parts (r' to $r' + \sqrt{\varepsilon}$ and $r'+\sqrt{\varepsilon}$ to t) and applying the bound of Lemma 4.4 into the second part by taking $a = \varepsilon^{-2}r$ and $b = \varepsilon^{-2}r'$. The second inequality come from applying Jensen's inequality

$$\Big\| \mathbb{E}\Big[\nabla^+ \widetilde{\mathcal{Z}}^{\varepsilon}_{\varepsilon^{-2}r}(x) \nabla^- \widetilde{\mathcal{Z}}^{\varepsilon}_{\varepsilon^{-2}r}(x) \Big| \mathcal{F}_{\varepsilon^{-2}r'}\Big]\Big\|_2 \le \big\| \nabla^+ \widetilde{\mathcal{Z}}^{\varepsilon}_{\varepsilon^{-2}r'}(x) \nabla^- \widetilde{\mathcal{Z}}^{\varepsilon}_{\varepsilon^{-2}r'}(x) \big\|_2,$$

and then using (4.9). Combining (4.10) and (4.9) to control the r.h.s. of (4.8) yields

$$\mathbb{E}[(\mathcal{R}^{\varepsilon})^2] \le C \sum_{y_1, y_2 \in \mathbb{Z}} (\varphi(\varepsilon y_1)\varphi(\varepsilon y_2))^2 e^{2(u+1)\varepsilon(|y_1|+|y_2|)} \Big(\varepsilon^{\frac{5}{2}} + \varepsilon^{\frac{13}{4}}\Big).$$

Due to the rapid decay of φ, the r.h.s. of the above inequality vanishes as $\varepsilon \to 0$, which completes the proof of (4.3c). □

Lemma 4.4. *There exists a constant $C = C(k, u, T) > 0$ such that*

$$\sup_{x\in\mathbb{Z}} e^{-2(u+1)\varepsilon|x|} \Big\| \mathbb{E}\Big[\nabla^+ \widetilde{\mathcal{Z}}^{\varepsilon}_{a}(x) \nabla^- \widetilde{\mathcal{Z}}^{\varepsilon}_{a}(x) \Big| \mathcal{F}_b\Big]\Big\|_{2k} \le C\Big(\frac{1}{\varepsilon(a-b)^{3/2}} + \varepsilon^{\frac{3}{2}}\Big), \tag{4.11}$$

for any $k \in \mathbb{N}$ and $0 < b < b + \varepsilon^{-\frac{3}{2}} \le a < \varepsilon^{-2}T$,
where $\mathcal{F}_b$ is the σ-algebra generated by $\{s_t\}_{t\in[0,b]}$.

Proof. Using (3.3), we can write

$$\widetilde{\mathcal{Z}}^{\varepsilon}_{a}(x) = \sum_{y\in\mathbb{Z}} \mathfrak{p}^{\varepsilon}_{a-b}(x-y) \widetilde{\mathcal{Z}}^{\varepsilon}_{b}(y) + \int_b^a \sum_{y\in\mathbb{Z}} \mathfrak{p}^{\varepsilon}_{a-c}(x-y) d\widetilde{\mathcal{M}}^{\varepsilon}_{c}(y). \tag{4.12}$$

It follows from (4.12) that $\mathbb{E}[\nabla^+ \widetilde{\mathcal{Z}}^{\varepsilon}_{a}(x) \nabla^- \widetilde{\mathcal{Z}}^{\varepsilon}_{a}(x) | \mathcal{F}_b] = \textbf{(I)} + \textbf{(II)}$, where

$$\textbf{(I)} := \sum_{y_1, y_2 \in \mathbb{Z}} \nabla^+ \mathfrak{p}^{\varepsilon}_{a-b}(x-y_1) \nabla^- \mathfrak{p}^{\varepsilon}_{a-b}(x-y_2) \widetilde{\mathcal{Z}}^{\varepsilon}_{b}(y_1) \widetilde{\mathcal{Z}}^{\varepsilon}_{b}(y_2),$$

$$\textbf{(II)} := \mathbb{E}\Big[\int_b^a \sum_{y\in\mathbb{Z}} \nabla^+ \mathfrak{p}^{\varepsilon}_{a-c}(x-y) \nabla^- \mathfrak{p}^{\varepsilon}_{a-c}(x-y) d\big\langle \widetilde{\mathcal{M}}^{\varepsilon}(y), \widetilde{\mathcal{M}}^{\varepsilon}(y)\big\rangle_c \Big| \mathcal{F}_b\Big].$$

For any $c \in [b, a]$, $k \in \mathbb{N}$ and $\kappa \in \mathbb{R}_{\ge 0}$, define

$$f^{(k)}_{\kappa}(c; b) := \sup_{x\in\mathbb{Z}} e^{-2\kappa\varepsilon|x|} \Big\| \mathbb{E}\Big[\nabla^+ \widetilde{\mathcal{Z}}^{\varepsilon}_{c}(x) \nabla^- \widetilde{\mathcal{Z}}^{\varepsilon}_{c}(x) \Big| \mathcal{F}_b\Big]\Big\|_{2k}.$$

$\alpha(c) := \varepsilon(1 \wedge (c-b)^{-\frac{1}{2}}) + \varepsilon^{\frac{3}{2}}, \quad \beta_1(c) :=(1 \wedge (c-b)^{-\frac{3}{2}}), \quad \beta_2(c) := (1 \wedge (a-c)^{-\frac{3}{2}}).$

In what follows, we will show that

$$\sup_{x\in\mathbb{Z}} e^{-2(u+1)\varepsilon|x|}\|(\mathbf{I})\|_{2k} \leq C_1\varepsilon^{-1}\beta_1(a) \tag{4.13}$$

and

$$\sup_{x\in\mathbb{Z}} e^{-2(u+1)\varepsilon|x|}\|(\mathbf{II})\|_{2k} \leq C_2\alpha(a) + C_3\int_b^a \beta_2(c) f^{(k)}_{u+1}(c;b)dc, \tag{4.14}$$

for some $C_1 = C_1(k,u,T) > 0$, $C_2 = C_2(k,u,T) > 0$ and $C_3 = C_3(k,u,T) > 0$. Assuming (4.13) and (4.14), we first complete the proof of (4.11). Since $f^{(k)}_{u+1}(a;b)$ is less than the sum of l.h.s. of (4.13) and (4.14), summing both sides of the inequalities in (4.13) and (4.14) yields

$$f^{(k)}_{u+1}(a;b) \leq C_1\varepsilon^{-1}\beta_1(a) + C_2\alpha(a) + C_3\int_b^a \beta_2(c) f^{(k)}_{u+1}(c;b)dc. \tag{4.15}$$

Note that (4.15) verifies the condition of the Gronwall's inequality[1] inside the interval $[a,b]$. Applying Gronwall's inequality, we write

$$f^{(k)}_{u+1}(a;b) \leq \Pi_{\alpha,\beta_1}(a) + C_3\int_b^a \Pi_{\alpha,\beta_1}(c)\beta_2(c)\exp\Big(C_3\int_b^c \beta_2(w)dw\Big)dc, \tag{4.16}$$

where $\Pi_{\alpha,\beta_1}(c)$ is the shorthand for $C_1\varepsilon^{-1}\beta_1(c) + C_2\alpha(c)$. Since $\varepsilon^{-1}(a-b)^{-\frac{3}{2}} \geq T^{-1}\varepsilon(a-b)^{-\frac{1}{2}}$ for all $0 < b < a < \varepsilon^{-2}T$, $\alpha(a)$ is less than $(1\wedge \varepsilon^{-1}(a-b)^{-\frac{3}{2}}) + \varepsilon^{\frac{3}{2}}$. Therefore, $\Pi_{\alpha,\beta_1}(a)$ is bounded above by $C'_1(\varepsilon^{-1}(a-b)^{-\frac{3}{2}} + \varepsilon^{\frac{3}{2}})$ for some $C'_1 = C'_1(T) > 0$. By a direct computation, $\int_b^c \beta_2(w)dw \leq \big(1-(c-b)^{-\frac{1}{2}}\big) \leq 1$. Substituting the upper bounds of $\Pi_{\alpha,\beta_1}(a)$ and $\exp(\int_b^c \beta_2(w)dw)$ into the r.h.s. of (4.16), we get

$$f^{(k)}_{u+1}(a;b) \leq C'_1(\varepsilon^{-1}(a-b)^{-\frac{3}{2}} + \varepsilon^{\frac{3}{2}}) + C'_2\int_b^a \Pi_{\alpha,\beta_1}(c)\beta_2(c)dc, \tag{4.17}$$

for some constant $C'_2 = C'_2(T) > 0$. Note that owing to (4.17), (4.11) will be proved once we show the following

$$\int_b^a \Pi_{\alpha,\beta_1}(c)\beta_2(c)dc \leq C\Big(\varepsilon^{-1}(a-b)^{-\frac{3}{2}} + \varepsilon^{\frac{3}{2}}\Big). \tag{4.18}$$

Thus we now proof (4.18). Since $\max\{(a-c),(c-b)\} \geq (a-b)/2$ for any $c \in [a,b]$, we have

$$\begin{aligned}\big(1\wedge(c-b)^{-\frac{3}{2}}\big)\big(1\wedge(a-c)^{-\frac{3}{2}}\big) &\leq \big(1\wedge 2^{\frac{3}{2}}(a-b)^{-\frac{3}{2}}\big)\Big(\big(1\wedge(a-c)^{-\frac{3}{2}}\big)\\ &\quad + \big(1\wedge(c-b)^{-\frac{3}{2}}\big)\Big).\end{aligned}$$

[1] Gronwall's inequality says that for any interval I of the form $[b,\infty)$, or $[b,a]$, or $[b,a)$ with $b < a$ and any real valued functions f, g and h with the negative part of f being integrable on every closed and bounded subinterval I, if h satisfies $h(c) \leq f(c) + \int_b^c g(r)h(r)dr$ for all $c \in I$, then, one has $h(a) \leq f(a) + \int_b^a f(r)g(r)\exp(\int_b^r g(w)dw)dr$.

Integrating both sides of the above inequality w.r.t. c yields

$$\int_b^a \big(1 \wedge (c-b)^{-\frac{3}{2}}\big)\big(1 \wedge (a-c)^{-\frac{3}{2}}\big)dc \le C(a-b)^{-\frac{3}{2}}, \tag{4.19}$$

for some constant $C = C(T) > 0$. By using the fact that $(c-b) \le \varepsilon^{-2}T$ for ant $c \in [a,b]$, we have $\varepsilon(1 \wedge (c-b)^{-\frac{1}{2}}) \le T\varepsilon^{-1}(1 \wedge (c-b)^{-\frac{3}{2}})$. Therefore, in a same way as in (4.19), we get

$$\int_b^a \varepsilon\big(1 \wedge (c-b)^{-\frac{1}{2}}\big)\big(1 \wedge (a-c)^{-\frac{3}{2}}\big)dc \le C(a-b)^{-\frac{3}{2}}. \tag{4.20}$$

Combining (4.19) and (4.20) with the fact that $\int_b^a (1 \wedge (a-c)^{-3/2})dc \le 1$ shows (4.18).

To complete the proof of this lemma, it boils down to proving (4.13) and (4.14) which do as follows. We first show (4.13). Observe that

$$\begin{aligned}
\|(\mathbf{I})\|_{2k} &\le \sum_{y_1,y_2\in\mathbb{Z}} |\nabla^+ \mathfrak{p}^\varepsilon_{a-b}(x-y_1)\nabla^- \mathfrak{p}^\varepsilon_{a-b}(x-y_2)|\|\widetilde{\mathcal{Z}}^\varepsilon_b(y_1)\|_{4k}\|\widetilde{\mathcal{Z}}^\varepsilon_b(y_2)\|_{4k} \\
&\le C\Big(\sum_{y\in\mathbb{Z}} \nabla^+\mathfrak{p}^\varepsilon_{a-b}(x-y)e^{u\varepsilon(|y|+1)}\Big)^2 \\
&\le C\sum_{y\in\mathbb{Z}} (\nabla^+\mathfrak{p}^\varepsilon_{a-b}(x-y))^2 e^{2(u+1)\varepsilon|y|}\sum_{y\in\mathbb{Z}} e^{-2\varepsilon|y|} \\
&\le Ce^{2(u+1)\varepsilon|x|}\varepsilon^{-1}(1\wedge(a-b)^{-\frac{3}{2}}),
\end{aligned} \tag{4.21}$$

for some $C = C(k,u,T) > 0$. The inequality in the first line follows from the triangle inequality of the L^{2k}-norm and Hölder's inequality (which bounds $\|\widetilde{\mathcal{Z}}^\varepsilon_b(y_1)\widetilde{\mathcal{Z}}^\varepsilon_b(y_2)\|_{2k}$ by the product of $\|\widetilde{\mathcal{Z}}^\varepsilon_b(y_1)\|_{4k}$ and $\|\widetilde{\mathcal{Z}}^\varepsilon_b(y_2)\|_{4k}$). The inequality in the second line is obtained by substituting the upper bound of $\|\widetilde{\mathcal{Z}}^\varepsilon_b(\bullet)\|$ from (3.5a). Applying Cauchy–Schwarz inequality, we obtain the inequality of the third line. The last inequality follows by applying the first inequality of (A.4b) to bound $|\nabla^+\mathfrak{p}^\varepsilon_{a-b}(x-y)|$ by $C(1\wedge(a-b)^{-\frac{3}{2}})$ and the second inequality of (A.4a) to bound the sum $\sum_{y\in\mathbb{Z}}\mathfrak{p}^\varepsilon_{a-b}(x-y)\exp(2(u+1)\varepsilon|x-y|)$ by a constant. Furthermore, the geometric sum $\sum_{y\in\mathbb{Z}} e^{-\varepsilon|y|}$ is bounded above by $C\varepsilon^{-1}$ for some constant $C > 0$. From (4.21), (4.13) follows by noting that the r.h.s. of (4.21) does not depend on x after dividing both sides by $\exp(2(u+1)\varepsilon|x|)$.

Now, we show (4.14). In a similar way as used to derive (4.6), we may use (2.6) and (2.15) to show that

$$\begin{aligned}
(\mathbf{II}) =& 2\varepsilon\int_b^a \sum_{y\in\mathbb{Z}} \nabla^+\mathfrak{p}^\varepsilon_{a-c}(x-y)\nabla^-\mathfrak{p}^\varepsilon_{a-c}(x-y)\Big(e^{2c\theta} - \mathbb{E}\big[\varepsilon^{-1}\nabla^+\widetilde{\mathcal{Z}}^\varepsilon_c(y)\nabla^-\widetilde{\mathcal{Z}}^\varepsilon_c(y) \\
&- \tfrac{1}{2}\varepsilon^{\frac{1}{2}}U^\varepsilon_c(x)|\mathcal{F}_b\big]\Big)dc,
\end{aligned}$$

where U^ε_t is defined in (4.7). We can write the expression above as a sum of three terms which we will denote by $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ and those are defined as follows:

$$\mathcal{W}_1 := 2\varepsilon\int_b^a e^{2c\theta}\sum_{y\in\mathbb{Z}} \nabla^+\mathfrak{p}^\varepsilon_{a-c}(x-y)\nabla^-\mathfrak{p}^\varepsilon_{a-c}(x-y)dc,$$

$$\mathcal{W}_2 := 2\int_b^a \sum_{y\in\mathbb{Z}} \nabla^+\mathfrak{p}^\varepsilon_{a-c}(x-y)\nabla^-\mathfrak{p}^\varepsilon_{a-c}(x-y)\mathbb{E}\big[\nabla^+\widetilde{\mathcal{Z}}^\varepsilon_c(y)\nabla^-\widetilde{\mathcal{Z}}^\varepsilon_c(y)\big|\mathcal{F}_b\big]dc,$$

$$\mathcal{W}_3 := \varepsilon^{\frac{3}{2}}\int_b^a \sum_{y\in\mathbb{Z}} \nabla^+\mathfrak{p}^\varepsilon_{a-c}(x-y)\nabla^-\mathfrak{p}^\varepsilon_{a-c}(x-y)\mathbb{E}\big[U^\varepsilon_c(x)\big|\mathcal{F}_b\big]dc.$$

Next, we show that the following holds:

$$\sup_{x\in\mathbb{Z}} e^{-2(u+1)|x|}|\mathcal{W}_1| \le C\varepsilon(1\wedge(a-b)^{-\frac{1}{2}}), \qquad \sup_{x\in\mathbb{Z}} e^{-2(u+1)\varepsilon|x|}\|\mathcal{W}_3\|_{2k} \le C\varepsilon^{\frac{3}{2}}, \tag{4.22}$$

$$\sup_{x\in\mathbb{Z}} e^{-2(u+1)\varepsilon|x|}\|\mathcal{W}_2\|_{2k} \le \int_b^a (1\wedge(a-c)^{-\frac{3}{2}})f^{(k)}_{u+1}(c;b)dc. \tag{4.23}$$

Since $\|(\mathbf{II})\|_{2k}$ is bounded above by $\|\mathcal{W}_1\|_{2k} + \|\mathcal{W}_2\|_{2k} + \|\mathcal{W}_3\|_{2k}$ via triangle inequality of the L^{2k}-norm, combining (4.22) and (4.23) yields (4.14).

Throughout the rest, we will prove the inequalities in (4.22) and (4.23). We start with proving the first inequality of (4.22). To prove this, we use the following 'key identity' of [BG97] (see also [CST18, Lemma 4.2] or (A.6) herein)

$$\int_0^\infty \sum_{y\in\mathbb{Z}} \nabla^+\mathfrak{p}^\varepsilon_r(x-y)\nabla^-\mathfrak{p}^\varepsilon_r(x-y)dr = 0. \tag{4.24}$$

Via the change of variable $c \mapsto a-c$ we may rewrite

$$\begin{aligned}\mathcal{W}_1 &= 2\varepsilon e^{-2a\theta}\int_0^{a-b} e^{2c\theta}\sum_{y\in\mathbb{Z}}\nabla^+\mathfrak{p}^\varepsilon_c(x-y)\nabla^-\mathfrak{p}^\varepsilon_c(x-y)dc\\ &= 2\varepsilon e^{-2a\theta}\int_0^{a-b}(e^{2c\theta}-1)\sum_{y\in\mathbb{Z}}\nabla^+\mathfrak{p}^\varepsilon_c(x-y)\nabla^-\mathfrak{p}^\varepsilon_c(x-y)dc\\ &\quad - 2\varepsilon e^{-2a\theta}\int_{a-b}^\infty \sum_{y\in\mathbb{Z}}\nabla^+\mathfrak{p}^\varepsilon_c(x-y)\nabla^-\mathfrak{p}^\varepsilon_c(x-y)dc,\end{aligned} \tag{4.25}$$

where the second equality follows by first splitting the integral into two parts by writing $e^{2c\theta}$ as $(e^{2c\theta}-1)+1$ and then, using (4.24) for the second part. We claim that

$$\varepsilon e^{-2a\theta}\int_0^{a-b}|e^{2c\theta}-1|\sum_{y\in\mathbb{Z}}\nabla^+\big|\mathfrak{p}^\varepsilon_c(x-y)\nabla^-\mathfrak{p}^\varepsilon_c(x-y)\big|dc \le C\varepsilon^3(a-b)^{\frac{1}{2}}, \tag{4.26}$$

while

$$\varepsilon e^{-2a\theta}\int_{a-b}^\infty \sum_{y\in\mathbb{Z}}\big|\nabla^+\mathfrak{p}^\varepsilon_c(x-y)\nabla^-\mathfrak{p}^\varepsilon_c(x-y)\big|dc \le C\varepsilon\big(1\wedge(a-b)^{-\frac{1}{2}}\big). \tag{4.27}$$

Note that the l.h.s. of (4.26) and (4.27) bound the two term on the r.h.s. (4.25) respectively (displayed in the last two lines of (4.25)). The first inequality of (4.22) follows immediately from this after recalling that $b < a \in [0, \varepsilon^{-2}T]$ and hence $\varepsilon^3(a-$

$b)^{\frac{1}{2}} + \varepsilon(a-b)^{-\frac{1}{2}} < C\varepsilon(a-b)^{-\frac{1}{2}}$ for a constant only depending on T. To show (4.27) we use that for any $T > 0$ there exists a constant C such that the bound $|e^{\omega} - 1| \le C\omega$ holds for all $\omega \in [0, T]$ to control $|e^{2c\theta} - 1| \le C\varepsilon^2 c$; we also apply the first inequality of (A.4b)

$$\left|\nabla^+ \mathfrak{p}^{\varepsilon}_c(x-y)\nabla^- \mathfrak{p}^{\varepsilon}_c(x-y)\right| \le \min\{1, c^{-\frac{3}{2}}\}\big(\mathfrak{p}^{\varepsilon}_c(x-y) + \mathfrak{p}^{\varepsilon}_c(x-y+1)\big). \tag{4.28}$$

Substituting (4.28) and the inequality $|e^{2c\theta}-1| \le C\varepsilon^2 c$ into the l.h.s. of (4.26), summing over y and integrating w.r.t. c, we get (4.26). In a similar way, we get (4.27) by using (4.28).

Starting with $\mathcal{W}_3$, by using triangle inequality of the L^{2k}-norm, we write

$$\|\mathcal{W}_3\|_{2k} \le \varepsilon^{\frac{3}{2}} \int_b^a \sum_{y\in\mathbb{Z}} |\nabla^+ \mathfrak{p}^{\varepsilon}_{a-c}(x-y)\nabla^- \mathfrak{p}^{\varepsilon}_{a-c}(x-y)| \big\|\mathbb{E}\big[U^{\varepsilon}_c(y)\big|\mathcal{F}_b\big]\big\|_{2k} dc.$$

Owing to the Jensen's inequality, we may bound $\big\|\mathbb{E}\big[U^{\varepsilon}_c(y)\big|\mathcal{F}_b\big]\big\|_{2k}$ by $\big\|U^{\varepsilon}_c(y)\big\|_{2k}$. Now, we can use the fact (shown soon after (4.7)) that for any $k \ge 1$, the expectation $\mathbb{E}[\|U^{\varepsilon}_t(y)\|_{2k}]$ after scaling by $e^{-2(u+1)\varepsilon|y|}$ is uniformly bounded in y as $\varepsilon \to 0$. Combining this last fact with the bound on $|\nabla^+ \mathfrak{p}^{\varepsilon}_{a-c}\nabla^- \mathfrak{p}^{\varepsilon}_{a-c}|$ from (4.28) yields that $e^{-2(u+1)\varepsilon|x|}\|\mathcal{W}_3\|_{2k}$ is bounded by

$$C\varepsilon^{\frac{3}{2}} \int_b^a \sum_{y\in\mathbb{Z}} (1 \wedge (a-c)^{-\frac{3}{2}})\big(\mathfrak{p}^{\varepsilon}_{a-c}(x-y) + \mathfrak{p}^{\varepsilon}_{a-c}(x-y+1)\big)e^{2(u+1)\varepsilon|x-y|} dc$$

for some constant C which does not depend on ε, T or x. By the second inequality of (A.4a), the sum $\sum_{y\in\mathbb{Z}} \mathfrak{p}^{\varepsilon}_{a-c}(x-y)\exp(2(u+1)\varepsilon|x-y|)$ and $\sum_{y\in\mathbb{Z}} \mathfrak{p}^{\varepsilon}_{a-c}(x-y+1)\exp(2(u+1)\varepsilon|x-y|)$ is bounded above by some positive constant which only depends on T. Moreover, $\int_b^a (1 \wedge (a-c)^{-3/2})dc$ is bounded above by 1. As a consequence the r.h.s. of the last display is bounded above by $C\varepsilon^{3/2}$ where C only depends on T and u. This proves the second inequality of (4.22).

We are left to show (4.23) which we prove as follow. Via the triangle inequality of the L^{2k}-norm, we may write

$$\|\mathcal{W}_2\|_{2k} \le 2\int_b^a \sum_{y\in\mathbb{Z}} \left|\nabla^+ \mathfrak{p}^{\varepsilon}_{a-c}(x-y)\nabla^- \mathfrak{p}^{\varepsilon}_{a-c}(x-y)\right| \times \left\|\mathbb{E}\left[\nabla^+ \widetilde{\mathcal{Z}}^{\varepsilon}_c(y)\nabla^- \widetilde{\mathcal{Z}}^{\varepsilon}_c(y)\middle|\mathcal{F}_b\right]\right\|_{2k} dc. \tag{4.29}$$

From the definition of $f^{(k)}_{\kappa}(\bullet; b)$,

$$\left\|\mathbb{E}\left[\nabla^+ \widetilde{\mathcal{Z}}^{\varepsilon}_c(y)\nabla^- \widetilde{\mathcal{Z}}^{\varepsilon}_c(y)\middle|\mathcal{F}_b\right]\right\|_{2k} \le e^{2(u+1)\varepsilon|y|} f^{(k)}_{u+1}(c; b).$$

Combining this last inequality with (4.28) (to bound $|\nabla^+ \mathfrak{p}^{\varepsilon}_{a-c}(x-y)\nabla^- \mathfrak{p}^{\varepsilon}_{a-c}(x-y)|$) and applying in the r.h.s. of (4.29) yields that the r.h.s. of (4.29) is bounded by

$$C\int_b^a (1 \wedge (a-c)^{-\frac{3}{2}}) f^{(k)}_{u+1}(c, b) \sum_{y\in\mathbb{Z}} e^{2(u+1)\varepsilon|y|}\big(\mathfrak{p}^{\varepsilon}_{a-c}(x-y) + \mathfrak{p}^{\varepsilon}_{a-c}(x-y+1)\big) dc.$$

Via triangle inequality, we bound $\exp(2(u+1)\varepsilon|y|)$ by $\exp(2(u+1)\varepsilon(|x-y|+|x|))$ in the r.h.s. of the above inequality. Owing to the second inequality of (A.4a), one can bound $\sum_{y\in\mathbb{Z}} e^{2(u+1)\varepsilon|x-y|}\mathfrak{p}^{\varepsilon}_{a-c}(x-y)$ and $\sum_{y\in\mathbb{Z}} e^{2(u+1)\varepsilon|x-y|}\mathfrak{p}^{\varepsilon}_{a-c}(x-y+1)$ by some constant $C = C(u, T) > 0$. Combining these estimates and substituting those into the last inequality in the above display we arrive at (4.23). This completes the proof. □

4.1. Proof of Proposition 4.1. Let $\mathcal{Z}$ be a limit of a subsequence $\{\mathcal{Z}^{\varepsilon}\}_{\varepsilon}$. Then, for a $\varphi \in \mathcal{C}_b^{\infty}(\mathbb{R}) \cap L^2(\mathbb{R})$, the random variables $\mathfrak{M}_T^{\varepsilon}(\varphi)$ and $\mathfrak{N}_T^{\varepsilon}(\varphi)$, defined in (4.2a)–(4.2b), converge to $\mathfrak{M}_T(\varphi)$ and $\mathfrak{N}_T(\varphi)$ from (4.1) in L^1 as $\varepsilon \to 0$. This implies that $\mathfrak{M}(\varphi)$ and $\mathfrak{N}(\varphi)$ are two local martingales, and hence $\mathcal{Z}$ solves the martingale problem associated to (1.2) with $A = 0$ and $B_T = e^{T/4}$. By [SV06, Ch. 8, Thm. 8.1.5], $\mathcal{Z}$ is the unique solution of (1.2) started from the initial data $\mathcal{Z}_0$, which is the weak limit of the sequence $\{\mathcal{Z}_0^{\varepsilon}\}_{\varepsilon}$.

5. Convergence of a Generalized Dynamic ASEP

In this section we prove Theorem 1.6, without relying on the duality relation of [BC17, Theorem 2.3]. To this end, we write a system of SDEs governing the evolution of the rescaled height function $\hat{s}^{\varepsilon}$, and use the Da Prato-Debussche trick [DPD03] to prove convergence of solutions. Since this method is quite robust, we can consider a more general evolution of the height functions (1.9), with a function $\mathfrak{f}$ satisfying Assumption 1.5.

5.1. Heuristics of the argument. We start with describing heuristics of our argument. For a height function $s \in \Omega_{\chi}^N$, we denote by $a^{\downarrow}(s, x)$ and $a^{\uparrow}(s, x)$ the down and up jumps rates in (1.9) respectively. (Recall, that the set Ω_{χ}^N contains periodic height functions s.) Let $\zeta_t^{\downarrow}(x)$ and $\zeta_t^{\uparrow}(x)$ be the processes describing down and up jumps of the height function $s_t(x)$. Then they are solutions of the system of SDEs

$$d\zeta_t^{\downarrow}(x) = 2\,\mathbb{1}_{\{s_t(x)>s_t(x-1)=s_t(x+1)\}} dQ_t^{\downarrow}(x), \quad d\zeta_t^{\uparrow}(x) = 2\,\mathbb{1}_{\{s_t(x)<s_t(x-1)=s_t(x+1)\}} dQ_t^{\uparrow}(x), \tag{5.1}$$

with the initial states $\zeta_0^{\downarrow}(x) = \zeta_0^{\uparrow}(x) = 0$, where $Q_t^{\downarrow}(x)$ and $Q_t^{\uparrow}(x)$ are Poisson processes with rates $a^{\downarrow}(s_t, x)$ and $a^{\uparrow}(s_t, x)$ respectively. To be more precise, we should use the left limits of s at time t on the r.h.s. of both equations in (5.1). However, we prefer not to indicate it, to make our notation less cumbersome. The evolution of the height function s is described by

$$ds_t(x) = d\zeta_t^{\uparrow}(x) - d\zeta_t^{\downarrow}(x). \tag{5.2}$$

To make the SDEs martingale-driven, we define the compensated Poisson processes to be the solutions of $d\tilde{Q}_t^{\downarrow}(x) = dQ_t^{\downarrow}(x) - a^{\downarrow}(s_t, x)dt$ and $d\tilde{Q}_t^{\uparrow}(x) = dQ_t^{\uparrow}(x) - a^{\uparrow}(s_t, x)dt$, starting from zeros at time $t = 0$. These new processes are càdlàg martingales with the predictable quadratic covariations given by $\langle \tilde{Q}^{\downarrow}(x), \tilde{Q}^{\uparrow}(y)\rangle_t \equiv 0$ and

$$\langle \tilde{Q}^{\downarrow}(x), \tilde{Q}^{\downarrow}(y)\rangle_t = \mathbb{1}_{x=y} \int_0^t a^{\downarrow}(s_r, x)dr, \qquad \langle \tilde{Q}^{\uparrow}(x), \tilde{Q}^{\uparrow}(y)\rangle_t = \mathbb{1}_{x=y} \int_0^t a^{\uparrow}(s_r, x)dr.$$

It is easy to check that the following two identities hold:

$$\mathbb{1}_{\{s(x)<s(x-1)=s(x+1)\}} = \frac{1}{4}(1 + \nabla^- s(x))(1 - \nabla^+ s(x)),$$
$$\mathbb{1}_{\{s(x)>s(x-1)=s(x+1)\}} = \frac{1}{4}(1 - \nabla^- s(x))(1 + \nabla^+ s(x)),$$

where $\nabla^{\pm}$ are discrete derivatives, defined in Section 1.3. We will often also use the following two functions

$$\varrho(s,x) := \frac{a^{\uparrow}(s,x) + a^{\downarrow}(s,x)}{2}, \qquad \lambda(s,x) := \frac{a^{\uparrow}(s,x) - a^{\downarrow}(s,x)}{2}.$$

Combining these identities with (5.2), we obtain the systems of SDEs describing the evolution of the height function s:

$$ds_t(x) = \varrho(s_t(x))\Delta s_t(x)dt + F(s_t,x)dt + dM_t(x), \tag{5.3}$$

where $\Delta := \nabla^+ - \nabla^-$ is the discrete Laplacian, the function F is given by

$$F(s,x) := \lambda(s,x)\big(1 - \nabla^- s(x)\, \nabla^+ s(x)\big),$$

and $t \mapsto M_t(x)$ is a càdlàg martingale, starting at 0, with jumps of size 1 and with the bracket process satisfying $\frac{d}{dt}\langle M(x), M(y)\rangle_t = \mathbb{1}_{x=y} C(s_t,x)$ where

$$C(s,x) := 2\varrho(s,x)\big(1 - \nabla^- s(x)\, \nabla^+ s(x)\big) + 2\lambda(s,x)\Delta s(x).$$

For two different points $x \neq y$, the martingales $M_t(x)$ and $M_t(y)$ almost surely do not make jumps at the same time. Moreover, the number of jumps of $t \mapsto M_t(x)$ at every finite interval $[0,T]$ has bounded moments uniformly in x and locally uniformly in T, which means that the martingale makes a.s. finitely many jumps on every bounded time interval.

We need to tilt the function to make it periodic. More precisely, under the diffusive scaling and after recentering define $\hat{s}^{\varepsilon}_t(x) := \sqrt{\varepsilon}(s(\varepsilon^{-2}t, \varepsilon^{-1}x) - \chi x)$, where χ is defined in the statement of Theorem 1.6. Then (5.3) becomes

$$\partial_t \hat{s}^{\varepsilon}_t(x) = \varrho_{\varepsilon}(\hat{s}^{\varepsilon}_t, x)\Delta_{\varepsilon}\hat{s}^{\varepsilon}_t(x) + F_{\varepsilon}(\hat{s}^{\varepsilon}_t, x) + \xi^{\varepsilon}(t,x), \tag{5.4}$$

where $\nabla^{\pm}_{\varepsilon}$ are the respective discrete derivatives and Δ_{ε} is the discrete Laplacian, defined in Section 1.3. The rescaled functions in (5.4) are

$$\varrho_{\varepsilon}(\hat{s}^{\varepsilon}, x) := \varrho\big(\varepsilon^{-1/2}\hat{s}^{\varepsilon}(x) + \chi x, \varepsilon^{-1}x\big), \tag{5.5a}$$

$$\lambda_{\varepsilon}(\hat{s}^{\varepsilon}, x) := \varepsilon^{-3/2}\lambda\big(\varepsilon^{-1/2}\hat{s}^{\varepsilon}(x) + \chi x, \varepsilon^{-1}x\big), \tag{5.5b}$$

$$F_{\varepsilon}(\hat{s}^{\varepsilon}, x) := \lambda_{\varepsilon}(\hat{s}^{\varepsilon}, x)\big(1 - \varepsilon\nabla^-_{\varepsilon}\hat{s}^{\varepsilon}(x)\, \nabla^+_{\varepsilon}\hat{s}^{\varepsilon}(x)\big). \tag{5.5c}$$

The noise in (5.4) is given by $\xi^{\varepsilon}(t,x) := dM^{\varepsilon}_t(x)$, where the rescaled martingales are $M^{\varepsilon}_t(x) := \sqrt{\varepsilon}M(\varepsilon^{-2}t, \varepsilon^{-1}x)$, have jumps of size $\sqrt{\varepsilon}$ and have the predictable quadratic covariation $\frac{d}{dt}\langle M^{\varepsilon}(x), M^{\varepsilon}(y)\rangle_t = \varepsilon^{-1}\mathbb{1}_{x=y}C_{\varepsilon}(\hat{s}^{\varepsilon}_t, x)$, where

$$C_{\varepsilon}(\hat{s}^{\varepsilon}, x) := 2\varrho_{\varepsilon}(\hat{s}^{\varepsilon}, x)\big(1 - \varepsilon\nabla^-_{\varepsilon}\hat{s}^{\varepsilon}(x)\, \nabla^+_{\varepsilon}\hat{s}^{\varepsilon}(x)\big) + 2\varepsilon^3\lambda_{\varepsilon}(\hat{s}^{\varepsilon}, x)\Delta_{\varepsilon}\hat{s}^{\varepsilon}(x). \tag{5.6}$$

Furthermore, properties of the martingales M imply that, for $x \neq y$, $M^{\varepsilon}_t(x)$ and $M^{\varepsilon}_t(y)$ a.s. do not jump together, and on every time interval $[0, \varepsilon^2 T]$ the martingale $M^{\varepsilon}_t(x)$ makes a.s. finitely many jumps.

Let us now take the asymmetry to be $q = e^{-\varepsilon}$, as in the statement of Theorem 1.6. Then, we have the following results for the functions ϱ_{ε} and λ_{ε}.

Lemma 5.1. *The functions ϱ_{ε} and λ_{ε}, defined in* (5.5a) *and* (5.5b), *have the properties:*

(1) *There is a constant* c_0 *such that* $|\varrho_\varepsilon(\hat{s}, x) - 1 + \varepsilon/2| \leq c_0\varepsilon^2$, *uniformly in* $x \in \mathbb{R}$, *where* γ *is from Assumption* 1.5.
(2) *Recall the constants* $\mathfrak{a}$ *and* γ, *defined in Assumption* 1.5. *Then one has the bounds* $|\lambda_\varepsilon(\hat{s}, x)| \leq c_1\varepsilon^{-1/2}$ *and* $|\lambda_\varepsilon(\hat{s}, x) + \frac{\mathfrak{a}\hat{s}(x)}{4}| \leq c_1\varepsilon^{1-\gamma}(1 + \sqrt{\varepsilon}|\hat{s}(x)|)^\gamma$ *uniformly in* $x \in \mathbb{R}$, *for some constant* c_1.

The constants c_0 *and* c_1 *are independent of* $\hat{s}$, ε *and* x.

Proof. Let us denote for brevity $s = \varepsilon^{-1/2}\hat{s}(x)$. Then, recalling the jump rates (1.9), the function ϱ_ε can be written as

$$\begin{aligned}\varrho_\varepsilon(\hat{s}, x) &= 1 - \frac{1 - e^{-\varepsilon}}{2} + \frac{1 - e^{-\varepsilon}}{2}\left(\frac{1}{1 + e^{\varepsilon\mathfrak{f}(s)+\varepsilon}} - \frac{1}{1 + e^{\varepsilon\mathfrak{f}(s)-\varepsilon}}\right)\\ &= 1 - \frac{1 - e^{-\varepsilon}}{2} + \frac{(1 - e^{-\varepsilon})(1 - e^{2\varepsilon})}{2}\frac{1}{1 + e^{\varepsilon\mathfrak{f}(s)+\varepsilon}}\frac{e^{\varepsilon\mathfrak{f}(s)-\varepsilon}}{1 + e^{\varepsilon\mathfrak{f}(s)-\varepsilon}}\ .\end{aligned}$$

The last two factors are bounded uniformly in s and ε, hence the bound in (1) on ϱ_ε follows.

Now, we will prove the bound in (2) on λ_ε. To this end, we rewrite it as

$$\lambda_\varepsilon(\hat{s}, x) = \frac{1 - e^{-\varepsilon}}{2\varepsilon^{3/2}}\left(\frac{1}{1 + e^{\varepsilon\mathfrak{f}(s)+\varepsilon}} + \frac{1}{1 + e^{\varepsilon\mathfrak{f}(s)-\varepsilon}} - 1\right).$$

The terms in the parenthesis are bounded by a constant, yielding $|\lambda_\varepsilon(\hat{s}, x)| \leq c_1\varepsilon^{-1/2}$. Further, using the Taylor expansion for $e^{-\varepsilon}$, we can write

$$\lambda_\varepsilon(\hat{s}, x) = \frac{1}{2\sqrt{\varepsilon}}\left(\frac{1}{1 + e^{\varepsilon\mathfrak{f}(s)+\varepsilon}} + \frac{1}{1 + e^{\varepsilon\mathfrak{f}(s)-\varepsilon}} - 1\right) + \lambda_\varepsilon^{(1)}(\hat{s}, x),$$

where $|\lambda_\varepsilon^{(1)}(\hat{s}, x)| \leq C\sqrt{\varepsilon}$. Next, we will replace $\mathfrak{f}(s)$ by $\mathfrak{a}s$:

$$\frac{1}{1 + e^{\varepsilon\mathfrak{f}(s)\pm\varepsilon}} = \frac{1}{1 + e^{\sqrt{\varepsilon}\mathfrak{a}\hat{s}\pm\varepsilon}} + \frac{e^{\sqrt{\varepsilon}\mathfrak{a}\hat{s}\pm\varepsilon}}{1 + e^{\sqrt{\varepsilon}\mathfrak{a}\hat{s}\pm\varepsilon}}\frac{1 - e^{\varepsilon(\mathfrak{f}(s)-\mathfrak{a}s)}}{1 + e^{\varepsilon\mathfrak{f}(s)\pm\varepsilon}},$$

where the last term can be bounded, using Assumption 1.5, by a multiple of $|1 - e^{\varepsilon(\mathfrak{f}(s)-\mathfrak{a}s)}| \leq C\varepsilon^{1-\gamma}(\sqrt{\varepsilon}|\hat{s}(x)| + 1)^\gamma$. Hence, we obtain

$$\lambda_\varepsilon(\hat{s}, x) = \frac{1}{2\sqrt{\varepsilon}}\left(\frac{1}{1 + e^{\sqrt{\varepsilon}\mathfrak{a}\hat{s}+\varepsilon}} + \frac{1}{1 + e^{\sqrt{\varepsilon}\mathfrak{a}\hat{s}-\varepsilon}} - 1\right) + \lambda_\varepsilon^{(2)}(\hat{s}, x),$$

where $|\lambda_\varepsilon^{(2)}(\hat{s}, x)| \leq C\varepsilon^{1-\gamma}(\sqrt{\varepsilon}|\hat{s}(x)|+1)^\gamma$. From this the required bound on λ_ε follows. □

Now, we will investigate the limit of the functions F_ε, defined in (5.5c). Lemma 5.1 yields

$$F_\varepsilon(\hat{s}^\varepsilon, x) = \left(-\frac{\mathfrak{a}\hat{s}^\varepsilon(x)}{4} + \hat{\lambda}_\varepsilon(\hat{s}^\varepsilon, x)\right)\big(1 - \varepsilon\nabla_\varepsilon^-\hat{s}^\varepsilon(x)\,\nabla_\varepsilon^+\hat{s}^\varepsilon(x)\big), \tag{5.7}$$

where $\hat{\lambda}_\varepsilon(\hat{s}^\varepsilon, x) := \lambda_\varepsilon(\hat{s}^\varepsilon, x) + \frac{\mathfrak{a}\hat{s}^\varepsilon(x)}{4}$, vanishing as $\varepsilon \to 0$ as soon as $\hat{s}^\varepsilon$ is bounded uniformly in ε. The product $\varepsilon\nabla_\varepsilon^-\hat{s}^\varepsilon\,\nabla_\varepsilon^+\hat{s}^\varepsilon$ is expected to vanish in the limit $\varepsilon \to 0$ in a space of discretized distributions, which suggests the following limit in a respective topology:

$$F_\varepsilon(\hat{s}^\varepsilon, x) + \frac{\mathfrak{a}\hat{s}^\varepsilon(x)}{4} \xrightarrow{\varepsilon\to 0} 0.$$

However, this limit is difficult to prove, because the function F_ε is non-linear in s^ε. This is one of the main difficulties in the proof of Theorem 1.6, which is resolved in Lemma 5.9 below.

The martingales M^ε, defining the random noise ξ^ε in (5.4), are expected to converge to the cylindric Wiener process, which implies that the limit of $\hat{s}^\varepsilon$ is the periodic solution of (1.2) with $B = 1$ and $A = -\frac{\mathfrak{a}}{4}$.

We split the actual proof of Theorem 1.1 into several steps: we rewrite the equation (5.4) in mild form, and then bound each term in the expression we get. Derivation of bounds on the non-linear function F_ε is the most difficult part in our analysis.

5.2. Reformulation of the problem. The non-linear part of the equation (5.4) makes it non-trivial to bound the solution. More precisely, we expect that ξ^ε converges to the space-time white noise. Which means that for every $t > 0$ the solution $\hat{s}^\varepsilon_t$ is expected to have spatial Hölder regularity $\frac{1}{2} - \kappa$, for any $\kappa > 0$. On the other hand, the function F_ε, defined in (5.7), contains the term $\nabla_\varepsilon^- \hat{s}^\varepsilon\, \nabla_\varepsilon^+ \hat{s}^\varepsilon$ which needs to be controlled as $\varepsilon \to 0$. We show below, that the factor ε in front of this term makes it vanish in a suitable topology. This seemingly easy fact is not straightforward to prove, and for this we use the idea of [DPD03].

We start with rewriting (5.4) in a mild form. To this end, we need to replace the non-constant multiplier ϱ_ε by 1. Using Lemma 5.1, we can write $\varrho_\varepsilon(\hat{s}^\varepsilon, x) = 1 + \hat{\varrho}_\varepsilon(\hat{s}^\varepsilon, x)$, where we have a good control over $\hat{\varrho}_\varepsilon$. Hence, we rewrite (5.4) as

$$\partial_t \hat{s}^\varepsilon_t(x) = \Delta_\varepsilon \hat{s}^\varepsilon_t(x) + \hat{F}_\varepsilon(\hat{s}^\varepsilon_t, x) + F_\varepsilon(\hat{s}^\varepsilon_t, x) + \xi^\varepsilon(t, x),$$

with a new function

$$\hat{F}_\varepsilon(\hat{s}^\varepsilon, x) := \hat{\varrho}_\varepsilon(\hat{s}^\varepsilon, x)\Delta_\varepsilon \hat{s}^\varepsilon(x). \tag{5.8}$$

Let $S^\varepsilon_t := e^{t\Delta_\varepsilon}$ be the semigroup of the linear operator $\frac{d}{dt} - \Delta_\varepsilon$. Then the last equation can be written in the mild form

$$\hat{s}^\varepsilon_t(x) = (S^\varepsilon_t \hat{s}^\varepsilon_0)(x) + \int_0^t (S^\varepsilon_{t-r}(\hat{F}_\varepsilon + F_\varepsilon)(\hat{s}^\varepsilon_r))(x) dr + \int_0^t (S^\varepsilon_{t-r} dM^\varepsilon_r)(x). \tag{5.9}$$

Here, for a function $F_\varepsilon(\hat{s}^\varepsilon, x)$ depending on $\hat{s}^\varepsilon$ and the spatial variable x, we use the notation $F_\varepsilon(\hat{s}^\varepsilon)(x) := F_\varepsilon(\hat{s}^\varepsilon, x)$. In particular, the semigroup S^ε in the middle term of (5.9) acts on the function $x \mapsto F_\varepsilon(\hat{s}^\varepsilon_r)(x)$. We will use the same notation for functions depending on some other quantity instead of $\hat{s}^\varepsilon$.

We will write (5.9) in a way which gives a better control on the non-linearity F_ε.

5.2.1. Definitions of auxiliary processes In this section we define some auxiliary processes, which enable us to obtain good control on the non-linearity in (5.9). For the martingales $(M^\varepsilon_t(x))_{t\geq 0}$ we define the processes

$$\hat{X}^\varepsilon_\tau(t,x) := \int_0^\tau (S^\varepsilon_{t-r} dM^\varepsilon_r)(x), \qquad X^\varepsilon_t(x) = \hat{X}^\varepsilon_t(t,x) + (S^\varepsilon_t \hat{s}^\varepsilon_0)(x), \tag{5.10}$$

where $\hat{s}^\varepsilon_0$ is the initial data of (5.4). We will write for brevity $\hat{X}^\varepsilon_t(x) = \hat{X}^\varepsilon_t(t,x)$. The process $\hat{X}^\varepsilon_\tau(t,x)$ is a martingale for $\tau \in [0,t]$, and we can write explicitly its predictable covariation. Then we introduce the function $u^\varepsilon := \hat{s}^\varepsilon - X^\varepsilon$ and derive from (5.9)

$$u^\varepsilon_t(x) = \int_0^t (S^\varepsilon_{t-r} \hat{F}_\varepsilon(X^\varepsilon_r + u^\varepsilon_r))(x) dr + \int_0^t (S^\varepsilon_{t-r} F_\varepsilon(X^\varepsilon_r + u^\varepsilon_r))(x) dr. \tag{5.11}$$

The *ansatz* is that u^ε can be bounded in a space of higher regularity than $\hat{s}^\varepsilon$. If we solve (5.11) for u^ε, then the solution $\hat{s}^\varepsilon$ of (5.9) is obtained by

$$\hat{s}^\varepsilon = X^\varepsilon + u^\varepsilon. \tag{5.12}$$

A problem with (5.11) is that the non-linearity F_ε contains the term $\nabla^-_\varepsilon \hat{X}^\varepsilon \nabla^+_\varepsilon \hat{X}^\varepsilon$ which has a nasty behaviour in the limit $\varepsilon \to 0$. However, we obtain good bounds in Lemma 5.3 on its “renormalized" version

$$Z^\varepsilon_t(x) := \nabla^-_\varepsilon \hat{X}^\varepsilon_t(x) \nabla^+_\varepsilon \hat{X}^\varepsilon_t(x) - \mathfrak{C}^\varepsilon_t(x), \tag{5.13}$$

where we write $\mathfrak{C}^\varepsilon_t(x) := \mathfrak{C}^\varepsilon_t(t,x)$ for the predictable quadratic covariation

$$\mathfrak{C}^\varepsilon_\tau(t,x) := \langle \nabla^-_\varepsilon \hat{X}^\varepsilon_\bullet(t,x), \nabla^+_\varepsilon \hat{X}^\varepsilon_\bullet(t,x) \rangle_\tau. \tag{5.14}$$

We will write a Wick-type product of the processes $\hat{s}^\varepsilon$ in the following way

$$\begin{aligned} \big(\nabla^-_\varepsilon \hat{s}^\varepsilon \diamond \nabla^+_\varepsilon \hat{s}^\varepsilon\big)_t(x) &:= \nabla^-_\varepsilon \hat{s}^\varepsilon_t(x) \nabla^+_\varepsilon \hat{s}^\varepsilon_t(x) - \mathfrak{C}^\varepsilon_t(x), \\ &= Z^\varepsilon_t(x) + \nabla^-_\varepsilon (X^\varepsilon + u^\varepsilon)_t(x) \nabla^+_\varepsilon u^\varepsilon_t(x) + \nabla^-_\varepsilon u^\varepsilon_t(x) \nabla^+_\varepsilon X^\varepsilon_t(x), \end{aligned} \tag{5.15}$$

in which the nasty product $\nabla^-_\varepsilon \hat{X}^\varepsilon \nabla^+_\varepsilon \hat{X}^\varepsilon$ is replaced by its renormalized version Z^ε. After that we replace the product $\nabla^-_\varepsilon \hat{s}^\varepsilon \nabla^+_\varepsilon \hat{s}^\varepsilon$ in (5.7) by the Wick-type product $\nabla^-_\varepsilon \hat{s}^\varepsilon \diamond \nabla^+_\varepsilon \hat{s}^\varepsilon$ by adding and subtracting the function $\mathfrak{C}^\varepsilon$, so that this renormalization does not change (5.11). More precisely, we define two new functions

$$\begin{aligned} F_\varepsilon(X^\varepsilon, Z^\varepsilon, u^\varepsilon)_t(x) &:= \lambda_\varepsilon(\hat{s}^\varepsilon_t, x)\big(1 - \varepsilon \nabla^-_\varepsilon \hat{s}^\varepsilon_t \diamond \nabla^+_\varepsilon \hat{s}^\varepsilon_t\big), \\ \tilde{F}_\varepsilon(X^\varepsilon, \hat{X}^\varepsilon, u^\varepsilon)_t(x) &:= \varepsilon \lambda_\varepsilon(\hat{s}^\varepsilon_t, x) \mathfrak{C}^\varepsilon_t(x), \end{aligned} \tag{5.16}$$

where on the r.h.s. we use the function $\hat{s}^\varepsilon$, which is defined in terms of X^ε and u^ε by (5.12). We note that the new function F_ε depends on Z^ε by (5.15), and $\tilde{F}_\varepsilon$ depends on $\hat{X}^\varepsilon$ by (5.14). Then (5.11) can be written as

$$\begin{aligned} u^\varepsilon_t(x) = \int_0^t (S^\varepsilon_{t-r} \hat{F}_\varepsilon(\hat{s}^\varepsilon_r))(x) dr + \int_0^t (S^\varepsilon_{t-r} \tilde{F}_\varepsilon(X^\varepsilon, \hat{X}^\varepsilon, u^\varepsilon)_r)(x) dr \\ + \int_0^t (S^\varepsilon_{t-r} F_\varepsilon(X^\varepsilon, Z^\varepsilon, u^\varepsilon)_r)(x) dr, \end{aligned} \tag{5.17}$$

This equation should be understood in the following way: we have defined two auxiliary processes, X^ε and Z^ε. Fixing them, we solve (5.17) for u^ε, and after that we recover the solution (5.12) for the initial problem (5.9). Adding these two auxiliary processes into the equation allows to obtain a better control on the non-linear term in (5.9). More precisely, for η as in Theorem 1.6, we fix any $\kappa_\star \in (0, \frac{1}{2})$ and $\hat{\kappa} \in (0, \frac{1}{2} - \kappa_\star)$, whose precise values will be specified in the proof of Theorem 1.6. Then, for any $T > 0$, we expect the following bounds to hold:

$$\|X^\varepsilon\|_{\mathcal{C}_T^\eta} \leq L, \qquad \|\hat{X}^\varepsilon\|_{\mathcal{C}_T^\eta} \leq L, \qquad \|u^\varepsilon\|_{\mathcal{C}_T^{2\eta}} \leq L, \qquad \|Z^\varepsilon\|_{\mathcal{C}_{T,\varepsilon}^{-1/2+\kappa_\star}} \leq \varepsilon^{-1/2-\kappa_\star-\hat{\kappa}} L, \tag{5.18}$$

where the constant $L > 0$ is independent of ε and T, and where the norms are defined in Section 1.3. Moreover, we expect the processes X^ε, $\hat{X}^\varepsilon$ and u^ε to converge in these topologies, as $\varepsilon \to 0$. Since we expect the regularity of $\hat{X}^\varepsilon$ be close to $\frac{1}{2}$, the definition (5.13) suggests that the regularity of Z^ε is close to -1. However, the process Z^ε is a discretization of a space-time distribution, i.e. the limit as $\varepsilon \to 0$ is not a function in the time variable. This explains why the sup-norm of Z^ε in the time variable is expected to explode in the limit. It will be advantageous to measure regularity of Z^ε close to $-\frac{1}{2}$, which makes the divergence in (5.18) stronger. Since the term Z^ε is multiplied by ε in (5.17), this divergence will be compensated (see the proof of Lemma 5.9).

In order to derive the claimed bounds and prove limits, we will follow the usual strategy: we first derive the respective result for a local in time solution, and then patch local solutions together to get the required result for the global solution. To this end, for any constant $L > 0$ we define the stopping time

$$\sigma_{L,\varepsilon} := \inf\bigl\{T \geq 0 : \|X^\varepsilon\|_{\mathcal{C}_T^\eta} \geq L \text{ or } \|u^\varepsilon\|_{\mathcal{C}_T^{2\eta}} \geq L, \text{ or } \|Z^\varepsilon\|_{\mathcal{C}_{T,\varepsilon}^{-1/2+\kappa_\star}} \geq \varepsilon^{-1/2-\kappa_\star-\hat{\kappa}} L\bigr\}, \tag{5.19}$$

which guarantees that on the random time interval $[0, \sigma_{L,\varepsilon}]$ the a priori bounds (5.18) hold.

In the following exposition we prefer to use '$\lesssim$', which means a bound '$\leq$' with a constant multiplier, independent of the relevant quantities. By these relevant quantities we will usually mean ε and the space-time variables.

5.2.2. Bounds on the auxiliary processes In this section we prove bounds on the auxiliary processes $\hat{X}^\varepsilon$, X^ε and Z^ε, defined in (5.10) and (5.13) respectively. We start by bounding the processes $\hat{X}^\varepsilon$ and X^ε.

Lemma 5.2. *In the setting of Theorem 1.6, for every $p \geq 1$ and $T > 0$ there is a constant $C = C(p) > 0$, such that the processes $\hat{X}^\varepsilon$ and X^ε, defined in (5.10), are bounded in the following way:*

$$\bigl(\mathbb{E}\bigl[\|\hat{X}^\varepsilon\|_{\mathcal{C}_T^\eta}^p\bigr]\bigr)^{\frac{1}{p}} \leq C, \qquad \bigl(\mathbb{E}\bigl[\|X^\varepsilon\|_{\mathcal{C}_T^\eta}^p\bigr]\bigr)^{\frac{1}{p}} \leq C + C\bigl(\mathbb{E}\bigl[\|s_0^\varepsilon\|_{\mathcal{C}^\eta}^p\bigr]\bigr)^{\frac{1}{p}}. \tag{5.20}$$

Proof. We note that the second bound in (5.20) follows from the first one and the properties of the heat semigroup provided in Lemma A.1. Indeed, from the triangle inequality we get

$$\|X^\varepsilon\|_{\mathcal{C}_T^\eta} \leq \|\hat{X}^\varepsilon\|_{\mathcal{C}_T^\eta} + \|S^\varepsilon \hat{s}_0^\varepsilon\|_{\mathcal{C}_T^\eta} \leq C(1 + \|s_0^\varepsilon\|_{\mathcal{C}^\eta}),$$

and the claim follows from the Minkowski inequality.

We now prove the first bound in (5.20). To this end, we will apply the Burkholder–Davis–Gundy inequality (Lemma B.2) to the martingale $\tau \mapsto \hat{X}^\varepsilon_\tau(t,x)$, for which we need to bound the quadratic covariation and jumps of the latter. Hence, the definition (5.10) and the formula for the quadratic variation of M^ε, provided above (5.6), yield

$$\begin{aligned}\langle \hat{X}^\varepsilon_\bullet(t,x), \hat{X}^\varepsilon_\bullet(t,x)\rangle_\tau &= \varepsilon^2 \sum_{y_1,y_2\in\varepsilon\mathbb{Z}} \int_{r=0}^{\tau} p^\varepsilon_{t-r}(x-y_1)p^\varepsilon_{t-r}(x-y_2)d\langle M^\varepsilon(y_1), M^\varepsilon(y_2)\rangle_r \\ &= \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^\tau p^\varepsilon_{t-r}(x-y)^2 C_\varepsilon(\hat{s}^\varepsilon_r, y)dr,\end{aligned}$$

where p^ε is the discrete heat kernel, generated by Δ_ε. Moreover, (5.6), Lemma 5.1 and the identity $|\nabla_\varepsilon \hat{s}^\varepsilon| = \varepsilon^{-1/2}$ imply that $C_\varepsilon(\hat{s}^\varepsilon_r, y)$ is bounded uniformly in ε, which yields

$$\langle \hat{X}^\varepsilon_\bullet(t,x), \hat{X}^\varepsilon_\bullet(t,x)\rangle_\tau \lesssim \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^\tau p^\varepsilon_{t-r}(x-y)^2 dr. \tag{5.21}$$

Now we turn to jumps of the martingales $\hat{X}^\varepsilon_\tau(t,x)$. The definition (5.10) yields

$$\Delta_r \hat{X}^\varepsilon_\bullet(t,x) = \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} p^\varepsilon_{t-r}(x-y)\Delta_r M^\varepsilon(y),$$

where $\Delta_r M^\varepsilon(y) := M^\varepsilon_r(y) - M^\varepsilon_{r-}(y)$ is the jump of $M^\varepsilon(y)$ at time r (and likewise for $\hat{X}^\varepsilon$). Since the martingales $M^\varepsilon_r(y)$ have jumps of size $\sqrt{\varepsilon}$, and $M^\varepsilon_r(y)$ and $M^\varepsilon_r(y')$, with $y \neq y'$, a.s. do not jump simultaneously, we obtain the simple bound

$$|\Delta_r \hat{X}^\varepsilon_\bullet(t,x)| \leq \varepsilon^{3/2} \sup_{y\in\varepsilon\mathbb{Z}} p^\varepsilon_{t-r}(x-y). \tag{5.22}$$

Applying now the Burkholder–Davis–Gundy inequality (Lemma B.2) to the martingales $\tau \mapsto \hat{X}^\varepsilon_\tau(t,x)$, and using (5.21) and (5.22), we obtain

$$\Big(\mathbb{E}|\hat{X}^\varepsilon(t,x)|^p\Big)^{\frac{2}{p}} \lesssim \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t p^\varepsilon_{t-r}(x-y)^2 dr + \varepsilon^3 \sup_{r\in[0,t]} \sup_{y\in\varepsilon\mathbb{Z}} p^\varepsilon_{t-r}(x-y)^2. \tag{5.23}$$

Using the first bound in (A.3a), the last term in the expression (5.23) can be bounded by $c\varepsilon$, where c is independent of ε, x and t. To bound the first term in (5.23), we apply the first estimate in (A.3a) and the fact that the heat kernel sums up to 1 in the spatial variable:

$$\varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t p^\varepsilon_{t-r}(x-y)^2 dr \lesssim \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t \frac{p^\varepsilon_{t-r}(x-y)}{\sqrt{t-r}} dr = \int_0^t \frac{dr}{\sqrt{t-r}} = 2\sqrt{t}.$$

Hence, we have the following bound, where the constant C depends on p and t:

$$\Big(\mathbb{E}|\hat{X}^\varepsilon(t,x)|^p\Big)^{\frac{1}{p}} \leq C. \tag{5.24}$$

Similarly, for two different points x_1 and x_2, the process $\tau \mapsto \hat{X}^\varepsilon_\tau(t, x_1) - \hat{X}^\varepsilon_\tau(t, x_2)$ is a martingale. Applying again the Burkholder–Davis–Gundy inequality (Lemma B.2), similarly to (5.23) we obtain

$$\Big(\mathbb{E}|\hat{X}^\varepsilon(t, x_1) - \hat{X}^\varepsilon(t, x_2)|^p\Big)^{\frac{2}{p}} \lesssim \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t \big|p^\varepsilon_{t-r}(x_1 - y) - p^\varepsilon_{t-r}(x_2 - y)\big|^2 dr \\ + \varepsilon^3 \sup_{r\in[0,t]} \sup_{y\in\varepsilon\mathbb{Z}} \big|p^\varepsilon_{t-r}(x_1 - y) - p^\varepsilon_{t-r}(x_2 - y)\big|^2. \tag{5.25}$$

Using (A.3b), we can bound the last term in (5.25) by a multiple of $\varepsilon^{1-2(\eta+\kappa)}|x_1 - x_2|^{2(\eta+\kappa)}$, for any $\kappa > 0$ sufficiently small. Since $\eta < \frac{1}{2}$, for $\kappa > 0$ small enough we have $1 - 2(\eta + \kappa) \geq 0$, and the last power of ε can be bounded by 1. For the first term in (5.25) we use the bound (A.3b) to estimate it by a multiple of

$$\varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t \big|p^\varepsilon_{t-r}(x_1 - y) - p^\varepsilon_{t-r}(x_2 - y)\big| \frac{|x_1 - x_2|^{2\eta+\kappa}}{(t-r)^{(1+2\eta+\kappa)/2}} dr \\ \lesssim \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t \big(p^\varepsilon_{t-r}(x_1 - y) + p^\varepsilon_{t-r}(x_2 - y)\big) \frac{|x_1 - x_2|^{2\eta+\kappa}}{(t-r)^{(1+2\eta+\kappa)/2}} dr \\ \lesssim \int_0^t \frac{|x_1 - x_2|^{2\eta+\kappa}}{(t-r)^{(1+2\eta+\kappa)/2}} dr \lesssim |x_1 - x_2|^{2\eta+\kappa},$$

which holds for $\eta > 0$ and $\kappa > 0$, such that $2\eta + \kappa < 1$. Here, we have used the fact that the heat kernel sums up to 1 in the spatial variable. Hence, we have the following bound, with a constant $C > 0$ depending on p and t:

$$\Big(\mathbb{E}|\hat{X}^\varepsilon(t, x_1) - \hat{X}^\varepsilon(t, x_2)|^p\Big)^{\frac{1}{p}} \leq C|x_1 - x_2|^{\eta+\kappa}. \tag{5.26}$$

Now, we turn to the proof of time continuity. For two time points $0 \leq t_1 < t_2$, such that $t_2 - t_2 \leq 1$, we can write

$$\hat{X}^\varepsilon(t_2, x) - \hat{X}^\varepsilon(t_1, x) = \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^{t_1} \big(p^\varepsilon_{t_2-r}(x - y) - p^\varepsilon_{t_1-r}(x - y)\big) dM^\varepsilon_r(y) \\ + \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_{t_1}^{t_2} p^\varepsilon_{t_2-r}(x - y) dM^\varepsilon_r(y) =: \mathcal{I}^\varepsilon_1(t_1) + \mathcal{I}^\varepsilon_2(t_2),$$

where the variables t_1 and t_2 in $\mathcal{I}^\varepsilon_1$ and $\mathcal{I}^\varepsilon_2$ refer to the upper bounds of the intervals of integration. The processes $\tau_1 \mapsto \mathcal{I}^\varepsilon_1(\varepsilon^{-2}1)$ and $\tau_2 \mapsto \mathcal{I}^\varepsilon_2(\tau_2)$ are martingales, for $\tau_1 \in [0, t_1]$ and $\tau_2 \in [t_1, t_2]$, and we are going to bound them separately.

Applying the Burkholder–Davis–Gundy inequality (Lemmas B.2) to $\mathcal{I}^\varepsilon_1$, similarly to (5.23)

$$\left(\mathbb{E}|\mathcal{I}_1^\varepsilon(t_1)|^p\right)^{\frac{2}{p}} \lesssim \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^{t_1} \left|p^\varepsilon_{t_1-r}(x-y) - p^\varepsilon_{t_2-r}(x-y)\right|^2 dr \\ + \varepsilon^3 \sup_{r\in[0,t_1]} \sup_{y\in\varepsilon\mathbb{Z}} \left|p^\varepsilon_{t_1-r}(x-y) - p^\varepsilon_{t_2-r}(x-y)\right|^2. \tag{5.27}$$

Using (A.3c), the last term in (5.27) is bounded by $\varepsilon^{1-2(\eta+\kappa)}(t_2-t_1)^{\eta+\kappa}$, where the power of ε is positive if $\eta < \frac{1}{2}$ and $\kappa > 0$ is sufficiently small. Furthermore, using (A.3c), the first term in (5.27) is bounded by a multiple of

$$\varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^{t_1} \left|p^\varepsilon_{t_1-r}(x-y) - p^\varepsilon_{t_2-r}(x-y)\right| \frac{(t_2-t_1)^{\eta+\kappa}}{(t_1-r)^{(1+2\eta+2\kappa)/2}} dr \\ \lesssim \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^{t_1} \left(p^\varepsilon_{t_1-r}(x-y) + p^\varepsilon_{t_2-r}(x-y)\right) \frac{(t_2-t_1)^{\eta+\kappa}}{(t_1-r)^{(1+2\eta+2\kappa)/2}} dr \\ \lesssim \int_0^{t_1} \frac{(t_2-t_1)^{\eta+\kappa}}{(t_1-r)^{(1+2\eta+2\kappa)/2}} dr \lesssim (t_2-t_1)^{\eta+\kappa},$$

for $\eta < \frac{1}{2}$ and $\kappa > 0$ sufficiently small. Here, as before we summed up the heat kernels to 1.

Similarly, we apply the Burkholder–Davis–Gundy inequality (Lemmas B.2) to $\mathcal{I}_2^\varepsilon$ and obtain

$$\left(\mathbb{E}|\mathcal{I}_2^\varepsilon(t_2)|^p\right)^{\frac{2}{p}} \lesssim \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_{t_1}^{t_2} p^\varepsilon_{t_2-r}(x-y)^2 dr + \varepsilon^3 \sup_{r\in[t_1,t_2]} \sup_{y\in\varepsilon\mathbb{Z}} p^\varepsilon_{t_2-r}(x-y)^2. \tag{5.28}$$

The last term in (5.28) can be bounded using (A.3a) by a multiple of ε. Applying (A.3a) to the first term in (5.28) and using the fact that the heat kernel is summed up to 1 in the spatial variable, we can bound it by a multiple of

$$\varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_{t_1}^{t_2} \frac{p^\varepsilon_{t_2-r}(x-y)}{\sqrt{t_2-r}} dr = \int_{t_1}^{t_2} \frac{dr}{\sqrt{t_2-r}} = 2\sqrt{t_2-t_1} \leq 2(t_2-t_1)^{\eta+\kappa},$$

where $\eta+\kappa \leq \frac{1}{2}$ and where we have used $t_2-t_1 \leq 1$.

Combining the derived bounds on $\mathcal{I}_1^\varepsilon$ and $\mathcal{I}_2^\varepsilon$ with the Minkowski inequality, we obtain

$$\left(\mathbb{E}|\hat{X}^\varepsilon(t_2,x) - \hat{X}^\varepsilon(t_1,x)|^p\right)^{\frac{1}{p}} \leq \left(\mathbb{E}|\mathcal{I}_1^\varepsilon(t_1)|^p\right)^{\frac{1}{p}} + \left(\mathbb{E}|\mathcal{I}_2^\varepsilon(t_2)|^p\right)^{\frac{1}{p}} \\ \leq C\left(|t_1-t_2|^{\eta+\kappa} + \varepsilon\right) \leq 2C(\sqrt{|t_2-t_1|} \vee \varepsilon)^{2(\eta+\kappa)}. \tag{5.29}$$

The required bound (5.20) now follows from (5.24), (5.26), (5.29) and the Kolmogorov continuity criterion [Kal02]. □

Furthermore, we can derive a bound on the process Z^ε in a certain space of distributions.

Lemma 5.3. *For any $\kappa_\star \in (0, \frac{1}{2})$, $\kappa > 0$ and $p \geq 1$ there is a constant $C = C(\kappa_\star, \kappa, p) > 0$ such that for any $T > 0$ the process Z^ε, defined in (5.13), satisfies the bound*

$$\big(\mathbb{E}\|Z^\varepsilon\|^p_{\mathcal{C}^{-1/2+\kappa_\star}_{T,\varepsilon}}\big)^{\frac{1}{p}} \leq C\varepsilon^{-1/2-\kappa_\star-\kappa}, \tag{5.30}$$

where the norm is defined in Section 1.3.

Proof. The process Z^ε can be written as $Z^\varepsilon_t(x) = \hat{Z}^\varepsilon_t(t,x)$, where

$$\hat{Z}^\varepsilon_\tau(t,x) := \nabla^-_\varepsilon \hat{X}^\varepsilon_\tau(t,x)\nabla^+_\varepsilon \hat{X}^\varepsilon_\tau(t,x) - \mathfrak{C}^\varepsilon_\tau(t,x),$$

and $\mathfrak{C}^\varepsilon$ has been defined in (5.14). The definition of the predictable quadratic covariation [JS03, Ch. I.4] implies that $\tau \mapsto \hat{Z}^\varepsilon_\tau(t,x)$ is a martingale, for $\tau \in [0,t]$. Moreover, we can use the Itô formula to write

$$\hat{Z}^\varepsilon_\tau = \int_0^\tau \nabla^-_\varepsilon \hat{X}^\varepsilon_{r-} d\, \nabla^+_\varepsilon \hat{X}^\varepsilon_r + \int_0^\tau \nabla^+_\varepsilon \hat{X}^\varepsilon_{r-} d\, \nabla^-_\varepsilon \hat{X}^\varepsilon_r + D^\varepsilon_\tau, \tag{5.31}$$

where the process $D^\varepsilon_\tau(t,x)$ is a martingale for $\tau \in [0,t]$, defined by

$$D^\varepsilon_\tau(t,x) := \big[\nabla^-_\varepsilon \hat{X}^\varepsilon_\bullet(t,x), \nabla^+_\varepsilon \hat{X}^\varepsilon_\bullet(t,x)\big]_\tau - \langle \nabla^-_\varepsilon \hat{X}^\varepsilon_\bullet(t,x), \nabla^+_\varepsilon \hat{X}^\varepsilon_\bullet(t,x)\rangle_\tau.$$

Here, as in Section 1.3, $[\bullet, \bullet]$ refers to the quadratic covariation [JS03, Thm. I.4.52]. We will prove (5.30) by bounding each term in (5.31) separately.

We start with analysis of the first integral in (5.31), which we denote by $\mathcal{Y}^\varepsilon_\tau(t,x)$. For a rescaled test function φ^λ_z, as in (1.11), we use the notation of Appendix B and write $\langle\!\langle \mathcal{Y}^\varepsilon_\tau(t), \varphi^\lambda_z\rangle\!\rangle_\varepsilon = \mathfrak{I}^\varepsilon_2 F^\varepsilon_\lambda(\tau)$, where $\mathfrak{I}^\varepsilon_2$ refers to the second order stochastic integral with the kernel

$$F^\varepsilon_\lambda(r_1, r_2; y_1, y_2) := \varepsilon \sum_{x \in \varepsilon\mathbb{Z}} \varphi^\lambda_z(x) \nabla^-_\varepsilon p^\varepsilon_{t-r_1}(x-y_1)\nabla^+_\varepsilon p^\varepsilon_{t-r_2}(x-y_2). \tag{5.32}$$

Applying Lemma B.3, we obtain the moment bound

$$\bigg(\mathbb{E}\Big[\sup_{0\leq\tau\leq t} \big|\mathfrak{I}^\varepsilon_2 F^\varepsilon_\lambda(\tau)\big|^p\Big]\bigg)^{\frac{2}{p}} \lesssim \mathscr{E}^{(1)}_{\varepsilon,\lambda} + \mathscr{E}^{(2)}_{\varepsilon,\lambda} + \mathscr{E}^{(3)}_{\varepsilon,\lambda}, \tag{5.33}$$

where the terms on the r.h.s. are given by

$$\mathscr{E}^{(1)}_{\varepsilon,\lambda} := \varepsilon^2 \sum_{y_1, y_2 \in \varepsilon\mathbb{Z}} \int_0^t \int_0^{r_1} F^\varepsilon_\lambda(r_1, r_2; y_1, y_2)^2 \, dr_2 \, dr_1, \tag{5.34a}$$

$$\mathscr{E}^{(2)}_{\varepsilon,\lambda} := \varepsilon^4 \sum_{y_2 \in \varepsilon\mathbb{Z}} \int_0^t \bigg(\sup_{\substack{r_1 \in [0,r_2] \\ y_1 \in \varepsilon\mathbb{Z}}} \big|F^\varepsilon_\lambda(r_1, r_2; y_1, y_2)\big|^p \bigg)^{\frac{2}{p}} dr_2, \tag{5.34b}$$

$$\mathscr{E}^{(3)}_{\varepsilon,\lambda} := \varepsilon^3 \bigg(\mathbb{E}\Big[\sup_{\substack{r_2 \in [0,t] \\ y_2 \in \varepsilon\mathbb{Z}}} \big|(\mathfrak{I}^\varepsilon_1 F^\varepsilon_\lambda)(r_2; y_2)\big|^p\Big]\bigg)^{\frac{2}{p}}, \tag{5.34c}$$

because the jump sizes of all martingales are $\sqrt{\varepsilon}$ and their predictable quadratic covariation (5.6) is bounded.

Our aim is to bound the three terms appearing on the r.h.s. of (5.33). For this, we use the results provided in Appendix C, and derive the required bounds from measuring "strength" of singularities of the involved kernels. We start with estimating the first term (5.34a). We may write this term as

$$\mathscr{E}^{(1)}_{\varepsilon,\lambda} := \varepsilon^2 \sum_{x_1, x_2 \in \varepsilon\mathbb{Z}} \varphi^\lambda_z(x_1)\varphi^\lambda_z(x_2)\mathcal{K}^{(\varepsilon,1)}_t(x_2 - x_1),$$

where the kernel $\mathcal{K}^{(\varepsilon,1)}_t$ is given by

$$\begin{aligned}\mathcal{K}^{(\varepsilon,1)}_t(x_2 - x_1) :=& \varepsilon^2 \sum_{y_1, y_2 \in \varepsilon\mathbb{Z}} \int_{r_1=0}^{t}\int_{r_2=0}^{r_1} \nabla^-_\varepsilon p^\varepsilon_{t-r_1}(x_1 - y_1)\nabla^+_\varepsilon p^\varepsilon_{t-r_2}(x_1 - y_2)\\ &\times \nabla^-_\varepsilon p^\varepsilon_{t-r_1}(x_2 - y_1)\nabla^+_\varepsilon p^\varepsilon_{t-r_2}(x_2 - y_2)\, dr_2\, dr_1.\end{aligned}$$

From Lemmas C.2 and C.1 we obtain $\mathcal{K}^{(\varepsilon,1)} \in \mathcal{S}^2_\varepsilon$, where we use the notation of Appendix C. We cannot apply Lemma C.3 directly, because the strength of singularity is too high. To overcome this difficulty, we simply notice that $\varepsilon^{1+\zeta}\mathcal{K}^{(\varepsilon,1)} \in \mathcal{S}^{1-\zeta}_\varepsilon$, for any $\zeta > 0$. Taking $\zeta = 2(\kappa_\star + \kappa)$, Lemma C.3 yields

$$|\mathscr{E}^{(1)}_{\varepsilon,\lambda}| \lesssim \varepsilon^{-1-2(\kappa_\star+\kappa)}\lambda^{-1+2(\kappa_\star+\kappa)}, \tag{5.35}$$

for $\lambda \geq \varepsilon$, and for any $\kappa_\star > 0$ and $\kappa > 0$.

Now, we will bound the second term (5.34b). From the definition (5.32) we obtain

$$\mathscr{E}^{(2)}_{\varepsilon,\lambda} \leq \varepsilon^4 \sum_{y_2 \in \varepsilon\mathbb{Z}} \int_{r_2=0}^{t} \Bigg(\sup_{\substack{r_1 \in [0, r_2]\\ y_1 \in \varepsilon\mathbb{Z}}} |\nabla^-_\varepsilon p^\varepsilon_{t-r_1}(y_1)| \Bigg)^2 \Bigg(\varepsilon \sum_{x \in \varepsilon\mathbb{Z}} \varphi^\lambda_z(x)\nabla^+_\varepsilon p^\varepsilon_{t-r_2}(x - y_2) \Bigg)^2 dr_2.$$

Using (A.3a), we can simply bound $|\nabla^-_\varepsilon p^\varepsilon_r(y)| \lesssim \varepsilon^{-2}$, which yields

$$\begin{aligned}\mathscr{E}^{(2)}_{\varepsilon,\lambda} &\lesssim \sum_{y \in \varepsilon\mathbb{Z}} \int_0^t \Bigg(\varepsilon \sum_{x \in \varepsilon\mathbb{Z}} |\varphi^\lambda_z(x)||\nabla^+_\varepsilon p^\varepsilon_{t-r}(x - y)| \Bigg)^2 dr\\ &\leq \varepsilon^2 \sum_{x_1, x_2 \in \varepsilon\mathbb{Z}} |\varphi^\lambda_z(x_1)||\varphi^\lambda_z(x_2)|\mathcal{K}^{(\varepsilon,2)}_t(x_2 - x_1),\end{aligned}$$

with the kernel $\mathcal{K}^{(\varepsilon,2)}$, given by

$$\mathcal{K}^{(\varepsilon,2)}_t(x_2 - x_1) := \sum_{y \in \varepsilon\mathbb{Z}} \int_0^t |\nabla^+_\varepsilon p^\varepsilon_{t-r}(x_1 - y)\nabla^+_\varepsilon p^\varepsilon_{t-r}(x_2 - y)|\, dr. \tag{5.36}$$

Lemmas C.2 and C.1 yield $\varepsilon\mathcal{K}^{(\varepsilon,2)} \in \mathcal{S}^1_\varepsilon$. Using the same trick as in (5.35) to "improve" singularity, Lemma C.3 yields

$$\mathscr{E}^{(2)}_{\varepsilon,\lambda} \lesssim \varepsilon^{-1-2(\kappa_\star+\kappa)}\lambda^{-1+2(\kappa_\star+\kappa)}, \tag{5.37}$$

for $\lambda \geq \varepsilon$, and for any $\kappa_\star > 0$ and $\kappa > 0$.

In order to bound the last term (5.34c), we use the definition (5.32) and we bound as before $|\nabla_\varepsilon^+ p_r^\varepsilon(y)| \lesssim \varepsilon^{-2}$, which yields

$$\big|(\mathfrak{I}_1^\varepsilon F_\lambda^\varepsilon)(r_2; y_2)\big| \lesssim \varepsilon^{-2}\big|(\mathfrak{I}_1^\varepsilon \hat{F}_\lambda^\varepsilon)(r_2; y_2)\big|,$$

with a new kernel $\hat{F}_\lambda^\varepsilon$, given by

$$\hat{F}_\lambda^\varepsilon(r; y) := \varepsilon \sum_{x\in\varepsilon\mathbb{Z}} |\varphi_z^\lambda(x)||\nabla_\varepsilon^- p_{t-r}^\varepsilon(x-y)|.$$

Applying this bound and then Lemma B.3, we obtain

$$\mathscr{E}_{\varepsilon,\lambda}^{(3)} \lesssim \varepsilon^{-1}\Big(\mathbb{E}\Big[\sup_{\substack{r_2\in[0,t]\\ y_2\in\varepsilon\mathbb{Z}}} \big|(\mathfrak{I}_1^\varepsilon \hat{F}_\lambda^\varepsilon)(r_2; y_2)\big|^p\Big]\Big)^{\frac{2}{p}} \lesssim \varepsilon^{-1}\big(\mathscr{E}_{\varepsilon,\lambda}^{(3,1)} + \mathscr{E}_{\varepsilon,\lambda}^{(3,2)}\big), \tag{5.38}$$

where the terms on the r.h.s. are given by

$$\mathscr{E}_{\varepsilon,\lambda}^{(3,1)} := \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t \hat{F}_\lambda^\varepsilon(r; y)^2\, dr, \qquad \mathscr{E}_{\varepsilon,\lambda}^{(3,2)} := \varepsilon^3 \sup_{\substack{r\in[0,t]\\ y\in\varepsilon\mathbb{Z}}} \big|\hat{F}_\lambda^\varepsilon(r; y)\big|^2. \tag{5.39}$$

The first term $\mathscr{E}_{\varepsilon,\lambda}^{(3,1)}$ we can write in the following way:

$$\mathscr{E}_{\varepsilon,\lambda}^{(3,1)} = \varepsilon^3 \sum_{x_1,x_2\in\varepsilon\mathbb{Z}} |\varphi_z^\lambda(x_1)||\varphi_z^\lambda(x_2)|\hat{\mathcal{K}}_t^\varepsilon(x_2-x_1),$$

where $\hat{\mathcal{K}}^\varepsilon$ is defined as $\mathcal{K}^{(\varepsilon,2)}$ in (5.36), but using the discrete derivative ∇_ε^- instead of ∇_ε^+. In particular, in the same way as in (5.37) we obtain $\mathscr{E}_{\varepsilon,\lambda}^{(3,1)} \lesssim \varepsilon^{-2(\kappa_\star+\kappa)}\lambda^{-1+2(\kappa_\star+\kappa)}$, which holds for any $\kappa_\star > 0$ and $\kappa > 0$.

In order to bound the term $\mathscr{E}_{\varepsilon,\lambda}^{(3,2)}$, we use Lemma C.2 and get $\varepsilon^{1+\zeta}\nabla_\varepsilon^- p^\varepsilon \in \mathcal{S}_\varepsilon^{1-\zeta}$, for any $\zeta > 0$. Using this fact, one can show that $\big|\hat{F}_\lambda^\varepsilon(r; y)\big| \lesssim \varepsilon^{-1-\zeta}\lambda^{-1+\zeta}$, for $\lambda \geq \varepsilon$. Taking $\zeta = 1/2 + \kappa_\star + \kappa$, from (5.39) we obtain $\mathscr{E}_{\varepsilon,\lambda}^{(3,2)} \lesssim \varepsilon^{-2(\kappa_\star+\kappa)}\lambda^{-1+2(\kappa_\star+\kappa)}$. From (5.38) and the above derived bounds we conclude

$$\mathscr{E}_{\varepsilon,\lambda}^{(3)} \lesssim \varepsilon^{-1-2(\kappa_\star+\kappa)}\lambda^{-1+2(\kappa_\star+\kappa)}. \tag{5.40}$$

Combining (5.33) with the bounds (5.35), (5.37) and (5.40), we obtain

$$\Big(\mathbb{E}\big|\langle\langle \mathcal{Y}_t^\varepsilon(t), \varphi_z^\lambda\rangle\rangle_\varepsilon\big|^p\Big)^{\frac{2}{p}} \leq \Big(\mathbb{E}\sup_{0\leq\tau\leq t}\big|\langle\langle \mathcal{Y}_\tau^\varepsilon(t), \varphi_z^\lambda\rangle\rangle_\varepsilon\big|^p\Big)^{\frac{2}{p}} \lesssim \varepsilon^{-1-2(\kappa_\star+\kappa)}\lambda^{-1+2(\kappa_\star+\kappa)},$$

which holds for any $\kappa_\star > 0$ and $\kappa > 0$. In a similar way we can prove the bound

$$\Big(\mathbb{E}\big|\langle\langle \mathcal{Y}_t^\varepsilon(t) - \mathcal{Y}_s^\varepsilon(s), \varphi_z^\lambda\rangle\rangle_\varepsilon\big|^p\Big)^{\frac{2}{p}} \lesssim |t-s|^{\kappa/2}\varepsilon^{-1-2(\kappa_\star+\kappa)}\lambda^{-1+2\kappa_\star+\kappa},$$

and by the Kolmogorov continuity criterion we conclude that the process $(t,x) \mapsto \mathcal{Y}_t^\varepsilon(t,x)$ satisfies the bound (5.30). Obviously, the same result holds true for the second term in (5.31).

Now, we turn to the martingale $D^\varepsilon_\tau(t,x)$ in (5.31). As before, we take a rescaled test function φ^λ_z and using the notation of Appendix B we write $\langle\!\langle D^\varepsilon_\tau(t), \varphi^\lambda_z\rangle\!\rangle_\varepsilon = (\mathfrak{I}^\varepsilon_1 \tilde F^\varepsilon_\lambda)(\tau)$, where the stochastic integral is with respect to the martingale $N^\varepsilon_t(x) := [M^\varepsilon(x)]_t - \langle M^\varepsilon(x)\rangle_t$ and where the kernel $\tilde F^\varepsilon_\lambda(r; y) := \varepsilon F^\varepsilon_\lambda(r, r; y, y)$ is defined using (5.32).

Since the martingale $M^\varepsilon_t(x)$ has bounded total variation, the jump size of $N^\varepsilon_t(x)$ at time t, if it happens, is equal to $\Delta_t N^\varepsilon(x) = (\Delta_t M^\varepsilon(x))^2 = \varepsilon$, because $M^\varepsilon(x)$ has jumps of size $\sqrt{\varepsilon}$ (see [JS03, Thm. I.4.52]). Moreover, the predictable quadratic covariation of $N^\varepsilon_t(x)$ is proportional to ε. Then Lemma B.3 yields the bound

$$\left(\mathbb{E}\Big[\sup_{0\le\tau\le t} \big|(\mathfrak{I}^\varepsilon_1 \tilde F^\varepsilon_\lambda)(\tau)\big|^p\Big]\right)^{\frac{2}{p}} \lesssim \varepsilon^2 \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t \tilde F^\varepsilon_\lambda(r; y)^2 dr + \varepsilon^4 \sup_{\substack{r\in[0,t]\\ y\in\varepsilon\mathbb{Z}}} \big|\tilde F^\varepsilon_\lambda(r; y)\big|^2$$
$$=: \tilde{\mathscr{E}}^{(1)}_{\varepsilon,\lambda} + \tilde{\mathscr{E}}^{(2)}_{\varepsilon,\lambda}. \tag{5.41}$$

We can write the first term in the following way:

$$\tilde{\mathscr{E}}^{(1)}_{\varepsilon,\lambda} := \varepsilon^4 \sum_{x_1,x_2\in\varepsilon\mathbb{Z}} \varphi^\lambda_z(x_1)\varphi^\lambda_z(x_2)\tilde{\mathcal{K}}^\varepsilon_t(x_2 - x_1),$$

where the kernel $\tilde{\mathcal{K}}^\varepsilon_t$ is given by

$$\tilde{\mathcal{K}}^\varepsilon_t(x_2 - x_1) := \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t \nabla^-_\varepsilon p^\varepsilon_{t-r}(x_1 - y)\nabla^+_\varepsilon p^\varepsilon_{t-r}$$
$$(x_1 - y)\nabla^-_\varepsilon p^\varepsilon_{t-r}(x_2 - y)\nabla^+_\varepsilon p^\varepsilon_{t-r}(x_2 - y)\, dr.$$

In the same way as we did above, we can "improve" the singularity by multiplying the kernel by a positive power of ε. Then Lemmas C.2 and C.1 yield $\varepsilon^{2(1+\zeta)}\tilde{\mathcal{K}}^\varepsilon \in \mathcal{S}^{3-2\zeta}_\varepsilon$, for any $\zeta > 0$. Applying Lemma C.3 with $\zeta = 1 + \kappa_\star + \kappa$, we obtain

$$|\tilde{\mathscr{E}}^{(1)}_{\varepsilon,\lambda}| \lesssim \varepsilon^{-1-2(\kappa_\star+\kappa)}\lambda^{-1+2(\kappa_\star+\kappa)}. \tag{5.42}$$

Now, we turn to the second term in (5.41). One can show that for any $\zeta > 0$ one has $\varepsilon^{2+\zeta}|\tilde F^\varepsilon_\lambda(r; y)| \lesssim \lambda^{-1+\zeta}$, where $\lambda \ge \varepsilon$. Hence, taking $\zeta = 2(\kappa_\star + \kappa)$, we obtain

$$\tilde{\mathscr{E}}^{(2)}_{\varepsilon,\lambda} \lesssim \varepsilon^{1-2(\kappa_\star+\kappa)}\lambda^{-1+2(\kappa_\star+\kappa)}. \tag{5.43}$$

Combining (5.41) with the bounds (5.42) and (5.43), we obtain

$$\left(\mathbb{E}\Big[\big|\langle\!\langle D^\varepsilon_t(t), \varphi^\lambda_z\rangle\!\rangle_\varepsilon\big|^p\Big]\right)^{\frac{2}{p}} \lesssim \left(\mathbb{E}\Big[\sup_{0\le\tau\le t} \big|\mathfrak{I}^\varepsilon_1 \tilde F^\varepsilon_\lambda(\tau)\big|^p\Big]\right)^{\frac{2}{p}} \lesssim \varepsilon^{-1-2(\kappa_\star+\kappa)}\lambda^{-1+2(\kappa_\star+\kappa)}.$$

In a similar way we can prove the following bound

$$\left(\mathbb{E}\Big[\big|\langle\!\langle D^\varepsilon_t(t) - D^\varepsilon_s(s), \varphi^\lambda_z\rangle\!\rangle_\varepsilon\big|^p\Big]\right)^{\frac{2}{p}} \lesssim |t-s|^{\kappa/2}\varepsilon^{-1-2(\kappa_\star+\kappa)}\lambda^{-1+2\kappa_\star+\kappa}.$$

The Kolmogorov continuity criterion implies that the process $(t, x) \mapsto D^\varepsilon_t(t, x)$ satisfies the bound (5.30), and this finishes the proof. □

5.2.3. Convergence of the processes X^ε We can prove convergence of the stopped processes $X^\varepsilon_{t\wedge\sigma_{L,\varepsilon}}$, where the stopping time $\sigma_{L,\varepsilon}$ is defined in (5.19).

Proposition 5.4. *Under the assumptions of Theorem 1.6, let us extend the processes* $t \mapsto X^\varepsilon_t$*, defined in (5.10), piece-wise linearly in the spatial variable to* $\mathbb{R}$*. Let furthermore* X *be the 1-periodic solution of (1.2) with* $A = 0$ *and* $B_T = 1$*, and with the initial state* $\mathcal{Z}_0$*. Finally, let us define the limit* $\sigma_L := \lim_{\varepsilon\to 0} \sigma_{L,\varepsilon}$ *in probability. Then the process* $t \mapsto X^\varepsilon_{t\wedge\sigma_{L,\varepsilon}}$ *converges weakly in* $D([0,\infty), \mathcal{C}(\mathbb{R}))$ *to* $X_{t\wedge\sigma_L}$*, as* $\varepsilon \to 0$.

Tightness of the processes X^ε are proved in Lemma 5.2, and in order to identify their limit we need the following bound on the bracket process of the martingales M^ε.

Lemma 5.5. *Under the assumptions of Theorem 1.6, let the initial state satisfy the bound* $\|\hat{s}^\varepsilon_0\|_{\mathcal{C}^\eta} \le L$*, for a constant* $L > 0$*. Then the function* C_ε*, defined in (5.6), can be bounded:*

$$\sup_{0\le t\le\sigma_{L,\varepsilon}} \|C_\varepsilon(\hat{s}^\varepsilon_t) - 2\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon} \le C\varepsilon^{\frac{1}{2}-\kappa_\star-\hat{\kappa}} L + C\varepsilon^{3\eta-1} t^{-\eta/2} L^2, \tag{5.44}$$

where the stopping time $\sigma_{L,\varepsilon}$ *is defined in (5.19), using the values* $\kappa_\star$*,* $\hat{\kappa}$ *and* η*.*

Proof. Lemma 5.1 yields $\varrho_\varepsilon(\hat{s}^\varepsilon, x) = 1-\varepsilon/2+\tilde{\varrho}_\varepsilon(\hat{s}^\varepsilon, x)$, where $|\tilde{\varrho}_\varepsilon(\hat{s}^\varepsilon, x)| \lesssim \varepsilon^2$. Using the definition (5.6), we can write

$$\begin{aligned} C_\varepsilon(\hat{s}^\varepsilon_t) - 2 =& 2\big(\varrho_\varepsilon(\hat{s}^\varepsilon_t) - 1\big) + 2\varepsilon^3\lambda_\varepsilon(\hat{s}^\varepsilon_t)\Delta_\varepsilon\hat{s}^\varepsilon_t + \varepsilon\big(\varepsilon - 2\tilde{\varrho}_\varepsilon(\hat{s}^\varepsilon_t)\big)\nabla^-_\varepsilon \hat{s}^\varepsilon_t \, \nabla^+_\varepsilon\hat{s}^\varepsilon_t \\ &- 2\varepsilon\nabla^-_\varepsilon \hat{s}^\varepsilon_t \, \nabla^+_\varepsilon \hat{s}^\varepsilon_t, \end{aligned}$$

and we denote these four terms by $C_{\varepsilon,1}$, $C_{\varepsilon,2}$, $C_{\varepsilon,3}$ and $C_{\varepsilon,4}$ respectively. Lemma 5.1 yields a bound on the first term: $|C_{\varepsilon,1}(\hat{s}^\varepsilon_t, x)| \lesssim \varepsilon$. Furthermore, the bound $|\lambda_\varepsilon(\hat{s}^\varepsilon_t, x)| \lesssim \varepsilon^{-1/2}$ yields a bound on the second term: $|C_{\varepsilon,2}(\hat{s}^\varepsilon_t, x)| \lesssim \sqrt{\varepsilon}\|\hat{s}^\varepsilon_t\|_{L^\infty} \lesssim \sqrt{\varepsilon}L$, where we consider $t \in [0, \sigma_{L,\varepsilon}]$. The estimate $|\nabla_\varepsilon\hat{s}^\varepsilon_t| \le \varepsilon^{\eta-1}\|\hat{s}^\varepsilon_t\|_{\mathcal{C}^\eta}$ yields the bound $|C_{\varepsilon,3}(\hat{s}^\varepsilon_t, x)| \lesssim \varepsilon^{2\eta}\|\hat{s}^\varepsilon_t\|^2_{\mathcal{C}^\eta} \lesssim \varepsilon^{2\eta}L^2$, for $t \in [0, \sigma_{L,\varepsilon}]$.

Now, we turn to the most complicated term $C_{\varepsilon,4}$. Using (5.12), and replacing the product $\nabla^-_\varepsilon \hat{X}^\varepsilon \nabla^+_\varepsilon \hat{X}^\varepsilon$ by its renormalized version (5.13), we obtain

$$\nabla^-_\varepsilon \hat{s}^\varepsilon_t \nabla^+_\varepsilon \hat{s}^\varepsilon_t = Z^\varepsilon_t + \mathfrak{C}^\varepsilon_t + \Big(\big(\nabla^-_\varepsilon \hat{s}^\varepsilon \nabla^+_\varepsilon \hat{s}^\varepsilon\big)_t - \nabla^-_\varepsilon \hat{X}^\varepsilon_t \nabla^+_\varepsilon \hat{X}^\varepsilon_t\Big). \tag{5.45}$$

On the time interval $t \in [0, \sigma_{L,\varepsilon}]$, by (5.19) the $(-1/2+\kappa_\star)$-norm of the first term in (5.45) is bounded by $\varepsilon^{-\frac{1}{2}-\kappa_\star-\hat{\kappa}}L$. For the last term in (5.45), we use (5.12) and bound the absolute value of the last term in (5.45) by a constant times

$$\varepsilon^{3\eta-2}\big(\|\hat{X}^\varepsilon_t\|_{\mathcal{C}^\eta} + \|S^\varepsilon_t \hat{s}_0\|_{\mathcal{C}^\eta}\big)\big(\|S^\varepsilon_t\hat{s}_0\|_{\mathcal{C}^{2\eta}} + \|u^\varepsilon_t\|_{\mathcal{C}^{2\eta}}\big) + \varepsilon^{4\eta-2}\|u^\varepsilon_t\|^2_{\mathcal{C}^{2\eta}}.$$

Lemma A.1 yields $\|S^\varepsilon_t\hat{s}_0\|_{\mathcal{C}^\eta} \lesssim \|\hat{s}_0\|_{\mathcal{C}^\eta}$ and $\|S^\varepsilon_t\hat{s}_0\|_{\mathcal{C}^{2\eta}} \lesssim t^{(\eta-2\eta)/2}\|\hat{s}_0\|_{\mathcal{C}^\eta}$. Moreover, for $t \in [0, \sigma_{L,\varepsilon}]$ the norms of all stochastic processes and of $\hat{s}_0$ are bounded by L. Thus, the last term in (5.45) is bounded by a multiple of $\varepsilon^{3\eta-2}t^{(\eta-2\eta)/2}L^2$.

Now we will estimate the remaining term $\mathfrak{C}^\varepsilon_t$ in (5.45). Define $K^\varepsilon_t(x) := \nabla^-_\varepsilon p^\varepsilon_t(x)\nabla^+_\varepsilon p^\varepsilon_t(x)$ (see (A.5)) and use the definition (5.14) to write

$$\mathfrak{C}^\varepsilon_t(x) = \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t K^\varepsilon_{t-r}(x-y) C_\varepsilon(\hat{s}^\varepsilon_r, y)dr, \tag{5.46}$$

where we made use of the function C_ε in (5.6). Furthermore, rewrite

$$\mathfrak{C}_t^\varepsilon(x) = 2\varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t K_{t-r}^\varepsilon(x-y)dr + \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t K_{t-r}^\varepsilon(y)\big(C_\varepsilon(\hat{s}_r^\varepsilon, x-y) - 2\big)dr.$$

Applying (A.6), the first term can be bounded by a constant, uniformly in ε.

Let us denote $\hat{C}_t^\varepsilon := \|C_\varepsilon(\hat{s}^\varepsilon_{t\wedge\sigma_{L,\varepsilon}}) - 2\|_{\mathcal{C}_\varepsilon^{-1/2+\kappa_\star}}$. Then, combining all these bounds, we obtain

$$\hat{C}_t^\varepsilon \le \gamma_\varepsilon(t) + \varepsilon^2 \sum_{x\in\varepsilon\mathbb{Z}} \int_0^t |K_{t-r}^\varepsilon(x)|\hat{C}_r^\varepsilon dr, \tag{5.47}$$

for the function

$$\gamma_\varepsilon(t) := c\varepsilon + c\varepsilon^{\frac{1}{2}-\kappa_\star-\hat{\kappa}} L + c\sqrt{\varepsilon}L + c\varepsilon^{3\eta-1}t^{-\eta/2}L^2,$$

where the constant $c > 0$ is independent of ε, L and t. Denoting $\hat{C}^\varepsilon_{[0,t]} := \sup_{r\in[0,t]} \hat{C}_r^\varepsilon$, and using the first bound in (A.7), from (5.47) we get

$$\hat{C}^\varepsilon_{[0,t]} \le \gamma_\varepsilon(t) + c_1\hat{C}^\varepsilon_{[0,t]},$$

where $c_1 \in (0,1)$. This yields $\hat{C}^\varepsilon_{[0,t]} \le \gamma_\varepsilon(t)/(1-c_1)$, which is the required result (5.44). □

With these results at hand, we are ready to prove Proposition 5.4.

Proof of Proposition 5.4. Tightness of the processes X^ε is proved in Lemma 5.2, and it is sufficient to prove that the weak limit of every converging subsequence equals to the solution of (1.2) with $A = 0$ and $B_T = \sqrt{2}$, and with the initial state $\mathcal{Z}_0$.

With a little abuse of notation, let X^ε be such convergent sequence, with the limit $\mathcal{Z}$. Then (5.10) implies that X^ε is the solution of

$$dX_t^\varepsilon = \Delta_\varepsilon X_t^\varepsilon dt + dM_t^\varepsilon, \qquad X_0^\varepsilon = \hat{s}_0^\varepsilon.$$

Similarly to (4.2), we can associate to this equation two martingales

$$\mathfrak{M}_t^\varepsilon(\varphi) := X_t^\varepsilon(\varphi) - X_0^\varepsilon(\varphi) - \int_0^t X_r^\varepsilon(\Delta_\varepsilon\varphi)dr, \tag{5.48a}$$

$$\mathfrak{N}_t^\varepsilon(\varphi) := \big(\mathfrak{M}_t^\varepsilon(\varphi)\big)^2 - \int_0^t \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \varphi(y)^2 d\big\langle M^\varepsilon(y), M^\varepsilon(y)\big\rangle_r, \tag{5.48b}$$

where for a function $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ we use the shorthand notation $X_t^\varepsilon(\varphi) := \langle\langle X_t^\varepsilon, \varphi\rangle\rangle_\varepsilon$, and where the bracket process of the martingale M^ε is given in (5.6). Using the a priori bounds (5.18), which hold on the time interval $t \in [0, \sigma_{L,\varepsilon}]$, by analogy to Lemma 4.3 we can prove the L^1 convergence of (5.48a) to

$$\mathfrak{M}_t(\varphi) := \mathcal{Z}_t(\varphi) - \mathcal{Z}_0(\varphi) - \int_0^t \mathcal{Z}_r(\varphi'')dr.$$

Moreover, Lemma 5.5 yields the L^1 convergence of (5.48b) to

$$\mathfrak{N}_t(\varphi) := \big(\mathfrak{M}_t(\varphi)\big)^2 - 2t\|\varphi\|_{L^2}^2.$$

Therefore, the weak limit of X^ε is the unique solution of the martingale problem associated to (1.2) with $A = 0$ and $B_T = \sqrt{2}$. □

Using Lemma 5.5 we can also bound the function $\mathfrak{C}^\varepsilon_t(x)$, defined in (5.14).

Lemma 5.6. *For the function $\mathfrak{C}^\varepsilon_t(x)$ (see (5.13) and (5.14)) and the stopping time $\sigma_{L,\varepsilon}$, defined in (5.19), the following bound holds:*

$$\sup_{0\le t\le \sigma_{L,\varepsilon}} \|\mathfrak{C}^\varepsilon_t\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon} \le C\varepsilon^{-\frac{1}{2}-\kappa_\star-\hat\kappa} L + C\varepsilon^{3\eta-2} t^{(\eta-2\eta)/2} L^2, \tag{5.49}$$

where the constant $C > 0$ is independent of ε and t, and where the values $\kappa_\star$, $\hat\kappa$ and η are from the definition of the stopping time $\sigma_{L,\varepsilon}$ in (5.19).

Proof. Let as before $K^\varepsilon_t(x) := \nabla^-_\varepsilon p^\varepsilon_t(x)\nabla^+_\varepsilon p^\varepsilon_t(x)$. Then, using (5.46), we obtain

$$\mathfrak{C}^\varepsilon_t(x) = 2\varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t K^\varepsilon_{t-r}(x-y)dr + \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t K^\varepsilon_{t-r}(y)\big(C_\varepsilon(\hat s^\varepsilon_r, x-y) - 2\big)dr.$$

Applying (A.6), the first term can be bounded by a constant, uniformly in ε. Hence, we get

$$\|\mathfrak{C}^\varepsilon_t\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon} \le c + \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t |K^\varepsilon_{t-r}(y)|\, \|C_\varepsilon(\hat s^\varepsilon_r) - 2\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon} dr.$$

The required bound (5.49) now follows from (5.44) and (A.7). □

5.2.4. Bounds on the mild solution In this section we derive bounds on the r.h.s. of (5.17). For this, we denote the three terms on the r.h.s. of (5.17) by $\mathcal{I}^\varepsilon_1(t,x)$, $\mathcal{I}^\varepsilon_2(t,x)$ and $\mathcal{I}^\varepsilon_3(t,x)$ respectively. We start with $\mathcal{I}^\varepsilon_1(t,x)$.

Lemma 5.7. *Let $\eta \in (\frac{1}{3}, \frac{1}{2})$ and let $\kappa \in (0, 1-2\eta)$. Then $\mathcal{I}^\varepsilon_1(t,x)$ satisfies the bound*

$$\sup_{0\le t\le T\wedge\sigma_{L,\varepsilon}} \|\mathcal{I}^\varepsilon_1(t)\|_{\mathcal{C}^{2\eta}} \le C\big(\varepsilon^\kappa T^{(1-\kappa-2\eta)/2} + \varepsilon^\eta T^{1-\eta/2}\big)L, \tag{5.50}$$

where the constant $C > 0$ is independent of ε and T.

Proof. We first derive a bound on the function $\hat F_\varepsilon$, defined in (5.8). Using Lemma 5.1 we can write $\hat\varrho_\varepsilon(\hat s^\varepsilon, x) = -\varepsilon/2 + \tilde\varrho_\varepsilon(\hat s^\varepsilon, x)$, for a function $\tilde\varrho$ satisfying $|\tilde\varrho(\hat s^\varepsilon, x)| \lesssim \varepsilon^2$. This allows to decompose $\hat F_\varepsilon = \hat F^{(1)}_\varepsilon + \hat F^{(2)}_\varepsilon$, where $\hat F^{(1)}_\varepsilon(\hat s^\varepsilon, x) := -\varepsilon\Delta_\varepsilon \hat s^\varepsilon(x)/2$ and $\hat F^{(2)}_\varepsilon(\hat s^\varepsilon, x) := \tilde\varrho_\varepsilon(\hat s^\varepsilon, x)\Delta_\varepsilon \hat s^\varepsilon(x)$. Using the above estimate on $\tilde\varrho$ and the definition of the discrete Laplacian, the second function can be simply bounded by

$$|\hat F^{(2)}_\varepsilon(\hat s^\varepsilon_t, x)| \le |\tilde\varrho_\varepsilon(\hat s^\varepsilon, x)||\Delta_\varepsilon \hat s^\varepsilon_t(x)| \lesssim \sup_{\substack{y\in\mathbb{R}:\\ |x-y|\le\varepsilon}} |\hat s^\varepsilon_t(y) - \hat s^\varepsilon_t(x)| \lesssim \varepsilon^\eta \|\hat s^\varepsilon_t\|_{\mathcal{C}^\eta}.$$

One can see that the function $\hat F^{(1)}_\varepsilon$ is not suitable for a uniform bound. On the contrary, it can be considered as a discretization of a distribution, whose convolution with a test function has a good bound. More precisely, for a rescaled and recentered function φ^λ_z, defined in Section 1.3, we write

$$\langle\!\langle \hat F^{(1)}_\varepsilon(\hat s^\varepsilon_t), \varphi^\lambda_z\rangle\!\rangle_\varepsilon = \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \hat F^{(1)}_\varepsilon(\hat s^\varepsilon_t, y)\varphi^\lambda_z(y) = -\frac{\varepsilon^2}{2} \sum_{y\in\varepsilon\mathbb{Z}} \Delta_\varepsilon \hat s^\varepsilon_t(y)\varphi^\lambda_z(y).$$

Applying summation by parts twice, the last expression can be written in the following way: $-\frac{\varepsilon^2}{2}\sum_{y\in\varepsilon\mathbb{Z}}\hat{s}_t^\varepsilon(y)\Delta_\varepsilon\varphi_z^\lambda(y)$. Thus, using the fact that φ is supported in a ball of radius $\lambda > 0$, we obtain

$$|\langle\!\langle \hat{F}_\varepsilon^{(1)}(\hat{s}_t^\varepsilon), \varphi_z^\lambda\rangle\!\rangle_\varepsilon| \lesssim \varepsilon^2\lambda^{-3}\|\hat{s}_t^\varepsilon\|_{L^\infty}\sum_{y\in\varepsilon\mathbb{Z}}\mathbb{1}_{\{|y-z|\le\lambda\}} \lesssim \varepsilon\lambda^{-2}\|\hat{s}_t^\varepsilon\|_{L^\infty} \lesssim \varepsilon^\kappa\lambda^{-1-\kappa}\|\hat{s}_t^\varepsilon\|_{L^\infty},$$

for $\kappa\in[0,1]$ and $\lambda\ge\varepsilon$. To estimate the sum in the second bound, we used the fact the number of terms in the sum is proportional to λ/ε.

Using the operator norm $\|\bullet\|_{V\to W}$ for two spaces V and W, introduced in Section 1.3, we get

$$\|\mathcal{I}_1^\varepsilon(t)\|_{\mathcal{C}^{2\eta}} \lesssim \int_0^t \|S_{t-r}^\varepsilon\|_{\mathcal{C}_\varepsilon^{-1-\kappa}\to\mathcal{C}^{2\eta}}\|\hat{F}_\varepsilon^{(1)}(\hat{s}_r^\varepsilon)\|_{\mathcal{C}_\varepsilon^{-1-\kappa}}dr + \int_0^t \|S_{t-r}^\varepsilon\|_{L^\infty\to\mathcal{C}^{2\eta}}\|\hat{F}_\varepsilon^{(2)}(\hat{s}_r^\varepsilon)\|_{L^\infty}dr.$$

Furthermore, combining the derived bounds on the functions $\hat{F}_\varepsilon^{(1)}$ and $\hat{F}_\varepsilon^{(2)}$ with Lemma A.1 yields

$$\|\mathcal{I}_1^\varepsilon(t)\|_{\mathcal{C}^{2\eta}} \lesssim \int_0^t \varepsilon^\kappa(t-r)^{-(1+\kappa+2\eta)/2}\|\hat{s}_r^\varepsilon\|_{L^\infty}dr + \int_0^t \varepsilon^\eta(t-r)^{-\eta}\|\hat{s}_r^\varepsilon\|_{\mathcal{C}^\eta}dr.$$

We use the bounds $\|\hat{s}_r^\varepsilon\|_{L^\infty}\le\|\hat{s}_r^\varepsilon\|_{\mathcal{C}^\eta}$ and $\|\hat{s}_r^\varepsilon\|_{\mathcal{C}^\eta}\le\|X_r^\varepsilon\|_{\mathcal{C}^\eta}+\|u_r^\varepsilon\|_{\mathcal{C}^\eta}$. Then combining the last two bounds, we conclude

$$\|\mathcal{I}_1^\varepsilon(t)\|_{\mathcal{C}^{2\eta}} \lesssim \big(\varepsilon^\kappa t^{(1-\kappa-2\eta)/2} + \varepsilon^\eta t^{1-\eta/2}\big)L,$$

which holds as soon as $\kappa < 1-2\eta$ and $\eta < 1$. From this the required bound (5.50) follows. □

Lemma 5.8. *Let the value η be as in Theorem 1.6, and let the value $\kappa_\star$ in (5.19) satisfies $\kappa_\star > \frac{1}{2}-\eta$. Then $\mathcal{I}_2^\varepsilon(t,x)$ satisfies the bound*

$$\sup_{0\le t\le T\wedge\sigma_{L,\varepsilon}}\|\mathcal{I}_2^\varepsilon(t)\|_{\mathcal{C}^{2\eta}} \le CT^{\hat{\beta}}\varepsilon^\beta L(1+L)^2, \tag{5.51}$$

for some constant $C>0$ independent of ε and T, for some value $\hat{\beta}>0$, depending on η, η, $\kappa_\star$ and $\hat{\kappa}$, and for $\beta = (\frac{1}{2}-\kappa_\star-\hat{\kappa})\wedge(3\eta-1)\wedge(1-\gamma-2\kappa_\star-\hat{\kappa})\wedge(2\eta-\gamma)$, where γ is from Assumption 1.5.

Proof. Using the definition (5.16) and Lemma 5.1, we can write $\tilde{F}_\varepsilon = \tilde{F}_\varepsilon^{(1)}+\tilde{F}_\varepsilon^{(2)}$, where $\tilde{F}_\varepsilon^{(1)}(t,x) := -\frac{\varepsilon\mathfrak{a}\hat{s}_t^\varepsilon(x)}{4}\mathfrak{C}_t^\varepsilon(x)$ and $\tilde{F}_\varepsilon^{(2)}(t,x) := \varepsilon\hat{\lambda}_\varepsilon(\hat{s}_t^\varepsilon,x)\mathfrak{C}_t^\varepsilon(x)$, where the following bound holds

$$|\hat{\lambda}_\varepsilon(\hat{s}_t^\varepsilon,x)| \lesssim \varepsilon^{1-\gamma}(1+\sqrt{\varepsilon}|\hat{s}_t^\varepsilon(x)|)^\gamma. \tag{5.52}$$

We start with estimating the function $\tilde{F}_\varepsilon^{(1)}$. If we chose $\kappa_\star > \frac{1}{2}-\eta$, then applying Lemmas A.2 and 5.6 we obtain

$$\|\tilde{F}_\varepsilon^{(1)}(t)\|_{\mathcal{C}_\varepsilon^{-1/2+\kappa_\star}} \lesssim \varepsilon\|\hat{s}_t^\varepsilon\|_{\mathcal{C}^\eta}\|\mathfrak{C}_t^\varepsilon\|_{\mathcal{C}_\varepsilon^{-1/2+\kappa_\star}} \lesssim \varepsilon^{\frac{1}{2}-\kappa_\star-\hat{\kappa}}L^2 + \varepsilon^{3\eta-1}t^{-\eta/2}L^3, \tag{5.53}$$

which holds for $t\in[0,\sigma_{L,\varepsilon}]$, and where we have estimated $\|\hat{s}_t^\varepsilon\|_{\mathcal{C}^\eta}\le\|X_t^\varepsilon\|_{\mathcal{C}^\eta}+\|u_t^\varepsilon\|_{\mathcal{C}^\eta}\le 2L$.

Now, we turn to the function $\tilde{F}_\varepsilon^{(2)}$. The bound (5.52) yields

$$|\tilde{F}_\varepsilon^{(2)}(t,x)| \le \varepsilon|\hat{\lambda}_\varepsilon(\hat{s}_t^\varepsilon, x)||\mathfrak{C}_t^\varepsilon(x)| \lesssim \varepsilon^{2-\gamma}(1+\sqrt{\varepsilon}|\hat{s}_t^\varepsilon(x)|)^\gamma |\mathfrak{C}_t^\varepsilon(x)|.$$

Furthermore, from the identity (5.13) we obtain

$$\begin{aligned}|\mathfrak{C}_t^\varepsilon(x)| \le |Z_t^\varepsilon(x)| + |\nabla_\varepsilon^- \hat{X}_t^\varepsilon(x)||\nabla_\varepsilon^+ \hat{X}_t^\varepsilon(x)| &\le \|Z_t^\varepsilon\|_{L^\infty} + \varepsilon^{-2} \sup_{\substack{y\in\mathbb{R}:\\ |x-y|\le\varepsilon}} |\hat{X}_t^\varepsilon(x) - \hat{X}_t^\varepsilon(y)|^2 \\ &\le \|Z_t^\varepsilon\|_{L^\infty} + \varepsilon^{2(\eta-1)}\|\hat{X}_t^\varepsilon\|_{\mathcal{C}^\eta}^2,\end{aligned}$$

where in the last line we used the definition of the Hölder norm. Hence, we have

$$|\tilde{F}_\varepsilon^{(2)}(t,x)| \le \varepsilon^{2-\gamma}(1+\sqrt{\varepsilon}|\hat{s}_t^\varepsilon(x)|)^\gamma \big(\|Z_t^\varepsilon\|_{L^\infty} + \varepsilon^{2(\eta-1)}\|\hat{X}_t^\varepsilon\|_{\mathcal{C}^\eta}^2\big).$$

On the time interval $t \in [0, \sigma_{L,\varepsilon}]$ we have $\|\hat{X}_t^\varepsilon\|_{\mathcal{C}^\eta} \le L$ and $|\hat{s}_t^\varepsilon(x)| \le\le \|X_t^\varepsilon\|_{\mathcal{C}^\eta} + \|u_t^\varepsilon\|_{\mathcal{C}^\eta} \le 2L$. Moreover, the definition (1.11) yields the bound

$$\|Z_t^\varepsilon\|_{L^\infty} \le \varepsilon^{-1/2-\kappa_\star}\|Z_t^\varepsilon\|_{\mathcal{C}_\varepsilon^{-1/2+\kappa_\star}} \le \varepsilon^{-1-2\kappa_\star-\hat{\kappa}}L.$$

Hence, for the function $\tilde{F}_\varepsilon^{(2)}$ we have the following bound

$$|\tilde{F}_\varepsilon^{(2)}(t,x)| \le \varepsilon^{2-\gamma}L(1+\sqrt{\varepsilon}L)^\gamma\big(\varepsilon^{-1-2\kappa_\star-\hat{\kappa}} + \varepsilon^{2(\eta-1)}L\big). \tag{5.54}$$

Combining the derived bounds (5.53) and (5.54), we obtain

$$\begin{aligned}\|\tilde{F}_\varepsilon(t)\|_{\mathcal{C}_\varepsilon^{-1/2+\kappa_\star}} &\lesssim \varepsilon^{\frac{1}{2}-\kappa_\star-\hat{\kappa}}L^2 + \varepsilon^{3\eta-1}t^{-\eta/2}L^3 + \varepsilon^{2-\gamma}L(1+\sqrt{\varepsilon}L)^\gamma\big(\varepsilon^{-1-2\kappa_\star-\hat{\kappa}} + \varepsilon^{2(\eta-1)}L\big) \\ &\lesssim \varepsilon^\beta t^{-\eta/2}L(1+L)^2,\end{aligned}$$

where $\beta = (\frac{1}{2} - \kappa_\star - \hat{\kappa}) \wedge (3\eta - 1) \wedge (1 - \gamma - 2\kappa_\star - \hat{\kappa}) \wedge (2\eta - \gamma)$. Then Lemma A.1 yields

$$\begin{aligned}\|\mathcal{I}_2^\varepsilon(t)\|_{\mathcal{C}^{2\eta}} &\lesssim \int_0^t \|S_{t-r}^\varepsilon\|_{\mathcal{C}_\varepsilon^{-1/2+\kappa_\star}\to\mathcal{C}^{2\eta}}\|\tilde{F}_\varepsilon(r,\bullet)\|_{\mathcal{C}_\varepsilon^{-1/2+\kappa_\star}}dr \\ &\lesssim \varepsilon^\beta L(1+L)^2\int_0^t (t-r)^{-(1/2-\kappa_\star+2\eta)/2}r^{-\eta/2}dr.\end{aligned}$$

Since $(1/2-\kappa_\star+2\eta)/2 < 1$ and $-\eta/2 > -1$, the last integral is of order $t^{\hat{\beta}}$, for some $\hat{\beta} > 0$. This is exactly the required result (5.51). □

To bound $\mathcal{I}_3^\varepsilon(t,x)$ we will compare it to

$$\hat{\mathcal{I}}_3^\varepsilon(t,x) := -\frac{\mathfrak{a}}{4}\int_0^t (S_{t-r}^\varepsilon\hat{s}_r^\varepsilon)(x)dr. \tag{5.55}$$

Lemma 5.9. *Let the value η be as in Theorem 1.6. Then $\mathcal{I}^\varepsilon_3(t,x)$ and $\hat{\mathcal{I}}^\varepsilon_3(t,x)$ satisfy the bounds*

$$\sup_{0\le t\le T\wedge\sigma_{L,\varepsilon}} \|\mathcal{I}^\varepsilon_3(t)\|_{\mathcal{C}^{2\eta}} \le CT^{\hat\beta}(L+\varepsilon^\beta(1+L)^3), \tag{5.56a}$$

$$\sup_{0\le t\le T\wedge\sigma_{L,\varepsilon}} \|(\mathcal{I}^\varepsilon_3-\hat{\mathcal{I}}^\varepsilon_3)(t)\|_{\mathcal{C}^{2\eta}} \le CT^{\hat\beta}\varepsilon^\beta(1+L)^3, \tag{5.56b}$$

for some constant $C>0$ independent of ε and T, for some $\hat\beta>0$, depending on η and $\kappa_\star$, and for the value $\beta=(2\eta-\gamma)\wedge(3\eta-1)\wedge(1/2-\kappa_\star-\hat\kappa)$, where γ is from Assumption 1.5.

Proof. Before deriving bounds on $\mathcal{I}^\varepsilon_3$, we estimate the function inside the integral in the definition of $\mathcal{I}^\varepsilon_3$. Using the expression (5.16), we can write this function as

$$F_\varepsilon(X^\varepsilon,Z^\varepsilon,u^\varepsilon)+\frac{\mathfrak{a}\hat{s}^\varepsilon}{4}=\hat\lambda_\varepsilon(\hat{s}^\varepsilon)\big(1-\varepsilon\nabla^-_\varepsilon\hat{s}^\varepsilon\diamond\nabla^+_\varepsilon\hat{s}^\varepsilon\big)+\frac{\varepsilon\mathfrak{a}}{4}\hat{s}^\varepsilon\big(\nabla^-_\varepsilon\hat{s}^\varepsilon\diamond\nabla^+_\varepsilon\hat{s}^\varepsilon\big),$$

and we denote the two terms on the r.h.s. by $F^{(1)}_\varepsilon$ and $F^{(2)}_\varepsilon$ respectively. For the first term, the simple bound $|\nabla^\pm_\varepsilon\hat{s}^\varepsilon_t(x)|\le\varepsilon^{\eta-1}\|\hat{s}^\varepsilon_t\|_{\mathcal{C}^\eta}$ and the bound on the error term $\hat\lambda_\varepsilon$, provided in Lemma 5.1, yields

$$|F^{(1)}_\varepsilon(t,x)|\lesssim\varepsilon^{1-\gamma}\big(1+\sqrt{\varepsilon}\|\hat{s}^\varepsilon_t\|_{L^\infty}\big)^\gamma\big(1+\varepsilon^{2\eta-1}\|\hat{s}^\varepsilon_t\|^2_{\mathcal{C}^\eta}\big)\lesssim\varepsilon^{1-\gamma}\big(1+\sqrt{\varepsilon}L\big)^\gamma\big(1+\varepsilon^{2\eta-1}L^2\big),$$

where the value γ is from Assumption 1.5, and where we consider $t\in[0,\sigma_{L,\varepsilon}]$. Here, we made use of $\|\hat{s}^\varepsilon\|_{\mathcal{C}^\eta_t}\le\|X^\varepsilon\|_{\mathcal{C}^\eta_t}+\|u^\varepsilon\|_{\mathcal{C}^{2\eta}_t}$.

Now, we turn to the function $F^{(2)}_\varepsilon$. Using the definition of the Wick-type product (5.15) we can write $F^{(2)}_\varepsilon=F^{(3)}_\varepsilon+F^{(4)}_\varepsilon$, where

$$F^{(3)}_\varepsilon:=\frac{\varepsilon\mathfrak{a}}{4}\hat{s}^\varepsilon\left(\nabla^-_\varepsilon X^\varepsilon\,\nabla^+_\varepsilon u^\varepsilon+\nabla^-_\varepsilon u^\varepsilon\,\nabla^+_\varepsilon\hat{s}^\varepsilon\right),\qquad F^{(4)}_\varepsilon:=\frac{\varepsilon\mathfrak{a}}{4}\hat{s}^\varepsilon Z^\varepsilon,$$

where we use the process Z^ε, defined in (5.13). For the function $F^{(3)}_\varepsilon$ we have the bound

$$|F^{(3)}_\varepsilon(t,x)|\lesssim\|\hat{s}^\varepsilon_t\|_{L^\infty}\left(\varepsilon^{3\eta-1}\|X^\varepsilon_t\|_{\mathcal{C}^\eta}\|u^\varepsilon_t\|_{\mathcal{C}^{2\eta}}+\varepsilon^{4\eta-1}\|u^\varepsilon_t\|^2_{\mathcal{C}^{2\eta}}\right)\lesssim\varepsilon^{3\eta-1}L^3,$$

on the time interval $t\in[0,\sigma_{L,\varepsilon}]$. For the term $F^{(4)}_\varepsilon$, we use Lemma A.2 and obtain

$$\|F^{(4)}_\varepsilon(t)\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon}\lesssim\varepsilon\|\hat{s}^\varepsilon_t\|_{\mathcal{C}^\eta}\|Z^\varepsilon_t\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon}\lesssim\varepsilon^{1/2-\kappa_\star-\hat\kappa}L^2,$$

which holds on the time interval $t\in[0,\sigma_{L,\varepsilon}]$.

Combining these bounds on the functions $F^{(1)}_\varepsilon$ and $F^{(2)}_\varepsilon$, we conclude

$$\|(F_\varepsilon+\mathfrak{a}\hat{s}^\varepsilon/4)(t)\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon}\le\|F^{(1)}_\varepsilon(t)\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon}+\|F^{(3)}_\varepsilon(t)\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon}+\|F^{(4)}_\varepsilon(t)\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon}.$$

The definition (1.11) yields $\|\cdot\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon}\le\|\cdot\|_{L^\infty}$, which yields

$$\|(F_\varepsilon+\mathfrak{a}\hat{s}^\varepsilon/4)(t)\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon}\le\|F^{(1)}_\varepsilon(t)\|_{L^\infty}+\|F^{(3)}_\varepsilon(t)\|_{L^\infty}+\|F^{(4)}_\varepsilon(t)\|_{\mathcal{C}^{-1/2+\kappa_\star}_\varepsilon}\lesssim\varepsilon^\beta(1+L)^3,$$

where $\beta = (2\eta - \gamma) \wedge (3\eta - 1) \wedge (1/2 - \kappa_\star - \hat{\kappa})$. Then Lemma A.1 gives

$$\begin{aligned} \|(\mathcal{I}_3^\varepsilon - \hat{\mathcal{I}}_3^\varepsilon)(t)\|_{\mathcal{C}^{2\eta}} &\lesssim \int_0^t \|S_{t-r}^\varepsilon\|_{\mathcal{C}_\varepsilon^{-1/2+\kappa_\star} \to \mathcal{C}^{2\eta}} \|(F_\varepsilon + \mathfrak{a}\hat{s}^\varepsilon/4)(r)\|_{\mathcal{C}_\varepsilon^{-1/2+\kappa_\star}} dr \\ &\lesssim \varepsilon^\beta (1+L)^3 \int_0^t (t-r)^{-(1/2-\kappa_\star+2\eta)/2} dr \lesssim \varepsilon^\beta (1+L)^3 t^{\hat{\beta}}, \end{aligned}$$

which holds for some $\hat{\beta} > 0$, because $(1/2 - \kappa_\star + 2\eta)/2 < 1$. This is the required bound (5.56b), and the bound (5.56a) follows from the triangle inequality. □

5.3. Proof of the main convergence result.

Proof of Theorem 1.6. Taking into account our restrictions on the values γ and η in Assumption 1.5 and Theorem 1.6, we can choose the values $\kappa_\star$ and $\hat{\kappa}$ in (5.19) to be such that the powers of ε in (5.51) and (5.56a) are strictly positive. Then (5.17) and Lemmas 5.7, 5.8 and 5.9 yield the bound

$$\sup_{0 \le t \le T \wedge \sigma_{L,\varepsilon}} \|u_t^\varepsilon\|_{\mathcal{C}^{2\eta}} \le C T^{\hat{\beta}} (1+L)^3,$$

for $T \in (0, 1]$, for a constant C, independent of ε and T, and for some value $\hat{\beta} > 0$. Taking $T = T_*$ sufficiently small and depending on L, we can write

$$\mathbb{E}\|u^\varepsilon\|_{\mathcal{C}^{2\eta}_{T_* \wedge \sigma_{L,\varepsilon}}} \le \hat{C},$$

for a different proportionality constant $\hat{C} > 0$. Similarly, the second bound in Lemma 5.9 yields

$$\mathbb{E}\|u^\varepsilon - \hat{\mathcal{I}}_3^\varepsilon\|_{\mathcal{C}^{2\eta}_{T_* \wedge \sigma_{L,\varepsilon}}} \le \hat{C}\varepsilon^\beta,$$

for some $\beta > 0$, where $\hat{\mathcal{I}}_3^\varepsilon$ is defined in (5.55). Iterating this procedure with a new initial data $u^\varepsilon(T_*)$, we obtain these bounds on any time interval $[0, T]$:

$$\mathbb{E}\|u^\varepsilon\|_{\mathcal{C}^{2\eta}_{T \wedge \sigma_{L,\varepsilon}}} \le \tilde{C}, \qquad \mathbb{E}\|u^\varepsilon - \hat{\mathcal{I}}_3^\varepsilon\|_{\mathcal{C}^{2\eta}_{T \wedge \sigma_{L,\varepsilon}}} \le \tilde{C}\varepsilon^\kappa, \tag{5.57}$$

with a new proportionality constant $\tilde{C} > 0$. One can see that in the case when $\sigma_{L,\varepsilon} \ge T > 0$, uniformly in ε, these bounds and Proposition 5.4 imply that $\hat{s}^\varepsilon := X^\varepsilon + u^\varepsilon$ converges weakly in $D([0, T], \mathcal{C}(\mathbb{R}))$ to the solution of (1.2) with $A = -\frac{\mathfrak{a}}{4}$ and $B = 1$, and with the initial state $\mathcal{Z}_0$. In order to prove convergence of $\hat{s}^\varepsilon$ on $[0, \infty)$, we define a new stopping time

$$\sigma^* := \lim_{L \to \infty} \inf\Big\{t \ge 0 : \limsup_{\varepsilon \to 0} \|u^\varepsilon\|_{\mathcal{C}_t^{2\eta}} \ge L\Big\},$$

which together with the first bound in (5.57) ensures

$$\mathbb{E}\|u^\varepsilon\|_{\mathcal{C}^{2\eta}_{T \wedge \sigma^*}} \le \tilde{C}. \tag{5.58}$$

Then for any $T > 0$ and $\kappa \in (0, \beta)$ we can estimate

$$\mathbb{P}\big[\|u^\varepsilon - \hat{\mathcal{I}}_3^\varepsilon\|_{\mathcal{C}^{2\eta}_{T\wedge\sigma^*}} \geq \varepsilon^\kappa\big] \leq \mathbb{P}\big[\|u^\varepsilon - \hat{\mathcal{I}}_3^\varepsilon\|_{\mathcal{C}^{2\eta}_{\sigma_{L,\varepsilon}}} \geq \varepsilon^\kappa\big] + \mathbb{P}[\sigma_{L,\varepsilon} < T \wedge \sigma^*]. \tag{5.59}$$

The desired result follows if we prove that the limits $\lim_{L\uparrow\infty} \lim_{\varepsilon\downarrow 0}$ of this expression vanish. The first term in (5.59) we can bound using the Chebyshev's inequality and the second estimate in (5.57) by

$$\mathbb{P}\big[\|u^\varepsilon - \hat{\mathcal{I}}_3^\varepsilon\|_{\mathcal{C}^{2\eta}_{\sigma_{L,\varepsilon}}} \geq \varepsilon^\kappa\big] \leq \frac{\mathbb{E}\|u^\varepsilon - \hat{\mathcal{I}}_3^\varepsilon\|_{\mathcal{C}^{2\eta}_{\sigma_{L,\varepsilon}}}}{\varepsilon^\kappa} \leq \tilde{C}\varepsilon^{\beta-\kappa},$$

which vanishes as $\varepsilon \to 0$, for every fixed L. To bound the second term in (5.59), we make use of the definition (5.19) and obtain

$$\mathbb{P}\big[\sigma_{L,\varepsilon} < T \wedge \sigma^*\big] \leq \mathbb{P}\big[\|X^\varepsilon\|_{\mathcal{C}^\eta_T} \geq L\big] + \mathbb{P}\big[\|Z^\varepsilon\|_{\mathcal{C}^{-1/2+\kappa_\star}_{T,\varepsilon}} \geq \varepsilon^{-1/2-\kappa_\star-\hat{\kappa}} L\big] + \mathbb{P}\big[\|u^\varepsilon\|_{\mathcal{C}^{2\eta}_{T\wedge\sigma^*}} \geq L\big]. \tag{5.60}$$

Lemmas 5.2 and 5.3 imply that the first two probabilities in (5.60) vanish as $L \to \infty$. The bound (5.58) guarantees that the last probability in (5.60) vanishes as $L \to \infty$.

Combining these limits together we obtain $\lim_{\varepsilon\to 0} \mathbb{P}\big[\|u^\varepsilon - \hat{\mathcal{I}}_3^\varepsilon\|_{\mathcal{C}^{2\eta}_{T\wedge\sigma^*}} = 0$, for any $T > 0$. Proposition 5.4 implies convergence of $\hat{s}^\varepsilon := X^\varepsilon + u^\varepsilon$ weakly in $D([0, T \wedge \sigma^*], \mathcal{C}(\mathbb{R}))$ to the solution $\mathcal{Z}$ of (1.2) with $A = -\frac{\mathfrak{a}}{4}$ and $B = 1$, and with the initial state $\mathcal{Z}_0$. Since the process $\mathcal{Z}_t$ is defined for $t \in [0, \infty)$, we conclude that the required convergence holds in $D([0, \infty), \mathcal{C}(\mathbb{R}))$. □

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Appendix A: Properties of heat kernels and semigroups

In this appendix we provide regularity properties of the continuous and discrete heat semigroups. Moreover, we list bounds on the discrete heat kernel, which are used in the article.

A.1. Bounds on heat kernels and semigroups. Let $S_t^\varepsilon = e^{t\Delta_\varepsilon}$ be the discrete heat semigroup, generated by the discrete Laplacian Δ_ε defined in Section 1.3. This semigroup has nice regularizing properties when acting on spaces of functions and distributions introduced in Section 1.3, which we provide below.

Lemma A.1. *Let us restrict the domain of S_t^ε to periodic functions/distributions on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then, for any $\alpha \leq 0$ and for any $\gamma > \alpha \vee 0$, there is a constant $C > 0$, independent of $\varepsilon \in (0, 1]$ and $t > 0$, such that the following bound holds:*

$$\|S_t^\varepsilon\|_{\mathcal{C}^\alpha_\varepsilon \to \mathcal{C}^\gamma} \leq C t^{(\alpha-\gamma)/2}. \tag{A.1}$$

Proof. The bound (A.1) can be proved in the same way as a more general result [CM18, Prop. 4.17]. More precisely, the semigroup S_t^ε is given by convolution with the discrete heat kernel $p_t^\varepsilon(x)$. Furthermore, the bound (1.11) holds also for rescaled Schwarz functions φ. Finally, the kernel $p_t^\varepsilon(x)$ is Schwarz in the x variable, and can be considered as a function rescaled by $\lambda = \sqrt{t}$. Then (A.1) follows from our definitions of the norms. □

The following is an analogue of the classical result [BCD11, Thm. 2.85] for our ε-dependent norms:

Lemma A.2. *Let $\varphi \in \mathcal{C}^\alpha$ and $\psi \in \mathcal{C}_\varepsilon^\beta$, where $\beta < 0 < \alpha < 1$ and $\alpha + \beta > 0$. Then there is a constant C, depending on α and β, such that*

$$\|\varphi\psi\|_{\mathcal{C}_\varepsilon^\beta} \leq C\|\varphi\|_{\mathcal{C}^\alpha}\|\psi\|_{\mathcal{C}_\varepsilon^\beta} .$$

The heat semigroups S_t and S_t^ε are given by convolutions with respectively the continuous and discrete heat kernels. The latter ones are the Green's functions of the operators $\partial_t - \partial_x^2$ and $\partial_t - \Delta_\varepsilon$, and are defined as the unique solutions of the equations:

$$\partial_t p_t(x) = \partial_x^2 p_t(x), \qquad \partial_t p_t^\varepsilon(y) = \Delta_\varepsilon p_t^\varepsilon(y), \qquad \text{for all } t \geq 0, x \in \mathbb{R}, y \in \varepsilon\mathbb{Z}, \tag{A.2}$$

with the respective initial values $p_0(x) = \delta_x$ and $p_0^\varepsilon(y) = \varepsilon^{-1}\mathbb{1}_{\{y=0\}}$, where δ_x is the Dirac delta function. For these kernels we have the following bounds.

Lemma A.3. *For any $u > 0$, any $v \in (0, \frac{1}{2})$ and any $T > 0$, there exists a constant C, depending on u and v, and independent of $\varepsilon > 0$ and T, such that the following bounds hold*

$$p_t^\varepsilon(x) \leq C\big(\sqrt{t} \vee \varepsilon\big)^{-1}, \qquad \varepsilon \sum_{x\in\varepsilon\mathbb{Z}} p_t^\varepsilon(x)|x|^\alpha e^{u|x|} \leq C(\sqrt{t} \vee \varepsilon)^\alpha, \tag{A.3a}$$

uniformly in $x \in \mathbb{R}$, $t \in [0, T]$ and $\varepsilon \in (0, 1]$, and for any $\alpha \geq 0$. Furthermore, the following bounds hold uniformly in $t \in [0, T]$ and $x, x_1, x_2 \in \mathbb{R}$:

$$|\nabla_\varepsilon^+ p^\varepsilon(x)| \leq C(\varepsilon^2 t^{-\frac{3}{2}} \wedge \varepsilon^{-1}), \qquad |p_t^\varepsilon(x_1) - p_t^\varepsilon(x_2)| \leq C\big(\sqrt{t} \vee \varepsilon\big)^{-1-v}|x_2 - x_1|^{2v}. \tag{A.3b}$$

Finally, for any points $x \in \mathbb{R}$ and $0 \leq t_1 < t_2 \leq T$ one had the bounds

$$p_{t_2}^\varepsilon(x) \leq Ce^{t_2-t_1} p_{t_1}^\varepsilon(x), \qquad |p_{t_1}^\varepsilon(x) - p_{t_2}^\varepsilon(x)| \leq C\big(\sqrt{t_1} \vee \varepsilon\big)^{-1-v}(t_2 - t_1)^{v/2}. \tag{A.3c}$$

Proof. These bounds follow from the respective bounds for the non-rescaled discrete heat kernel $p_t^0(x)$, proved in [BG97], and the identity

$$p_t^\varepsilon(x) = \frac{1}{\varepsilon} p_{\varepsilon^{-2}t}^0(\varepsilon^{-1}x).$$

More precisely, the first bound in (A.3a) follows from [BG97, Eq. 4.22], and the second one can be proved similarly to [BG97, Eq. 4.14]. A proof of the bound (A.3b) can be found in [BG97, Eq. 4.39]. The bounds in (A.3c) are proved in [BG97, Eqs. A.5, 4.44]. □

In Section 3, we often use the microscopic heat kernel $\{\mathfrak{p}_t^\varepsilon\}_{t\in\mathbb{R}_{\geq 0}}$ as defined in (3.4). It is worth noting that p_t^ε of (A.2) is related to $\mathfrak{p}_t^\varepsilon$ via $\mathfrak{p}_t^\varepsilon(x) = \varepsilon p^\varepsilon_{\varepsilon^{-2}\sqrt{q}t}(\varepsilon^{-1}x)$. In the following result, we write the bounds on $\mathfrak{p}_t^\varepsilon$ by translating Lemma A.3 using the relation between $\mathfrak{p}_t^\varepsilon$ and p_t^ε.

Lemma A.4. *Fix any $u > 0$, any $v \in [0, \frac{1}{2})$ and any $T > 0$, there exists a constant $C = C(u, v) > 0$ such that the following bounds hold*

$$\mathfrak{p}_t^\varepsilon(x) \leq C\big(\sqrt{t} \vee 1\big)^{-1}, \qquad \sum_{x\in\mathbb{Z}} \mathfrak{p}_t^\varepsilon(x)(\varepsilon|x|)^\alpha e^{u\varepsilon|x|} \leq C(\sqrt{t} \vee 1)^\alpha, \tag{A.4a}$$

uniformly in $x \in \mathbb{Z}$, $t \in [0, \varepsilon^{-2}T]$ and $\varepsilon \in (0, 1]$, and for any $\alpha \geq 0$. Furthermore, the following bounds hold uniformly in $t \in [0, \varepsilon^{-2}T]$ and $x, x_1, x_2 \in \mathbb{Z}$:

$$|\nabla^+\mathfrak{p}^\varepsilon(x)| \leq C(t^{-\frac{3}{2}} \wedge 1), \qquad |\mathfrak{p}_t^\varepsilon(x_1) - \mathfrak{p}_t^\varepsilon(x_2)| \leq C\big(\sqrt{t} \vee 1\big)^{-1-v}(\varepsilon|x_2 - x_1|)^{2v}. \tag{A.4b}$$

Finally, for any points $x \in \mathbb{Z}$ and $0 \leq t_1 < t_2 \leq \varepsilon^{-2}T$ one had the bounds

$$\mathfrak{p}_{t_2}^\varepsilon(x) \leq Ce^{\varepsilon^2(t_2-t_1)}\mathfrak{p}_{t_1}^\varepsilon(x), \quad |\mathfrak{p}_{t_1}^\varepsilon(x) - \mathfrak{p}_{t_2}^\varepsilon(x)| \leq C\big(\sqrt{t_1} \vee 1\big)^{-1-v}\big(\varepsilon^2(t_2 - t_1)\big)^{v/2}. \tag{A.4c}$$

A.2. Properties of some discrete kernels. Let $\nabla_\varepsilon^\pm$ be the discrete derivatives, defined in Section 1.3, p_t^ε from (A.2), and define

$$K_t^\varepsilon(x) := \nabla_\varepsilon^- p_t^\varepsilon(x)\nabla_\varepsilon^+ p_t^\varepsilon(x). \tag{A.5}$$

We collect important properties of this kernel in the following lemma.

Lemma A.5. *There is a constant $c_0 > 0$, such that for every $T > 0$ the following hold*

$$\Big|\sum_{x\in\varepsilon\mathbb{Z}} \int_0^T K_t^\varepsilon(x)dt\Big| \leq \frac{c_0\varepsilon^{-1}}{\sqrt{T \vee \varepsilon^2}}, \qquad \sum_{x\in\varepsilon\mathbb{Z}} \int_0^\infty K_t^\varepsilon(x)dt = 0. \tag{A.6}$$

Moreover, for every $T > 0$ and every $a \geq 0$, there exist values $\varepsilon_0 > 0$, $c_1 \in (0, 1)$ and $c_2 > 0$ such that the following bounds hold uniformly in $\varepsilon \in (0, \varepsilon_0)$:

$$\sum_{x\in\varepsilon\mathbb{Z}} \int_0^T |K_t^\varepsilon(x)|e^{a|x|}dt \leq \frac{c_1}{\varepsilon^2}, \qquad \sum_{x\in\varepsilon\mathbb{Z}} \int_0^T |K_t^\varepsilon(x)|e^{a|x|}(T-t)^{-\frac{1}{2}}dt \leq \frac{c_2}{\varepsilon^2}. \tag{A.7}$$

Proof. The second identity in (A.6) follows from the first bound in the limit $T \to +\infty$. In order to prove the first bound in (A.6) we write $K_t^\varepsilon(x) = \varepsilon^{-4}K_{\varepsilon^{-2}t}(\varepsilon^{-1}x)$, where the kernel K equals K^ε with $\varepsilon = 1$. Furthermore, using (A.2) and (A.5), we can write the Fourier integral

$$K_t(x) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} dk \int_{-\pi}^{\pi} d\ell\, e^{i(k+\ell)x}\big(1 - e^{-ik}\big)\big(e^{i\ell} - 1\big)e^{-t(2-\cos k - \cos \ell)}.$$

Using the identity $\sum_{x\in\mathbb{Z}} e^{ikx} = 2\pi\delta_k$, we conclude that

$$\sum_{x\in\varepsilon\mathbb{Z}} \int_0^T K_t^\varepsilon(x)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\, e^{ik}\big(1-e^{-2T(1-\cos k)}\big) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} dk\, e^{ik}e^{-2T(1-\cos k)}.$$

The absolute value of the last expression can be bounded by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-2T(1-\cos k)}dk \le \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-Tk^2}dk = \frac{1}{\pi\sqrt{T}}\int_0^{\pi\sqrt{T}} e^{-k^2}dk \le \frac{c_0}{\sqrt{T\vee 1}},$$

for some constant $c_0 > 0$, independent of T. The required bound (A.6) follows now from the last one after rescaling.

The bounds (A.7) follow immediately from [BG97, Lem. A.3] and [BG97, Lem. A.4] respectively. □

Appendix B: Bounds on iterated stochastic integrals

In this appendix we recall some properties of càdlàg martingales and provide moment bounds for iterated stochastic integrals.

B.1. Iterated stochastic integrals. Assume we have a family of square integrable martingales $(M_t^{\varepsilon,\ell}(x))_{t\ge 0}$, parameterized by $\ell \ge 1$ and $x \in \varepsilon\mathbb{Z}$, which have the following properties: the martingale $M_t^{\varepsilon,\ell}(x)$ is of bounded total variation and has jumps of size $\varepsilon^{\delta_\ell}$ for $\delta_\ell \ge \frac{1}{2}$; $M_t^{\varepsilon,\ell}(x)$ and $M_t^{\varepsilon,\ell}(y)$ a.s. do not jump together, if $x \neq y$; on every time interval $[0, \varepsilon^2 T]$ the martingale $M_t^{\varepsilon,\ell}(x)$ makes a.s. finitely many jumps. In this case, the quadratic variation can be written as

$$\langle M^{\varepsilon,\ell}(x), M^{\varepsilon,\ell}(y)\rangle_t = \varepsilon^{-1}\mathbb{1}_{x=y}\int_0^t C_r^{\varepsilon,\ell}(x)dr,$$

for some function $C_r^{\varepsilon,\ell}(x)$, adapted to the underlying filtration in the variable $r \ge 0$. Moreover, we assume that the function $C_r^{\varepsilon,\ell}(x)$ is a.s. bounded by a constant $c_{\varepsilon,\ell}$, uniformly in r and x.

When working with such martingales, the following two forms of the Burkholder–Davis–Gundy inequality will be used. The first form is standard.

Lemma B.1. *For all $p \ge 1$ there exists a constant $C_p > 0$ such that for all martingales M_t, and for all $T > 0$,*

$$\mathbb{E}\big[\sup_{t\in[0,T]} |M_t - M_0|^p\big] \le C_p\mathbb{E}\big[[M,M]_T^{p/2}\big].$$

The second form of the inequality is adapted to the setting described.

Lemma B.2. *Let M^ε be a martingale with the just described properties. Then, for any $p \ge 1$, there exists a constant $C = C(p)$, such that for every $T > 0$ one has*

$$\mathbb{E}\Big[\sup_{0\le t\le T} |M^\varepsilon(t)|^p\Big] \le C\bigg(\mathbb{E}\Big[\langle M^\varepsilon, M^\varepsilon\rangle_T^{\frac{p}{2}}\Big] + \mathbb{E}\Big[\sup_{0\le t\le T} |\Delta_t M^\varepsilon|^p\Big]\bigg), \tag{B.1}$$

where $\Delta_t M^\varepsilon := M^\varepsilon(t) - M^\varepsilon(t-)$ denotes the jump of M^ε at time t.

Proof. This formula can be proved by discrete time approximations of the martingales M^ε, and using an analogous formula [HH80, Thm. 2.11] in discrete time. □

For a continuous function $F\colon [0,\infty)^n \times \varepsilon\mathbb{Z}^n \to \mathbb{R}$, we define iterated stochastic integrals $(\mathfrak{I}_n^\varepsilon F)(t)$ as follows: for $n = 1$ we set $(\mathfrak{I}_1^\varepsilon F)(t) := \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t F(r,y)\, dM_r^{\varepsilon,1}(y)$; for $n \geq 2$ we define recursively

$$(\mathfrak{I}_n^\varepsilon F)(t) := \varepsilon \sum_{y\in\varepsilon\mathbb{Z}} \int_0^t (\mathfrak{I}_{n-1}^\varepsilon F^{(r,y)})(r-)\, dM_r^{\varepsilon,n}(y), \tag{B.2}$$

where $r-$ is the left limit at r, and the function $F^{(r,y)}\colon [0,\infty)^{n-1} \times (\varepsilon\mathbb{Z})^{n-1}$ is defined as

$$F^{(r,y)}(r_1,\ldots,r_{n-1}; y_1,\ldots,y_{n-1}) := F(r_1,\ldots,r_{n-1},r; y_1,\ldots,y_{n-1},y).$$

To make notation shorter, in what follows we write $(\mathfrak{I}_\ell^\varepsilon F)(r_{\ell+1},\ldots,r_n; y_{\ell+1},\ldots,y_n)$ for the iterated stochastic integral, taken with respect to the variables $r_1,\ldots,r_\ell$ and $y_1,\ldots,y_\ell$ and evaluated at time $t = r_{\ell+1}$, where $r_{\ell+1},\ldots,r_n$ and $y_{\ell+1},\ldots,y_n$ are treated as free parameters. With this definition, the process $t \mapsto (\mathfrak{I}_n^\varepsilon F)(t)$ is a martingale, and moreover we have the following moment bounds for it, which is a modification of [MW17, Lem. 4.1].

Lemma B.3. *Let $n \geq 1$ and let $F\colon [0,\infty)^n \times (\varepsilon\mathbb{Z})^n \to \mathbb{R}$ be continuous and deterministic. Then for any $p \geq 1$, there exists a constant $C = C(n,p)$ such that*

$$\left(\mathbb{E}\Big[\sup_{0\le t\le T} \big|(\mathfrak{I}_n^\varepsilon F)(t)\big|^p\Big]\right)^{2/p} \le C\left(\prod_{\ell=1}^n c_{\varepsilon,\ell}\right)\varepsilon^n \sum_{\mathbf{y}\in(\varepsilon\mathbb{Z})^n} \int_{r_n=0}^T \cdots \int_{r_1=0}^{r_2} F(\mathbf{r};\mathbf{y})^2 d\mathbf{r} + \mathscr{E}_n^\varepsilon(T), \tag{B.3}$$

where we use the shorthand notation $\mathbf{r} = (r_1,\ldots,r_n)$ and $d\mathbf{r} = dr_1\cdots dr_n$, and where the error term $\mathscr{E}_n^\varepsilon$ is given by

$$\mathscr{E}_n^\varepsilon(T) := C\sum_{\ell=1}^n \varepsilon^{n-\ell+2(1+\delta_\ell)}\left(\prod_{k=\ell+1}^n c_{\varepsilon,k}\right) \times \sum_{\mathbf{y}_{\ell+1,n}\in(\varepsilon\mathbb{Z})^{n-\ell}} \int_{r_n=0}^T \cdots \int_{r_{\ell+1}=0}^{r_{\ell+2}} \left(\mathbb{E}\Big[\sup_{\substack{r_\ell\in[0,r_{\ell+1}]\\ y_\ell\in\varepsilon\mathbb{Z}}} \big|(\mathfrak{I}_{\ell-1}^\varepsilon F)(\mathbf{r}_{\ell,n};\mathbf{y}_{\ell,n})\big|^p\Big]\right)^{2/p} d\mathbf{r}_{\ell+1,n},$$

where $\mathbf{r}_{\ell,n} = (r_\ell,\ldots,r_n)$, $\mathbf{y}_{\ell,n} = (y_\ell,\ldots,y_n)$, $d\mathbf{r}_{\ell+1,n} = dr_{\ell+1}\cdots dr_n$ and $r_{n+1} = T$.

Proof. We first prove the bound (B.3) in the case $n = 1$. In order to apply the Burkholder–Davis–Gundy inequality (B.1), we need to bound the quadratic variation and the jump size of the martingale $(\mathfrak{I}_1 F)(t)$. Since the jump of the martingale $M^{\varepsilon,1}$ equals ε^{δ_1}, a jump of $(\mathfrak{I}_1 F)(t)$ is bounded by $\sup_{r\in[0,T]}\sup_{y\in\varepsilon\mathbb{Z}} \varepsilon^{1+\delta_1}|F(r,y)|$. For the predictable covariation we have

$$\begin{aligned}\langle \mathfrak{I}_1 F, \mathfrak{I}_1 F\rangle_T &= \varepsilon^2 \sum_{y,y'\in\varepsilon\mathbb{Z}} \int_{r=0}^T F(r,y)F(r,y')d\langle M^{\varepsilon,1}(y), M^{\varepsilon,1}(y')\rangle_r \\ &= \varepsilon\sum_{y\in\varepsilon\mathbb{Z}} \int_0^T F(r,y)^2 C_r^{\varepsilon,1}(y)dr \le c_{\varepsilon,1}\varepsilon\sum_{y\in\varepsilon\mathbb{Z}} \int_0^T F(r,y)^2 dr.\end{aligned}$$

Hence, the Burkholder–Davis–Gundy inequality (B.1) yields for every $p \geq 1$ the bound

$$\Big(\mathbb{E}\Big[\sup_{0\leq t\leq T}\big|(\mathfrak{I}_1 F)(r)\big|^p\Big]\Big)^{2/p} \lesssim c_{\varepsilon,1}\varepsilon\sum_{y\in\varepsilon\mathbb{Z}}\int_0^T F(r,y)^2 dr + \varepsilon^{2(1+\delta_1)}\sup_{r\in[0,T]}\sup_{y\in\varepsilon\mathbb{Z}}|F(r,y)|^2,$$

where the proportionality constant depends on p. This is exactly the required bound (B.3).

Now, we will proceed by induction. To this end, we assume that (B.3) holds for $n-1$, and will prove it for $n \geq 2$. As before, we prove the martingale $(\mathfrak{I}_n F)(t)$ using the Burkholder–Davis–Gundy inequality, for which we need to bound the jumps and quadratic covariation of $(\mathfrak{I}_n F)(t)$. Using the recursive definition (B.2) and properties of the martingales $M^{\varepsilon,n}$, the jump of $(\mathfrak{I}_n F)(t)$ can be bounded by

$$\varepsilon^{1+\delta_n}\sup_{r\in[0,t]}\sup_{y\in\varepsilon\mathbb{Z}}\big|(\mathfrak{I}^\varepsilon_{n-1}F^{(r,y)})(r-)\big|.$$

Furthermore, the quadratic covariation can be written as

$$\begin{aligned}\big\langle\mathfrak{I}_n F,\mathfrak{I}_n F\big\rangle_T &= \varepsilon^2\sum_{y,y'\in\varepsilon\mathbb{Z}}\int_{r=0}^T(\mathfrak{I}^\varepsilon_{n-1}F^{(r,y)})(r-)(\mathfrak{I}^\varepsilon_{n-1}F^{(r,y')})(r-)d\langle M^{\varepsilon,n}(y),M^{\varepsilon,n}(y')\rangle_r\\ &= \varepsilon\sum_{y\in\varepsilon\mathbb{Z}}\int_0^T(\mathfrak{I}^\varepsilon_{n-1}F^{(r,y)})(r-)^2 C^{\varepsilon,n}_r(y)dr\\ &\leq c_{\varepsilon,n}\varepsilon\sum_{y\in\varepsilon\mathbb{Z}}\int_0^T(\mathfrak{I}^\varepsilon_{n-1}F^{(r,y)})(r-)^2 dr.\end{aligned}$$

Combining these bounds with the Burkholder–Davis–Gundy inequality (B.1), we obtain

$$\begin{aligned}\Big(\mathbb{E}\Big[\sup_{0\leq t\leq T}\big|(\mathfrak{I}^\varepsilon_n F)(t)\big|^p\Big]\Big)^{2/p} &\lesssim c_{\varepsilon,n}\varepsilon\sum_{y\in\varepsilon\mathbb{Z}}\int_0^T\Big(\mathbb{E}\Big[(\mathfrak{I}^\varepsilon_{n-1}F^{(r,y)})(r-)^p\Big]\Big)^{2/p}dr\\ &\quad+\varepsilon^{2(1+\delta_n)}\Big(\mathbb{E}\Big[\sup_{\substack{r\in[0,t]\\ y\in\varepsilon\mathbb{Z}}}\big|(\mathfrak{I}^\varepsilon_{n-1}F^{(r,y)})(r-)\big|^p\Big]\Big)^{2/p},\end{aligned}$$

which holds for every $p \geq 1$ with a proportionality constant depending on p. Applying now the induction hypothesis to the stochastic integrals $(\mathfrak{I}^\varepsilon_{n-1}F^{(r,y)})(r-)$ on the r.h.s., we arrive at (B.3). □

Appendix C: Bounds on singular kernels

In this appendix, we provide some bounds on singular kernels, which are used in the proof of Lemma 5.3. For this, we follow the idea of [Hai14, Sec. 10.3] (or rather of [HM18, Sec. 6 and 7.1], since all the kernels are discrete), and introduce a "strength" of singularities of kernels.

More precisely, we consider a kernel $P^\varepsilon : \mathbb{R}\times(\varepsilon\mathbb{Z}) \to \mathbb{R}$, and for $\alpha > 0$ we will write $P^\varepsilon \in \mathcal{S}^\alpha_\varepsilon$ if P^ε can be written as $P^\varepsilon = K^\varepsilon + R^\varepsilon$, where the kernels K^ε and R^ε have the following properties:

1. The kernel K^ε is supported in a ball centered at the origin, and satisfies the bound

$$|K^\varepsilon(t,x)| \leq C(\sqrt{|t|} \vee |x| \vee \varepsilon)^{-\alpha}, \tag{C.1}$$

   uniformly in $t \in \mathbb{R}$, $x \in \varepsilon\mathbb{Z}$ and $\varepsilon \in (0,1]$, where the constant C is independent of t, x and ε.
2. For every fixed t, the kernel $R^\varepsilon(t,x)$ decays at infinity faster than any polynomial in the variable x, i.e. for every $n \in \mathbb{N}$ one has $\lim_{|x|\to\infty} R^\varepsilon(t,x)/|x|^n = 0$.
3. Moreover, $R^\varepsilon(t,x)$ is bounded uniformly in ε and is integrable over $\mathbb{R} \times (\varepsilon\mathbb{Z})$.

The value of α measures the "strength" of singularity of the kernel P^ε. Moreover, the following lemma shows that such singular kernels preserve these properties under multiplication and convolution.

Lemma C.1. *Let $P^\varepsilon \in \mathcal{S}_\varepsilon^\alpha$ and $\bar{P}^\varepsilon \in \mathcal{S}_\varepsilon^{\bar{\alpha}}$ be two singular kernels, for $\alpha, \bar{\alpha} > 0$. Then one has*

1. *the product $P^\varepsilon \bar{P}^\varepsilon$ is a singular kernel and $P^\varepsilon \bar{P}^\varepsilon \in \mathcal{S}_\varepsilon^{\alpha+\bar{\alpha}}$;*
2. *if $\alpha \vee \bar{\alpha} < 3$ and $\alpha + \bar{\alpha} > 3$, then the space-time convolution of these kernels satisfies $P^\varepsilon * \bar{P}^\varepsilon \in \mathcal{S}_\varepsilon^{\alpha+\bar{\alpha}-3}$.*

Proof. These results is a simple version of [HM18, Lem. 7.3]. □

The discrete heat kernel p^ε, defined in (A.2), is a singular kernel in the sense introduced above. More precisely, the following result holds:

Lemma C.2. *One has $p^\varepsilon \in \mathcal{S}_\varepsilon^1$ and $\nabla_\varepsilon^\pm p^\varepsilon \in \mathcal{S}_\varepsilon^2$, where $\nabla_\varepsilon^\pm$ are the discrete derivatives, defined in Section 1.3.*

Proof. This results follow from [HM18, Lem. 5.4]. □

The following lemma shows how such singular kernels behave under convolutions with scaled test functions.

Lemma C.3. *Let φ_z^λ be a scaled test function, as in (1.11), and let $P^\varepsilon \in \mathcal{S}_\varepsilon^\alpha$, for some $\alpha \in (0,1)$. Then the following bound holds:*

$$\varepsilon^2 \sum_{x_1,x_2 \in \varepsilon\mathbb{Z}} \left|\varphi_z^\lambda(x_1)\varphi_z^\lambda(x_2) P^\varepsilon(t, x_2 - x_1)\right| \leq C(\lambda \vee \varepsilon)^{-\alpha},$$

uniformly in $\lambda \in (0,1]$, $t \in \mathbb{R}$, $z \in \varepsilon\mathbb{Z}$ and $\varepsilon \in (0,1]$.

Proof. Combining the bound (C.1) with the definition of the rescaled function φ_z^λ, we obtain

$$\begin{aligned} \varepsilon^2 \sum_{x_1,x_2\in\varepsilon\mathbb{Z}} \left|\varphi_z^\lambda(x_1)\varphi_z^\lambda(x_2) P^\varepsilon(t,x_2-x_1)\right| &\lesssim \varepsilon^2 \sum_{x_1,x_2\in\varepsilon\mathbb{Z}} \left|\varphi_z^\lambda(x_1)\varphi_z^\lambda(x_2)\right| (|x_2-x_1| \vee \varepsilon)^{-\alpha} \\ \lesssim \varepsilon^2\lambda^{-2} \sum_{\substack{x_1,x_2\in\varepsilon\mathbb{Z}:\\ |x_1|\leq\lambda, |x_2|\leq\lambda}} (|x_2-x_1|\vee\varepsilon)^{-\alpha} &\lesssim \varepsilon\lambda^{-1} \sum_{\substack{x\in\varepsilon\mathbb{Z}:\\ |x|\leq 2\lambda}} (|x|\vee\varepsilon)^{-\alpha}, \end{aligned}$$

where in the last bound, we simply changed the variables of summation $x := x_2 - x_1$. If $\alpha < 1$, then the last sum is bounded by a multiple of $(\lambda \vee \varepsilon)^{-\alpha}$, uniformly in λ and ε. □

Appendix D: Proofs of Lemma 1.3 and Corollary 1.4

Proof of Lemma 1.3. For the duration of this proof, we will drop the subscript 0 and write s and $\hat{s}^\varepsilon$ in place of s_0 or $\hat{s}_0^\varepsilon$ (recall, these are only functions of space since they are initial data).

Part 1: Substituting $q = e^{-\varepsilon}$ into (1.7), we show by direct computation that $\hat{s}^\varepsilon(0)$ converges to a standard Gaussian random variable (centered with variance one). This can be done, for instance by observing that

$$\mathbb{P}\big(\hat{s}^\varepsilon(0) \le w\big) = \mathbb{P}\big(s(0) \le \varepsilon^{-\frac{1}{2}} w + \log_q \alpha\big) = \sum_{n=-\infty}^{N} \frac{\alpha^{-2n} q^{n(2n-1)}(1+\alpha^{-1}q^{2n})}{(-\alpha^{-1}, -q\alpha, q; q)_\infty},$$

where $N = \left\lfloor \frac{1}{2}\varepsilon^{-\frac{1}{2}} w + \frac{1}{2}\log_q \alpha \right\rfloor$. The last sum can be approximated by the Gaussian integral. By stationarity in the spatial coordinate, it follows that $\hat{s}^\varepsilon(x)$ likewise has a Gaussian limit (for each x). In the same manner, we may show that $\hat{s}^\varepsilon(x)$ satisfies (1.5a) for any x.

We turn to showing (1.5b). Thanks to the spatial stationarity, it suffices to show (1.5b) for $x_1 = x$ and $x_2 = 0$ where $x < 0$. If $x \le -1$, then by the triangle inequality, the stationarity of $\hat{s}^\varepsilon(x)$ and the bound (1.5a), there exists a positive absolute constant $C = C(k)$ such that

$$\|\hat{s}^\varepsilon(x) - \hat{s}^\varepsilon(0)\|_{2k} \le \|\hat{s}^\varepsilon(x)\|_{2k} + \|\hat{s}^\varepsilon(0)\|_{2k} \le 2C.$$

This proves (1.5b) when $x < -1$. It only remains to show that (1.5b) holds when $x \in (-1, 0)$.

Define $M : \mathbb{Z} \to \mathbb{R}$ such that $M(0) = 0$ and

$$M(x) = \begin{cases} \sum_{i=1}^{x} \big(s(i-1) - \mathbb{E}[s(i-1)\big|s(i)]\big), & x > 0, \\ \sum_{i=-x}^{-1} \big(s(i) - \mathbb{E}[s(i)\big|s(i+1)]\big), & x \le -1. \end{cases}$$

Using M we may write a difference equation for $s(x)$, with $x \in \mathbb{Z}$:

$$\nabla^- s(x) = s(x) - \mathbb{E}[s(x-1)\big|s(x)] - \nabla^- M(x) = -\frac{q^{s(x)} - \alpha}{q^{s(x)} + \alpha} - \nabla^- M(x). \quad \text{(D.1)}$$

For any $x \in \varepsilon\mathbb{Z}$, define $M^\varepsilon(x) := \sqrt{\varepsilon} M(\varepsilon^{-1}x)$ and extend to all $x \in \mathbb{R}\backslash\varepsilon\mathbb{Z}$ by linear interpolation. Thanks to (D.1),

$$\nabla_\varepsilon^- \hat{s}^\varepsilon(x) = -\sqrt{\varepsilon}\frac{q^{\varepsilon^{-\frac{1}{2}}\hat{s}^\varepsilon(x)} - 1}{q^{\varepsilon^{-\frac{1}{2}}\hat{s}^\varepsilon(x)} + 1} - \nabla_\varepsilon^- M^\varepsilon(x). \quad \text{(D.2)}$$

Taylor expanding with respect to ε,

$$\sqrt{\varepsilon}\frac{q^{\varepsilon^{-\frac{1}{2}}\hat{s}^\varepsilon(x)} - 1}{q^{\varepsilon^{-\frac{1}{2}}\hat{s}^\varepsilon(x)} + 1} = -\frac{1}{2}\varepsilon\hat{s}^\varepsilon(x) + \varepsilon^{\frac{3}{2}}\mathcal{B}^\varepsilon(x), \quad \text{(D.3)}$$

where $|\mathcal{B}^\varepsilon(x)| < (1 + |\hat{s}^\varepsilon(0)|)^2$ for all $|x| \le 1$. Plugging (D.3) into the r.h.s. of (D.2) and solving the discrete difference equation (D.2) yields

$$\hat{s}^\varepsilon(x) = \big(1 - \frac{\varepsilon}{2}\big)^{-\varepsilon^{-1}x}\hat{s}^\varepsilon(0) + \big(1 - \frac{\varepsilon}{2}\big)^{-\varepsilon^{-1}x} \sum_{\ell=\varepsilon^{-1}x}^{-1} \Big(1 - \frac{\varepsilon}{2}\Big)^{\ell} \Big(\nabla_\varepsilon^- M^\varepsilon(\varepsilon\ell) + \varepsilon^{\frac{3}{2}}\mathcal{B}^\varepsilon(\varepsilon\ell)\Big). \tag{D.4}$$

It is easy to check that this indeed solves (D.1) for all $x \in \varepsilon\mathbb{Z}_{<0}$.

We now claim that for any $x \in (-1, 0)$ and any $u > 0$

$$\mathbb{E}\Big[\exp\big(u|\hat{s}^\varepsilon(x) - \hat{s}^\varepsilon(0)|\big)\Big] \le 2\exp\Big(\frac{u^2}{2}\int_0^{|x|} e^{y-|x|}dy\Big). \tag{D.5}$$

Before verifying this, let us see how to complete the proof of (1.5b). Fix any $\beta \in (0, \frac{1}{4})$ and $u > 0$. Note that the integral inside exp on the r.h.s. of (D.5) is bounded above by $C' \min\big\{|x|^{4\beta}, 1\big\}$ for some constant $C' = C'(\beta) > 0$. Combining this with Markov's inequality, we find that

$$\big\|\hat{s}^\varepsilon(x) - \hat{s}^\varepsilon(0)\big\|_{2k} \le C'|x|^{2\beta},$$

which completes the proof of (1.5b).

We must now prove (D.5). It suffices to do this for $x \in \varepsilon\mathbb{Z}_{<0} \cap (-1, 0)$ since $|\hat{s}^\varepsilon(x) - \hat{s}^\varepsilon([x]_\varepsilon)| \le \sqrt{\varepsilon}$ where $[x]_\varepsilon$ is an element of $\varepsilon\mathbb{Z}_{<0}$ nearest to x. Let $x = \varepsilon^{-1}x$ where $x \in \varepsilon\mathbb{Z}_{<0} \cap (-1, 0)$. $\nabla_\varepsilon^- M^\varepsilon(x)$ is a centered Bernoulli random variable scaled by $\sqrt{\varepsilon}$. Hence, by using Hoeffding's inequality, we find that

$$\mathbb{E}\Big[\exp\big(u\nabla_\varepsilon^- M(x)\big)\big|s(x+1), s(x+2)\dots\Big] \le \exp(\varepsilon u^2/2).$$

In a similar way, we successively bound

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(u\big(1 - \frac{\varepsilon}{2}\big)^{\ell-\varepsilon^{-1}x}\nabla_\varepsilon^- M(\varepsilon\ell)\Big)\big|s(\ell+1), s(\ell+2), \dots\Big]\\ &\le \exp\Big(\frac{\varepsilon u^2}{2}\big(1 - \frac{\varepsilon}{2}\big)^{2(\ell-\varepsilon^{-1}x)}\Big),\end{aligned}$$

for all $\ell = \varepsilon^{-1}x + 1, \dots, -1$. Combining these bounds with the bound (1.5a) for $\hat{s}^\varepsilon(0)$, we see via (D.4) that

$$\mathbb{E}\Big[\exp\Big(u\big(\hat{s}^\varepsilon(x) - \hat{s}^\varepsilon(0)\big)\Big)\Big] \le \exp\Big(\frac{u^2}{2}\varepsilon\sum_{\ell=\varepsilon^{-1}x}^{-1}\big(1 - \frac{\varepsilon}{2}\big)^{2(\ell-\varepsilon^{-1}x)}\Big). \tag{D.6}$$

Summing the above inequality with u and $-u$ we find that l.h.s. (D.5) $\le 2\times$ r.h.s. (D.6). Recalling that $(1 - \varepsilon/2) \le e^{-\varepsilon/2}$, we may upper bound the sum on the r.h.s. of (D.6) by the integral in the r.h.s. of (D.5), completing the proof of (D.5).

Part 2: Owing to the Part 1 of this lemma and the Kolmogorov-Centsov criterion of tightness (see [Kal02, Theorem 2.23]), $\{\hat{s}^\varepsilon\}_\varepsilon$ is a tight sequence in $\mathcal{C}(\mathbb{R})$. Let us assume that we have some subsequential limit $\varepsilon_k \to 0$ along which $\{\hat{s}^{\varepsilon_k}(x)\}_{x\in\mathbb{R}}$ converges weakly to some limit $\{\mathcal{Z}_0(x)\}_{x\in\mathbb{R}}$ in $\mathcal{C}(\mathbb{R})$. We have already shown that for any x, marginally $\mathcal{Z}_0(x)$ must be standard Gaussian. Thus, it suffices to show that $\mathcal{Z}_0$ satisfies (1.8). By uniqueness of that solution, this will then show that all subsequential limits are the same, and hence we have convergence as desired.

So, we seek to prove that $\mathcal{Z}_0$ satisfies (1.8). Let us abuse notation and write ε in place of ε_k. Using similar argument to that used in proving (D.5), there exists $C > 0$ such that

$$\mathbb{E}\Big[\exp\big(u|M^\varepsilon(x_1) - M^\varepsilon(x_2)|\big)\Big] \le \exp\big(Cu^2|x_1 - x_2|\big),$$

for any $x_1, x_2 \in \mathbb{R}$ and $u > 0$. This shows the tightness of $\{M^\varepsilon\}_\varepsilon$ in $\mathcal{C}(\mathbb{R})$. Moreover, the weak convergence of $\hat{s}^\varepsilon$, (D.1) and the tightness of $\{M^\varepsilon\}_\varepsilon$ imply that M^ε also weakly converges to some limiting spatial process W taking values in $\mathcal{C}(\mathbb{R})$. Notice that $\varepsilon^{-\frac{1}{2}}(q^{\varepsilon^{-1/2}\hat{s}^\varepsilon(x)} - 1)/(q^{\varepsilon^{-1/2}\hat{s}^\varepsilon(x)} + 1) \Rightarrow -\frac{1}{2}\mathcal{Z}_0(x)$ in $\mathcal{C}(\mathbb{R})$. Combining this with (D.2) yields

$$d\mathcal{Z}_0(x) = -\frac{1}{2}\mathcal{Z}_0(x)dx + dW(x).$$

Now, it suffices to show that W is a two sided Brownian motion up to some scaling. To prove this, for any $\psi \in L^2(\mathbb{R})$, we define

$$\mathcal{M}^\varepsilon_\psi := \sum_{x\in\mathbb{Z}} \psi(\varepsilon x)\nabla^-_\varepsilon M^\varepsilon(\varepsilon x).$$

Owing to the decay of ψ, the weak convergence of M^ε to W and the moment bound of M^ε from (D.5), we see that $\mathcal{M}^\varepsilon_\psi$ converges weakly to $\int_\mathbb{R} \psi(x)dW(x)$. Furthermore, we have

$$\begin{aligned}\mathbb{E}\big[(\mathcal{M}^\varepsilon_\psi)^2\big] &= \sum_{x,y\in\mathbb{Z}} \psi(\varepsilon x)\psi(\varepsilon y)\mathbb{E}[\nabla^-_\varepsilon M^\varepsilon(\varepsilon x)\nabla^-_\varepsilon M^\varepsilon(\varepsilon y)] \\ &= \sum_{x\in\mathbb{Z}} \psi(\varepsilon x)\psi(\varepsilon x)\mathbb{E}[(\nabla^-_\varepsilon M^\varepsilon(x))^2], \end{aligned} \tag{D.7}$$

where the last line follows by noting that $\mathbb{E}[\nabla^-_\varepsilon M^\varepsilon(\varepsilon x)\nabla^-_\varepsilon M^\varepsilon(\varepsilon y)] = 0$ if $x \neq y$. Furthermore, $\mathbb{E}[(\nabla^-_\varepsilon M^\varepsilon(x))^2]$ is equal to $\mathbb{E}[q^{\varepsilon^{-1/2}\hat{s}^\varepsilon(\varepsilon x)}/(q^{\varepsilon^{-1/2}\hat{s}^\varepsilon(\varepsilon x)}+1)^2]$ which converges to $1/4$ as $\varepsilon \to 0$. Combining this with the decays of ψ shows that the r.h.s. of (D.7) converges to $\frac{1}{4}\int_R \psi^2(x)dx$ via dominated convergence theorem. Thus, for any $\psi \in L^2(\mathbb{R})$, we have $\mathbb{E}[(\int \psi(x)dW(x))^2] = \frac{1}{4}\int_\mathbb{R} \psi^2(x)dx$. Owing to this and $W(0) = 0$, W must be $1/2$ times a two sided Brownian motion. This shows that $\mathcal{Z}_0$ satisfies (1.8) as desired.
□

Proof of Corollary 1.4. The first part of Lemma 1.3 shows that the initial data $\{\hat{s}^\varepsilon_0\}$ satisfies (1.5a) and (1.5b) whereas the second part of the lemma shows that the initial data converges weakly to the stationary solution of the spatial Ornstein–Uhlenbeck process (1.8). Combining both parts of Lemma 1.3 with Theorem 1.1 completes the proof of this corollary. □

References

[ABB18] Aggarwal, A., Borodin, A., Bufetov, A.: Stochasticization of solutions to the Yang–Baxter equation. arXiv:1810.04299 (2018)

[Agg18] Aggarwal, A.: Dynamical stochastic higher spin vertex models. Selecta Mathematica, 24(3), 2659–2735, 2018

[AV87] Andjel, E.D., Vares, M.E.: Hydrodynamic equations for attractive particle systems on Z. J. Statist. Phys., 47(1–2):265–288, 1987

[BC17] Borodin, A., Corwin, I.: Dynamic ASEP, duality and continuous q^{-1}-Hermite polynomials. arXiv:1705.01980 (2017)

[BCD11] Bahouri, H., Chemin, J.-Y., Danchin, R.: Fourier analysis and nonlinear partial differential equations, Grundlehren der Mathematischen Wissenschaften, volume 343 . Springer, Heidelberg (2011)

[BG97] L. Bertini and G. Giacomin. Stochastic Burgers and KPZ equations from particle systems. Comm. Math. Phys., 183(3):571–607, 1997

[BG18] Borodin, A., Gorin, V.: A stochastic telegraph equation from the six-vertex model. arXiv:1803.09137 (2018)

[BGRS02] C Bahadoran, H Guiol, K Ravishankar, E Saada (2002) A constructive approach to Euler hydrodynamics for attractive processes Application to k-step exclusion. Stoch. Process. Appl. 99(1):1-3

[Bil99] Billingsley, P.: Convergence of probability measures Wiley Series in Probability and Statistics: Probability and Statistics. Wiley, New York, 2nd edition (1999)

[BM18] A. Bufetov and K. Matveev. Hall-Littlewood RSK field. Selecta Mathematica, 24(5), 4839–4884, 2018

[Bor17] Borodin, A.: Symmetric elliptic functions, IRF models, and dynamic exclusion processes. arXiv:1701.05239 (2017)

[BP16] Borodin, A., Petrov, L.: Lectures on integrable probability: stochastic vertex models and symmetric functions. arXiv:1605.01349 (2016)

[BP18] A. Borodin and L. Petrov (2018) Higher spin six vertex model and symmetric rational functions. *Selecta Math.*, 24(2):751–874,

[CM18] G. Cannizzaro and K. Matetski. Space-time discrete KPZ equation. Comm. Math. Phys., 358(2):521–588, 2018

[CP16] I. Corwin and L. Petrov. Stochastic higher spin vertex models on the line. Comm. Math. Phys., 343(2):651–700, 2016

[CST18] I. Corwin, H. Shen, and L. C. Tsai. ASEP(q, j) converges to the KPZ equation. Ann. Inst. Henri Poincaré Probab. Stat., 54(2):995–1012, 2018

[DG91] P. Dittrich and J. Gärtner. A central limit theorem for the weakly asymmetric simple exclusion process. Math. Nachr., 151:75–93, 1991

[DMPS89] A. De Masi, E. Presutti, and E. Scacciatelli. The weakly asymmetric simple exclusion process. Ann. Inst. H. Poincaré Probab. Statist., 25(1):1–38, 1989

[DPD03] G. Da Prato and A. Debussche. Strong solutions to the stochastic quantization equations. Ann. Probab., 31(4):1900–1916, 2003

[DPZ14] Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions, Encyclopedia of Mathematics and its Applications, 2nd edition. Cambridge University Press, Cambridge (2014)

[DT16] A. Dembo and L.-C. Tsai. Weakly asymmetric non-simple exclusion process and the Kardar-Parisi-Zhang equation. Comm. Math. Phys., 341(1):219–261, 2016

[FKS91] P. A. Ferrari, C. Kipnis, and E. Saada. Microscopic structure of travelling waves in the asymmetric simple exclusion process. Ann. Probab., 19(1):226–244, 1991

[Gär88] J. Gärtner. Convergence towards Burgers' equation and propagation of chaos for weakly asymmetric exclusion processes. Stoch. Proc. Appl., 27(2):233–260, 1988

[GJ12] Patrícia Gonçalves and Milton Jara. Crossover to the KPZ equation. Ann. Henri Poincaré, 13(4), 813–826, 2012

[Hai09] Hairer, M.: An introduction to stochastic PDEs. arXiv:1605.01349 (2009)

[Hai14] M. Hairer. A theory of regularity structures. Invent. Math., 198(2):269–504, 2014

[HH80] Hall, P., Heyde, C.C.: Martingale Limit Theory and Its Application. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London (1980). Probability and Mathematical Statistics

[HL18] M. Hairer and C. Labbé. Multiplicative stochastic heat equations on the whole space. J. Eur. Math. Soc., 20(4):1005–1054, 2018

[HM18] M. Hairer and K. Matetski. Discretisations of rough stochastic PDEs. Ann. Probab., 46(3):1651–1709, 2018

[JS03] Jacod, J., Shiryaev, A.N.: Limit theorems for stochastic processes, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 288. Springer, Berlin,(2003)

[Kal02] Kallenberg, O.: Foundations of modern probability Probability and its Applications (New York). Springer, New York, 2nd edition (2002)

[MW17] J. C. Mourrat and H. Weber. Convergence of the two-dimensional dynamic Ising-Kac model to Φ^4_2. Comm. Pure Appl. Math., 70(4):717–812, 2017

[PR18] Perkowski, N., Rosati, T.-C.: The KPZ equation on the real line. arXiv:1808.00354 (2018)
[SV06] Stroock, D.W., Varadhan S.R.S.: Multidimensional diffusion processes. Classics in Mathematics. Springer, Berlin (2006). Reprint of the 1997 edition
[Wal86] Walsh,J.B.: An introduction to stochastic partial differential equations. In: École d'été de probabilités de Saint-Flour, XIV—1984, volume 1180 of Lecture Notes in Mathematics, pp. 265–439. Springer, Berlin (1986)

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