Finite-Time Distributed Convex Optimization for Continuous-Time Multiagent Systems With Disturbance Rejection

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***Abstract*—This paper presents continuous distributed algorithms to solve the finite-time distributed convex optimization problems of multiagent systems in the presence of disturbances. The objective is to design distributed algorithms such that a team of agents seeks to minimize the sum of local objective functions in a finite-time and robust manner. Specifically, a distributed optimization algorithm, combined with a continuous integral sliding-mode control scheme, is proposed to solve this finite-time optimization problem, while rejecting local disturbance signals. The developed algorithm is further applied to solve economic dispatch and resource allocation problems, and proven that under proposed schemes, the optimal solution can be achieved in finite time, while satisfying both global equality and local inequality constraints. Examples and numerical simulations are provided to show the effectiveness of the proposed methods.**

***Index Terms*—Distributed convex optimization, disturbance rejection, finite-time convergence, multiagent system.**

I. Introduction

In recent years, there is increasing attention devoted to the distributed optimization problem, where a team of agents cooperatively minimizes the sum of agents' local objective functions in a distributed way, i.e.,

$$\min_{\theta} \mathrm{F}(\theta) = \sum_{i=1}^{N} f_i(\theta),\ i = 1, 2, \ldots, N \tag{1}$$

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where $\theta \in \mathbb{R}^n$ is a global decision variable and $f_i(\theta) : \mathbb{R}^n \rightarrow \mathbb{R}$ represents a local objective function for agent i. This problem arises in many applications involving multiagent systems, including parameter estimation and source localization in sensor networks [1], energy and thermal comfort optimization in smart building [2], demand response [3], and economic dispatch [4] in smart grid, and resource allocation in the multicell network [5], to just list a few. In the existing literature, gradient-based methods were widely employed to solve convex optimization problems. In particular, the authors in [6] presented distributed gradient-based optimization algorithms to minimize cost functions. By using the consensus design, a subgradient scheme was proposed in [7] to obtain an approximately optimal solution with a constant step size. A projected distributed subgradient method was developed in [8] to handle a set constraint. It was extended to the dual problem with constraints in [9]. Zero-gradient-sum (ZGS) algorithms were designed in [10] from a control point to enable the convergence to an optimal solution.

Most of the existing results build on consensus algorithms described by either discrete-time dynamics or continuous-time dynamics to find the optimal solution. By including a quadratic penalty in the Lagrangian problem, the authors in [11] employed saddle-point dynamics with proportional–integral (PI) like consensus schemes to find the optimal solution with an undirected graph. The authors in [12] extended the design to a discrete-time communication case. Furthermore, the event-triggered distributed optimization was studied in [13]. One observation is that all of the aforementioned discrete-time and continuous-time optimization algorithms were based on linear algorithms, which enabled the optimal solutions asymptotic or exponential. That is, the optimal solutions were achieved over an infinite-time horizon, which only provided the suboptimal solutions for practical applications. Thus, it is highly desirable to achieve the optimal solution in finite time.

In multiagent systems, many techniques have been proposed to obtain finite-time consensus. A discontinuous sliding-mode scheme via a signum function was developed in [14] to achieve finite-time consensus. However, this nonsmooth algorithm was undesired in reality due to chattering behaviors. Hence, continuous finite-time designs were developed. Tools from the homogeneity theory were utilized in [15] for finite-time consensus, while the analysis made the estimation of settling time difficult. Besides, distributed protocols with odd functions were provided in [16], where each agent was required to obtain its neighbors' inputs simultaneously, causing a control-loop problem. Further, a finite-time protocol via a smoothing factor was presented in [17] to remove the chattering. Although these smooth designs enable finite-time convergence, robustness against disturbances cannot be achieved. Recently, the integral sliding-mode control (ISMC) schemes were widely used to provide robustness in multiagent coordination. In [18], the ISMC scheme was adopted to achieve robustication of average consensus. The authors in [19] also adopted the ISMC scheme with event-triggered sampling to achieve formation tracking. In [20], finite-time consensus was obtained based on the ISMC scheme for second-order multiagent systems with disturbances. Mismatched disturbances were further considered in [21] and a supertwisting-based ISMC scheme motivated by [22] and [23] was adopted to achieve finite-time formation tracking.

So far, finite-time convergence and robustness against uncertainties/disturbances have not been fully addressed in distributed optimization, which are important in practice. Take the power system for instance. Since there are many renewable energy resources introduced to the power system, frequent and severe changes of operating conditions require a faster convergence rate of a distributed optimal solution to meet the challenges of power system developments [24]. In addition, the lack of robust designs against disturbances may result in unreliable dispatches and instability of the power system. Due to the nonlinearity of the cost function and disturbances, the existing distributed optimization techniques may not be directly applied. Recently, some results have been reported on distributed convex optimization with either finite-time convergence or disturbance rejection. In particular, the signum function-based algorithms were developed to achieve finite-time distributed optimization of multiagent systems with first-order dynamics [25], linear dynamics [26], and a convex constraint [27]. However, those nonsmooth algorithms are usually undesired in practice. In the absence of disturbances, the finite-time distributed optimization was achieved in [28] via a smooth factor. In the presence of disturbances, the authors in [29] studied the "bounded" optimization error with disturbances and proposed algorithms to enable the convergence to a neighborhood of optimization solutions. Based on an internal model design, distributed optimization was considered for multiagent systems with known-frequency disturbances [30] or unknown-frequency disturbances [31], where the convergence to an optimization point was achieved in a semiglobal sense. In contrast, our preliminary work in [32] first employed a smooth ISMC scheme to solve a distributed convex optimization problem with the simultaneous finite-time and robust properties, while it only considered quadratic optimization.

In this paper, we investigate a finite-time distributed optimization problem for continuous-time multiagent systems subject to disturbances. By using a supertwisting-based ISMC strategy, finite-time distributed optimization algorithms are proposed to solve this problem with finite-time convergence and disturbance rejection. Specifically, to guarantee that the disturbance does not influence the search of an optimal solution, the ISMC scheme is adopted so that the equivalent system of the original system can reside on the sliding manifold in a finite time with disturbance rejection. Then, distributed optimization can be achieved on the equivalent system via a finite-time distributed gradient protocol. Compared with the existing works, the main contributions of this paper are as follows.

1) Distributed optimization algorithms with a supertwisting-based ISMC scheme are presented to search for the optimal solutions in a finite-time and robust fashion for distributed quadratic and nonquadratic optimization, respectively. The proposed optimization algorithms are continuous and distributed, which avoid the chattering phenomenon and/or the control loop issue that widely exists in the finite-time consensus works (e.g., [14], [16], [18], [19]).
2) In contrast to the algorithms in [25]–[31] that are designed to achieve finite-time convergence or disturbance rejection separately, the proposed algorithms in this paper guarantee the finite-time convergence of the optimal solution for distributed optimization, and meanwhile enable the complete disturbance-rejecting property. The aforementioned design limitations in [25]–[31] are also removed. Moreover, unlike [29]–[31], the developed algorithms are allowed for any arbitrary disturbances that only satisfy mild and reasonable smoothness and boundedness properties.
3) The proposed distributed optimization algorithms are further employed to solve the economic dispatch and resource allocation problems, respectively. The penalty function and saddle-point dynamics are leveraged to search for the optimal solutions, respectively. The proposed algorithms can handle both global equality and local inequality constraints. Convergence is proven by using Lyapunov analysis.

Organization: preliminaries are provided in Section II, and a distributed optimization problem is formulated in Section III. Sections IV and V present finite-time algorithms to search for the optimal solutions for distributed quadratic and nonquadratic optimization, respectively. The economic dispatch and resource allocation problems are further solved. Numerical examples are given in Section VI, while Section VII concludes this paper.

II. Preliminaries

A. Notation

$\mathbb{R}$ and $\mathbb{R}^{N\times N}$ denote the sets of reals and $N \times N$ matrices, respectively. Let $\mathrm{col}(x_1,\ldots,x_N)$ and $\mathrm{diag}\{a_1,\ldots,a_N\}$ represent a column vector with entries x_i and a diagonal matrix with a_i, $i = 1, 2, \ldots, N$, respectively. The symbol $\otimes$ denotes the Kronecker product. Define $1_N = \mathrm{col}(1,\ldots,1) \in \mathbb{R}^N$. For a matrix $P = P^T$, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote its minimum and maximum eigenvalues, respectively. For a scalar $x_i \in \mathbb{R}$, we define $\mathrm{sig}^\theta(x_i) = |x_i|^\theta \mathrm{sign}(x_i)$, where $\theta \in (0, 1)$, $\mathrm{sign}(x_i)$ is the signum function, and $|x_i|$ is the absolute value of $x_i \in \mathbb{R}$. For a vector $x_i \in \mathbb{R}^n$, we define

$$\mathrm{sig}^\theta(x_i) = \|x_i\|^\theta\, \mathrm{sign}(x_i) \tag{2}$$

where $||x_i|| \in \mathbb{R}$ denotes the two-norm of a vector $x_i \in \mathbb{R}^n$.

B. Graph Theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represent a graph where $\mathcal{V} \in \{1, 2, \ldots, N\}$ denotes the set of vertices. Every agent is represented by a vertex. The set of edges is denoted as $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We assume that there is no self-loop in the graph, that is, $(i,i) \notin \mathcal{E}$. $\mathcal{N}_i(\mathcal{G}) = \{j \in \mathcal{V} \,|(j,i) \in \mathcal{E}\}$ denotes the neighborhood set of vertex i. Graph $\mathcal{G}$ is said to be undirected if for any edge $(i,j) \in \mathcal{E}$, edge $(j,i) \in \mathcal{E}$. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ denotes the adjacency matrix of $\mathcal{G}$, where $a_{ij} > 0$ if and only if $(j,i) \in \mathcal{E}$, else $a_{ij} = 0$. The Laplacian matrix of $\mathcal{G}$ is denoted by $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$, where $l_{ii} = \sum_{j=1}^N a_{ij}$ and $l_{ij} = -a_{ij}$ if $i \neq j$. Let $\mathcal{L} = D - \mathcal{A}$ with the diagonal matrix given by $D = \text{diag}\{\sum_{j=1}^N a_{ij}\}$.

C. Saddle-Point Dynamics

A pair (x^*, y^*) is a min–max saddle point of $F(x,y)$ if for all (x,y), the following inequality is satisfied [36]:

$$F(x^*, y) \leq F(x^*, y^*) \leq F(x, y^*).$$

Saddle-point dynamics is a method that seeks the saddle point of a continuously differentiable function $F(x,y)$ that is strictly convex in x and concave in y. The idea is to minimize $F(x,y)$ with respect to x and maximize it with respect to y. For $k_1, k_2 > 0$, the saddle-point dynamics are given by

$$\dot{x} = -k_1 \frac{\partial F(x,y)}{\partial x},\ \dot{y} = k_2 \frac{\partial F(x,y)}{\partial y}. \tag{3}$$

Suppose that the optimal solution of this optimization problem exists and is finite. Let x^* and y^* be the optimal solution of the primal and dual problems, respectively. Then, (3) enables (x,y) to converge to (x^*, y^*) asymptotically [36].

D. Finite-Time Stability

Lemma 1: [17] If $\xi_1, \xi_2, \ldots, \xi_N \geq 0$ and $0 < p \leq 1$, then

$$\left(\sum_{i=1}^N \xi_i\right)^p \leq \sum_{i=1}^N \xi_i^p. \tag{4}$$

Lemma 2: [20] Consider a continuous function $\dot{x} = f(x,t)$ with $f(0,t) = 0$, $x \in \mathcal{D} \subset \mathbb{R}^n$. Suppose that there exists a $\mathcal{C}^1$ function $V(x)$ defined on a neighborhood of the origin, and real numbers $a > 0$, $0 < \beta < 1$ such that $V(x) \geq 0$ and

$$\dot{V}(x) + aV^\beta(x) \leq 0 \tag{5}$$

then the origin of the system is finite-time stable, that is, $V(x)$ will reach zero in a finite time with the settling time $t^* \leq \frac{V^{1-\beta}(x(0))}{a(1-\beta)}$ and $V(x) = 0$ for all $t > t^*$.

III. Problem Formulation

Consider a network of N agents interacting over the graph $\mathcal{G}$. Suppose that each agent generates a local estimate $\mathbf{x}_i(t)$ on the optimal solution to the problem in (1) according to the following single-integrator agent dynamics:

$$\dot{\mathbf{x}}_i(t) = u_i(t) + \omega_i(t),\ i = 1, 2, \ldots, N \tag{6}$$

where $\mathbf{x}_i(t), u_i(t) \in \mathbb{R}^n$ denote, respectively, the state and control input of agent i, and $\omega_i(t) \in \mathbb{R}^n$ are bounded disturbances. Since all agents can communicate only with their neighbors in the network, local gradient information and relative decision variables will be utilized to find the optimal solution.

Problem 1. *Fast Distributed Convex Optimization:* Each agent produced by the distributed algorithm (6) seeks the optimal solution of (1) in a finite time by solving the following distributed convex optimization problem:

$$\min_{\mathbf{x}}\ F(\mathbf{x}) = \sum_{i=1}^N f_i(\mathbf{x}_i), \text{subject to } \tilde{\mathcal{L}}\mathbf{x} = 0,\ \tilde{\mathcal{L}} = \mathcal{L} \otimes I_n \tag{7}$$

where $\mathbf{x} = \text{col}(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is a collective vector of $\mathbf{x}_i$.

It follows from [11] that if the optimal solution of the problem in (1) is $\theta^* = \arg\min_\theta \sum_{i=1}^N f_i(\theta)$, then $1_N \otimes \theta^*$ is an optimal solution to (7) [11]. The objective of this paper is thus to design a distributed updating algorithm u_i so that

1) u_i, depending on local gradient information and relative decision variables, is robust against disturbances ω_i;
2) $\mathbf{x}_i$ fastly converges to the optimal solution under ω_i.

Remark 1: Notice that the formulated problem can cover the (finite-time) consensus/distributed optimization, and distributed optimization with disturbance rejection as special cases. Clearly, without assigning the cost functions to the agent network, then it becomes a finite-time consensus problem studied in [14]–[20]. When the agent dynamics are not subject to disturbances, the studied problem is reduced to a distributed optimization problem considered in [8]–[13] or a finite-time distributed optimization problem studied in [25]–[28]. If the finite-time convergence is not required, it is reduced to the studied distributed optimization problem with disturbance rejection in [29]–[31].

Remark 2: Solving Problem 1 in a fast and distributed manner is important and challenging. To the best of our knowledge, there are only a few works in [25]–[31] that study the finite-time and robust convergence, separately, for distributed optimization. This paper develops a smooth distributed optimization algorithm such that the system is capable of achieving finite-time and robust convergence simultaneously. Equation (7) may also be formulated as a fixed terminal time (the desired finite-time convergence time) optimal control problem where (6) contains the state dynamics. The solution is not necessarily distributed, which is why we adopt *a priori* distributed formulation.

To solve this problem, we make the following assumptions.

Assumption 1: Each agent can communicate with its neighbors through an undirected and connected graph $\mathcal{G}$.

Assumption 2: The disturbance $\omega_i(t)$ and its time derivative are bounded by known constants (i.e., $\omega_i(t), \dot{\omega}_i(t) \in \mathcal{L}_\infty$).[1]

Remark 3: Assumption 1 on the communication graph has been widely utilized in the existing papers (e.g., [6]–[8], [10], [26], [31]) that solve distributed convex optimization problems. Assumption 2 has also been widely adopted in the existing work (e.g., see [20] and [21] for just an example) to deal with continuous and differentiable disturbances. Many types of practical disturbances satisfy this assumption, including constant, ramp, and sinusoidal disturbances. Besides, harmonic disturbances in [30] and [31] can be covered as a special case.

IV. Finite-Time Distributed Quadratic Optimization

A. Solve Problem 1 for Quadratic Objective Functions

Assumption 3: $f_i(\theta)$ are quadratic objective functions.

From Assumption 3, suppose that $f_i(\mathbf{x}_i)$ in (7) is given by

$$f_i(\mathbf{x}_i) = \mathbf{x}_i^T A_i \mathbf{x}_i + B_i^T \mathbf{x}_i + C_i \tag{8}$$

where $\mathbf{x}_i \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}(A_i > 0)$, $B_i \in \mathbb{R}^n$, and $C_i \in \mathbb{R}$.

[1] Adaptive controller designs in [40] might be adopted in future work to remove the need for these known upper bounds.

A penalty function-based design is used to find an approximate optimal solution of Problem 1. Define

$$P(\mathbf{x}) = \sum_{i=1}^{N} f_i(\mathbf{x}_i) + \frac{\gamma}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\|\mathbf{x}_i - \mathbf{x}_j\|^2 \tag{9}$$

where $\gamma > 0$ is a penalty parameter and the second term denotes the penalty for violations on the constraint $\tilde{\mathcal{L}}\mathbf{x} = 0$. The penalty term is equal to 0 if and only if $\tilde{\mathcal{L}}\mathbf{x} = 0$. Let $\mathbf{x}_i^*$ be the optimal solution of (9). By the Karush–Kuhn–Tucker (KKT) condition, $P(\mathbf{x})$ reaches the minimum if $2A_i\mathbf{x}_i^* + B_i + \gamma\sum_{j=1}^{N} l_{ij}\mathbf{x}_j^* = 0$.

A finite-time distributed algorithm is thus proposed as

$$u_i = u_i^o + u_i^r,\ i = 1, 2, \ldots, N \tag{10a}$$

$$u_i^o = -\mathrm{sig}^{\alpha}\left(\nabla f_i(\mathbf{x}_i) + \gamma\sum_{j=1}^{N} a_{ij}(\mathbf{x}_i - \mathbf{x}_j)\right) \tag{10b}$$

$$u_i^r = -k_{1i}\mathbf{sig}^{\frac{1}{2}}(s_i) + \phi_i,\ \dot{\phi}_i = -k_{2i}\mathrm{sign}(s_i) \tag{10c}$$

$$s_i = \mathbf{x}_i - \mathbf{x}_i(0) - \int_0^t u_i^o(\tau)d\tau \tag{10d}$$

where $\nabla f_i(\mathbf{x}_i)$ is the gradient of the objective function, $\gamma > 0$ is the penalty parameter, $\mathrm{sig}^{\alpha}(\cdot)$ is defined in (2), $\alpha \in (0, 1)$, $\mathbf{sig}^{\frac{1}{2}}(s_i) = \mathrm{col}(\mathrm{sig}^{\frac{1}{2}}(s_{i1}), \ldots, \mathrm{sig}^{\frac{1}{2}}(s_{in}))$ with $\mathrm{sig}^{\frac{1}{2}}(s_{ik}) = |s_{ik}|^{\frac{1}{2}}\mathrm{sign}(s_{ik})$, $k = 1, \ldots, n$, and $k_{1i}, k_{2i} > 0$ are constants.

Remark 4: To solve Problem 1, the distributed optimization algorithm has been proposed by adopting a supertwisting-based ISMC scheme in (10c) and (10d), which is capable of achieving chattering avoidance, disturbance rejection, and finite-time convergence simultaneously. In particular, the proposed algorithm includes two parts: the continuous distributed optimal controller u_i^o and the continuous ISMC controller u_i^r. To make sure that the disturbance does not influence the achievement of distributed optimization, we employ u_i^r that makes the equivalent system of the original system reside on the sliding manifold in finite time with disturbance rejection. As a result, the finite-time distributed optimization can be achieved on the equivalent system via the designed continuous optimal controller u_i^o.

Remark 5: The proposed distributed algorithm in (10) can represent many protocols in the existing literature. For example, if the disturbances are not considered in (6), the controller u_i^r is removed, and the system becomes $\dot{\mathbf{x}}_i = -\mathrm{sig}^{\alpha}(\nabla f_i(\mathbf{x}_i) + \gamma\sum_{j=1}^{N} a_{ij}(\mathbf{x}_i - \mathbf{x}_j))$. It can be seen that if we set $\alpha = 1$, it becomes the typical nonlinear optimization protocol, and in this case, it can solve an asymptotic distributed convex optimization problem in [6]–[13]. Moreover, if we set $\alpha = 0$, it becomes the discontinuous distributed optimization algorithm studied in [25]–[27]. Furthermore, when the agents are not assigned with the local gradients of the objective functions, it becomes the typical discontinuous consensus protocol in [14] for $\alpha = 0$, the finite-time consensus protocol in [17] for $\alpha \in (0, 1)$, and the consensus protocol in [40] for $\alpha = 1$.

Theorem 1: Under Assumptions 1–3, the proposed updating algorithm in (10) enables the agents' strategies to converge to an approximate optimal solution of Problem 1 in a finite time, that is, $\lim_{t\to T_1}\mathbf{x}_i = \mathbf{x}_i^*,\ i = 1, 2, \ldots, N$.

Proof: The proof includes two steps:

Step 1: prove that $\lim_{t\to T_0} s_i = \dot{s}_i = 0$;

Step 2: prove that $\lim_{t\to T_1}\mathbf{x}_i = \mathbf{x}_i^*$, where T_0, T_1 are to be determined in the subsequent analysis.

i) Taking the time derivative of (10d) and submitting the updating law into the agent dynamics (6) yield

$$\dot{s}_i = u_i^r + \omega_i = -k_{1i}\mathbf{sig}^{\frac{1}{2}}(s_i) + \varphi_i \tag{11a}$$

$$\dot{\varphi}_i = -k_{2i}\mathrm{sign}(s_i) + \dot{\omega}_i \tag{11b}$$

$$\varphi_i = \phi_i + \omega_i,\ i = 1, 2, \ldots, N. \tag{11c}$$

Inspired by [23], define a variable $\xi_{ik} = \mathrm{col}(\mathrm{sig}^{\frac{1}{2}}(s_{ik}), \varphi_{ik})$ where s_{ik}, φ_{ik} is the kth element of s_i, φ_i, respectively. Select a Lyapunov function candidate $V(t) = \sum_{i=1}^{N}\sum_{k=1}^{n} V_{ik}(t) = \sum_{i=1}^{N}\sum_{k=1}^{n}\xi_{ik}^T P_{ik}\xi_{ik}$, where $P_{ik} \in \mathbb{R}^{2\times 2}$ is a constant, symmetric, and positive definite matrix. Due to the term $\mathrm{sig}^{\frac{1}{2}}(s_{ik})$, V is absolutely continuous (AC) but not locally Lipschitz on the set $\Xi = \{(s_{ik}, \varphi_{ik}) \in \mathbb{R}^2 | s_{ik} = 0\}$. This violates the classical Lyapunov theorem, which requires the Lyapunov function to be continuously differentiable, or at least locally Lipschitz. As illustrated in [23], $V(t)$ can still be used for stability analysis thanks to the Zubov theorem, which only requires a Lyapunov function to be continuous. Similarly, it can be checked that $V(t)$ is an AC function of t, and thus, its time derivative is defined almost everywhere [40]–[42].

The time derivative of ξ_{ik} can be expressed as

$$\dot{\xi}_{ik} = \frac{1}{2}|s_{ik}|^{-\frac{1}{2}}\begin{bmatrix} -k_{1i}\mathrm{sig}^{\frac{1}{2}}(s_{ik}) + \varphi_{ik} \\ -2[k_{2i} - \dot{\omega}_{ik}\mathrm{sign}(s_{ik})]\mathrm{sig}^{\frac{1}{2}}(s_{ik}) \end{bmatrix}.$$

Then, the time derivative of $V(t)$ along (11) is given by

$$\dot{V} = \sum_{i=1}^{N}\sum_{k=1}^{n}|s_{ik}|^{-\frac{1}{2}}\xi_{ik}^T\left(R_{ik}^T P_{ik} + P_{ik}R_{ik}\right)\xi_{ik} \tag{12}$$

where $R_{ik} = \begin{bmatrix} -\frac{1}{2}k_{1i} & \frac{1}{2} \\ -[k_{2i} - \dot{\omega}_{ik}\mathrm{sign}(s_{ik})] & 0 \end{bmatrix}$ is Hurwitz if and only if $k_{1i} > 0$, $k_{2i} > \|\dot{\omega}_i\|_\infty$ by Assumption 2.

Since R_{ik} is Hurwitz, there exists a unique solution P_{ik} to the following algebraic Lyapunov equation $R_{ik}^T P_{ik} + P_{ik}R_{ik} = -Q_{ik}, i = 1, 2, \ldots, N, k = 1, 2, \ldots, n$ for each symmetric and positive define matrix Q_{ik}, such that for the constructed strict Lyapunov function $V(t)$ [23], we can further obtain $\dot{V}(t) = -\sum_{i=1}^{N}\sum_{k=1}^{n}|s_{ik}|^{-\frac{1}{2}}\xi_{ik}^T Q_{ik}\xi_{ik} \le 0$. In addition, $|s_{ik}|^{\frac{1}{2}} = |\mathrm{sig}^{\frac{1}{2}}(s_{ik})| \le |\xi_{ik}| \le \lambda_{\min}^{-\frac{1}{2}}(P_{ik})V_{ik}^{\frac{1}{2}}$. Then, it follows from (4) that for $\alpha_0 = \min_{i,k}\{\lambda_{\min}^{\frac{1}{2}}(P_{ik})\lambda_{\min}(Q_{ik})/\lambda_{\max}(P_{ik})\}$:

$$\begin{aligned}\dot{V} &\le -\sum_{i=1}^{N}\sum_{k=1}^{n}\lambda_{\min}^{\frac{1}{2}}(P_{ik})V_{ik}^{-\frac{1}{2}}\frac{\lambda_{\min}(Q_{ik})}{\lambda_{\max}(P_{ik})}V_{ik} \\ &\le -\alpha_0\sum_{i=1}^{N}\sum_{k=1}^{n}V_{ik}^{\frac{1}{2}} \le -\alpha_0\left(\sum_{i=1}^{N}\sum_{k=1}^{n}V_{ik}\right)^{\frac{1}{2}} = -\alpha_0 V^{\frac{1}{2}}.\end{aligned} \tag{13}$$

Therefore, it is obtained that $V \in \mathcal{L}_\infty$ and $s_i(t)$ converges to zero in a finite time with $T_0 = \frac{2}{\alpha_0}V^{\frac{1}{2}}(0)$ based on Lemma 2. Moreover, it follows from (10d) that $s_i(0) = 0$, which means that $s_i(t)$ in (10d) starts on it at the initial time and afterwards, the multiagent system will not go away in sequential time. By Lemma 2, the accurate finite-time convergence is achieved despite disturbances, that is, $\lim_{t\to T_0} s_i = \dot{s}_i = 0$.

Remark 6: With the signum function $\text{sign}(s_i)$ in the proposed algorithm, the right-hand sides of $\dot{\phi}_i$ in (10c) are discontinuous and their solutions should be investigated in terms of differential inclusions via the nonsmooth analysis [40]–[42]. However, since the signum function is measurable and locally essentially bounded, the Filippov solutions of the closed-loop dynamics always exist. Besides, the adopted Lyapunov function candidate is AC. Thus, its set-valued Lie derivative is a Singleton at the discontinuous points and the proof still holds. To avoid symbol redundancy, the differential inclusions are not utilized. Further, the Filippov solutions are AC curves [42], which means that the agents' states are continuous.

ii) By Step 1, $\dot{s}_i = \dot{\mathbf{x}}_i - u_i^o = 0, t \geq T_0$, i.e.,

$$\dot{\mathbf{x}}_i = -\text{sig}^{\alpha}\left(\nabla f_i(\mathbf{x}_i) + \gamma \sum_{j=1}^{N} a_{ij}(\mathbf{x}_i - \mathbf{x}_j)\right), 0 < \alpha < 1. \tag{14}$$

Next, we will prove $\lim_{t\to T_1} \mathbf{x}_i = \mathbf{x}_i^*$. Define $e_i = \mathbf{x}_i - \mathbf{x}_i^*$. By using the first-order optimal condition

$$\begin{aligned}\frac{\partial P(\mathbf{x}_i)}{\partial \mathbf{x}_i} &= 2A_i(e_i + \mathbf{x}_i^*) + B_i + \gamma \sum_{j=1}^{N} l_{ij}(e_j + \mathbf{x}_j^*) \\ &= 2A_i e_i + \gamma \sum_{j=1}^{N} a_{ij}(e_i - e_j).\end{aligned} \tag{15}$$

The updating law in (14) is thus rewritten as $\dot{\mathbf{x}}_i = \dot{e}_i = -\text{sig}^{\alpha}(2A_i e_i + \gamma \sum_{j=1}^{N} a_{ij}(e_i - e_j))$. Define an error variable $\vartheta_i = 2A_i e_i + \gamma \sum_{j=1}^{N} a_{ij}(e_i - e_j)$. Then, $\dot{e}_i = -\text{sig}^{\alpha}(\vartheta_i)$. Let $e, \vartheta, \text{sign}(\vartheta)$ be the stack vectors of e_i, ϑ_i, and $\text{sign}(\vartheta_i)$, respectively. Thus, $\vartheta = He$ and $\dot{e} = -(\text{diag}\{\|\vartheta_i\|^{\alpha}\} \otimes I_n)\text{sign}(He)$, where $H = \mathbf{A} + \gamma(\mathcal{L} \otimes I_n)$ with $\mathbf{A} = \text{diag}\{2A_i\}$ being a positive definite diagonal matrix by Assumption 3.

As the graph is undirected and connected by Assumption 1, H is invertible [13]. Define a Lyapunov function candidate as

$$W = \frac{1}{2}\left(\frac{\partial P(\mathbf{x})}{\partial \mathbf{x}}\right)^T H^{-1}\left(\frac{\partial P(\mathbf{x})}{\partial \mathbf{x}}\right) = \frac{1}{2}\vartheta^T H^{-1}\vartheta \tag{16}$$

where $\partial P(\mathbf{x})/\partial \mathbf{x}$ is the collective form of $\partial P(\mathbf{x}_i)/\partial \mathbf{x}_i$. Then, the time derivative of W along (15) is given by

$$\begin{aligned}\dot{W} &= \vartheta^T H^{-1}\dot{\vartheta} = -(He)^T(\text{diag}\{\|\vartheta_i\|^{\alpha}\} \otimes I_n)\text{sign}(He) \\ &= -\begin{bmatrix}\vartheta_1 \\ \vdots \\ \vartheta_N\end{bmatrix}^T \begin{bmatrix}\|\vartheta_1\|^{\alpha} I_n & & \\ & \ddots & \\ & & \|\vartheta_N\|^{\alpha} I_n\end{bmatrix}\begin{bmatrix}\text{sign}(\vartheta_1) \\ \vdots \\ \text{sign}(\vartheta_N)\end{bmatrix} \\ &\leq -\sum_{i=1}^{N}\|\vartheta_i\|^{\alpha+1} = -\sum_{i=1}^{N}\left(\|\vartheta_i\|^2\right)^{\frac{\alpha+1}{2}}.\end{aligned} \tag{17}$$

Since $(\sum_{k=1}^{n} \varepsilon_k)^p \leq \sum_{k=1}^{n} \varepsilon_k^p$ holds for $\varepsilon_k \geq 0$ and $0 < p \leq 1$, then for $\alpha_1 = [2/\lambda_{\max}(H^{-1})]^{\frac{\alpha+1}{2}}$, (17) can be rewritten as

$$\begin{aligned}\dot{W} &\leq -\left\{\sum_{i=1}^{N}\left(2A_i e_i + \gamma \sum_{j=1}^{N} a_{ij}(e_i - e_j)\right)^2\right\}^{\frac{\alpha+1}{2}} \\ &\leq -[2/\lambda_{\max}(H^{-1})]^{\frac{\alpha+1}{2}} W^{\frac{\alpha+1}{2}} = -\alpha_1 W^{\frac{\alpha+1}{2}}.\end{aligned} \tag{18}$$

According to the finite-time stability lemma 2, $W \in \mathcal{L}_{\infty}$ and $e_i(t)$ converge to zero in a finite time with the settling time given by $T_1 = T_0 + \frac{2}{\alpha_1(1-\alpha)} W^{\frac{1-\alpha}{2}}(T_0)$. Thus, by (14) and (16), $e_i(t) \to 0$ as $t \to T_1$, which implies $\lim_{t\to T_1} \mathbf{x}_i = \mathbf{x}_i^*$. ■

B. Application to an Economic Dispatch Problem

By the penalty method, Problem 1 is solved in Section A to obtain an approximately optimal solution. Next, the proposed algorithm will be applied to solve the following economic dispatch problem where each generator has a local cost function $f_i(x_i) = a_i x_i^2 + b_i x_i + c_i,\ a_i > 0, b_i, c_i \geq 0, x_i \in \mathbb{R}$ [4]. By saddle-point dynamics, distributed algorithms will be developed to solve this problem with global and local constraints.

$$\begin{aligned}&\min \sum_{i=1}^{N} f_i(x_i),\ i = 1, 2, \ldots, N \\ &\text{subject to} \sum_{i=1}^{N} x_i = d \text{ and } x_i^{\min} \leqslant x_i \leqslant x_i^{\max}\end{aligned} \tag{19}$$

where $d \in \mathbb{R}$ is the total power demand, and $x_i^{\min}$, $x_i^{\max}$ with $x_i^{\min} < x_i^{\max}$ are the lower and upper generation bounds.

When the capacity constraints are not considered first, define a Lagrangian function as $L(x, \lambda_0) = \sum_{i=1}^{N} f_i(x_i) + \lambda_0(d - \sum_{i=1}^{N} x_i)$, where λ_0 is the Lagrange multiplier and x is the collective vector of x_i. By using saddle-point dynamics, there exists a pair (x^*, λ_0^*) such that x^* is the optimal solution to (19). Thus, (x, λ_0) converges to (x^*, λ_0^*) by a centralized algorithm

$$\dot{x}_i = -\frac{\partial L(x, \lambda_0)}{\partial x_i} = -\frac{\partial f_i(x_i)}{\partial x_i} + \lambda_0 \tag{20a}$$

$$\dot{\lambda}_0 = \frac{\partial L(x, \lambda_0)}{\partial \lambda_0} = d - \sum_{i=1}^{N} x_i. \tag{20b}$$

Based on the first-order optimal condition, the optimal solution to (20a) and (20b) is given by

$$x_i^* = (\lambda_0^* - b_i)/(2a_i) \text{ and } \lambda_0^* = \left(d + \sum_{i=1}^{N}\frac{b_i}{2a_i}\right) \Bigg/ \sum_{i=1}^{N}\frac{1}{2a_i}. \tag{21}$$

1) Distributed Solution without Capacity Constraints

Inspired by (10), a distributed algorithm is proposed as

$$\dot{x}_i = u_i + \omega_i,\ u_i = u_i^o + u_i^r,\ i = 1, \ldots, N \tag{22a}$$

$$u_i^o = -\sum_{j=1}^{N} a_{ij}\text{sig}^{\beta}(\lambda_i - \lambda_j) \tag{22b}$$

$$\dot{\lambda}_i = 2a_i\left(u_i^o + \text{sig}^{\alpha}\left(x_i - \frac{\lambda_i - b_i}{2a_i}\right)\right) \tag{22c}$$

$$u_i^r = -k_{1i}\text{sig}^{\frac{1}{2}}(s_i) + \phi_i,\ \dot{\phi}_i = -k_{2i}\text{sign}(s_i) \tag{22d}$$

$$s_i = x_i - x_i(0) - \int_0^t u_i^o(\tau)d\tau. \tag{22e}$$

Let $\hat{x}_i = (\lambda_i - b_i)/2a_i,\ \delta_i = \lambda_i - \sum_{j=1}^{N}\lambda_j/N,$ and $\eta_i = x_i - \hat{x}_i$. Next, the goal is to show $\delta_i = 0$ and $\eta_i = 0$ in a finite time T, that is, $\lambda_i = \lambda_0^*$ and $x_i = x_i^*$ after a finite time T, for certain settling time $T > 0$.

Theorem 2: Suppose that Assumptions 1 and 2 hold. Then, the distributed algorithm (22) enables (x_i, λ_i) to converge to the optimal solution (x_i^*, λ_0^*) in a finite time, i.e.,

$$\lim_{t\to T} x_i = x_i^* \text{ and } \lim_{t\to T} \lambda_i = \lambda_0^*,\ i = 1, 2, \ldots, N. \quad (23)$$

Proof: The proof includes the following steps:

Step 1: prove that $\lim_{t\to T_0} s_i = \dot{s}_i = 0$;

Step 2: prove that $\lim_{t\to T_1} x_i = \hat{x}_i$;

Step 3: prove that $\lim_{t\to T_2} \lambda_i = \lambda_0^*$;

Step 4: by Step 3, $\hat{x}_i = x_i^*$. Thus, $\lim_{t\to T} x_i = x_i^*, T = T_2$.

i) According to (11)–(13) in Step 1 of Theorem 1, it is not difficult to obtain $\lim_{t\to T_0} s_i = \dot{s}_i = 0$ in a similar way.

ii) Define a Lyapunov function candidate: $V = \frac{1}{2}\sum_{i=1}^N \eta_i^2 = \frac{1}{2}\sum_{i=1}^N (x_i - \hat{x}_i)^2$. Then, the time derivative of V is given by

$$\begin{aligned}\dot{V} &= \sum_{i=1}^N (x_i - \hat{x}_i)(\dot{x}_i - \dot{\hat{x}}_i) = \sum_{i=1}^N (x_i - \hat{x}_i)\left(u_i^o - \frac{1}{2a_i}\dot{\lambda}_i\right)\\ &= -\sum_{i=1}^N (x_i - \hat{x}_i)\mathrm{sig}^\alpha\left(x_i - \frac{\lambda_i - b_i}{2a_i}\right)\\ &= -\sum_{i=1}^N (x_i - \hat{x}_i)\mathrm{sign}(x_i - \hat{x}_i)|x_i - \hat{x}_i|^\alpha\\ &\leqslant -\left(\sum_{i=1}^N (x_i - \hat{x}_i)^2\right)^{\frac{\alpha+1}{2}} = -(2V)^{\frac{\alpha+1}{2}}. \end{aligned} \quad (24)$$

Thus, $\dot{V} + (2V)^{\frac{\alpha+1}{2}} \leqslant 0$. According to the finite-time stability lemma and the selected Lyapunov function, $\lim_{t\to T_1} x_i = \hat{x}_i$ with the settling time $T_1 = \frac{V^{\frac{1-\alpha}{2}}(T_0)}{(\sqrt{2})^{\alpha-1}(1-\alpha)} + T_0$.

iii) Since $x_i = \frac{\lambda_i - b_i}{2a_i}$ as $t \geqslant T_1$ shown in Step 2, we have

$$\dot{\lambda}_i = -2a_i \sum_{j=1}^N a_{ij}\mathrm{sig}^\beta(\lambda_i - \lambda_j),\ t \geqslant T_1. \quad (25)$$

Since $a_{ij} = a_{ji}$ by Assumption 1, $\sum_{i=1}^N \frac{1}{2a_i}\dot{\lambda}_i(t) = 0$. That is, $\sum_{i=1}^N \frac{1}{2a_i}\lambda_i(t)$ is invariable as $t \geqslant T_1$, which yields $\sum_{i=1}^N \frac{1}{2a_i}\lambda_i(t) = \sum_{i=1}^N \frac{1}{2a_i}\lambda_i(T_1)$. In addition, $\lim_{t\to T_1} x_i = \hat{x}_i$ implies $x_i(T_1) = \frac{\lambda_i(T_1) - b_i}{2a_i}$. Therefore, it is not difficult to obtain $\sum_{i=1}^N \frac{1}{2a_i}\lambda_i(t) = \sum_{i=1}^N (x_i(T_1) + \frac{b_i}{2a_i})$.

Based on (25), next we will show $\lim_{t\to T_2} \lambda_i = \lambda_j$.

Define a Lyapunov function candidate: $W = \sum_{i=1}^N \frac{1}{4a_i}\delta_i^2(t)$. Then, the time derivative of W along (25) is given by

$$\begin{aligned}\dot{W} &= \sum_{i=1}^N \frac{1}{2a_i}\delta_i(t)\left(-2a_i\sum_{j=1}^N a_{ij}\mathrm{sig}^\beta(\delta_i(t) - \delta_j(t))\right)\\ &= -\frac{1}{2}\sum_{i,j=1}^N a_{ij}(\delta_i(t) - \delta_j(t))\mathrm{sig}^\beta(\delta_i(t) - \delta_j(t))\\ &= -\frac{1}{2}\sum_{i,j=1}^N a_{ij}|\delta_i - \delta_j|^{\beta+1} \leqslant -\beta_0 W^{\frac{\beta+1}{2}} \end{aligned} \quad (26)$$

where $\beta_0 = \frac{1}{2}\varepsilon^{\frac{\beta+1}{2}}$, $\varepsilon = 8\lambda_2(\mathcal{L}_\beta)\tilde{a}_0$, and $\mathcal{L}_\beta$ has the same structure with $\mathcal{L}$, where a_{ij} is replaced by $a_{ij}^{\frac{2}{1+\beta}}$, and $\tilde{a}_0 = \min_{i\in\mathcal{V}} a_i$.

Hence, $\dot{W} + \beta_0 W^{\frac{\beta+1}{2}} \leqslant 0$. Applying the finite-time stability lemma yields $\lim_{t\to T_2} \lambda_i = \lambda_j$ with the settling time described as $T_2 = T_1 + 2W^{\frac{1-\beta}{2}}(T_1)/(\beta_0(1-\beta))$.

iv) Since $\lambda_i = \lambda_j$ as $t \geq T_2$, there exists a constant $\tilde{\lambda}$ such that $\lambda_i = \tilde{\lambda},\ t \geq T_2$. By $\sum_{i=1}^N \frac{1}{2a_i}\lambda_i(t) = \sum_{i=1}^N (x_i(T_1) + \frac{b_i}{2a_i})$

$$\tilde{\lambda} = \sum_{i=1}^N \left(x_i(T_1) + \frac{b_i}{2a_i}\right) \Big/ \sum_{i=1}^N \frac{1}{2a_i}. \quad (27)$$

It follows from (22e) and $a_{ij} = a_{ji}$ by Assumption 1 that:

$$\sum_{i=1}^N x_i = \sum_{i,j=1}^N a_{ij}\int_0^t \mathrm{sig}^\beta(\lambda_j(s) - \lambda_i(s))ds + \sum_{i=1}^N x_i(0) = d. \quad (28)$$

According to (27) and (28)

$$\tilde{\lambda} = \left(d + \sum_{i=1}^N \frac{b_i}{2a_i}\right) \Big/ \sum_{i=1}^N \frac{1}{2a_i} = \lambda_0^*. \quad (29)$$

Thus, this consensus value is $\tilde{\lambda} = \lambda_0^*$, where λ_0^* is the optimal solution shown in (21). Since $\lim_{t\to T_2} \lambda_i = \lambda_j$ by Step 3, $\lim_{t\to T_2} \lambda_i = \lambda_0^*$ holds. Moreover, $\lim_{t\to T_1} x_i = \hat{x}_i = \frac{\lambda_i - b_i}{2a_i}$ by Step 2. That is, $x_i = \frac{\lambda_i^* - b_i}{2a_i} = x_i^*$ (i.e., $x_i = x_i^*$ as $t \geq T_2$). In conclusion, it proves that $\lambda_i = \lambda_0^*$ and $x_i = x_i^*$ after a finite time T_2, and the proof is thus completed. ■

Remark 7: From (28), it can be seen that $\sum_{i=1}^N x_i(0) = d$ as widely used in many existing papers, is employed to achieve the optimal solution. That is, the sum of initial states is required to satisfy the power demand condition. In future work, we will develop algorithms to remove this initial state requirement and meanwhile to enable finite-time and robust convergence.

2) Distributed Solution With Capacity Constraints

To avoid violations of capacity constraints, based on (22), a fast economic dispatch algorithm is presented as follows.

Theorem 3: Suppose that Assumptions 1 and 2 hold. Under the proposed Algorithm 1, the economic dispatch problem with global and local constraints can be solved in a finite time.

Proof: When considering the capacity constraints, we define the incremental cost as: $\lambda_i = 2a_i x_i + b_i$. The well-known solution is the equal incremental cost criterion [4]

$$\begin{cases} 2a_i x_i + b_i = \lambda^*, & \text{if } x_i^{\min} < x_i < x_i^{\max}\\ 2a_i x_i + b_i > \lambda^*, & \text{if } x_i = x_i^{\min}\\ 2a_i x_i + b_i < \lambda^*, & \text{if } x_i = x_i^{\max}. \end{cases} \quad (36)$$

By the defined Ξ in Algorithm 1, (36) can be rewritten as

$$\begin{aligned}\lambda^* = 2a_i x_i + b_i &= \left(d - \sum_{i\in\Xi} x_i + \sum_{i\notin\Xi}\frac{b_i}{2a_i}\right) \Big/ \sum_{i\notin\Xi}\frac{1}{2a_i}\\ &= \frac{\left(\sum_{i=1}^N \frac{1}{2a_i}\right)\frac{d + \sum_{i=1}^N \frac{b_i}{2a_i}}{\left(\sum_{i=1}^N \frac{1}{2a_i}\right)} - \sum_{i\in\Xi} x_i - \sum_{i\in\Xi}\frac{b_i}{2a_i}}{\sum_{i\notin\Xi}\frac{1}{2a_i}}\\ &= \frac{d + \sum_{i=1}^N \frac{b_i}{2a_i}}{\sum_{i=1}^N \frac{1}{2a_i}} + \frac{\sum_{i\in\Xi}\left(\frac{\lambda_0^* - 2a_i x_i - b_i}{2a_i}\right)}{\sum_{i\notin\Xi}\frac{1}{2a_i}}. \end{aligned} \quad (37)$$

Algorithm 1: Finite-Time Economic Dispatch Algorithm.

1. Run the distributed algorithm in (22) to obtain the optimal solution (x_i^*, λ_0^*) of (19) without capacity constraints.
2. Check the capacity constraint violations. Denote x_i as

$$x_i = \begin{cases} x_i^{\min}, & \text{if } x_i^* < x_i^{\min}, \\ x_i^{\max}, & \text{if } x_i^* > x_i^{\max}. \end{cases} \quad (30)$$

Define Ξ as the set of generators with $x_i = x_i^{\min}$ or $x_i = x_i^{\max}$.

3. Determine the optimal solution (x_i^*, λ^*) of (19) with capacity constraints by the following algorithm:

$$x_i^* = \begin{cases} \frac{\lambda^* - b_i}{2a_i}, & \text{if } i \notin \Xi, \\ x_i^{\min} \text{ or } x_i^{\max}, & \text{if } i \in \Xi, \end{cases} \quad (31)$$

$$\lambda^* = \lambda_0^* + \rho_i/\eta_i, \ i = 1, 2, \ldots, N, \quad (32)$$

$$\dot{\rho}_i = -\sum_{j=1}^N a_{ij}\text{sig}^\kappa(\rho_i - \rho_j), \dot{\eta}_i = -\sum_{j=1}^N a_{ij}\text{sig}^\kappa(\eta_i - \eta_j), \quad (33)$$

where $\kappa \in (0, 1)$, and ρ_i, η_i are two auxiliary variables with

$$\rho_i = \begin{cases} \frac{\lambda_0^* - 2a_i x_i - b_i}{2a_i}, & \text{if } i \in \Xi, \\ 0, & \text{if } i \notin \Xi, \end{cases} \quad (34)$$

$$\eta_i = \begin{cases} \frac{1}{2a_i}, & \text{if } i \notin \Xi, \\ 0, & \text{if } i \in \Xi. \end{cases} \quad (35)$$

4. When the capacity constraints are violated by the optimal generation, $\lambda_0^* = \lambda^*$ and go back to step 2; otherwise, end.

In light of (21), we can express (37) as

$$\lambda^* = \lambda_0^* + \sum_{i \in \Xi}\left(\frac{\lambda_0^* - 2a_i x_i - b_i}{2a_i}\right) \Big/ \sum_{i \notin \Xi}\frac{1}{2a_i}. \quad (38)$$

It follows from (33)–(35) that after a finite time:

$$\rho_i \to \frac{1}{N}\sum_{i \in \Xi}\frac{\lambda_0^* - 2a_i x_i - b_i}{2a_i} \text{ and } \eta_i \to \frac{1}{N}\sum_{i \notin \Xi}\frac{1}{2a_i}. \quad (39)$$

As a result, the optimal incremental cost is $\lambda^* = \lambda_0^* + \rho_i/\eta_i$, which implies (38), and x_i^* is thus obtained in (31). ∎

V. Finite-Time Distributed Nonquadratic Optimization

A. Solve Problem 1 for Nonquadratic Objective Functions

Assumption 4: $f_i(\theta)$ are twice continuously differentiable, strongly convex, and have a locally Lipschitz matrix $\nabla^2 f_i(\theta)$.

Motivated by the ZGS algorithm in [10] and [38], a new distributed algorithm is proposed as

$$u_i = u_i^o + u_i^r, \ i = 1, 2, \ldots, N \quad (40a)$$

$$u_i^o = -(\nabla^2 f_i(\mathbf{x}_i))^{-1}\sum_{j=1}^N a_{ij}\text{sig}^\alpha(\mathbf{x}_i - \mathbf{x}_j) \quad (40b)$$

$$u_i^r = -k_{1i}\text{sig}^{\frac{1}{2}}(s_i) + \phi_i, \ \dot{\phi}_i = -k_{2i}\text{sign}(s_i) \quad (40c)$$

$$s_i = \mathbf{x}_i - \mathbf{x}_i(0) - \int_0^t u_i^o(\tau)d\tau \quad (40d)$$

where $\nabla^2 f_i(\mathbf{x}_i)$ is the Hessian matrix of the local cost function and $\mathbf{x}_i(0) = \mathbf{x}_i^*$ with $\mathbf{x}_i^*$ being a minimizer of $f_i(\theta)$ [10], [38].

Remark 8: The above design in (40) is inspired by the ZGS algorithm in [10] and [38], and is combined with the continuous-time ISMC algorithm given in (10). Notice that $\lim_{t\to 0} s_i = \dot{s}_i = 0$ if $\omega_i(0) = 0$. That is, $\dot{\mathbf{x}}_i = -(\nabla^2 f_i(\mathbf{x}_i))^{-1}\sum_{j=1}^N a_{ij}\text{sig}^\alpha(\mathbf{x}_i - \mathbf{x}_j)$ for $t \geq 0$. Then, $\sum_{i\in\mathcal{V}}\nabla^2 f_i(\mathbf{x}_i)\dot{\mathbf{x}}_i = 0$ for $t \geq 0$. This implies that $\sum_{i\in\mathcal{V}}\nabla f_i(\mathbf{x}_i(t))$ is constant, which, together with $\sum_{i\in\mathcal{V}}\nabla f_i(\mathbf{x}_i(0)) = 0$, yields $\sum_{i\in\mathcal{V}}\nabla f_i(\mathbf{x}_i(t)) = 0$ for $t \geq 0$. Besides, $\sum_{j=1}^N a_{ij}\text{sig}^\alpha(\mathbf{x}_i - \mathbf{x}_j)$ gives a consensus value $\tilde{\mathbf{x}}$ in a finite time T. Thus, $\lim_{t\to T}\mathbf{x}_i(t) = \tilde{\mathbf{x}}$ and $\sum_{i\in\mathcal{V}}\nabla f_i(\tilde{\mathbf{x}}) = 0$ imply that finite-time distributed optimization is achieved.

Theorem 4: Under Assumptions 1, 2, and 4, the proposed distributed algorithm (40) enables $\mathbf{x}_i$ to converge to the optimal solution (i.e., θ^*) of the problem in (1) in a finite time, i.e.,

$$\lim_{t\to T}\mathbf{x}_i = \mathbf{x}_j = \theta^*, \ i = 1, 2, \ldots, N. \quad (41)$$

Proof: By (11)–(13) in Step 1 of Theorem 1, $\lim_{t\to T_0} s_i = \dot{s}_i = 0$. Thus, $\dot{s}_i = \dot{\mathbf{x}}_i - u_i^o = 0, t \geq T_0$, i.e.,

$$\dot{\mathbf{x}}_i = -(\nabla^2 f_i(\mathbf{x}_i))^{-1}\sum_{j=1}^N a_{ij}\text{sig}^\alpha(\mathbf{x}_i - \mathbf{x}_j), \ 0 < \alpha < 1. \quad (42)$$

From (42), $\sum_{i\in\mathcal{V}}\nabla^2 f_i(\mathbf{x}_i)\dot{\mathbf{x}}_i = 0$ for $t \geq 0$ by Remark 8, which implies that $\sum_{i\in\mathcal{V}}\nabla f_i(\mathbf{x}_i(t))$ is constant for $t \geq 0$ and $\sum_{i\in\mathcal{V}}\nabla f_i(\mathbf{x}_i^*) = 0$. Thus, $\sum_{i\in\mathcal{V}}\nabla f_i(\mathbf{x}_i(t)) = 0$ for $t \geq 0$. Denote $\varsigma = \frac{1}{N}\sum_{i\in\mathcal{V}}\mathbf{x}_i(t)$ and let $\tilde{\mathbf{x}}^* = \text{col}(\mathbf{x}^*, \mathbf{x}^*, \ldots, \mathbf{x}^*)$, $\mathbf{x} = \text{col}(x_1, x_2, \ldots, x_N) \in \mathbb{R}^{nN}$ be the minimizer ($\mathbf{x}^* = \theta^*$) and state vector, respectively. Then, $\sum_{i\in\mathcal{V}} f_i(\mathbf{x}^*) \leq \sum_{i\in\mathcal{V}} f_i(\varsigma)$ and for a constant $\Theta_i > 0$, $V(\mathbf{x}) = \sum_{i\in\mathcal{V}} f_i(\varsigma) - f_i(\mathbf{x}_i) - \nabla f_i(\mathbf{x}_i)^T(\varsigma - \mathbf{x}_i) \leq \frac{\max\{\Theta_i\}}{2}\mathbf{x}^T(\mathcal{L}\otimes I_n)\mathbf{x}$.

Next, choose a Lyapunov function candidate [10], [38]

$$V(\mathbf{x}) = \sum_{i\in\mathcal{V}} f_i(\mathbf{x}^*) - f_i(\mathbf{x}_i) - \nabla f_i(\mathbf{x}_i)^T(\mathbf{x}^* - \mathbf{x}_i). \quad (43)$$

Let $\mathcal{L}_f$ have the same structure with $\mathcal{L}$, where a_{ij} is replaced by $a_{ij}^{\frac{2}{1+\alpha}}$. The time derivative of $V(\mathbf{x})$ along (42) is

$$\dot{V}(\mathbf{x}) = -\frac{1}{2}\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i(\mathcal{G})} a_{ij}(\mathbf{x}_j - \mathbf{x}_i)^T\text{sig}^\alpha(\mathbf{x}_j - \mathbf{x}_i)$$
$$\leq -2^{\frac{\alpha-1}{2}}[\mathbf{x}^T(\mathcal{L}_f\otimes I_n)\mathbf{x}]^{\frac{\alpha+1}{2}}. \quad (44)$$

Then, the remainder of the proof is similar to [38] to achieve finite-time convergence ($\lim_{t\to T}\mathbf{x}_i = \mathbf{x}^*$) and, thus, is omitted. ∎

B. Application to A Resource Allocation Problem

In the above section, Problem 1 has been solved for nonquadratic objective functions. Next, the proposed algorithm will be applied to solve the following resource allocation problem with global and local constraints, where each agent has a local nonquadratic objective function

$$\min \sum_{i=1}^{N} f_i(x_i),\ i=1,2,\dots,N \tag{45a}$$

$$\text{subject to} \sum_{i=1}^{N} x_i = l \text{ and } x_i^{\min} \le x_i \le x_i^{\max} \tag{45b}$$

where $l \in \mathbb{R}^n$ is the total network resource, and $x_i^{\min}$, $x_i^{\max}$ ($x_i^{\min} < x_i^{\max}$) are the lower and upper bounds of x_i.

Assumption 5: The functions $f_i(x_i)$ are continuously differentiable, strongly convex, and have locally Lipschitz gradients.

We aim to eliminate the inequality constraints, while keeping the equality constraints intact. Thus, a modified penalty design is given with a smooth ϵ-exact penalty function [34]

$$x_{\epsilon i}(h_i(x_i)) = \begin{cases} 0, & \text{if } h_i(x_i) < 0 \\ \gamma h_i^2(x_i)/(2\epsilon), & \text{if } 0 \le h_i(x_i) \le \epsilon \\ \gamma(h_i(x_i) - \epsilon/2), & \text{if } h_i(x_i) > \epsilon \end{cases} \tag{46}$$

where $h_i(x_i) = (x_i^{\min} - x_i)^T(x_i^{\max} - x_i) + \epsilon$, $\epsilon > 0$ is a small constant, and γ is the penalty parameter.

Next, we solve the following optimization problem:

$$\min\ f_\epsilon(x) = \sum_{i=1}^{N} f_{\epsilon i}(x_i) = \sum_{i=1}^{N}(f_i(x_i) + x_{\epsilon i}(h_i(x_i))) \tag{47a}$$

$$\text{subject to} \sum_{i=1}^{N} x_i = l,\ i=1,2,\dots,N. \tag{47b}$$

Assume that $x^* = \text{col}(x_1^*,\dots,x_N^*)$ is the optimal solution of (45) and $\hat{x}^* = \text{col}(\hat{x}_1^*,\dots,\hat{x}_N^*)$ is the optimal solution of (47). By Propositions 3 and 4 in [35], the relationship between the optimal solution of (45) and (47) is $0 \le f(x^*) - f_\epsilon(\hat{x}^*) \le \epsilon\gamma N$, where $\gamma = \gamma^*(1-N)/(1-\sqrt{N})$, $\gamma^* > \max\{\lambda_1^*,\dots,\lambda_N^*\}$ with λ_i^* denoting the Lagrange multiplier vector satisfying the KKT condition, and the upper bound of λ_i^* is given by [35]

$$\max\{\lambda_i^*\}_{i=1}^N \le \frac{2\max\{\max_{x_i \in x_{fea,i}} \|\nabla f_i(x_i)\|\}_{i=1}^N}{\min\{\|x_i^{\max} - x_i^{\min}\|\}_{i=1}^N} \tag{48}$$

where $x_{fea,i} = \{x_i \in \mathbb{R}^n \,|\, \sum_{i=1}^N x_i = l \text{ and } h_i(x_i) \le 0\}$.

Based on the ϵ-exact penalty function, a finite-time distributed algorithm is proposed as

$$\dot{x}_i = u_i + \omega_i,\ u_i = u_i^o + u_i^r,\ i=1,\dots,N \tag{49a}$$

$$u_i^o = -\sum_{j=1}^{N} a_{ij}\text{sig}^\alpha(\nabla f_{\epsilon i}(x_i) - \nabla f_{\epsilon j}(x_j)) \tag{49b}$$

$$u_i^r = -k_{1i}\textbf{sig}^{\frac{1}{2}}(s_i) + \phi_i,\ \dot{\phi}_i = -k_{2i}\text{sign}(s_i) \tag{49c}$$

$$s_i = x_i - x_i(0) - \int_0^t u_i^o(\tau)d\tau. \tag{49d}$$

Remark 9: Unlike using projection methods [8], [9], and [37] to handle constraints where the local feasible sets are explicitly obtained, a modified ϵ-exact penalty function is employed here. The algorithm in (49) is distributed and enables finite-time and robust convergence. If we set $\alpha = 1$ and do not consider the disturbances, the algorithm is reduced to the protocol in [33].

Theorem 5: Under Assumptions 1, 2, and 5, the distributed algorithm in (49) makes the feasible set x_{fea} time-invariant, and any trajectory starting from x_{fea} converges to the solution set of (45) approximately in a finite time.

Proof: By (11)–(13) in Step 1 of Theorem 1, $\lim_{t\to T_0} s_i = \dot{s}_i = 0$. Thus, $\dot{s}_i = \dot{x}_i - u_i^o = 0, t \ge T_0$, i.e.,

$$\dot{x}_i = -\sum_{j=1}^{N} a_{ij}\text{sig}^\alpha(\nabla f_{\epsilon i}(x_i) - \nabla f_{\epsilon j}(x_j)),\ 0<\alpha<1. \tag{50}$$

By (50), $\sum_{i=1}^N \dot{x}_i = 0$. Therefore, the total network resource $\sum_{i=1}^N x_i$ is conserved and the feasible set x_{fea} is time-invariant. Now, we prove that the trajectories starting from x_{fea} fastly converge to the optimal solution set. Uniqueness of the solution to (47) follows from the strong convexity implying strict convexity. Choose a Lyapunov function candidate

$$V_\epsilon = \sum_{i=1}^{N}(f_{\epsilon i}(x_i) - f_{\epsilon i}(x_i^*)),\ i=1,2,\dots,N \tag{51}$$

where $V_\epsilon \ge 0$ by Assumption 5, and $V_\epsilon = 0$ if $x_i = x_i^*$.

The time derivative of V_ϵ is expressed as

$$\begin{aligned}\dot{V}_\epsilon &= -\sum_{i=1}^{N}\nabla f_{\epsilon i}^T(x_i)\sum_{j=1}^{N}a_{ij}\text{sig}^\alpha(\nabla f_{\epsilon i}(x_i) - \nabla f_{\epsilon j}(x_j)) \\ &= -\frac{1}{2}\sum_{i=1}^{N}(\nabla f_{\epsilon i}(x_i) - \nabla f_{\epsilon j}(x_j))^T\sum_{j=1}^{N}a_{ij}\text{sig}^\alpha \\ &\quad \times(\nabla f_{\epsilon i}(x_i) - \nabla f_{\epsilon j}(x_j)) \\ &\le -\frac{1}{2}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}^{\frac{2}{1+\alpha}}(\nabla f_{\epsilon i}(x_i) - \nabla f_{\epsilon j}(x_j))^2\right)^{\frac{1+\alpha}{2}} \\ &= -\frac{1}{2}\left(2\nabla f_\epsilon^T(x)\mathcal{L}_f\nabla f_\epsilon(x)\right)^{\frac{1+\alpha}{2}}\end{aligned} \tag{52}$$

where $\nabla f_\epsilon(x) = \text{col}(f_{\epsilon 1}(x_1),\dots,f_{\epsilon N}(x_N))$.

Since f_i is strongly convex by Assumption 5, $f_{\epsilon i}$ is strongly convex. Then, for $\hat{x} = \text{col}(\hat{x}_1,\dots,\hat{x}_N)$, and $\theta > 0$

$$\begin{aligned}f_\epsilon(\hat{x}) - f_\epsilon(x) &\ge \nabla f_\epsilon^T(x)\left((I_N - \frac{1_N^T 1_N}{N})\otimes I_n\right)(\hat{x} - x) \\ &\quad + \frac{\theta}{2}\|\hat{x} - x\|^2.\end{aligned} \tag{53}$$

For the fixed x, it follows from (53) that $f_\epsilon(\hat{x}) \ge f_\epsilon(x) - \frac{1}{2\theta}\|((I_N - \frac{1_N^T 1_N}{N})\otimes I_n)\nabla f_\epsilon(x)\|^2$. Hence

$$\left\|\left(\left(I_N - \frac{1_N^T 1_N}{N}\right)\otimes I_n\right)\nabla f_\epsilon(x)\right\|^2 \ge 2\theta(f_\epsilon(x) - f_\epsilon(\hat{x})). \tag{54}$$

In addition, by Assumption 1

$$\begin{aligned}&\nabla f_\epsilon^T(x)\mathcal{L}_f\nabla f_\epsilon(x) \\ &\quad\ge \lambda_2(\mathcal{L}_f)\left\|\left(\left(I_N - \frac{1_N^T 1_N}{N}\right)\otimes I_n\right)\nabla f_\epsilon(x)\right\|^2.\end{aligned}$$

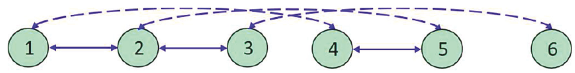


Fig. 1. Communication topology for a team of six agents.

Define $\theta_0 = 0.5(4\theta\lambda_2(\mathcal{L}_f))^{\frac{1+\alpha}{2}}$. Thus, for $x^* = \hat{x}$

$$\dot{V}_\epsilon \leq -0.5\left(4\theta\lambda_2(\mathcal{L}_f)V_\epsilon\right)^{\frac{1+\alpha}{2}} = -\theta_0 V_\epsilon^{\frac{1+\alpha}{2}}. \quad (55)$$

By Lemma 2 and the Comparison Lemma, $\lim_{t\to T} V_\epsilon(t) = 0$ with the settling time given by $T = T_0 + \frac{2}{\theta_0(1-\alpha)} V_\epsilon^{\frac{1-\alpha}{2}}(T_0)$. Since $V_\epsilon(t) \to 0$, $\sum_{i=1}^{N}(f_{\epsilon i}(x_i) - f_{\epsilon i}(x_i^*)) = 0$, which implies that $\nabla f_{\epsilon i}(x_i) = \nabla f_{\epsilon j}(x_j)$. Thus, for $x^* = \hat{x}$, $\lim_{t\to T} x_i = x_i^*$, $i = 1, \ldots, N$. Moreover, since $\sum_{i=1}^{N} x_i = l$ by hypothesis, the state trajectories starting from x_{fea} converge to the optimal solution set of (45) approximately. ■

Remark 10: Finite-time distributed convex optimization algorithms have been presented in Sections IV and V for quadratic and nonquadratic cost functions, respectively. The proposed algorithms are further applied to solve the economic dispatch and resource allocation problems, where each agent is assigned with a local cost function $f_i(x_i)$. That is, the consensus constraint of Problem 1 is thus not required, which simplifies the design of algorithms. As a first attempt to investigate the distributed convex optimization problem that takes the chattering avoidance, finite-time convergence, and disturbance rejection into account, we focus on the distributed unconstrained optimization problem.

VI. Numerical Simulation

In this section, different cases are provided to illustrate the effectiveness of the proposed distributed optimization algorithms. In particular, Case 1 considers finite-time distributed quadratic optimization for continuous-time multiagent systems subject to disturbances. The proposed algorithm is then applied to solve an economic dispatch problem in Case 2. Case 3 studies finite-time distributed nonquadratic optimization, while the algorithm is applied to solve a resource allocation problem in Case 4.

Case 1. Finite-Time Distributed Quadratic Optimization: Consider a multiagent system with six agents described by (6) in $\mathbb{R}^3$. The team of agents aims to achieve finite-time distributed optimization with disturbance rejection. The communication graph is depicted in Fig. 1. The network objective function is

$$\mathrm{F}(x) = \sum_{i=1}^{6} f_i(x) = x^T A x + B^T x \quad (56)$$

where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 2 & 6 \end{bmatrix} \text{ and } B = [0.3 \ -0.5 \ 0.8]^T.$$

In this simulation, the objective functions of the agents are $f_1(x) = \frac{1}{6}(x^T Ax + B^T x)$, $f_2(x) = \frac{1}{3}(x^T Ax + B^T x)$, $f_3(x) = \frac{1}{12}(x^T Ax + B^T x)$, $f_4(x) = \frac{1}{12}x^T Ax + \frac{1}{4}B^T x$, $f_5(x) = \frac{1}{12} x^T Ax$, $f_6(x) = \frac{1}{4}x^T Ax + \frac{1}{6}B^T x$. By direct calculation, it can be derived that the optimal solution is given by $[x_1^*, x_2^*, x_3^*]^T = [-0.0521, 0.0965, -0.0901]^T$. Each agent estimates this optimal solution based on the dynamics in (6), where the disturbances are expressed as $\omega_i(t) = 0.5i\mathrm{col}(\sin(t), \cos(2t)\text{-}1, \sin(3t))$. Thus, $\|\omega_i(t)\|_\infty = 0.5i$ and $\|\dot{\omega}_i(t)\|_\infty = 1.5i$. By Theorem 1, the parameters of the proposed algorithm in (10) are selected as $k_{1i} = 1.5i$, $k_{2i} = 3i$, and $\alpha = 0.3$. The simulation result is shown in Fig. 2, where the proposed design enables the agents' states to converge toward the optimal solution in a finite time. To better demonstrate the finite-time and robust convergence of (10), we make a comparison with the nonsmooth distributed gradient (NDG) algorithm in [25]–[27], and standard distributed gradient (SDG) algorithm in [11] and [29], respectively. Then, these algorithms are performed under the same environment, and the simulation results are depicted in Figs. 3 and 4. In particular, Fig. 3 shows that under the disturbances, although consensus is achieved in finite time via the NDG algorithm, the agents' states converge to the wrong optimal solution as shown in Fig. 3(a), while Fig. 3(b) shows the chattering phenomenon. Under the SDG algorithm, Fig. 4 shows the agent's responses without and with disturbances. From Fig. 4(a), the agents' states converge to the optimal solution asymptotically, while under the disturbances, there exist bounded errors as shown in Fig. 4(b) (consistent with the

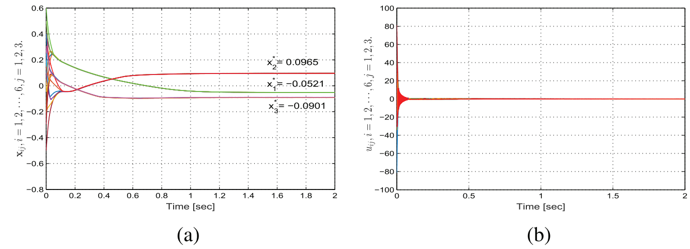


Fig. 2. Finite-time distributed optimization by the proposed algorithm in (10) with disturbances. (a) Estimated states on the optimal solution. (b) Control input.

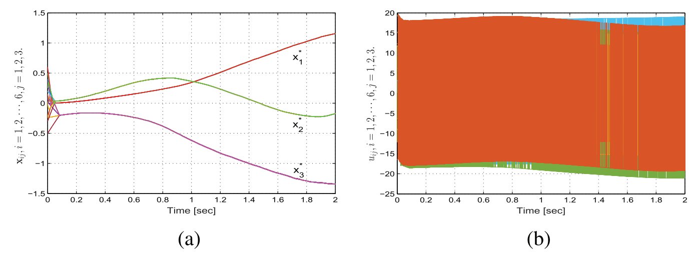


Fig. 3. Distributed optimization by the NDG algorithm in [25]–[27] with disturbances. (a) Estimated states on the optimal solution. (b) Control input.

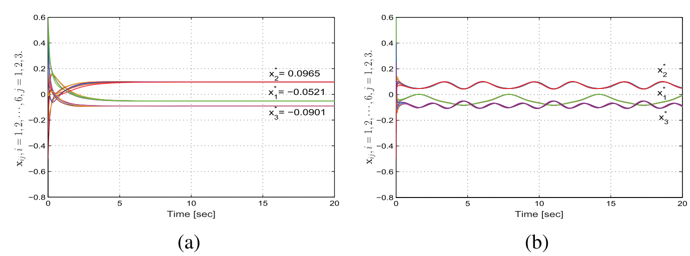


Fig. 4. Estimated states on the optimal solution by the SDG algorithm in [11], [29]. (a) Without disturbances. (b) With disturbances.

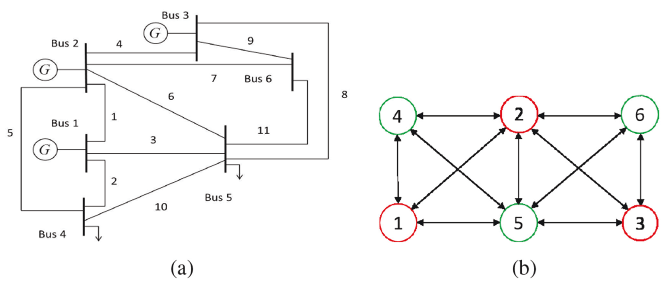


Fig. 5. IEEE 6-bus test system. (a) Single-line diagram. (b) Corresponding communication graph where the circles in red color represent generator buses, while the green ones are load buses.

TABLE I
IEEE 6-BUS TEST SYSTEM GENERATOR PARAMETERS [4]

Bus	a_i	b_i	c_i	x_i^{min} (MW)	x_i^{max} (MW)
1	0.00533	11.699	213.1	50	200
2	0.00889	10.333	200	37.5	130
3	0.00741	10.8333	240	45	220

observation in [29]). In comparison with the NDG and SDG algorithms, the proposed algorithm in (10) shows chattering avoidance, finite-time convergence, and robustness properties.

Case 2. Finite-Time Economic Dispatch (FED): In this case study, a power system with six buses and three generators is considered as shown in Fig. 5. The single-line diagram of the power system is depicted in Fig. 5(a), while the corresponding communication graph is shown in Fig. 5(b). The cost function parameters and generation capabilities are listed in Table I. The power demand of loads located in buses 4–6 is 100, 120, 200 MW, respectively. Thus, the total load demand is 420 MW. By direct calculation for $f_i(x_i) = a_i x_i^2 + b_i x_i + c_i$, it can be derived that the optimal generations are $x_1^* = 124.39$ MW, $x_2^* = 149.72$ MW, and $x_3^* = 145.89$ MW.

A. Without Capacity Limits

In this simulation, the capacity limits are not considered. Choose $\alpha = 0.7$ and $\beta = 0.8$. The simulation results are shown in Fig. 6 by performing (22). Fig. 6 shows the power generation x_i, estimated marginal costs λ_i, and total power output, respectively. It can be seen that x_i converges to the optimal generation x_i^*, λ_i converges to the marginal cost $\lambda^* = 13.0$ \$/MWh, and the sum of output generations satisfies the generation-demand equality constraint. However, the second generator violates the limit $x_2^{\max} = 120$ MW.

B. With Capacity Limits

In this simulation, the generator capacity limits are considered. Hence, Algorithm 1 is performed. First, by (22), $x_1^* = 124.39$ MW, $x_2^* = 149.72.39$ MW, and $x_3^* = 145.89$ MW. Notice that $x_2^* > x_{\max}^*$. Then, let $x_2^* = x_{\max}^*$ and $\Xi = \{2\}$ by (30). Based on (34) and (35), $\rho_i(0) = \mathrm{col}(0, 30, 0)$, and $\eta_i(0) = \mathrm{col}(93.81, 0, 67.48)$. By running (31)–(35), the optimal incremental cost is $\lambda^* = 13.18$ \$/MWh. Thus, the optimal generator outputs are $x_1^* = 141.68$ MW, $x_2^* = 120$ MW, and $x_3^* = 158.32$ MW. The simulation results are shown in Fig. 7,

TABLE II
IEEE 30-BUS TEST SYSTEM GENERATOR PARAMETERS [39]

Bus	a_i	b_i	c_i	$x_i^{min}(MW)$	$x_i^{max}(MW)$
1	0.04	2.0	561	50	190
2	0.03	3.0	310	35	200
3	0.035	4.0	78	45	230
4	0.042	4.0	561	50	250
5	0.045	3.0	310	45	220
6	0.05	2.0	78	55	210

where it is concluded that Algorithm 1 guarantees that the capacity constraints are not violated.

C. Convergence Performance Comparison

The IEEE 30-bus system as shown in Fig. 8 is chosen to test the scalability of the proposed FED algorithm, and make a comparison with the incremental cost consensus (ICC) algorithm in [39]. Buses 1, 2, 5, 8, 11, and 13 contain generators numbered 1 to 6. The data and the generator parameters are shown in Table II. If a bus only contains loads, the power generation is set to zero. The total load demand is set as $d = 850$ MW. Assume that at $t = 50$ s and $t = 100$ s, the load demand is increased by 30% and deduced by 20%, respectively. To give a marked comparison, the proposed FED algorithm and the ICC algorithm in [39] are performed under the same environment. The simulation results are shown in Fig. 9, where the comparative evaluation of the marginal cost update is depicted. It can be seen that the proposed algorithm leads to faster convergence compared with the ICC algorithm in [39].

Case 3. Finite-Time Distributed Nonquadratic Optimization: Consider a multiagent system with six agents described by (6) in $\mathbb{R}$. This example solves a finite-time distributed optimization problem for nonquadratic objective functions

$$f_i(x) = \frac{1}{2}\left(x - \frac{i}{2}\right)^2 + \frac{3}{4}\left(x - \frac{i}{2}\right)^4 + \frac{5}{8}\left(x - \frac{i}{2}\right)^6 \quad (57)$$

where $i = 1, 2, \ldots, 6$, and $x \in \mathbb{R}$ denotes the global variable.

Hence, it can be derived that $\mathbf{x}_i^* = \frac{i}{2}$, $i = 1, 2, \ldots, 6$, and the optimal value of cost function $\sum_{i=1}^{6} f_i(x)$ is given by 1.75 via calculation. In this simulation, each agent estimates this optimal solution based on the dynamics in (6), where the disturbances are given as $\omega_i(t) = 0.5i\sin(3t)$. Thus, $\|\omega_i(t)\|_\infty = 0.5i$ and $\|\dot{\omega}_i(t)\|_\infty = 1.5i$. The communication graph and parameters of the proposed finite-time distributed optimization algorithm in (40) are the same as those in Case 1. By Theorem 4, the simulation result is provided in Fig. 10, where the algorithm in (40) can guarantee the finite-time convergence of the optimal solution in the absence/presence of disturbances.

In order to better show the validity of the proposed algorithm in (40), we make a comparison with the ZGS algorithm in [10] and the SDG algorithm in [11] and [29]. The simulation results are shown in Figs. 11 and 12 to illustrate the agents' response on the optimal solution with/without the disturbance, and Table III shows the performance comparison of different algorithms. It can be seen that 1) in the absence of disturbances, it takes more time to achieve convergence by the ZGS and SDG algorithms, and 2) in the presence of disturbances, their optimal solutions cannot be exactly estimated.

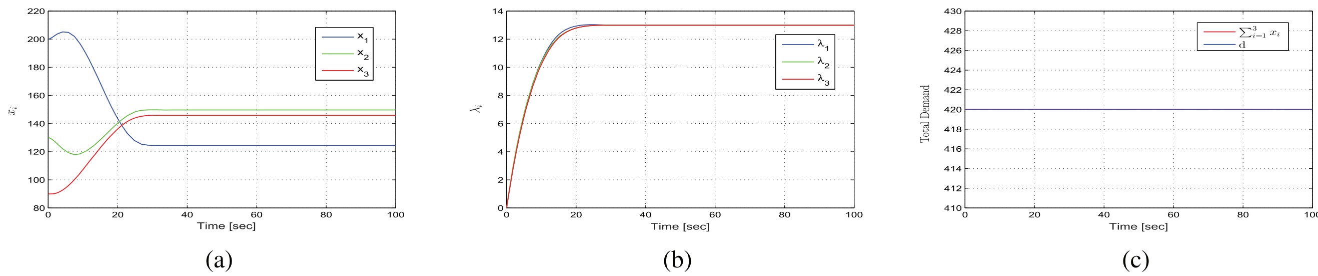


Fig. 6. IEEE 6-bus test system without capacity limits. (a) Power generation of generators. (b) Estimated marginal cost. (c) Power balance.

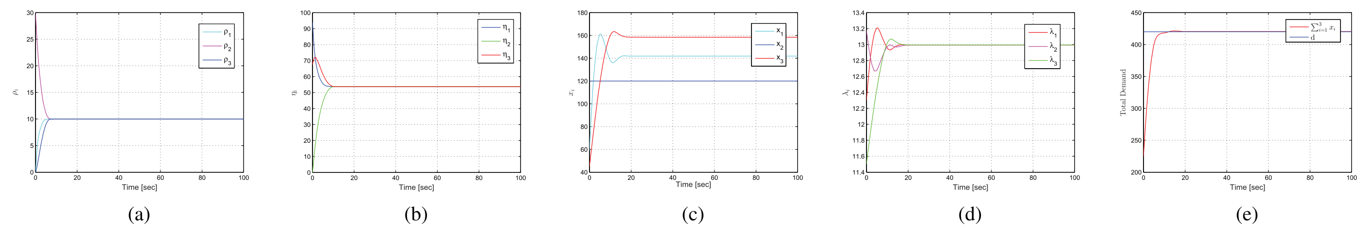


Fig. 7. IEEE 6-bus test system based on the proposed Algorithm 1 with capacity limits. (a) Variable ρ_i. (b) Variable η_i. (c) Power generation of generators. (d) Estimated marginal cost. (e) Power balance.

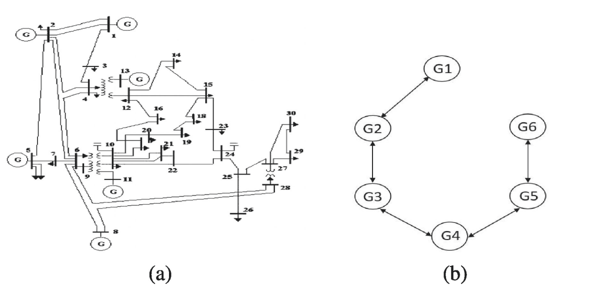


Fig. 8. IEEE 30-bus test system. (a) Single-line diagram. (b) Corresponding communication graph among generators.

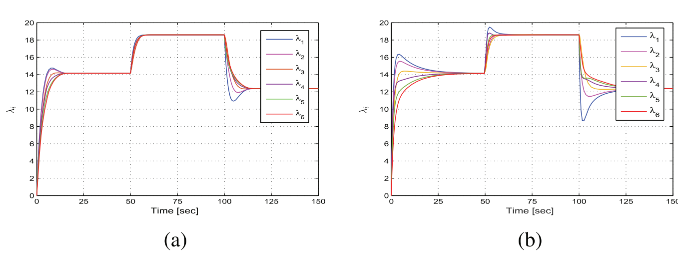


Fig. 9. Marginal cost update λ_i under the test system with time-varying demand. (a) Proposed FED algorithm. (b) ICC algorithm in [39].

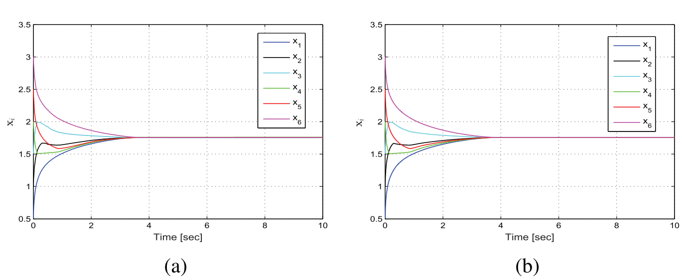


Fig. 10. Agents' finite-time estimation on the optimal solution by the proposed algorithm in (40). (a) Without disturbances. (b) With disturbances.

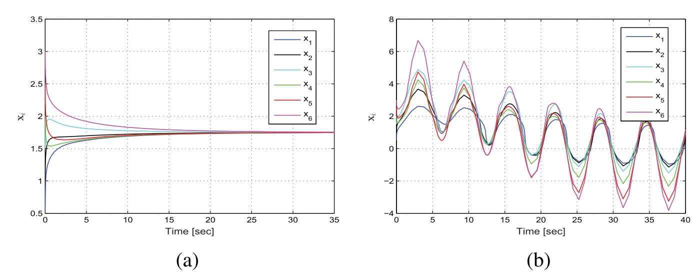


Fig. 11. Agents' estimation on the optimal solution by the ZGS algorithm in [10]. (a) Without disturbances. (b) With disturbances.

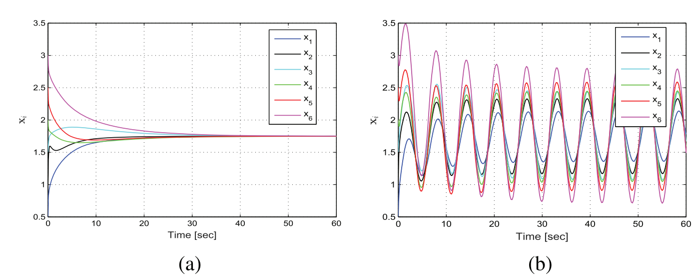
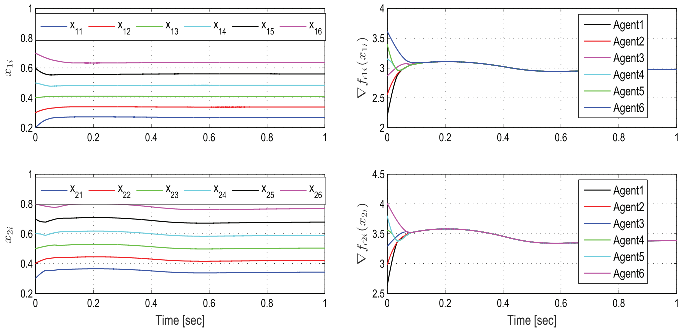


Fig. 12. Agents' estimation on the optimal solution by the SDG algorithm in [11] and [29]. (a) Without disturbances. (b) With disturbances.

TABLE III
PERFORMANCE COMPARISON OF DIFFERENT ALGORITHMS

Different algorithms	Convergence time (s)	Robustness
ISMC-based algorithm in (40)	4.0	yes
ZGS algorithm in [10], [28]	30.0	no
SDG algorithm in [11], [29]	45.0	no

Case 4. Finite-Time Resource Allocation: This example solves a resource allocation problem for nonquadratic



(a) (b)

Fig. 13. Fast resource allocation results using the algorithm (49). (a) Evolution of agents' resource allocations. (b) Agreement of agents' gradient updates.

objective functions

$$f_i(x_i) = x_i^T A x_i + B^T x_i + e^{B^T x_i} \tag{58}$$

where $x_i \in \mathbb{R}^2$, $A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ and $B = [0.1\ 0.2]^T$.

The local constraints for agents are given by: $0.1 * i \leq x_{1i} \leq 1.5$ and $0.1 * (i+1) \leq x_{2i} \leq 1.5$. The proposed algorithm (49) is performed with $\epsilon = 0.01$, and the simulation results are shown in Fig. 13. Fig. 13(a) shows the agents' optimal allocations and they always remain within the corresponding constraints, while Fig. 13(b) depicts the agreement of agents' gradient updates.

VII. Conclusion

In this paper, finite-time distributed algorithms have been proposed to address convex optimization problems for continuous-time multiagent systems in the presence of disturbances. The proposed distributed optimization algorithms combine a supertwisting-based and continuous ISMC scheme to deal with disturbances and to search for the optimal solution within a finite time. The designs are further applied to solve the economic dispatch and resource allocation problems with both global equality and local inequality constraints, respectively. It can be proven that the presented algorithms can find the optimal solution in a finite-time and robust manner.

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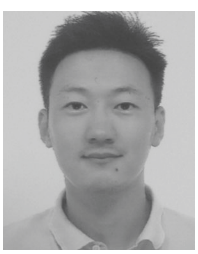

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