The Spectrum Problem for Two Multigraphs with Four Vertices and Seven Edges

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Abstract

Let G be one of the two multigraphs obtained from K_4-e by replacing two edges with a double-edge while maintaining a minimum degree of 2. We find necessary and sufficient conditions on n and λ for the existence of a G-decomposition of ${}^{\lambda}K_n$.

1 Introduction

Throughout this paper, we may refer to a multigraph as a graph; however, our graphs contain no loops. If we wish to emphasize that a given graph does not contain parallel edges, then we refer to it as a simple graph. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set (or multiset) of G, respectively. For a simple graph G and a positive integer λ, we use ${}^{\lambda}G$ to denote the graph obtained from G by replacing each edge in $E(G)$ with λ parallel edges. Alternatively, we let λG denote the graph consisting of λ vertex-disjoint copies of G. For edge-disjoint graphs G and H, we use $G \cup H$ to represent the graph with edge set (or multiset) $E(G) \cup E(H)$ and vertex set $V(G) \cup V(H)$. We define the *join* of two vertex-disjoint graphs G and H, denoted $G \vee H$, as the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \big\{ \{a,b\} : a \in V(G), b \in V(H) \big\}$. We use $K_{s\times t}$ to denote the complete multipartite simple graph with s parts

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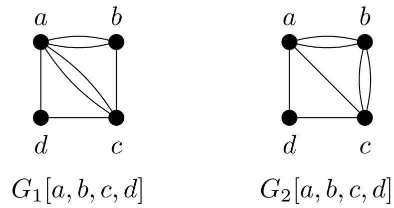

Figure 1: The two multigraphs consisting of $K_4 - e$ with two double edges and minimum degree 2.

of size t, and we use $K_{r,\, s\times t}$ to denote the complete multipartite simple graph with one part of size r and s parts of size t. If G is a subgraph of H, we use $H \setminus G$ to denote the graph obtained from H by removing $E(G)$ from $E(H)$.

1.1 The Spectrum Problem

Let K and G be graphs with G a subgraph of K. A *G-decomposition* of K is a set (or multiset) $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K such each $G_i \in \Delta$ is isomorphic to G and such that each edge of K appears in exactly one such G_i. Similarly, if G and H are each subgraphs or K, then a $\{G, H\}$-decomposition of K is defined to be a set $\{H_1, H_2, \ldots, H_t\}$ of subgraphs of K such that each $H_i \in \Delta$ is isomorphic to either G or H and such that each edge of K appears in exactly one such H_i. A G-decomposition of K is also known as a *(K, G)-design* or, if K is the complete graph on n vertices, a G-design of order n.

A classic problem in the study of graph designs is to find necessary and sufficient conditions for the existence of a G-decomposition of ${}^{\lambda}K_n$. This is known as the *spectrum problem* for G because the set of all such n is called the *spectrum for G-designs of index λ*. The spectrum for G-designs of index 1 has been determined for several classes of graphs including cycles, paths, stars, and simple graphs with at most 5 vertices (see [2]).

In recent years, there have been some investigations of G-designs of index λ where G is a multigraph with edge multiplicity at least 2. For example, in [4] Carter determined the spectra for G-designs of index λ for all connected cubic multigraphs G of order at most 6. The spectra for G-designs of index λ have been investigated for various multigraphs G of small order (see for example [8], [3], and [9]). In this paper we consider two multigraphs with 7 edges and minimum degree 2 obtained by replacing two edges of $K_4 - e$ with a double edge (see Figure 1). We settle the spectrum problem for these multigraphs.

1.2 Some Basic Results

The necessary conditions for the existence of a G-decomposition of ${}^{\lambda}K_n$ include the following:

- $|V(G)| \leq n$,
- $|E(G)|$ divides $|E({}^{\lambda}K_n)| = \lambda n(n-1)$, and
- $\gcd\{\deg(v) : v \in V(G)\}$ divides $\lambda(n-1)$.

Applying these necessary conditions to the two multigraphs under consideration, we obtain the following necessary conditions on their spectra.

Lemma 1.1. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G_1-decomposition of ${}^{\lambda}K_n$ only if the following hold:*

- *if* $\gcd(\lambda, 7) = 1$, *then* $n \equiv 0$ *or* $1 \pmod 7$*;*
- *if* $\gcd(\lambda, 7) = 7$, *then* $n \geq 4$*.*

Proof. Let $\lambda \geq 2$ and $n \geq 4$ and suppose there exists a G_1-decomposition of ${}^{\lambda}K_n$. Since $|E(G_1)|$ must divide $|E({}^{\lambda}K_n)|$ for such a G_1-decomposition to exist, we must have that $7 \mid \lambda n(n-1)/2$, and thus $14 \mid \lambda n(n-1)$. First, if $\gcd(\lambda, 7) = 1$, then $14 \mid n(n-1)$, and thus $n \equiv 0$ or $1 \pmod 7$. Finally, if $\gcd(\lambda, 7) = 7$, then $2 \mid n(n-1)$, which is true for any $n \geq 4$. ■

Lemma 1.2. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G_2-decomposition of ${}^{\lambda}K_n$ only if the following hold:*

- *if* $\gcd(\lambda, 14) = 1$, *then* $n \equiv 1$ *or* $7 \pmod{14}$*;*
- *if* $\gcd(\lambda, 14) = 2$, *then* $n \equiv 0$ *or* $1 \pmod 7$*;*
- *if* $\gcd(\lambda, 14) = 7$, *then* $n \equiv 1 \pmod 2$*;*
- *if* $\gcd(\lambda, 14) = 14$, *then* $n \geq 4$*.*

Proof. Let $\lambda \geq 2$ and $n \geq 4$ and suppose there exists a G_2-decomposition of ${}^{\lambda}K_n$. Since $|E(G_2)|$ must divide $|E({}^{\lambda}K_n)|$ for such a G_2-decomposition to exist, we must have that $7 \mid \lambda n(n-1)/2$, and thus $14 \mid \lambda n(n-1)$. Also, since all the vertices of G_2 have even degree, each vertex of ${}^{\lambda}K_n$ must similarly have even degree; thus, $2 \mid \lambda(n-1)$. First, if $\gcd(\lambda, 14) = 1$, then $14 \mid n(n-1)$ and $2 \mid (n-1)$, and thus $n \equiv 1$ or $7 \pmod{14}$. Second, if $\gcd(\lambda, 14) = 2$, then λ is even and $7 \mid n(n-1)$, and thus $n \equiv 0$ or $1 \pmod 7$. Third, if $\gcd(\lambda, 14) = 7$, then $2 \mid n(n-1)$ but also $2 \mid (n-1)$, and thus $n \equiv 1 \pmod 2$. Finally, if $\gcd(\lambda, 14) = 14$, then $14 \mid \lambda$, and thus there are no further restrictions on n. ■

The following theorems on decompositions of complete graphs and complete multipartite graphs are used extensively in proving our main results. All of these results can be found in the *Handbook of Combinatorial Designs* [5] (see [1], [6], and [7]).

Theorem 1.3. *If n is an odd positive integer, then there exists a $\{K_3, K_5\}$-decomposition of K_n.*

Theorem 1.4. *The necessary and sufficient conditions for the existence of a K_3-decomposition of $K_{t\times m}$ are (i) $t \geq 3$, (ii) $(t-1)m \equiv 0 \pmod 2$, and (iii) $t(t-1)m^2 \equiv 0 \pmod 6$.*

Theorem 1.5. *If $t \geq 3$ and $t \equiv 0 \pmod 3$, then there exists a K_3-decomposition of $K_{4,\,t\times 2}$.*

Combining the previous two results, we have the following corollary that is more directly applicable in our general constructions.

Corollary 1.6. *Let $t \geq 3$. There exists a K_3-decomposition of $K_{t\times 2}$ if $t \equiv 0$ or $1 \pmod 3$ and of $K_{4,\,(t-2)\times 2}$ if $t \equiv 2 \pmod 3$.*

The following is a well-known result that is a special case of Wilson's Fundamental Construction (see [7]).

Theorem 1.7. *Let m, n, r, s, and t be positive integers. If there exists a $(K_{t\times m}, K_n)$-design, then there exists a $(K_{t\times ms}, K_{n\times s})$-design. Similarly, if there exists a $(K_{r,\,t\times m}, K_n)$-design, then there exists a $(K_{rs,\,t\times ms}, K_{n\times s})$-design.*

2 Some Small Examples

In this section we present G_1- and G_2-decompositions of various graphs that are needed for the constructions used in Section 3. Let $G \in \{G_1, G_2\}$. Then $G[a, b, c, d]$ denotes the graph with vertex set $\{a, b, c, d\}$ and edge set as represented in Figure 1. For example, $G_2[0, 1, 2, 3]$ denotes the graph with vertex set $\{0, 1, 2, 3\}$ and edge multiset $\{\{0, 1\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 2\}, \{2, 3\}\}$. Given the graphs represented by the notation $G[a, b, c, d]$ and some $i \in \mathbb{Z}_n$, we define $G[a, b, c, d] + i = G[a+i, b+i, c+i, d+i]$ where all addition is performed in $\mathbb{Z}_n$. By convention, define $\infty + 1 = \infty$.

2.1 Small Designs of Index 2

Example 2.1. Let $V\left({}^2K_7\right) = \mathbb{Z}_6 \cup \{\infty\}$ and let $\Delta_1 = \{G_1[1, 0, 3, \infty] + i : i \in \mathbb{Z}_6\}$ and $\Delta_2 = \{G_2[0, 1, 3, \infty] + i : i \in \mathbb{Z}_6\}$. Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^2K_7$.

Example 2.2. Let $V\left({}^2K_8\right) = \mathbb{Z}_8$ and let $\Delta_1 = \{G_1[0, 2, 3, 7] + i : i \in \mathbb{Z}_8\}$ and $\Delta_2 = \{G_2[1, 0, 3, 5] + i : i \in \mathbb{Z}_8\}$. Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^2K_8$.

Example 2.3. Let $V\left({}^2K_{14}\right) = \mathbb{Z}_{13} \cup \{\infty\}$ and let $\Delta_1 = \{G_1[5,3,0,\infty]+i : i \in \mathbb{Z}_{13}\} \cup \{G_1[0,4,1,7]+i : i \in \mathbb{Z}_{13}\}$ and $\Delta_2 = \{G_2[0,3,1,\infty]+i : i \in \mathbb{Z}_{13}\} \cup \{G_2[0,5,1,7]+i : i \in \mathbb{Z}_{13}\}$. Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^2K_{14}$.

Example 2.4. Let $V\left({}^2K_{15}\right) = \mathbb{Z}_{15}$ and let $\Delta_1 = \{G_1[5,0,1,3]+i : i \in \mathbb{Z}_{15}\} \cup \{G_1[7,0,1,4]+i : i \in \mathbb{Z}_{15}\}$ and $\Delta_2 = \{G_2[0,3,1,7]+i : i \in \mathbb{Z}_{15}\} \cup \{G_2[0,5,1,7]+i : i \in \mathbb{Z}_{15}\}$. Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^2K_{15}$.

Example 2.5. Let $V\left({}^2K_{28}\right) = \mathbb{Z}_{27} \cup \{\infty\}$ and let

$$\begin{aligned}\Delta_1 = &\{G_1[17,0,11,\infty]+i : i \in \mathbb{Z}_{27}\} \cup \{G_1[1,0,15,23]+i : i \in \mathbb{Z}_{27}\}\\ &\cup \{G_1[9,0,12,17]+i : i \in \mathbb{Z}_{27}\} \cup \{G_1[4,11,0,2]+i : i \in \mathbb{Z}_{27}\},\\ \Delta_2 = &\{G_2[0,3,1,\infty]+i : i \in \mathbb{Z}_{27}\} \cup \{G_2[0,8,1,14]+i : i \in \mathbb{Z}_{27}\}\\ &\cup \{G_2[0,11,5,9]+i : i \in \mathbb{Z}_{27}\} \cup \{G_2[10,0,15,6]+i : i \in \mathbb{Z}_{27}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^2K_{28}$.

Example 2.6. Let $V\left({}^2K_{29}\right) = \mathbb{Z}_{29}$ and let

$$\begin{aligned}\Delta_1 = &\{G_1[0,2,8,5]+i : i \in \mathbb{Z}_{29}\} \cup \{G_1[0,14,11,4]+i : i \in \mathbb{Z}_{29}\}\\ &\cup \{G_1[0,12,13,6]+i : i \in \mathbb{Z}_{29}\} \cup \{G_1[0,10,9,5]+i : i \in \mathbb{Z}_{29}\},\\ \Delta_2 = &\{G_2[14,2,0,3]+i : i \in \mathbb{Z}_{29}\} \cup \{G_2[13,4,0,5]+i : i \in \mathbb{Z}_{29}\}\\ &\cup \{G_2[5,6,0,13]+i : i \in \mathbb{Z}_{29}\} \cup \{G_2[3,10,0,14]+i : i \in \mathbb{Z}_{29}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^2K_{29}$.

Example 2.7. Let $V\left({}^2K_{3\times7}\right) = \mathbb{Z}_{21}$ with partition $\big\{\{i \in \mathbb{Z}_{21} : i \equiv j \pmod 3\} : j \in \mathbb{Z}_3\big\}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[1,5,0,8]+i : i \in \mathbb{Z}_{21}\} \cup \{G_1[0,10,2,7]+i : i \in \mathbb{Z}_{21}\},\\ \Delta_2 &= \{G_2[0,8,7,2]+i : i \in \mathbb{Z}_{21}\} \cup \{G_2[0,11,7,2]+i : i \in \mathbb{Z}_{21}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^2K_{3\times7}$.

Example 2.8. Let $V\left({}^2K_{5\times7}\right) = \mathbb{Z}_{35}$ with partition $\big\{\{i \in \mathbb{Z}_{35} : i \equiv j \pmod 5\} : j \in \mathbb{Z}_5\big\}$ and let

$$\begin{aligned}\Delta_1 = &\{G_1[9,12,0,1]+i : i \in \mathbb{Z}_{35}\} \cup \{G_1[0,21,13,1]+i : i \in \mathbb{Z}_{35}\}\\ &\cup \{G_1[6,13,2,0]+i : i \in \mathbb{Z}_{35}\} \cup \{G_1[0,19,17,11]+i : i \in \mathbb{Z}_{35}\},\\ \Delta_2 = &\{G_2[0,11,4,16]+i : i \in \mathbb{Z}_{35}\} \cup \{G_2[0,6,4,13]+i : i \in \mathbb{Z}_{35}\}\\ &\cup \{G_2[0,17,9,12]+i : i \in \mathbb{Z}_{35}\} \cup \{G_2[0,14,13,16]+i : i \in \mathbb{Z}_{35}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^2K_{5\times7}$.

2.2 Small Designs of Index 3

Example 2.9. Let $V(^3K_7) = \mathbb{Z}_2 \times \mathbb{Z}_3 \cup \{\infty\}$ and let

$$\begin{aligned}\Delta_1 = &\{G_1[(0,0+i),(1,0+i),(1,2+i),(0,1+i)] : i \in \mathbb{Z}_3\}\\ &\cup \{G_1[(1,2+i),\infty,(1,1+i),(0,1+i)] : i \in \mathbb{Z}_3\}\\ &\cup \{G_1[(0,2+i),\infty,(0,1+i),(1,0+i)] : i \in \mathbb{Z}_3\},\\ \Delta_2 = &\{G_2[\infty,(0,1+i),(0,0+i),(1,1+i)] : i \in \mathbb{Z}_3\}\\ &\cup \{G_2[(1,1+i),(0,1+i),(1,0+i),\infty] : i \in \mathbb{Z}_3\}\\ &\cup \{G_2[(0,0+i),(1,1+i),(1,2+i),(0,2+i)] : i \in \mathbb{Z}_3\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of $^3K_7$.

Example 2.10. Let $V(^3K_8) = \mathbb{Z}_8$ and let

$$\begin{aligned}\Delta_1 = \{&G_1[4,1,3,6], G_1[2,6,0,3], G_1[2,3,7,0], G_1[0,1,3,4],\\ &G_1[0,4,5,1], G_1[0,6,7,5], G_1[1,6,5,2], G_1[2,4,1,7],\\ &G_1[3,5,7,1], G_1[4,7,5,2], G_1[6,3,5,2], G_1[6,4,7,1]\}.\end{aligned}$$

Then Δ_1 is a G_1-decomposition of $^3K_8$.

Example 2.11. Let $V(^3K_{14}) = \mathbb{Z}_{13} \cup \{\infty\}$ and let $\Delta_1 = \{G_1[0,\infty,10,4] + i : i \in \mathbb{Z}_{13}\} \cup \{G_1[0,2,5,1] + i : i \in \mathbb{Z}_{13}\} \cup \{G_1[0,1,6,2] + i : i \in \mathbb{Z}_{13}\}$. Then Δ_1 is a G_1-decomposition of $^3K_{14}$.

Example 2.12. Let $V(^3K_{15}) = \mathbb{Z}_{15}$ and let

$$\begin{aligned}\Delta_1 = &\{G_1[0,2,5,6] + i : i \in \mathbb{Z}_{15}\} \cup \{G_1[0,4,7,5] + i : i \in \mathbb{Z}_{15}\}\\ &\cup \{G_1[1,7,0,4] + i : i \in \mathbb{Z}_{15}\},\\ \Delta_2 = &\{G_2[0,1,7,2] + i : i \in \mathbb{Z}_{15}\} \cup \{G_2[0,2,7,3] + i : i \in \mathbb{Z}_{15}\}\\ &\cup \{G_2[0,3,7,1] + i : i \in \mathbb{Z}_{15}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of $^3K_{15}$.

Example 2.13. Let $V(^3K_{28}) = \mathbb{Z}_{27} \cup \{\infty\}$ and let

$$\begin{aligned}\Delta_1 = &\{G_1[0,\infty,13,26] + i : i \in \mathbb{Z}_{27}\} \cup \{G_1[0,12,16,5] + i : i \in \mathbb{Z}_{27}\}\\ &\cup \{G_1[0,10,6,12] + i : i \in \mathbb{Z}_{27}\}\\ &\cup \{G_1[0,9,5,8] + i : i \in \mathbb{Z}_{27}\} \cup \{G_1[0,7,8,10] + i : i \in \mathbb{Z}_{27}\}\\ &\cup \{G_1[0,3,2,9] + i : i \in \mathbb{Z}_{27}\}.\end{aligned}$$

Then Δ_1 is a G_1-decomposition of $^3K_{28}$.

Example 2.14. Let $V\left({}^3K_{29}\right) = \mathbb{Z}_{29}$ and let

$$\begin{aligned}\Delta_1 = &\{G_1[0,15,13,10]+i : i \in \mathbb{Z}_{29}\} \cup \{G_1[0,12,10,14]+i : i \in \mathbb{Z}_{29}\}\\ &\cup \{G_1[0,11,9,18]+i : i \in \mathbb{Z}_{29}\} \cup \{G_1[0,8,7,13]+i : i \in \mathbb{Z}_{29}\}\\ &\cup \{G_1[0,6,5,12]+i : i \in \mathbb{Z}_{29}\} \cup \{G_1[0,4,3,8]+i : i \in \mathbb{Z}_{29}\},\\ \Delta_2 = &\{G_2[0,13,27,14]+i : i \in \mathbb{Z}_{29}\} \cup \{G_2[0,12,2,19]+i : i \in \mathbb{Z}_{29}\}\\ &\cup \{G_2[0,11,2,20]+i : i \in \mathbb{Z}_{29}\} \cup \{G_2[0,8,1,22]+i : i \in \mathbb{Z}_{29}\}\\ &\cup \{G_2[0,6,1,24]+i : i \in \mathbb{Z}_{29}\} \cup \{G_2[0,4,1,26]+i : i \in \mathbb{Z}_{29}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^3K_{29}$.

Example 2.15. Let $V\left({}^3K_{3\times 7}\right) = \mathbb{Z}_{21}$ with partition $\big\{\{i \in \mathbb{Z}_{21} : i \equiv j \pmod 3\} : j \in \mathbb{Z}_3\big\}$ and let

$$\begin{aligned}\Delta_1 = &\{G_1[0,5,10,14]+i : i \in \mathbb{Z}_{21}\} \cup \{G_1[0,1,14,13]+i : i \in \mathbb{Z}_{21}\}\\ &\cup \{G_1[0,4,2,10]+i : i \in \mathbb{Z}_{21}\},\\ \Delta_2 = &\{G_2[0,8,7,11]+i : i \in \mathbb{Z}_{21}\} \cup \{G_2[0,2,7,8]+i : i \in \mathbb{Z}_{21}\}\\ &\cup \{G_2[0,11,7,2]+i : i \in \mathbb{Z}_{21}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^3K_{3\times 7}$.

Example 2.16. Let $V\left({}^3K_{5\times 7}\right) = \mathbb{Z}_{35}$ with partition $\big\{\{i \in \mathbb{Z}_{35} : i \equiv j \pmod 5\} : j \in \mathbb{Z}_5\big\}$ and let

$$\begin{aligned}\Delta_1 = &\{G_1[0,17,16,14]+i : i \in \mathbb{Z}_{35}\} \cup \{G_1[0,13,14,1]+i : i \in \mathbb{Z}_{35}\}\\ &\cup \{G_1[0,12,9,16]+i : i \in \mathbb{Z}_{35}\} \cup \{G_1[0,8,11,17]+i : i \in \mathbb{Z}_{35}\}\\ &\cup \{G_1[0,7,4,12]+i : i \in \mathbb{Z}_{35}\} \cup \{G_1[0,6,2,11]+i : i \in \mathbb{Z}_{35}\},\\ \Delta_2 = &\{G_2[0,4,1,7]+i : i \in \mathbb{Z}_{35}\} \cup \{G_2[0,9,1,4]+i : i \in \mathbb{Z}_{35}\}\\ &\cup \{G_2[0,7,1,9]+i : i \in \mathbb{Z}_{35}\} \cup \{G_2[0,13,2,14]+i : i \in \mathbb{Z}_{35}\}\\ &\cup \{G_2[0,18,2,13]+i : i \in \mathbb{Z}_{35}\} \cup \{G_2[0,14,2,18]+i : i \in \mathbb{Z}_{35}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^3K_{5\times 7}$.

2.3 Small Designs of Index 7

Example 2.17. Let $V\left({}^7K_4\right) = \mathbb{Z}_3 \cup \{\infty\}$ and let $\Delta_1 = \{G_1[\infty,0,1,2]+i : i \in \mathbb{Z}_3\} \cup \{G_1[0,1,2,\infty]+i : i \in \mathbb{Z}_3\}$. Then Δ_1 is a G_1-decomposition of ${}^7K_4$.

Example 2.18. Let $V\left({}^7K_5\right) = \mathbb{Z}_5$ and let $\Delta_1 = \{G_1[0,4,3,1]+i : i \in \mathbb{Z}_5\} \cup \{G_1[0,3,1,2]+i : i \in \mathbb{Z}_5\}$ and $\Delta_2 = \{G_2[0,3,2,1]+i : i \in \mathbb{Z}_5\} \cup \{G_2[0,2,3,1]+i : i \in \mathbb{Z}_5\}$. Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^7K_5$.

Example 2.19. Let $V({}^7K_6) = \mathbb{Z}_5 \cup \{\infty\}$ and let $\Delta_1 = \{G_1[0,1,2,\infty]+i : i \in \mathbb{Z}_5\} \cup \{G_1[\infty,0,1,3]+i : i \in \mathbb{Z}_5\} \cup \{G_1[0,2,3,4]+i : i \in \mathbb{Z}_5\}$. Then Δ_1 is a G_1-decomposition of ${}^7K_6$.

Example 2.20. Let $V({}^7K_9) = \mathbb{Z}_9$ and let

$$\begin{aligned}\Delta_1 = \{&G_1[0,1,3,4]+i : i \in \mathbb{Z}_9\} \cup \{G_1[0,2,4,3]+i : i \in \mathbb{Z}_9\}\\ &\cup \{G_1[0,5,6,8]+i : i \in \mathbb{Z}_9\} \cup \{G_1[0,6,5,7]+i : i \in \mathbb{Z}_9\},\\ \Delta_2 = \{&G_2[2,0,6,5]+i : i \in \mathbb{Z}_9\} \cup \{G_2[1,0,4,8]+i : i \in \mathbb{Z}_9\}\\ &\cup \{G_2[1,0,2,3]+i : i \in \mathbb{Z}_9\} \cup \{G_2[3,0,5,8]+i : i \in \mathbb{Z}_9\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^7K_9$.

Example 2.21. Let $V({}^7K_{10}) = \mathbb{Z}_9 \cup \{\infty\}$ and let

$$\begin{aligned}\Delta_1 = \{&G_1[0,\infty,3,4]+i : i \in \mathbb{Z}_9\} \cup \{G_1[2,0,\infty,1]+i : i \in \mathbb{Z}_9\}\\ &\cup \{G_1[0,1,4,3]+i : i \in \mathbb{Z}_9\} \cup \{G_1[0,4,3,1]+i : i \in \mathbb{Z}_9\}\\ &\cup \{G_1[4,6,0,2]+i : i \in \mathbb{Z}_9\}.\end{aligned}$$

Then Δ_1 is a G_1-decomposition of ${}^7K_{10}$.

Example 2.22. Let $V({}^7K_{11}) = \mathbb{Z}_{11}$ and let

$$\begin{aligned}\Delta_1 = \{&G_1[0,1,8,9]+i : i \in \mathbb{Z}_{11}\} \cup \{G_1[0,4,3,9]+i : i \in \mathbb{Z}_{11}\}\\ &\cup \{G_1[0,3,5,1]+i : i \in \mathbb{Z}_{11}\} \cup \{G_1[0,5,4,2]+i : i \in \mathbb{Z}_{11}\}\\ &\cup \{G_1[0,2,5,4]+i : i \in \mathbb{Z}_{11}\},\\ \Delta_2 = \{&G_2[1,0,2,6]+i : i \in \mathbb{Z}_{11}\} \cup \{G_2[3,0,4,9]+i : i \in \mathbb{Z}_{11}\}\\ &\cup \{G_2[4,0,5,7]+i : i \in \mathbb{Z}_{11}\} \cup {}^2\{G_2[2,0,3,7]+i : i \in \mathbb{Z}_{11}\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^7K_{11}$.

Example 2.23. Let $V({}^7K_{3\times 2}) = \mathbb{Z}_6$ with partition $\{\{0,3\},\{1,4\},\{2,5\}\}$ and let

$$\begin{aligned}\Delta_1 &= \{G_1[0,5,1,2]+i : i \in \mathbb{Z}_6\} \cup \{G_1[0,4,2,1]+i : i \in \mathbb{Z}_6\},\\ \Delta_2 &= \{G_2[0,1,2,4], G_2[0,1,5,4], G_2[0,2,1,5], G_2[0,5,4,2],\\ &\quad G_2[3,4,2,1], G_2[3,4,5,1], G_2[3,2,4,5], G_2[3,5,1,2],\\ &\quad G_2[1,3,2,0], G_2[1,3,5,0], G_2[4,0,2,3], G_2[4,0,5,3]\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^7K_{3\times 2}$.

Example 2.24. Let $V\left({}^7K_6 \setminus {}^7K_2\right) = \mathbb{Z}_6$ with $V\left({}^7K_2\right) = \{0,1\}$ and let

$$\begin{aligned}\Delta_1 = \{&G_1[5,3,2,4], G_1[3,2,4,5], G_1[2,1,3,5], G_1[0,2,3,4], G_1[0,2,3,4],\\ &G_1[0,2,4,3], G_1[0,3,5,4], G_1[1,3,5,2], G_1[1,3,5,2], G_1[4,1,5,0],\\ &G_1[4,2,1,3], G_1[4,2,1,3], G_1[5,0,2,1], G_1[5,0,4,1]\}.\end{aligned}$$

Then Δ_1 is a G_1-decomposition of ${}^7K_6 \setminus {}^7K_2$.

Example 2.25. Let $V\left({}^7K_7 \setminus {}^7K_3\right) = \mathbb{Z}_7$ with $V\left({}^7K_3\right) = \{0,1,2\}$ and let

$$\begin{aligned}\Delta_1 = \{&G_1[0,6,4,3], G_1[6,3,2,5], G_1[0,6,3,4], G_1[0,3,4,5], G_1[0,3,4,5],\\ &G_1[1,3,4,5], G_1[1,3,4,5], G_1[1,3,5,4], G_1[2,3,5,4], G_1[2,4,6,3],\\ &G_1[2,4,6,3], G_1[5,0,6,1], G_1[5,0,6,3], G_1[5,2,3,1], G_1[5,3,2,4],\\ &G_1[6,1,4,2], G_1[6,1,4,3], G_1[6,1,5,0]\},\end{aligned}$$

$$\begin{aligned}\Delta_2 = \{&G_2[2,5,4,6], G_2[2,3,6,4], G_2[2,5,3,4], G_2[0,3,4,5], G_2[0,3,4,5],\\ &G_2[0,3,4,5], G_2[0,4,5,6], G_2[0,6,5,3], G_2[2,3,6,4], G_2[3,1,5,2],\\ &G_2[3,1,5,2], G_2[3,1,5,6], G_2[3,6,5,1], G_2[4,0,6,1], G_2[4,1,6,2],\\ &G_2[4,1,6,2], G_2[4,1,6,2], G_2[5,0,6,2]\}.\end{aligned}$$

Then Δ_1 and Δ_2 are respectively G_1- and G_2-decompositions of ${}^7K_7 \setminus {}^7K_3$.

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Example 2.26. Let $V\left({}^{14}K_4\right) = \mathbb{Z}_4$ and let $\Delta_2 = \{G_2[0,1,2,3] + i : i \in \mathbb{Z}_4\} \cup {}^2\{G_2[0,2,1,3] + i : i \in \mathbb{Z}_4\}$. Then Δ_2 is a G_2-decomposition of ${}^{14}K_4$.

Example 2.27. Let $V\left({}^{14}K_6\right) = \mathbb{Z}_5 \cup \{\infty\}$ and let $\Delta_2 = {}^3\{G_2[\infty,0,2,4]+i : i \in \mathbb{Z}_5\} \cup \{G_2[0,1,2,\infty] + i : i \in \mathbb{Z}_5\} \cup {}^2\{G_2[0,1,2,3] + i : i \in \mathbb{Z}_6\}$. Then Δ_2 is a G_2-decomposition of ${}^{14}K_6$.

Example 2.28. Let $V\left({}^{14}K_{10}\right) = \mathbb{Z}_9 \cup \{\infty\}$ and let

$$\begin{aligned}\Delta_2 = {}^2&\{G_2[0,4,\infty,2] + i : i \in \mathbb{Z}_9\} \cup \{G_2[0,2,\infty,4] + i : i \in \mathbb{Z}_9\}\\ &\cup \{G_2[0,2,4,\infty] + i : i \in \mathbb{Z}_9\} \cup \{G_2[0,4,2,7] + i : i \in \mathbb{Z}_9\}\\ &\cup \{G_2[0,1,4,2] + i : i \in \mathbb{Z}_9\} \cup {}^4\{G_2[0,1,4,3] + i : i \in \mathbb{Z}_9\}.\end{aligned}$$

Then Δ_2 is a G_2-decomposition of ${}^{14}K_{10}$.

Example 2.29. Let $V\left({}^{14}K_6 \setminus {}^{14}K_2\right) = \mathbb{Z}_4 \cup \{\infty_1, \infty_2\}$ with $V\left({}^{14}K_2\right) = \{\infty_1, \infty_2\}$ and let

$$\begin{aligned}\Delta_2 = {}^3&\{G_2[0+i, \infty_1, 2+i, \infty_2] : i \in \mathbb{Z}_4\}\\ &\cup \{G_2[0+i, \infty_2, 2+i, \infty_1] : i \in \mathbb{Z}_4\}\\ &\cup \{G_2[0+i, \infty_2, 2+i, 1+i] : i \in \mathbb{Z}_4\}\\ &\cup {}^2\{G_2[0+i, 1+i, 2+i, 3+i] : i \in \mathbb{Z}_4\}.\end{aligned}$$

Then Δ_2 is a G_2-decomposition of ${}^{14}K_6 \setminus {}^{14}K_2$.

3 Main Results

Through judicious use of the examples from the previous section, we show that the necessary conditions on G_1- and G_2-designs are sufficient for any index $\lambda \geq 2$. In the following constructions, we make extensive use of the join of complete graphs. Of special note is our use of the null graph K_0, which has an empty vertex set. For example, $K_7 \vee K_0$ is simply K_7. Similarly, $K_7 \setminus K_0$ is also K_7. On the other hand, $K_7 \vee K_1 = K_8$, but $K_7 \setminus K_1 = K_7$.

First, we now settle the spectra for G_1- and G_2-designs of certain indices.

Lemma 3.1. *Let $G \in \{G_1, G_2\}$. There exists a G-decomposition of ${}^2K_n$ if $n \equiv 0$ or $1 \pmod 7$.*

Proof. Let $G \in \{G_1, G_2\}$ and let $n = 7r + t$ for some positive integer r and $t \in \{0, 1\}$. If (r, t) is $(1, 0)$, $(1, 1)$, $(2, 0)$, $(2, 1)$, $(4, 0)$, or $(4, 1)$, then the result follows from Examples 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6, respectively. The remainder of the proof breaks into two cases.

Case 1: r is odd with $r \geq 3$.
By Theorem 1.3 there exists a $\{K_3, K_5\}$-decomposition of K_r. Thus by Theorem 1.7 there exists a $\{K_{3\times 7}, K_{5\times 7}\}$-decomposition of $K_{r\times 7}$. Since there exist G-decompositions of both ${}^2K_{3\times 7}$ and ${}^2K_{5\times 7}$ (by Examples 2.7 and 2.8, respectively), a G-decomposition of ${}^2K_{r\times 7}$ also exists by transitivity. Finally, we note that $K_{7r+t} = (rK_7 \vee K_t) \cup K_{r\times 7} = K_{r\times 7} \cup \bigcup_{i=1}^{r} K_{7+t}$. Thus ${}^2K_{7r+t} = {}^2K_{r\times 7} \cup \bigcup_{i=1}^{r} {}^2K_{7+t}$, and the result follows from the existence of G-decompositions of ${}^2K_{r\times 7}$, ${}^2K_7$, and ${}^2K_8$.

Case 2: r is even with $r \geq 6$.
Let $r = 2s$ for some integer $s \geq 3$; hence, $n = 14s+t$. By Corollary 1.6 there exists a K_3-decomposition either of $K_{s\times 2}$ if $s \not\equiv 2 \pmod 3$ or of $K_{4,\,(s-2)\times 2}$ otherwise. Thus by Theorem 1.7 there exists a $K_{3\times 7}$-decomposition of either $K_{s\times 14}$ or $K_{28,\,(s-2)\times 14}$. Since there exists a G-decomposition of ${}^2K_{3\times 7}$ (by Example 2.7), a G-decomposition of either ${}^2K_{s\times 14}$ or ${}^2K_{28,\,(s-2)\times 14}$ also exists by transitivity. Finally, we note that K_{14s+t} can be described as either $(sK_{14} \vee K_t) \cup K_{s\times 14} = K_{s\times 14} \cup \bigcup_{i=1}^{s} K_{14+t}$ or $\big((K_{28} \cup (s-2)K_{14}) \vee K_t\big) \cup K_{28,\,(s-2)\times 14} = K_{28,\,(s-2)\times 14} \cup K_{28+t} \cup \bigcup_{i=1}^{s-2} K_{14+t}$. Thus, we describe ${}^2K_{14s+t}$ as ${}^2K_{s\times 14} \cup \bigcup_{i=1}^{s} {}^2K_{14+t}$ when $s \not\equiv 2 \pmod 3$ and as ${}^2K_{28,\,(s-2)\times 14} \cup {}^2K_{28+t} \cup \bigcup_{i=1}^{s-2} {}^2K_{14+t}$ when $s \equiv 2 \pmod 3$, and the result follows from the existence of G-decompositions of ${}^2K_{s\times 14}$ or ${}^2K_{28,\,(s-2)\times 14}$, ${}^2K_{14}$, ${}^2K_{15}$, ${}^2K_{28}$, and ${}^2K_{29}$. ∎

Lemma 3.2. *There exists a G_1-decomposition of ${}^3K_n$ if $n \equiv 0$ or $1 \pmod 7$.*

Proof. Let $n = 7r + t$ for some positive integer r and $t \in \{0, 1\}$. If (r, t) is $(1, 0)$, $(1, 1)$, $(2, 0)$, $(2, 1)$, $(4, 0)$, or $(4, 1)$, then the result follows from Examples 2.9, 2.10, 2.11, 2.12, 2.13, and 2.14, respectively. The proof then follows as in the proof of Lemma 3.1, where the requisite G_1-decompositions of the multipartite graphs ${}^3K_{3\times 7}$ and ${}^3K_{5\times 7}$ can be found in Examples 2.15 and 2.16, respectively. ■

Lemma 3.3. *There exists a G_2-decomposition of ${}^3K_n$ if $n \equiv 1$ or $7 \pmod{14}$.*

Proof. If n is 7, 15, or 29, then the result follows from Examples 2.9, 2.12, and 2.14, respectively. The remainder of the proof breaks into two cases.

Case 1: $n \equiv 1 \pmod{14}$ with $n \geq 43$.
Let $n = 14r + 1$ for some integer $r \geq 3$; hence, $n = 7(2r) + 1$. By Corollary 1.6 there exists a K_3-decomposition either of $K_{r\times 2}$ if $r \not\equiv 2 \pmod 3$ or of $K_{4,\,(r-2)\times 2}$ otherwise. Thus by Theorem 1.7 there exists a $K_{3\times 7}$-decomposition of either $K_{r\times 14}$ or $K_{28,\,(r-2)\times 14}$. Since there exists a G_2-decomposition of ${}^3K_{3\times 7}$ (by Example 2.15), a G_2-decomposition of either ${}^3K_{r\times 14}$ or ${}^3K_{28,\,(r-2)\times 14}$ also exists by transitivity. Finally, we note that K_{14r+1} can be described as either $(rK_{14} \vee K_1) \cup K_{r\times 14} = K_{r\times 14} \cup \bigcup_{i=1}^{r} K_{15}$ or $\big((K_{28} \cup (r-2)K_{14}) \vee K_1\big) \cup K_{28,\,(r-2)\times 14} = K_{28,\,(r-2)\times 14} \cup K_{29} \cup \bigcup_{i=1}^{r-2} K_{15}$. Thus, we describe ${}^3K_{14r+1}$ as ${}^3K_{r\times 14} \cup \bigcup_{i=1}^{r} {}^3K_{15}$ when $r \not\equiv 2 \pmod 3$ and as ${}^3K_{28,\,(r-2)\times 14} \cup {}^3K_{29} \cup \bigcup_{i=1}^{r-2} {}^3K_{15}$ when $r \equiv 2 \pmod 3$, and the result follows from the existence of G_2-decompositions of ${}^3K_{r\times 14}$ or ${}^3K_{28,\,(r-2)\times 14}$, ${}^3K_{15}$, and ${}^3K_{29}$.

Case 2: $n \equiv 7 \pmod{14}$ with $n \geq 21$.
Let $n = 14r + 7$ for some positive integer r; hence, $n = 7(2r + 1)$. By Theorem 1.3 there exists a $\{K_3, K_5\}$-decomposition of K_{2r+1}. Thus by Theorem 1.7 there exists a $\{K_{3\times 7}, K_{5\times 7}\}$-decomposition of $K_{(2r+1)\times 7}$. Since there exist G_2-decompositions of both ${}^3K_{3\times 7}$ and ${}^3K_{5\times 7}$ (by Examples 2.15 and 2.16, respectively), a G_2-decomposition of ${}^3K_{(2r+1)\times 7}$ also exists by transitivity. Finally, we note that $K_{14r+7} = K_{(2r+1)\times 7} \cup (2r + 1)K_7 = K_{(2r+1)\times 7} \cup \bigcup_{i=1}^{2r+1} K_7$. Thus ${}^3K_{14r+7} = {}^3K_{(2r+1)\times 7} \cup \bigcup_{i=1}^{2r+1} {}^3K_7$, and the result follows from the existence of G_2-decompositions of ${}^3K_{(2r+1)\times 7}$ and ${}^3K_7$. ■

Lemma 3.4. *There exists a G_1-decomposition of ${}^7K_n$ if $n \geq 4$.*

Proof. Let $n = 4r + t$ for some positive integer r and $t \in \{0, 1, 2, 3\}$. If (r, t) is $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, or $(2, 3)$, then the result follows from Examples 2.17, 2.18, 2.19, 2.20, 2.21, and 2.22, respectively. If (r, t) is $(1, 3)$ or $(2, 0)$, then $n \equiv 0$ or $1 \pmod 7$, and the result follows from 2 copies of a G_1-decomposition of ${}^2K_n$ (see Lemma 3.1) and

1 copy of a G_1-decomposition of ${}^3K_n$ (see Lemma 3.2). For the remainder of the proof, we assume $r \geq 3$. By Corollary 1.6 there exists a K_3-decomposition either of $K_{r\times 2}$ if $r \not\equiv 2 \pmod 3$ or of $K_{4,\, r\times 2}$ otherwise. Thus by Theorem 1.7 there exists a $K_{3\times 2}$-decomposition of either $K_{r\times 4}$ or $K_{8,\, r\times 4}$. Since there exists a G_1-decomposition of ${}^7K_{3\times 2}$ (by Example 2.23), a G_1-decomposition of either ${}^7K_{r\times 4}$ or ${}^7K_{8,\,(r-2)\times 4}$ also exists by transitivity. Finally, we note that K_{4r+t} can be described as either $(rK_4 \vee K_t) \cup K_{r\times 4} = K_{r\times 4} \cup K_{4+t} \cup \bigcup_{i=1}^{r-1}(K_{4+t} \setminus K_t)$ or $\big((K_8 \cup (r-2)K_4) \vee K_t\big) \cup K_{8,\,(r-2)\times 4} = K_{8,\,(r-2)\times 4} \cup K_{8+t} \cup \bigcup_{i=1}^{r-2}(K_{4+t} \setminus K_t)$. Thus, we describe ${}^7K_{4r+t}$ as ${}^7K_{r\times 4} \cup {}^7K_{4+t} \cup \bigcup_{i=1}^{r-1}({}^7K_{4+t} \setminus {}^7K_t)$ when $r \not\equiv 2 \pmod 3$ and as ${}^7K_{8,\,(r-2)\times 4} \cup {}^7K_{8+t} \cup \bigcup_{i=1}^{r-2}({}^7K_{4+t} \setminus K_t)$ when $r \equiv 2 \pmod 3$, and the result follows from the existence of G_1-decompositions of ${}^7K_{r\times 4}$ or ${}^7K_{8,\,(r-2)\times 4}$, ${}^7K_4$, ${}^7K_5$, ${}^7K_6$, ${}^7K_7$, ${}^7K_8$, ${}^7K_9$, ${}^7K_{10}$, ${}^7K_{11}$, ${}^7K_6 \setminus {}^7K_2$, and ${}^7K_7 \setminus {}^7K_3$, where the latter two decompositions are shown to exist in Examples 2.24 and 2.25, respectively. ■

Lemma 3.5. *There exists a G_2-decomposition of ${}^7K_n$ if $n \geq 5$ and n is odd.*

Proof. Let $n = 4r + t$ for some positive integer r and $t \in \{1, 3\}$. If (r,t) is $(1,1)$, $(2,1)$, or $(2,3)$, then the result follows from Examples 2.18, 2.20, and 2.22, respectively. If (r,t) is $(1,3)$, then $n = 7$, and the result follows from 2 copies of a G_2-decomposition of ${}^2K_7$ (see Lemma 3.1) and 1 copy of a G_2-decomposition of ${}^3K_7$ (see Lemma 3.3). For the remainder of the proof, we assume $r \geq 3$, and the proof then follows as in the proof of Lemma 3.4. ■

Lemma 3.6. *There exists a G_2-decomposition of ${}^{14}K_n$ if $n \geq 4$.*

Proof. If n is odd, then the result follows from 2 copies of a G_2-decomposition of ${}^7K_n$ (see Lemma 3.5). For the remainder of the proof, we assume n is even. Let $n = 4r + t$ for some positive integer r and $t \in \{0, 2\}$. If (r,t) is $(1,0)$, $(1,2)$, or $(2,2)$, then the result follows from Examples 2.26, 2.27, and 2.28, respectively. If (r,t) is $(2,0)$, then $n = 8$, and the result follows from 7 copies of a G_2-decomposition of ${}^2K_n$ (see Lemma 3.1). For the remainder of the proof, we assume $r \geq 3$. By Corollary 1.6 there exists a K_3-decomposition either of $K_{r\times 2}$ if $r \not\equiv 2 \pmod 3$ or of $K_{4,\, r\times 2}$ otherwise. Thus by Theorem 1.7 there exists a $K_{3\times 2}$-decomposition of either $K_{r\times 4}$ or $K_{8,\, r\times 4}$. Since there exists a G_2-decomposition of ${}^7K_{3\times 2}$ (by Example 2.23), a G_2-decomposition of either ${}^{14}K_{r\times 4}$ or ${}^{14}K_{8,\,(r-2)\times 4}$ also exists by transitivity. Finally, we note that K_{4r+t} can be described as either $(rK_4 \vee K_t) \cup K_{r\times 4} = K_{r\times 4} \cup K_{4+t} \cup \bigcup_{i=1}^{r-1}(K_{4+t} \setminus K_t)$ or $\big((K_8 \cup (r-2)K_4) \vee K_t\big) \cup K_{8,\,(r-2)\times 4} = K_{8,\,(r-2)\times 4} \cup K_{8+t} \cup \bigcup_{i=1}^{r-2}(K_{4+t} \setminus K_t)$. Thus, we describe ${}^{14}K_{4r+t}$ as ${}^{14}K_{r\times 4} \cup {}^{14}K_{4+t} \cup \bigcup_{i=1}^{r-1}({}^{14}K_{4+t} \setminus {}^{14}K_t)$ when $r \not\equiv 2 \pmod 3$ and as ${}^{14}K_{8,\,(r-2)\times 4} \cup {}^{14}K_{8+t} \cup \bigcup_{i=1}^{r-2}({}^{14}K_{4+t} \setminus K_t)$ when $r \equiv 2 \pmod 3$, and the result follows from the existence of G_2-decompositions of ${}^{14}K_{r\times 4}$ or ${}^{14}K_{8,\,(r-2)\times 4}$, ${}^{14}K_4$, ${}^{14}K_6$, ${}^{14}K_8$, ${}^{14}K_{10}$, and ${}^{14}K_6 \setminus {}^{14}K_2$, where the latter decompositions is shown to exist in Example 2.29. ■

Finally, we settle the spectra for G_1- and G_2-designs of any index λ (at least 2).

Theorem 3.7. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G_1-decomposition of ${}^{\lambda}K_n$ if and only if the following hold:*

- *if* $\gcd(\lambda, 7) = 1$*, then* $n \equiv 0$ *or* $1 \pmod 7$*;*
- *if* $\gcd(\lambda, 7) = 7$*, then* $n \geq 4$*.*

Proof. The necessity of the given conditions is established in Lemma 1.1. We now show sufficiency. Let $n \geq 4$ and let $\lambda = 7r + t$ for some integers $r \geq 0$ and $t \in \{2, 3, \ldots, 8\}$. In the case where $t = 7$, the result follows from $r + 1$ copies of a G_1-decomposition of ${}^{7}K_n$ (see Lemma 3.4). For the remainder of the proof, we assume $n \equiv 0$ or $1 \pmod 7$. In the case where t is even, the result follows from r copies of a G_1-decomposition of ${}^{7}K_n$ (see Lemma 3.4) and $t/2$ copies of a G_1-decomposition of ${}^{2}K_n$ (see Lemma 3.1). In the case where t is odd, the result follows from r copies of a G_1-decomposition of ${}^{7}K_n$ (see Lemma 3.4), 1 copy of a G_1-decomposition of ${}^{3}K_n$ (see Lemma 3.2), and $(t-3)/2$ copies of a G_1-decomposition of ${}^{2}K_n$ (see Lemma 3.1). ■

Theorem 3.8. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G_2-decomposition of ${}^{\lambda}K_n$ if and only if the following hold:*

- *if* $\gcd(\lambda, 14) = 1$*, then* $n \equiv 1$ *or* $7 \pmod{14}$*;*
- *if* $\gcd(\lambda, 14) = 2$*, then* $n \equiv 0$ *or* $1 \pmod 7$*;*
- *if* $\gcd(\lambda, 14) = 7$*, then* $n \equiv 1 \pmod 2$*;*
- *if* $\gcd(\lambda, 14) = 14$*, then* $n \geq 4$*.*

Proof. The necessity of the given conditions is established in Lemma 1.2. We now show sufficiency. Let $n \geq 4$ and let $\lambda = 14r + t$ for some integers $r \geq 0$ and $t \in \{2, 3, \ldots, 15\}$. In the case where $t = 14$, the result follows from $r + 1$ copies of a G_2-decomposition of ${}^{14}K_n$ (see Lemma 3.6). In the case where $t = 7$, we assume that n is odd, and the result follows from $2r + 1$ copies of a G_2-decomposition of ${}^{7}K_n$ (see Lemma 3.5). For the remainder of the proof, we assume $n \equiv 0$ or $1 \pmod 7$. In the case where t is even, the result follows from r copies of a G_2-decomposition of ${}^{14}K_n$ (see Lemma 3.6) and $t/2$ copies of a G_2-decomposition of ${}^{2}K_n$ (see Lemma 3.1).

In the case where t is odd, we assume that n is odd, and the result follows from r copies of a G_2-decomposition of ${}^{14}K_n$ (see Lemma 3.6), 1 copy of a G_2-decomposition of ${}^{3}K_n$ (see Lemma 3.3), and $(t-3)/2$ copies of a G_2-decomposition of ${}^{2}K_n$ (see Lemma 3.1). ■

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