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A new bound for smooth spline spaces

Hal Schenck*, Mike Stillman** and Beihui Yuan**

Abstract. For a planar simplicial complex $\Delta \subseteq \mathbb{R}^2$, Schumaker proves in [22] that a lower bound on the dimension of the space $C^r_k(\Delta)$ of planar splines of smoothness r and degree k on Δ is given by a polynomial $P_\Delta(r,k)$, and Alfeld–Schumaker show in [2] that $P_\Delta(r,k)$ gives the correct dimension when $k \geq 4r+1$. Examples due to Morgan–Scott, Tohaneanu, and Yuan show that the equality $\dim C^r_k(\Delta) = P_\Delta(r,k)$ can fail for $k \in \{2r, 2r+1\}$. In this note we prove that the equality $\dim C^r_k(\Delta) = P_\Delta(r,k)$ cannot hold in general for $k \leq (22r+7)/10$.

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1. Introduction

Let Δ be a triangulation of a simply connected polygonal domain in $\mathbb{R}^2$ having f_1 interior edges and f_0 interior vertices. A landmark result in approximation theory is the 1979 paper of Schumaker [22], showing that for any triangulation Δ, any smoothness r and any degree k, the dimension of the vector space $C^r_k(\Delta)$ of splines of smoothness r and degree at most k is bounded below by

$$P_\Delta(r,k) = \binom{k+2}{2} + \binom{k-r+1}{2} f_1 - \left(\binom{k+2}{2} - \binom{r+2}{2} \right) f_0 + \sigma, \quad (1.1)$$

where

$$\sigma = \sum \sigma_i, \quad \sigma_i = \sum_j \max\left\{ \big(r+1+j(1-n(v_i))\big), 0 \right\},$$

and $n(v_i)$ is the number of distinct slopes at an interior vertex v_i. In [2], Alfeld–Schumaker prove for $k \geq 4r+1$,

$$\dim C^r_k(\Delta) = P_\Delta(r,k).$$

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Hong [12] shows equality holds for $k \geq 3r+2$, and [2] shows equality for $k \geq 3r+1$ and generic Δ.

When the degree k is small compared to the order of smoothness, formula (1.1) can fail to give the correct value for $\dim C_k^r(\Delta)$: a 1975 example of Morgan–Scott shows it fails for $(r,k) = (1,2)$. In [19] it was conjectured that $\dim C_k^r(\Delta) = P_\Delta(r,k)$ for $k \geq 2r+1$, but a recent example [25] shows that equality fails for $(r,k) = (2,5)$. In 1974, Strang [26] conjectured that for $(r,k) = (1,3)$ the formula holds for a generic triangulation.

In [3], Billera used algebraic methods to prove Strang's conjecture, winning the Fulkerson prize for his work. A number of subsequent papers [4, 5, 7, 14, 15, 20, 21, 24, 28] use tools from algebraic geometry to study splines. The translation to algebraic geometry takes the set of splines of all polynomial degrees k, and packages it as a vector bundle $\mathcal{C}^r(\Delta)$ on $\mathbb{P}^2$. The discrepancy between $P_\Delta(r,k)$ and the actual dimension in degree k is then captured by the dimension $h^1(\mathcal{C}^r(\Delta)(k))$ of the first cohomology of $\mathcal{C}^r(\Delta)$.

The examples above do not preclude the possibility that $\dim C_k^r(\Delta) = P_\Delta(r,k)$ holds for every triangulation Δ if $k \geq 2r+2$. Our main result shows this is impossible:

Theorem 1.1. *There is no constant c so that $\dim C_k^r(\Delta) = P_\Delta(r,k)$ for all Δ and all $k \geq 2r + c$. In particular, there exists a planar simplicial complex Δ for which*

$$h^1(\mathcal{C}^r(\Delta)(k)) \neq 0 \quad \textit{for all } k \leq \frac{22r+7}{10}.$$

This shows there exists a simplicial complex Δ such that $\dim C_k^r(\Delta) > P_\Delta(r,k)$ for all $k \leq \frac{22r+7}{10}$. For formula (1.1) to yield the correct value for $\dim C_k^r(\Delta)$ for every triangulation Δ, we must have

$$k > \frac{22r+7}{10} > 2.2r.$$

2. Algebraic preliminaries

Billera's construction in [3] computes the C^1 splines as the top homology module of a certain chain complex. An introduction to homology and chain complexes aimed at a general audience appears in [18], so the presentation below is terse. The paper [20] introduces a modification of Billera's construction, allowing a precise splitting of the contributions to $\dim C_k^r(\Delta)$ into parts depending, respectively, on local and global geometry.

Definition 2.1. For a planar simplicial complex Δ, let Δ_i be the set of interior faces of dimension i (all triangles are considered interior). For $\tau \in \Delta_1$, let l_τ be a linear form vanishing on τ, and for $v \in \Delta_0$, let $J(v)$ be the ideal generated by l_τ^{r+1}, with τ ranging over all interior edges containing v. Construct a complex of $R = \mathbb{R}[x_1, x_2, x_3]$ modules as below, with differential ∂_i the usual boundary operator in relative (modulo $\partial(\Delta)$) homology.

$$\mathcal{R}/\mathcal{J}\colon\ 0 \longrightarrow \bigoplus_{\sigma\in\Delta_2} R \xrightarrow{\partial_2} \bigoplus_{\tau\in\Delta_1} R/l_\tau^{r+1} \xrightarrow{\partial_1} \bigoplus_{v\in\Delta_0} R/J(v) \longrightarrow 0.$$

By construction, $H_2(\mathcal{R}/\mathcal{J})$ is a graded R-module, consisting of the set of splines of all degrees, and defines the sheaf $\mathcal{C}^r(\Delta)$. It is easy to show that

$$H_0(\mathcal{R}/\mathcal{J}) = 0 \quad \text{and} \quad H_1(\mathcal{R}/\mathcal{J}) = \bigoplus_{k\geq 0} H^1(\mathcal{C}^r(\Delta)(k)).$$

In particular,

$$\dim C_k^r(\Delta) = P_\Delta(r,k) + \dim_{\mathbb{R}} H^1(\mathcal{C}^r(\Delta)(k)).$$

Recall that a *syzygy* on an ideal $\langle f_1, \ldots, f_k\rangle$ is a polynomial relation on the f_i. For an interior vertex v, $J(v) = \langle l_{\tau_1}^{r+1}, \ldots, l_{\tau_n}^{r+1}\rangle$, so a syzygy on $J(v)$ is of the form

$$\sum_{i=1}^n s_i \cdot l_{\tau_i}^{r+1} = 0.$$

A main result of [20] is:

Theorem 2.2. *The module $H_1(\mathcal{R}/\mathcal{J})$ is given by generators and relations as*

$$H_1(\mathcal{R}/\mathcal{J}) \simeq \Big(\bigoplus_{\tau\in\Delta_1^o} R(-r-1)\Big)/S,$$

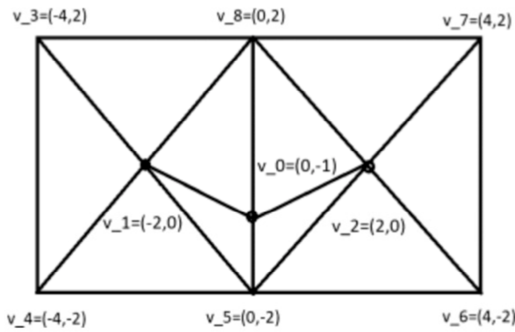
where

- *The set Δ_1^o consists of totally interior edges τ: neither vertex of τ is in $\partial(\Delta)$.*
- *$S = \bigoplus_{v\in\Delta_0} Syz(v)$: the direct sum of the syzygies on $J(v)$ at each interior vertex.*

Hence $H_1(\mathcal{R}/\mathcal{J})$ is the quotient of a free module with a generator for each totally interior edge τ by vectors of polynomials of the form $(s_1, \ldots, s_n)$. Note that if two totally interior edges τ_1, τ_2 with the same slope meet at a vertex, then there is a degree zero syzygy between them, and S will have a column with nonzero constant entries.

3. Proof of theorem

Following [25], we consider the simplicial complex Δ below.



By Theorem 2.2, the discrepancy module $H_1(\mathcal{R}/\mathcal{J})$ has two generators. There are three interior vertices, and we need to quotient by the syzygies at each vertex. Note that each vertex has only three edges with distinct slopes attached, hence we must compute the syzygies on ideals of the form

$$\langle l_1^{r+1}, l_2^{r+1}, l_3^{r+1} \rangle.$$

The key is that this is a local question, so after translating a vertex so it lies at the origin, we have an ideal in two variables (recall that because we homogenized the problem, our points now lie in $\mathbb{P}^2$, so the linear forms defining edges are homogeneous in three variables). The paper [10] gives a precise description of the syzygies on any ideal generated by powers of linear forms in two variables. In the case of three forms as above there are only two syzygies, in degrees

$$\left\lfloor \frac{r+1}{2} \right\rfloor \quad \text{and} \quad \left\lceil \frac{r+1}{2} \right\rceil.$$

Specializing to the case where $r+1 = 4j$, we see that there are two syzygies, both of degree $2j$. Next, we note that two of the three vertices are connected to one totally interior edge and two edges which touch the boundary, so writing the six relations (two syzygies on each of the three interior vertices) as a matrix, we see that

$$H_1(\mathcal{R}/\mathcal{J}) \simeq R^2(-r-1)/S,$$

where

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & 0 & 0 \\ 0 & 0 & s_{23} & s_{24} & s_{25} & s_{26} \end{bmatrix}.$$

As noted above, the rows correspond to the generators for $H_1(\mathcal{R}/\mathcal{J})$: the first row corresponds to the totally interior edge $\overline{v_0 v_1}$ and the second row to the totally interior edge $\overline{v_0 v_2}$; let l_{ij} denote a nonzero linear form vanishing on $\overline{v_i v_j}$.

The first two columns of S correspond to the two syzygies at vertex v_1, the second two columns to the syzygies at vertex v_0, and the last two columns to the syzygies at vertex v_2. Since the syzygies at v_0 are on the ideal

$$\langle l_{01}^{r+1}, l_{02}^{r+1}, l_{08}^{r+1}\rangle,$$

the third and fourth columns of S have no zero entries, because the syzygies involve both generating edges $\overline{v_0v_1}, \overline{v_0v_2}$. In contrast, the syzygies at v_1 are on the ideal

$$\langle l_{01}^{r+1}, l_{13}^{r+1}, l_{14}^{r+1}\rangle.$$

Hence in the matrix S, only the component of the syzygy involving l_{01}^{r+1} appears — there is no part of the syzygy involving l_{02}^{r+1}. This also explains why the rightmost two columns of S have nonzero entry only in the second row. For the next lemma, we need some concepts from commutative algebra.

Definition 3.1. An ideal $I = \langle f_1, \ldots, f_k\rangle \subseteq R$ with k minimal generators is a *complete intersection* if each f_i is not a zero divisor on $R/\langle f_1, \ldots, f_{i-1}\rangle$. Equivalently, the map

$$R/\langle f_1, \ldots, f_{i-1}\rangle \xrightarrow{\cdot f_i} R/\langle f_1, \ldots, f_{i-1}\rangle$$

is an inclusion.

From a geometric standpoint, being a complete intersection means that the locus $V(f_1, \ldots, f_k)$ where the f_j simultaneously vanish has codimension equal to k. In particular, an ideal I minimally generated by k elements is a compete intersection if it has codimension k, and an *almost complete intersection* if it has codimension $k-1$.

Definition 3.2. Let I, J be ideals in a ring R. Then the *colon ideal*

$$I : J = \{f \in R \mid f \cdot j \in I \text{ for all } j \in J\}.$$

There is a nice connection of colon ideals to syzygies: if $I = \langle f_1, \ldots, f_k\rangle$ and

$$\sum_{i=1}^{k} a_i f_i = 0$$

is a syzygy on I, then $a_k \in \langle f_1, \ldots, f_{k-1}\rangle : \langle f_k\rangle$. We shall make use of this in the next lemma.

Lemma 3.3. *The ideals*

$$I_1 = \langle s_{11}, s_{12}\rangle \quad \text{and} \quad I_2 = \langle s_{25}, s_{26}\rangle$$

are complete intersections.

Proof. An ideal with two generators f, g is a complete intersection when f and g are relatively prime, or equivalently when the unique minimal syzygy on f, g is given by

$$f \cdot g - g \cdot f = 0.$$

The ideal $\langle l_1^{r+1}, l_2^{r+1}, l_3^{r+1} \rangle$ is an almost complete intersection, which means that two generators, say $\{l_1^{r+1}, l_2^{r+1}\}$ are a complete intersection. Proposition 5.2 in [6] proves an almost complete intersection is directly linked to a Gorenstein ideal. In this case the linked ideal is

$$\langle l_1^{r+1}, l_2^{r+1} \rangle : l_3^{r+1} = \langle s_{11}, s_{12} \rangle.$$

A homogeneous Gorenstein ideal in two variables is a complete intersection, so the result follows. □

We're now ready to put the pieces together. Define

$$\phi = \begin{bmatrix} s_{13} & s_{14} \\ s_{23} & s_{24} \end{bmatrix}.$$

Then $H_1(\mathcal{R}/\mathcal{J})$ may be presented as the cokernel of the map

$$R^2(-6j) \xrightarrow{\phi} R(-4j)/I_1 \bigoplus R(-4j)/I_2.$$

The Hilbert function of a graded module M takes as input an integer t, and gives as output the dimension of the vector space M_t. Since I_i is a complete intersection with two generators in degree $2j$, there are exact sequences:

$$0 \longrightarrow R(-4j) \longrightarrow R(-2j)^2 \longrightarrow R \longrightarrow R/I_i \longrightarrow 0.$$

Tensoring this exact sequence with $R(-4j)$ yields a sequence whose rightmost term is a direct summand of the target of the map ϕ. When $k \geq 2r + 2 = 8j$ (so that all the modules in the exact sequence above contribute), taking the Euler characteristic of the sequence and using that

$$HF\big(R(-i), k\big) = \binom{k - i + 2}{2}$$

yields

$$\begin{aligned} HF\big(R^2(-6j), k\big) &= (k - 6j + 2)(k - 6j + 1), \\ HF\big(R(-4j)/I_1 \bigoplus R(-4j)/I_2, k\big) &= (k - 4j + 2)(k - 4j + 1) \\ &\quad - 2(k - 6j + 2)(k - 6j + 1) \\ &\quad + (k - 8j + 2)(k - 8j + 1). \end{aligned}$$

Therefore the Hilbert function of the target of ϕ minus the Hilbert function of the source of ϕ is

$$-k^2 + (12j - 3)k - 28j^2 + 18j - 2,$$

which has two real roots, the larger at

$$k = 6j - 3/2 + \frac{\sqrt{32j^2+1}}{2} > (6 + 2\sqrt{2})j - 3/2$$
$$> 8.8j - 2.2 + .7 = \frac{22r+7}{10}.$$

We have been working with the assumption that $r + 1 = 4j$; for $r \geq 7$ the condition $k \geq 8j$ holds and we've shown the cokernel of ϕ must be nonzero in degree $\leq \frac{22r+7}{10}$. For $r = 3$ and $j = 1$ the condition that $k \geq 8$ fails–the larger root is at approximately 7.4. In this case, a direct computation verifies that $\mathrm{coker}(\phi)$ is nonzero in degree 7. The same line of argument works with a minor modification for $(r + 1 \bmod 4) \in \{1, 2, 3\}$, with no change in the bound, and concludes the proof. □

Remarks and open questions. The triangulation Δ appearing in §3 is the only known triangulation for which

$$P_\Delta(r, 2r+1) \neq \dim C^r_{2r+1}(\Delta).$$

For $r \leq 70$, computations show that the maximal value for which $H_1(\mathcal{R}/\mathcal{J}_\Delta) \neq 0$ is

$$\left\lfloor \frac{9r+2}{4} \right\rfloor = \left\lfloor \frac{45r+10}{20} \right\rfloor \geq \left\lfloor \frac{44r+14}{20} \right\rfloor = \left\lfloor \frac{22r+7}{10} \right\rfloor.$$

In particular the bound of Theorem 1.1 is quite close to optimal for Δ. This raises two interesting questions.

(1) Is it possible to lower the value of k such that $\dim C^r_k(\Delta) = P_\Delta(r, k)$ holds for all Δ?

(2) Is it possible to raise the value of k such that $\dim C^r_k(\Delta) > P_\Delta(r, k)$ holds for some Δ?

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H. Schenck, Department of Mathematics and Statistics, Auburn University,
Auburn, AL 36849, USA

E-mail: hks0015@auburn.edu

M. Stillman, Department of Mathematics, Cornell University,
Ithaca, NY 14850, USA

E-mail: mike@math.cornell.edu

B. Yuan, Department of Mathematics, Cornell University,
Ithaca, NY 14850, USA

E-mail: by238@math.cornell.edu