0.1. Rees Algebras, Syzygies, and Implicitization

Our goal in this section is to describe in more depth the concepts introduced earlier in the chapter:

- Rees and Symmetric Algebras, Jacobian dual.
- Fitting ideals and Annihilators.
- Free resolutions and McRae invariant.
- The Approximation complex.
- Multigraded implicitization.
- Extensions and Future directions.

The setup is that the ideal

$$I = \langle f_1, \ldots, f_d \rangle \subseteq R = k[x_1, \ldots, x_n]$$

is generated by homogeneous elements of degree m, which define a map of rings

$$S = k[y_1, \ldots, y_d] \xrightarrow{\phi} R,\ y_i \mapsto f_i.$$

This in turn gives rise to a corresponding map of varieties

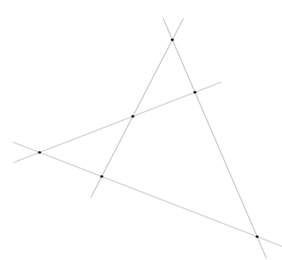
$$\mathbb{P}^{n-1} \setminus B \xrightarrow{\Phi} \mathbb{P}^{d-1},$$

where the *base locus* $B = \mathbf{V}(I)$ consists of the points where Φ is undefined. Our focus will be on the case where $d = n+1$ and Φ generically finite onto it's image, so the image of Φ is a hypersurface $H = \mathbf{V}(F)$. We begin with a motivating example:

Example 0.1. Let $l = x_1 + x_2 + x_3$ and

$$I = \langle x_1x_2x_3, x_1x_2l, x_1x_3l, x_2x_3l \rangle.$$

The four lines $\mathbf{V}(x_1), \mathbf{V}(x_2), \mathbf{V}(x_3), \mathbf{V}(l)$ depicted below



determine the base locus B, which consists of the six intersection points

$$B = \{[0:1:0], [0:0:1], [0:1:-1], [1:0:-1], [1:-1:0], [1:0:0]\}.$$

The ideal I has a presentation given by

$$0 \longrightarrow R(-4)^3 \xrightarrow{\begin{bmatrix} l & 0 & 0 \\ -x_3 & x_3 & 0 \\ 0 & -x_2 & x_2 \\ 0 & 0 & -x_1 \end{bmatrix}} R(-3)^4 \xrightarrow{\begin{bmatrix} x_1x_2x_3 & x_1x_2l & x_1x_3l & x_2x_3l \end{bmatrix}} I \to 0.$$

The leftmost matrix consists of the *syzygies* on the ideal I: polynomial relations on the 1×4 matrix of polynomials generating I. If we denote the basis vectors for R^4 as $\{y_1, \dots y_4\}$ then we may rewrite the syzygies as

$$y_1 l - y_2 x_3, y_2 x_3 - y_3 x_2, y_3 x_2 - y_4 x_1.$$

Now we turn this inside out, and write the three expressions above (recall that $l = x_1 + x_2 + x_3$) as column vectors with respect to the basis $\{x_1, x_2, x_3\}$ of R_1:

$$B = \begin{bmatrix} y_1 & 0 & -y_4 \\ y_1 & -y_3 & y_3 \\ y_1 - y_2 & y_2 & 0 \end{bmatrix}$$

Let

$$F = \det(B) = y_2 y_3 y_4 - y_1 y_3 y_4 - y_1 y_2 y_4 - y_1 y_2 y_3.$$

An easy computation shows that $F(f_1, f_2, f_3, f_4) = 0$, so that the determinant of B is the implicit equation for the hypersurface H.

It is instructive to examine the geometry of this example more closely. It is a classical fact ([**107**], §V.3) that the blow up X of $\mathbb{P}^2$ at six general points may be embedded as a smooth cubic surface in $\mathbb{P}^3$, using the divisor

$$D = 3E_0 - \sum_{i=1}^{6} E_i,$$

where E_0 is the proper transform of a line and E_i is the exceptional curve over a blown up point. The intersection pairing on X satisfies

$$E_0^2 = 1 \quad E_i^2 = -1 \text{ if } i \neq 0 \quad E_i E_j = 0 \text{ if } i \neq j$$

In our example, the points are not general: for example, the line $L = \mathbf{V}(x_1)$ contains the 3 basepoints (say p_1, p_2, p_3), and we compute

$$D \cdot L = (3E_0 - \sum_{i=1}^{6} E_i) \cdot (E_0 - E_1 - E_2 - E_3) = 3 - 3 = 0.$$

This computation shows that any of the four lines which contain three points of the base locus is contracted to a point; in particular we expect $\mathbf{V}(F)$ to have four singular points. The Jacobian ideal of F is generated by

$$\begin{bmatrix} y_2 y_3 + y_2 y_4 + y_3 y_4 \\ y_1 y_3 + y_1 y_4 - y_3 y_4 \\ y_1 y_2 + y_1 y_4 - y_2 y_4 \\ y_1 y_2 + y_1 y_3 - y_2 y_3 \end{bmatrix}$$

which has zeroes at exactly the four coordinate points of $\mathbb{P}^3$, confirming our expectation that $H = \mathbf{V}(F)$ has four singular points. ◁▷

The remainder of this section will be devoted to understanding the tools and techniques that are used to produce the implicit equation F, in particular to explain the mysterious switch between the matrix of syzygies of I, and the matrix B.

Rees and Symmetric algebras. The Rees algebra $R(I)$ of an ideal $I = \langle f_1, \ldots, f_d \rangle \subseteq R$ is defined as the image of the map

$$R[y_1, \ldots, y_d] \xrightarrow{\beta} R[t] \text{ via } y_i \mapsto f_i \cdot t.$$

If R is a domain, then since $R(I)$ is isomorphic to a subring of $R[t]$, then $R(I)$ will also be a domain. If we grade by the variables y_i, we have

$$\begin{array}{lcll} \ker(\beta)_1 & = & \sum a_i y_i \mapsto t \cdot \sum a_i f_i = 0 & \text{iff } (a_i) \in \mathrm{syz}(I) \\ \ker(\beta)_2 & = & \sum a_{ij} y_i y_j \mapsto t^2 \cdot \sum a_{ij} f_i f_j = 0 & \text{iff } (a_{ij}) \in \mathrm{syz}(I^2) \\ \ker(\beta)_3 & = & \vdots & \text{iff } (a_{ijk}) \in \mathrm{syz}(I^3) \end{array}$$

It follows from the definition that $R(I) \simeq R[y_1, \ldots, y_d]/\ker(\beta)$. A close cousin of $R(I)$ is the symmetric algebra $S(I)$, defined as $R[y_1, \ldots, y_d]/\ker(\beta)_1$. It turns out that computing $R(I)$ is a difficult problem, discussed at length at the end of the chapter. The symmetric algebra $S(I)$ is simpler; in the special case where $\ker(\beta) = \ker(\beta)_1$, the ideal I is said to be of *linear type*.

EXAMPLE 0.2. Example `FILL IN` computes the ideal defining the Rees algebra $R(I)$ of $I = \langle t^4, s^2t^2, s^4 - st^3 \rangle$. The `Macaulay2` package `ReesAlgebra` by David Eisenbud [**78**] allows us to compute $R(I)$ as follows (some output suppressed)

```
i1 : R=ZZ/31991[s,t];

i2 : I=ideal(t^4,s^2*t^2,s^4-s*t^3);

i3 : syz(gens I)
o3 = {4} | s2  -st |
     {4} | -t2 s2  |
     {4} | 0   -t2 |

              3       2
o3 : Matrix R  <--- R

i4 : RI=reesIdeal I;

o4 : Ideal of R[w , w , w ]
                 0   1   2

i5 : transpose gens RI

o5 = {-1, -6}  | w_0st-w_1s2+w_2t2                    |
     {-1, -6}  | w_0s2-w_1t2                          |
     {-2, -9}  | w_0w_1t-w_1^2s+w_0w_2s               |
     {-2, -9}  | w_0^2s-w_1^2t+w_0w_2t                |
     {-4, -16} | w_0^3w_1-w_1^4+2w_0w_1^2w_2-w_0^2w_2^2 |

i6 : isLinearType RI

o6 = false
```

The two syzygies on I appear as the first two generators for the ideal defining $R(I)$, and we see that $S(I)$ and $R(I)$ differ. ◁▷

It is a useful exercise (or see [**77**] Exercise A2.6) to show that if R is a domain, then the kernel of the surjection from $S(I)$ to $R(I)$ consists of those elements of $S(I)$ which are annihilated by R: the kernel of $S(I) \longrightarrow R(I)$ is the R-torsion of the symmetric algebra.

In general the linear type condition is rare: $S(I)$ typically has R-torsion. Busé-Jouanolou proved that if the base locus B is a zero-dimensional local complete intersection, then the R-torsion is rather mild, which can be phrased in terms of *local cohomology*:

Definition 0.3. For an ideal I and R-module M, the zeroth local cohomology is

$$H^0_I(M) = \{m \in M \mid I^j \cdot m = 0 \text{ for some } j \in \mathbb{N}\}.$$

Letting the role of I be played by the ideal $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$, there is a standard four term sequence in local cohomology (Theorem A4.1 of [**77**]):

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow \bigoplus_{i \in \mathbb{Z}} H^0(\widetilde{M}(i)) \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow 0.$$

If we use $S(I)$ for M in the above sequence, then if I is linear type outside $\mathfrak{m}$, this means $R(I)$ and $S(I)$ define the same sheaf outside $\mathbf{V}(\mathfrak{m})$, and $R(I)$ is the third term in the sequence above. The surjection from $S(I)$ to $R(I)$ forces $H^1_{\mathfrak{m}}(S(I))$ to vanish, and so we obtain the exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(S(I)) \longrightarrow S(I) \longrightarrow R(I) \longrightarrow 0. \tag{0.1}$$

Recall that the annihilator $\mathrm{Ann}(M)$ of an R-module M is the ideal

$$\{r \in R \mid r \cdot m = 0 \text{ for all } m \in M\}.$$

A pair of theorems due to Busé-Jouanolou are the key to using syzygies to study implicitization. Regarding $R(I)$ as a graded module over R, we have:

Theorem 0.4. [**39**] *The kernel of ϕ is $\mathrm{Ann}_S(R(I)_0)$.*

Proof. First, note that

$$\ker(\phi) = \ker(\beta) \cap S.$$

This follows since if $F \in S$ satisfies

$$F(f_1 t, \ldots, f_d t) = 0 \in R[t],$$

then specializing to $t = 1$ shows $\ker(\beta) \cap S \subseteq \ker(\phi)$. On the other hand, since the f_i are homogeneous, this means that $\ker(\phi)$ is also homogeneous, so $F(f_1 t, \ldots, f_d t) = t^m \cdot F(f_1, \ldots, f_d) = 0$, which implies $\ker(\phi) \subseteq \ker(\beta) \cap S$. Since $R(I) \simeq R \otimes S / \ker(\beta)$, $F \in S$ annihilates the degree zero (in R) component of $R(I)$ exactly if

$$F \cdot 1 \in \ker(\beta),$$

which holds exactly when $F \in \ker(\beta) \cap S = \ker(\phi)$. □

Caveat: if we do not assume that R is a polynomial ring, then an additional hypothesis, that $H^0_{\mathfrak{m}}(R) = 0$, is necessary for the theorem to hold. Similar arguments yield

Theorem 0.5. [**39**] *If $H^0_{\mathfrak{m}}(S(I))_b = 0$ for all $b \geq a$, then*

$$\mathrm{Ann}_S(S(I)_b) \quad \subseteq \quad \ker(\phi) \qquad \textit{for all } b \geq a,$$

and equality holds if I is of linear type outside $\mathfrak{m}$.

If I is not of linear type, then the next best case is when the defining ideal of $R(I)$ has the *expected form*. To explain this terminology, we need the notion of the *Jacobian dual*:

DEFINITION 0.6. Let N denote the matrix of first syzygies on $I = \langle f_1, \ldots, f_d \rangle \subseteq R = k[x_1, \ldots, x_n]$, and write

$$[y_1, \ldots, y_d] \cdot N = [x_1, \ldots, x_n] \cdot B,$$

where the entries of B are linear in the y_i variables. The matrix B is the Jacobian dual of N. If the defining ideal of $R(I)$ is generated by $[y_1, \ldots, y_d] \cdot N$ and the $d \times d$ minors of B, then $R(I)$ is said to be of *expected form*.

The mystery matrix B that appeared in Example 0.1 is exactly the Jacobian dual. In Theorem 0.5, the hypothesis that I is linear type outside $\mathfrak{m}$ means that in high enough degree b in the R variables,

$$H^0_{\mathfrak{m}}(S(I))_b = 0$$

and hence in high enough degree, $S(I)$ and $R(I)$ agree. Theorem 0.5 provides a path to connect implicitization problems to Fitting ideals and the McRae invariant, and as we shall see, gives a concrete means to efficiently compute the implicit equation F. Example 0.1 is of expected form; Example 0.17 is not.

Fitting Ideals and the Annihilator of a module. A standard construction in algebra is that of the Fitting ideal of a module. For us, the context is that of a polynomial ring R and finitely generated R-module M. Then

DEFINITION 0.7. Given a presentation

$$R^n \xrightarrow{\psi} R^m \longrightarrow M \longrightarrow 0.$$

for M, $\mathrm{Fitt}_i(M)$ is the ideal of $m-i$ by $m-i$ minors of ψ; in particular $\mathrm{Fitt}_0(M)$ is the ideal of $m \times m$ minors of ψ.

It is a good exercise to show that the Fitting ideals of M are independent of choice of ψ; they capture information about the annihilator of the module.

THEOREM 0.8. $\mathrm{Fitt}_0(M) \subseteq \mathrm{Ann}(M)$.

PROOF. If $n < m$ then since R^m is a free module, the map ψ splits and M has a direct summand which is a free module; if R is a domain then $\mathrm{Ann}(M) = 0$, as are the $m \times m$ minors. So assume $n \geq m$. Choose an $m \times m$ submatrix A of ψ such that $\det(A) \neq 0$, yielding a commuting diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
R^m & \xrightarrow{A} & R^m & \longrightarrow & M' & \longrightarrow & 0 \\
\downarrow & & \downarrow \simeq & & \downarrow & & \\
R^n & \xrightarrow{\psi} & R^m & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Since M is a quotient of M', relations on M' are also relations on M and

$$\mathrm{Ann}(M') \subseteq \mathrm{Ann}(M).$$

We need to show that $\det(A)$ kills M'. If $a \in M' = \operatorname{coker}(A)$, then

$$\det(A) \cdot a = 0 \iff \det(A) \cdot a = Ax \text{ for some } x,$$

and such an x can be found by Cramer's rule. □

By Proposition 20.7 of [**77**], if M can be generated by q elements, then we have that $\operatorname{Ann}(M)^q \subseteq \operatorname{Fitt}_0(M)$. This implies that the radical ideals satisfy

$$\sqrt{\operatorname{Fitt}_0(M)} = \sqrt{\operatorname{Ann}(M)}.$$

Theorem 0.4 and Theorem 0.5 show that finding the implicit equation is a problem of computing annihilators, hence Fitting ideals will play a role.

Free resolutions and the McRae invariant.

DEFINITION 0.9. A sequence of modules C_i and morphisms d_i

$$\mathcal{C}: \ \cdots \longrightarrow C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots$$

is a *complex* if $\operatorname{im}(d_{i+1}) \subseteq \ker(d_i)$ for every i. It is *exact* at position i if $\operatorname{im}(d_{i+1}) = \ker(d_i)$. The i^{th} homology module is defined as $H_i(\mathcal{C}) = \ker(d_i)/\operatorname{im}(d_{i+1})$, hence $H_i(\mathcal{C}) = 0$ iff $\mathcal{C}$ is exact at position i.

An important special case occurs when the complex has the form

$$\mathcal{F}: \ 0 \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0,$$

where the modules F_i are free of rank r_i, and the complex is everywhere exact. In this case $\mathcal{F}$ is called a *free resolution* of M.

By the Hilbert syzygy theorem (Corollary 19.7, [**77**]), a finitely generated module M over the polynomial ring R has a finite free resolution, which stops in at most n steps, where n is the number of variables of R.

EXAMPLE 0.10. Consider the ideal $I = \langle x^2, y^2, z^2 \rangle \subseteq R = k[x, y, z]$. Then the free resolution of R/I takes the form

$$0 \longrightarrow R(-6) \xrightarrow{\begin{bmatrix} z^2 \\ -y^2 \\ x^2 \end{bmatrix}} R^3(-4) \xrightarrow{\begin{bmatrix} y^2 & z^2 & 0 \\ -x^2 & 0 & z^2 \\ 0 & -x^2 & -y^2 \end{bmatrix}} R^3(-2) \xrightarrow{\begin{bmatrix} x^2 & y^2 & z^2 \end{bmatrix}} R \longrightarrow R/I$$

Notice that the first syzygies are spanned by the obvious relations: for every pair (f, g) in I, there is a relation $g \cdot f - f \cdot g = 0$. Such a syzygy is called a *Koszul syzygy*, and the free resolution above has a simple construction in terms of exterior algebra.

DEFINITION 0.11. For any set of polynomials $\{f_1, \ldots, f_d\} \subseteq R$, the Koszul complex is given by

$$0 \longrightarrow \Lambda^d(R^d) = R \longrightarrow \Lambda^{d-1}(R^d) \longrightarrow \cdots \Lambda^2(R^d) \xrightarrow{d_2} \Lambda^1(R^d) = R^d \xrightarrow{d_1} R,$$

with $d_i(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{k=1}^{i} (-1)^k f_k e_{j_1} \wedge \cdots \widehat{e_{j_k}} \wedge e_{j_i})$. A straightforward computation shows that $d_{i-1} \circ d_i = 0$.

The Koszul complex is exact iff the f_i are a *regular sequence*: for all i, multiplication by f_i defines an injection on $R/\langle f_1, \ldots, f_{i-1} \rangle$. Geometrically, this means that each $\mathbf{V}(f_1, \ldots, f_{i-1})$ has codimension $i-1$ or equivalently intersecting $\mathbf{V}(f_1, \ldots, f_{i-1})$ with $\mathbf{V}(f_i)$ drops the dimension by exactly one. For this reason, a variety whose ideal is defined by a regular sequence is known as a complete intersection.

Now we turn to a special situation: let M be a module over a polynomial ring R, such that M is *torsion*: for each $m \in M$, there is some nonzero $r \in R$ such that $r \cdot m = 0$. Let $\mathcal{F}$ be a free resolution for M. Since R is a domain we may localize $\mathcal{F}$ at the zero ideal, yielding an exact sequence of vector spaces, with the d_i having entries in the field of fractions of R. Since the rank of F_n is r_n we may factor d_n as $\begin{bmatrix}\alpha_n \\ \beta_n\end{bmatrix}$, with $\mathrm{rank}(\alpha_n) = r_n$.

By exactness the image of d_n is the kernel of d_{n-1}, and so we may write d_{n-1} as

$$\begin{bmatrix}\beta_{n-1} & * \\ * & \alpha_{n-1}\end{bmatrix}$$

with β_{n-1} is of rank r_n and α_{n-1} is of rank $(r_{n-1} - r_n)$. Iterate this procedure. Since M is torsion iff it has rank zero, we also have that

$$\sum_{i=0}^{n} (-1)^i r_i = 0.$$

Thus in the factorization of $d_1 = [\beta_1 \mid \alpha_1]$ the rank of $\alpha_1 = r_1 - r_2 + r_3 - \cdots = r_0$.

Definition 0.12. The McRae invariant of M is

$$\mathcal{S}(M) = \frac{\det(\alpha_1) \cdot \det(\alpha_3) \cdots}{\det(\alpha_2) \cdot \det(\alpha_4) \cdots} = \prod \det(\alpha_i)^{(-1)^{i-1}}.$$

This definition seems to depend on the choice of the α_i. However, just as in the case of Fitting ideals, it turns out that the McRae invariant is well-defined. For a proof of this, and the remarkable properties of the McRae invariant below, we refer to the book of Northcott [**162**].

Theorem 0.13. *The McRae invariant is*

- *Independent of the choices of* α_i.
- *Multiplicative on short exact sequences.*
- *Is an element of* R*, rather than the field of fractions.*
- *Is the smallest principal ideal containing* $\mathrm{Fitt}_0(M)$.

If we hope to apply Theorem 0.13 above in conjunction with Theorem 0.4 or Theorem 0.5, we will need to obtain a resolution of $R(I)_0$ or $S(I)_b$ as an S-module. This takes us to our next key tool.

Remark 0.14. If $\mathcal{F}_\bullet$ is not exact but simply a chain complex, it is possible to define the determinant of $\mathcal{F}_\bullet$, see [**94**]. We will not need this level of generality.

The Approximation Complex. The canonical example of a chain complex is the Koszul complex which appeared in Example 0.10. The Koszul complex takes center stage as we describe the work of Herzog-Simis-Vasconcelos in [**110**], [**111**] on approximation complexes. Roughly speaking, the idea is to build a hybrid complex using $S = k[y_1, \ldots, y_d]$ and the ideal $I = \langle f_1, \ldots, f_d \rangle \subseteq R = k[x_1, \ldots, x_n]$.

Definition 0.15. In the setting above, let d_i^f be the i^{th} Koszul differential on $\{f_1, \ldots, f_d\}$, Z_i the kernel of d_i^f, and d_i^S the Koszul differential on $\{y_1, \ldots, y_d\}$. The approximation complex $\mathcal{Z}$ is

$$\mathcal{Z}: \ \cdots \longrightarrow Z_{i+1} \otimes S \xrightarrow{d_{i+1}^S} Z_i \otimes S \xrightarrow{d_i^S} Z_{i-1} \otimes S \xrightarrow{d_{i-1}^S} \cdots$$

Like the Koszul complex, $\mathcal{Z}$ depends only on I and not the choice of generators. To see the maps are well defined, note that $d^f d^S + d^S d^f = 0$. Let $\gamma \in \ker(d_i^f) = Z_i$; since $\gamma \in \ker(d_i^f)$, $d^S d^f(\gamma) = 0$. Combining this with $(d^f d^S + d^S d^f)(\gamma) = 0$, we see that $d^f d^S(\gamma) = 0$; hence

$$d^S(\gamma) \in Z_{i-1} \otimes S.$$

Consider the rightmost homology of the approximation complex. Since Z_1 consists exactly of the syzygies on I, and Z_0 is R, we find

$$H_0(\mathcal{Z}) = R \otimes_k S/\operatorname{syz}(I) \simeq S(I).$$

If $H_i(\mathcal{Z}) = 0$ for all $i \geq 1$, then the approximation complex will give a free resolution for $S(I)$, and we will be able to apply the results on Fitting ideals and the McRae invariant to obtain information about the implicit equation.

THEOREM 0.16. [**110**] *The approximation complex $\mathcal{Z}$ is acyclic iff*

$$f_{i+1} \cdot H_j(K(f_1, \ldots, f_i)) = 0 \text{ for all } i \in \{0, \ldots, d-1\}, j > 0,$$

In [**37**], Busé-Chardin use a spectral sequence argument to prove that if $\mathbf{V}(I)$ is zero dimensional, then the approximation complex is acyclic outside $\mathbf{V}(\mathfrak{m})$ iff I is locally an *almost* complete intersection (ACI), that is, generated by at most one more equation than the codimension. In this case, Theorem 0.16 yields a free resolution of $S(I)$. In the ACI case an extraneous factor appears in the McRae invariant [**36**]. Of course, we do not need the whole free resolution; by Theorem 0.5 we need the subresolution of $S(I)_a$, where $H^0_{\mathfrak{m}}(S(I)_b) = 0$ for all $b \geq a$.

Results of Busé-Jouanolou [**39**] and Busé-Chardin [**37**] show that when the base locus is empty, taking $a \geq (n-1)(m-1)$ suffices, where m is the degree of the f_i, and if the base locus is nonempty, we can actually choose $a \geq (n-1)(m-1) - \epsilon$, where ϵ is the minimal degree hypersurface containing the base locus.

In Example 0.1, we had $n = 3 = m$, so if I were basepoint free we would need the degree $(3-1)(3-1) = 4$ piece of the approximation complex. However, since the minimal degree curve through the six base points is a cubic, it suffices to look at the degree $4 - 3 = 1$ graded piece of $S(I)$. This explains what happened in Example 0.1; the key takeaway is that *basepoints make the computation simpler!*

Our next example also has a local complete intersection base locus, but illustrates that typically we need more than two steps of the approximation complex to determine the implicit equation.

EXAMPLE 0.17. The ideal $I = \langle x_1^2, x_2^3 \rangle$ is a local complete intersection, and since I has a generator in degree two we need the degree two piece of the approximation complex. The map ϕ is determined by the cubics in I: $\begin{bmatrix} x_1^3 & x_1^2x_2 & x_2^3 & x_1^2x_3 \end{bmatrix}$ whose syzygies are generated by the columns of

$$\begin{bmatrix} -x_2 & -x_3 & 0 & 0 \\ x_1 & 0 & -x_3 & -x_2^2 \\ 0 & 0 & 0 & x_1^2 \\ 0 & x_1 & x_2 & 0 \end{bmatrix}$$

which has degree two component generated by the image of

$$\begin{bmatrix} -x_1x_2 & -x_2^2 & -x_2x_3 & -x_1x_3 & -x_2x_3 & -x_3^2 & 0 & 0 & 0 \\ x_1^2 & x_1x_2 & x_1x_3 & 0 & 0 & 0 & -x_2x_3 & -x_3^2 & -x_2^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1^2 \\ 0 & 0 & 0 & x_1^2 & x_1x_2 & x_1x_3 & x_2^2 & x_2x_3 & 0 \end{bmatrix}$$

Multiplying $[y_1, y_2, y_3, y_4]$ against M yields nine elements

$$[-y_1x_1x_2 + y_2x_1^2, \ldots, -y_2x_2^2 + y_3x_1^2],$$

and contracting against the quadrics in $\{x_1, x_2, x_3\}$ shows that $(Z_1)_2$ is generated by the columns of

$$M((Z_1)_2) = \begin{bmatrix} y_2 & 0 & 0 & y_4 & 0 & 0 & 0 & 0 & y_3 \\ -y_1 & y_2 & 0 & 0 & y_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2 & -y_1 & 0 & y_4 & 0 & 0 & 0 \\ 0 & -y_1 & 0 & 0 & 0 & 0 & y_4 & 0 & -y_2 \\ 0 & 0 & -y_1 & 0 & -y_1 & 0 & -y_2 & y_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -y_1 & 0 & -y_2 & 0 \end{bmatrix}$$

Next, we need the term Z_2 in the approximation complex. There are two ways to do this: on the one hand, because I is a local complete intersection, we could simply compute the kernel of the matrix above. Or we could consider the kernel of the second Koszul differential on I, which is generated by the columns of

$$\begin{bmatrix} x_3 & x_2^3 & 0 & 0 \\ 0 & -x_1^2x_2 & 0 & x_1^2x_3 \\ 0 & x_1^3 & x_1^2x_3 & 0 \\ -x_2 & 0 & 0 & -x_2^3 \\ x_1 & 0 & -x_2^3 & 0 \\ 0 & 0 & x_1^2x_2 & x_1^3 \end{bmatrix}$$

The quadratic component of the image is generated by

$$x_3(e_1 \wedge e_2) - x_2(e_1 \wedge e_4) + x_1(e_2 \wedge e_4).$$

Pushing this forward via the Koszul differential yields

$$x_3(x_1^3e_2 - x_1^2x_2e_1) - x_2(x_1^3e_4 - x_1^2x_3e_1) + x_1(x_1^2x_2e_4 - x_1^2x_2e_2).$$

Multiplying this by $\{x_1, x_2, x_3\}$ to obtain the quadratic component and expressing the result in terms of the ordered basis above shows that Z_2 has quadratic component spanned by the columns of

$$\begin{bmatrix} y_4 & 0 & 0 \\ 0 & y_4 & 0 \\ -y_1 & 0 & y_4 \\ -y_2 & 0 & 0 \\ y_1 & -y_2 & 0 \\ 0 & 0 & -y_2 \\ 0 & y_1 & 0 \\ 0 & 0 & y_1 \\ 0 & 0 & 0 \end{bmatrix}$$

The determinant of the topmost 3×3 block of this is y_4^3, and the determinant of the complementary (hence, rightmost) block of the matrix $M((Z_1)_2)$ is $y_4^3 \cdot (y_2^3 - y_1^2y_3)$, so the implicit equation is given by

$$\frac{y_4^3 \cdot (y_2^3 - y_1^2y_3)}{y_4^3} = y_2^3 - y_1^2y_3,$$

which is easily checked to be correct. ⊲⊳

Multigraded Implicitization. A case which has attracted much interest in geometric modeling occurs when the ring R is multigraded. The hypersurface case which is the focus here was studied by Botbol in [**25**], and we now describe this in more detail.

The most familiar example of a multigraded map occurs when the map ϕ is given by monomials. In this case the image is a *toric variety* and it is possible to compute the implicit equation (or equations if the image is not a hypersurface) using simplicial homology. Let

$$\mathcal{A} = \mathbb{N}\{\mathbf{a}_1, \ldots, \mathbf{a}_d\} \subseteq \mathbb{Z}^n$$

be a semigroup, and A the matrix with i^{th} column $\mathbf{a}_i$. The toric ideal of $\mathcal{A}$ is

$$I_{\mathcal{A}} = \langle y^{\alpha} - y^{\beta} \mid \alpha, \beta \in \mathbb{N}^d \text{ and } \alpha - \beta \in ker(A) \rangle \subseteq S = k[y_1, \ldots, y_d].$$

To any $\mathbf{m} \in \mathbb{Z}^n$ we associate a simplicial complex

$$\Delta_{\mathbf{m}} = \{J \subseteq \{1, \ldots, d\} \mid \mathbf{m} - \sum_{i \in J} \mathbf{a}_i \text{ lies in } \mathcal{A}\}.$$

A result of Hochster [**116**] shows that

$$\widetilde{H}_j(\Delta_{\mathbf{m}}, k) = \mathrm{Tor}_j^S(I_{\mathcal{A}}, k)_{\mathbf{m}}.$$

In particular, the generators of the ideal of the implicitization correspond to elements of $\mathrm{Tor}_0^S(I_{\mathcal{A}}, k)$, so are given by the zeroth reduced homology of certain simplicial complexes built from the semigroup $\mathcal{A}$.

Example 0.18. Let $\mathcal{A} \subseteq \mathbb{Z}^2$ be generated by

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

We think of these lattice points as vertices $v_1, \ldots, v_4$. Then a computation shows that for

$$\mathbf{m} \in \left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right\}$$

we have $\widetilde{H}_0(\Delta_{\mathbf{m}}, k) \simeq k$. For example $\Delta_{\binom{2}{4}}$ has vertices v_1, v_2, v_3 and edge $v_1 v_3$. Computing, we see that the ideal $I_{\mathcal{A}}$ has three generators

$$\begin{array}{ll} y_2^2 - y_1 y_3 & \text{in degree } \binom{2}{4} \\ y_1 y_4 - y_2 y_3 & \text{in degree } \binom{3}{3} \\ y_3^2 - y_2 y_4 & \text{in degree } \binom{4}{2} \end{array}$$

The first syzygies of $I_{\mathcal{A}}$ may be computed in similar fashion, and a computation shows that

$$\widetilde{H}_1(\Delta_{\mathbf{m}}, k) = k$$

iff

$$\mathbf{m} \in \left\{ \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix} \right\}$$

For more details and an in-depth exposition, see [**156**]. ◁▷

In the more general setting of a multigraded ideal I which is not generated by monomials, the approximation complex machinery works by taking a sufficiently high multidegree subcomplex of $\mathcal{Z}$. The multigraded structure means that the local cohomology must be computed with respect to a different ideal.

In geometric modeling, the most common surfaces are *triangular* surfaces, which correspond geometrically to $\mathbb{P}^2$, and *tensor product surfaces*, which correspond geometrically to $\mathbb{P}^1 \times \mathbb{P}^1$. Algebraically, a tensor product surface comes from a bigrading on $R = k[s,t,u,v]$, with s,t of degree $(1,0)$ and u,v of degree $(0,1)$. To make multigraded implicitization concrete, we focus on tensor product surfaces.

Let $R_{m,n}$ denote the graded piece of R in bidegree (m,n). A regular map $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\Phi} \mathbb{P}^3$ is defined by four polynomials

$$\{p_0, p_1, p_2, p_3\} \subseteq R_{m,n}$$

with no common zeros on $\mathbb{P}^1 \times \mathbb{P}^1$. The empty locus on $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by the bigraded ideal

$$\mathbf{m} = \langle s,t\rangle \cap \langle u,v\rangle = \langle su, sv, tu, tv\rangle.$$

Therefore the map Φ is basepoint free map iff the ideal

$$I = \langle p_0, p_1, p_2, p_3\rangle$$

satisfies $\sqrt{I} = \mathbf{m}$.

Example 0.19. Consider the ideal

$$I = \langle s^2u, s^2v, t^2u, t^2v + stv\rangle,$$

generated by four elements of bidegree $(2,1)$. The syzygies on I are generated by the columns of

$$\begin{bmatrix} -v & -t^2 & 0 & 0 & -tv \\ u & 0 & -st-t^2 & 0 & 0 \\ 0 & s^2 & 0 & -sv-tv & -sv \\ 0 & 0 & s^2 & tu & su \end{bmatrix}$$

which we encode as

$$\begin{array}{rcl} uy_1 - vy_0 & = & 0 \\ s^2y_2 - t^2y_0 & = & 0 \\ s^2y_3 - (st+t^2)y_1 & = & 0 \\ tuy_3 - (sv+tv)y_2 & = & 0 \\ suy_3 - svy_2 - tvy_0 & = & 0 \end{array}$$

If we were in the singly graded case, we would need to consider degree 2, and a basis for $\mathcal{Z}_1^2$ consists of $\{s,t,u,v\} \cdot uy_1 - vy_0$, and the remaining four relations. With respect to the ordered basis $\{s^2, st, t^2, su, sv, tu, tv, u^2, uv, v^2\}$ for R_2 the matrix for $d_1^2 : \mathcal{Z}_1^2 \longrightarrow \mathcal{Z}_0^2$ is

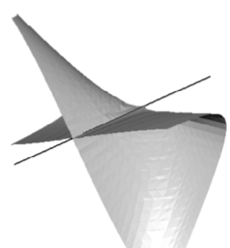
$$\begin{bmatrix} 0 & 0 & 0 & 0 & y_2 & y_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -y_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y_0 & -y_1 & 0 & 0 \\ y_1 & 0 & 0 & 0 & 0 & 0 & 0 & y_3 \\ -y_0 & 0 & 0 & 0 & 0 & 0 & -y_2 & -y_2 \\ 0 & y_1 & 0 & 0 & 0 & 0 & y_3 & 0 \\ 0 & -y_0 & 0 & 0 & 0 & 0 & -y_2 & -y_0 \\ 0 & 0 & y_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -y_0 & y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y_0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix represents *all* the first syzygies of total degree two. The multigraded setting gives us extra structure to work with, and in Corollary 14 of [**28**], Botbol-Dickenstein-Dohm give a bound in terms of the multidegree; in the case of this example the bound tells us that it suffices to work with the subcomplex of the approximation complex of bidegree $(1, 1)$. The first syzygies of bidegree $(1, 1)$ correspond to the submatrix whose rows are indexed by $\{su, sv, tu, tv\}$, which is given by

$$\begin{bmatrix} y_1 & 0 & 0 & y_3 \\ -y_0 & 0 & -y_2 & -y_2 \\ 0 & y_1 & y_3 & 0 \\ 0 & -y_0 & -y_2 & -y_0 \end{bmatrix}$$

The image of ϕ is the determinantal hypersurface

$$H = \mathbf{V}(y_0 y_1^2 y_2 - y_1^2 y_2^2 + 2y_0 y_1 y_2 y_3 - y_0^2 y_3^2).$$



For curves, Sederberg-Goldman-Du [**179**] and Cox-Hoffman-Wang [**56**] show that the structure of the syzygy matrix is closely connected to the behavior of the singular locus of the implicit curve. This also holds for tensor product surfaces in $\mathbb{P}^3$ of bidegree $(2, 1)$: in [**174**] Schenck-Seceleanu-Validashti show that there are six possible structures for the syzygy matrix, and describe the codimension one singular locus for each case. In our example above (from [**174**]) there is a unique syzygy of bidegree $(0, 1)$, which in turn forces the codimension one singular locus of H to be the union of three lines $\mathbf{V}(y_0, y_2) \cup \mathbf{V}(y_1, y_3) \cup \mathbf{V}(y_0, y_1)$.

Since the generators of I are of bidegree $(2,1)$, the bidegree $(0,1)$ syzygy occurs in total degree $(2,2)$, and the variables w_i are of degree $(1,0,0)$. Therefore the generator $w_1u - w_3v$ of the Rees ideal occurs in degree $(1,2,2)$, explaining the labelling of the last matrix below.

```
i1 : R=ZZ/31991[s,t,u,v,Degrees=>{{1,0},{1,0},{0,1},{0,1}}];

i2 : I=ideal(s^2*u,s^2*v,t^2*u,t^2*v+s*t*v);

i3 : rI=(res coker gens I).dd

           1                                4
o3 = 0 : R  <-------------------------- R  : 1
                | s2u t2u s2v stv+t2v |

           4                                                5
     1 : R  <---------------------------------------- R  : 2
               {2, 1} | -v -t2 0     0      0      |
               {2, 1} | 0  s2  0     -sv-tv -tv    |
               {2, 1} | u  0   st+t2 0      tu     |
               {2, 1} | 0  0   -s2   tu     -su+tu |

           5                         2
     2 : R  <--------------------- R  : 3
               {2, 2} | -t2 0   |
               {4, 1} | v   0   |
               {4, 1} | 0   -u  |
               {3, 2} | s-t -t  |
               {3, 2} | t   s+t |

           2
     3 : R  <----- 0 : 4
                0
i4 : transpose gens reesIdeal I

o4 = {-1, -2, -2} | w_1u-w_3v                                     |
     {-1, -4, -1} | w_2s2-w_3t2                                   |
     {-1, -3, -2} | w_0tu-w_2sv-w_2tv                             |
     {-1, -3, -2} | w_0su-w_2sv-w_3tv                             |
     {-1, -4, -1} | w_0s2-w_1st-w_1t2                             |
     {-2, -5, -2} | w_1w_2t+w_0w_3s-w_0w_3t-w_1w_3t               |
     {-2, -5, -2} | w_1w_2s-w_0w_3s+w_1w_3t                       |
     {-2, -4, -4} | w_0^2u2-2w_0w_2uv+w_2^2v2-w_2w_3v2             |
     {-3, -6, -4} | w_1w_2^2v+w_0^2w_3u-2w_0w_2w_3v-w_1w_2w_3v    |
     {-4, -8, -4} | w_1^2w_2^2-2w_0w_1w_2w_3-w_1^2w_2w_3+w_0^2w_3^2 |
```

Tensor product surfaces are a type of ruled surfaces, on which there is an extensive literature, ranging from the classical work of Edge [**76**] and Salmon [**172**] to contemporary work motivated by geometric modeling. We give pointers to relevant papers in the next section. ◁▷

Directions for future research. There are a myriad of open questions at the interface of geometric modeling and algebraic geometry. A central problem is how to leverage results from situations where the free resolution of I is known to obtain results on the Rees algebra. For an example along these lines, if I is codimension two and Cohen-Macaulay, then I has a Hilbert-Burch resolution: $I = \langle f_1, \dots f_d \rangle$ is generated by the $(d-1) \times (d-1)$ minors of a $d \times d-1$ matrix, whose columns are the syzygies of I. In [**158**], Morey-Ulrich determine the equations for the Rees algebra, when all the syzygies are linear, and [**24**] gives an answer when exactly one syzygy is nonlinear; however the general case remains open. Some additional questions:

- Rees algebras of rational curves and connections to singularities. In addition to the works [**179**] and [**56**] mentioned earlier, there is a large literature on the interplay between syzygies and singularities of rational curves. In [**57**] Cox-Kustin-Polini-Ulrich carry out a comprehensive investigation; see also Cortadellas Benítez-D'Andrea [**50**] [**51**].
- Tensor product surfaces have been extensively studied, but there remain many open questions. See [**69**], [**74**], [**79**], [**86**], [**213**] for some work on low degree cases; more generally study toric implicitization [**25**], [**28**],[**68**].
- Matrix representations: rather than finding the actual implicit equation, focus on a matrix which drops rank at points of the implicitization. As Botbol-Busé-Chardin write in [**26**] "matrices have to be seen as implicit representations on their own, without relying on the more classical implicit equation". For more work along these lines, see [**28**], and [**27**].
- Codimension three Gorenstein ideals have a structure theorem, and in [**142**], Kustin-Polini-Ulrich determine the equations for the Rees algebra. Is there a similar result for Gorenstein deviation two ideals, where there is a structure theory due to Huneke-Ulrich [**121**] and the resolutions are known from Kustin [**141**]?
- The resolution of ideals obtained from the submaximal minors of a matrix of variables may be constructed via representation theory and Bott-Borel-Weyl, see Weyman [**208**] for details. What can one say about the Rees algebra of such examples? About the simplest case of an Eagon-Northcott resolution?
- While Hochster's method provides a way to compute the implicit equations for the image of a monomial map, computing minimal generators for $R(I)$ is nontrivial: see Lin [**149**] for a beautiful application of Alexander duality to the initial ideal of the maximal minors of a generic matrix, and Fouli-Lin [**84**] for additional results on squarefree monomial ideals.
- When $R(I)$ is of quadratic type (determined by $\mathrm{syz}(I)$ and $\mathrm{syz}(I^2)$, [**38**] shows that half the degrees where the torsion is nonzero are understood.
- Methods from tropical geometry: if one can determine the degree of the implicit equation of a hypersurface, then the problem of actually describing it is linear algebra; in [**72**] and [**191**] tropical geometry is used to shed light on the problem.