TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 373, Number 11, November 2020, Pages 7639–7668
https://doi.org/10.1090/tran/8210
Article electronically published on September 14, 2020

STRICHARTZ ESTIMATES AND STRAUSS CONJECTURE ON NON-TRAPPING ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

YANNICK SIRE, CHRISTOPHER D. SOGGE, CHENGBO WANG, AND JUNYONG ZHANG

ABSTRACT. We prove global-in-time Strichartz estimates for the shifted wave equations on non-trapping asymptotically hyperbolic manifolds. The key tools are the spectral measure estimates from [Ann. Inst. Fourier, Grenoble 68 (2018), pp. 1011–1075] and arguments borrowed from [Analysis PDE 9 (2016), pp. 151–192], [Adv. Math. 271 (2015), pp. 91–111]. As an application, we prove the small data global existence for any power $p \in (1, 1 + \frac{4}{n-1})$ for the shifted wave equation in this setting, involving nonlinearities of the form $\pm|u|^p$ or $\pm|u|^{p-1}u$, which answers partially an open question raised in [Discrete Contin. Dyn. Syst. 39 (2019), pp. 7081–7099].

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

The purpose of this paper is to study the dispersive behaviour of the linear wave equation on non-trapping asymptotically hyperbolic manifolds, which is a class of manifolds with variable curvature, and its application to the small data global existence for the nonlinear Cauchy problem with power nonlinearities.

Received by the editors October 6, 2019.
2010 *Mathematics Subject Classification.* Primary 47J35, 35L71, 35L05, 35S30.
Key words and phrases. Strichartz estimate, asymptotically hyperbolic manifold, spectral measure, Strauss conjecture, shifted wave.
The first author was partially supported by the Simons Foundation.
The second author was supported by the NSF and the Simons Foundation.
The third author was supported in part by NSFC 11971428 and the National Support Program for Young Top-Notch Talents.
The fourth author was supported by NSFC Grants (11771041, 11831004) and H2020-MSCA-IF-2017(790623).

1.1. **Background on Strichartz estimates.** The dispersive decay and Strichartz estimates are known to play an important role in the study of the behaviour of solutions to nonlinear Schrödinger equations, nonlinear wave equations, and other nonlinear dispersive equations; e.g. see Tao [31]. The first aim of this article is to prove global-in-time Strichartz estimates for the wave equation on non-trapping asymptotically hyperbolic manifolds.

Let (M°, g) be a Riemannian manifold of dimension $n \geq 2$, and let $I \subset \mathbb{R}$ be a time interval. Suppose $u(t,z)$: $I \times M^\circ \to \mathbb{R}$ is a solution of the wave equation

$$\partial_t^2 u - \Delta_g u = F, \quad u(0) = u_0(z), \quad \partial_t u(0) = u_1(z),$$

where Δ_g denotes the Laplace-Beltrami operator on (M°, g). The general Strichartz estimates show that

$$\begin{aligned} &\|u(t,z)\|_{L_t^q(I;L_z^r(M^\circ))} + \|u(t,z)\|_{C(I;\dot{H}^s(M^\circ))} \\ &\quad \lesssim \|u_0\|_{\dot{H}^s(M^\circ)} + \|u_1\|_{\dot{H}^{s-1}(M^\circ)} + \|F\|_{L_t^{\tilde{q}'}(I;L_z^{\tilde{r}'}(M^\circ))}, \end{aligned} \tag{1.1}$$

where $\dot{H}^s$ denotes the homogeneous L^2-Sobolev space over M° and the pairs $(q,r), (\tilde{q}, \tilde{r}) \in [2,\infty]^2$ satisfy the wave-admissible condition

$$\frac{2}{q} + \frac{n-1}{r} \leq \frac{n-1}{2}, \quad (q,r,n) \neq (2,\infty,3), \tag{1.2}$$

and the gap condition

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2. \tag{1.3}$$

It is well known that (1.1) holds for $(M^\circ, g) = (\mathbb{R}^n, \delta)$ with $I = \mathbb{R}$ and $r, \tilde{r} < \infty$, and the result is sharp; see Strichartz [25], Ginibre-Velo [12], Keel-Tao [16], and references therein. There is a huge literature about Strichartz inequalities on Euclidean space or manifolds, and it is beyond the scope of this introduction to review all of it. We instead mention a few of the most relevant papers about Strichartz estimates for the wave equation on the real hyperbolic spaces. On the real hyperbolic spaces $\mathbb{H}^n$, Anker-Pierfelice [1], Anker-Pierfelice-Vallarino [2], Metcalfe-Taylor [23, 24] and Tataru [29] have showed better dispersive estimates and hence stronger results than in the Euclidean space. More precisely, they can obtain results with (q,r) exterior of the range (1.2). Our first results will generalize their results to any non-trapping asymptotically hyperbolic space, i.e. a non-compact Riemannian manifold with variable curvature in which conjugate points can possibly appear, causing the failure of the usual dispersive estimate.

1.2. **The setting.** In this paper, we work on an n-dimensional complete non-compact Riemannian manifold (M°, g) where the metric g is an asymptotically hyperbolic metric. This setting is the same as in Chen-Hassell [7, 8], Mazzeo [21], and Mazzeo-Melrose [22]. Let x be a boundary-defining function for the compactification M of M°. We say a metric g is conformally compact if $x^2 g$ is a Riemannian metric and extends smoothly up to the boundary ∂M. Mazzeo [21] showed that its sectional curvature tends to $-|dx|^2_{x^2 g}$ as $x \to 0$. In particular, if the limit is such that $-|dx|^2_{x^2 g} = -1$, we say that the conformally compact metric g is asymptotically hyperbolic. More specifically, let $y = (y_1, \dots, y_{n-1})$ be local coordinates on

$Y = \partial M$, and let (x, y) be the local coordinates on M near ∂M. The metric g in a collar neighborhood $[0, \epsilon)_x \times \partial M$ takes the form

$$g = \frac{dx^2}{x^2} + \frac{h(x,y)}{x^2} = \frac{dx^2}{x^2} + \frac{\sum h_{jk}(x,y)dy^j dy^k}{x^2}, \tag{1.4}$$

where $x \in C^\infty(M)$ is a boundary-defining function for ∂M and h is a smooth family of metrics on $Y = \partial M$. In addition, if every geodesic in M reaches ∂M both forwards and backwards, we say M is non-trapping. The Poincaré disc $(\mathbb{B}^n, g)$ is a typical example of such a manifold. Indeed, considering the ball $\mathbb{B}^n = \{z \in \mathbb{R}^n : |z| < 1\}$ endowed with the metric

$$g = \frac{4dz^2}{(1-|z|^2)^2}, \tag{1.5}$$

one can take $x = (1-|z|)(1+|z|)^{-1}$ as the boundary-defining function and ω as the coordinates on $\mathbb{S}^{n-1}$. Then the Poincaré metric takes the form

$$g = \frac{dx^2}{x^2} + \frac{\frac{1}{4}(1-x^2)^2 d\omega^2}{x^2},$$

where $d\omega^2$ is the standard metric on the sphere $\mathbb{S}^{n-1}$. Another typical example is the real hyperbolic space $\mathbb{H}^n$, which is a complete simply connected manifold of constant negative curvature -1. Since the curvature is a negative constant, $\mathbb{H}^n$ is automatically non-trapping and has no conjugate points.

1.3. **The main result about Strichartz estimates.** Consider the wave equation associated to the Laplace-Beltrami operator Δ_g on the non-trapping asymptotically hyperbolic manifold (M°, g):

$$\begin{cases} \partial_t^2 u - \Delta_g u = F, \\ u(0) = u_0(z),\ \partial_t u(0) = u_1(z). \end{cases} \tag{1.6}$$

From Mazzeo-Melrose [22], the continuous spectrum of $-\Delta_g$ is contained in $[\frac{(n-1)^2}{4}, +\infty)$, while the point spectrum is contained in $(0, \frac{(n-1)^2}{4})$. When $-\Delta_g$ has no point spectrum, it is natural to consider a family of Klein-Gordon equations

$$\begin{cases} \partial_t^2 u(t,z) - \Delta_g u(t,z) + mu(t,z) = F(t,z), \\ u(0) = u_0(z),\ \partial_t u(0) = u_1(z), \end{cases} \tag{1.7}$$

with the constant

$$m \geq -\rho^2 := -(n-1)^2/4. \tag{1.8}$$

In particular for $m = -\rho^2$, the equation is named the *shifted* wave equation. In this paper, we focus on the shifted wave equation on any non-trapping asymptotically hyperbolic manifold, motivated by the problem of small data global existence raised in [26]. Another motivation is to continue the study of dispersive equations on manifolds with variable curvature. As mentioned above, there possibly exist conjugate points in the variable curvature setting and they cause the failure of the usual dispersive estimates, but not of the Strichartz estimates. For example, on non-trapping asymptotically conic manifolds whose curvature tends to zero as the boundary-defining function $x \to 0$, Hassell and the last author [13, 33] established the global-in-time Strichartz estimates for Schrödinger and wave equations, which are the same as in Euclidean space. While on a non-trapping asymptotically hyperbolic manifold whose sectional curvature tends to -1, Chen [6] showed Strichartz estimates for the Schrödinger equation, which are stronger than the Euclidean result. The crucial point in these papers is to use the microlocal method to deal with the conjugate points of the manifold. If the manifold has non-positive curvature, e.g. the hyperbolic space $\mathbb{H}^n$ considered in [1,2,23,24,29], then there are no conjugate points. In the Euclidean space, the Strichartz estimates usually are proved by interpolating an L^2-estimate and a dispersive estimate. For Schrödinger equations, the dispersive estimate directly follows from the representation of the solution, while for the wave equation, the dispersive estimate requires a more complicated argument, which typically involves Littlewood-Paley theory. However, in the hyperbolic setting, the usual Littlewood-Paley theory is missing; see Bouclet [5]. To get around this, in the real hyperbolic space with constant sectional curvature -1, Metcalfe-Taylor [23] made use of Sobolev spaces based on BMO-spaces and interpolation results from [30]; Anker-Pierfelice [1] and Anker-Pierfelice-Vallarino [2] used a good representation of the fundamental solution of the wave equation and a complex interpolation argument. Before these works, Tataru [29] obtained Strichartz estimates for $\mathbb{H}^n$ using complex interpolation.

For the variable curvature setting, we do not know such precise results. Of course, a standard replacement (which is very often sufficient) can be to use the Littlewood-Paley-Stein theory based on heat semi-groups; see e.g. [17,20]. We refer the reader in particular to the recent work [18], where the authors develop a systematic treatment of Littlewood-Paley theory using the heat flow for the (shifted) Laplace-Beltrami operator on hyperbolic spaces (see also [19]). In this case also, we could not overcome the technical issues. We take then a new approach. Our approach consists of splitting the solution space into low and high frequencies. We derive general Strichartz estimates, of independent interest, and use part of them (high frequencies) to obtain the global well-posedness for power-type nonlinearities. The argument crucially uses a microlocalized spectral measure estimate, which is a replacement for the argument involving restriction theorem (like the Stein-Tomas theorem) for the Euclidean case.

Now we state our main result on the Strichartz estimate. Before doing so, we introduce some notation. Let $H = -\Delta_g - \rho^2$ and let $\chi \in \mathcal{C}_c^\infty([0,\infty)$ such that $\chi(\lambda) = 1$ for $\lambda \leq 1$ and vanishes when $\lambda \geq 2$. Define the norm of $H_c^{a,b}$ by

$$\|f\|_{H_c^{a,b}} = \|(1-\chi)(\sqrt{H})H^{\frac{a}{2}}f\|_{L^c} + \|\chi(\sqrt{H})H^{\frac{b}{2}}f\|_{L^c}. \tag{1.9}$$

In the particular case $c = 2$, we write briefly $H^{a,b}$. The space introduced here is an analogue of the usual Sobolev space but with separated regularity corresponding to high and low frequencies. Next we define the sets related with the admissible conditions:

$$\Lambda_w = \Big\{(q,r,\mu) \in [2,\infty] \times (2,\infty] \times \mathbb{R} : \frac{2}{q} \leq (n-1)(\frac{1}{2} - \frac{1}{r}), \quad \mu > s_w\Big\}, \tag{1.10}$$

where

$$s_w = n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q} \tag{1.11}$$

and

$$\Lambda_e = \left\{(q, r, \mu) \in [2, \infty] \times (2, \infty] \times \mathbb{R} : \frac{2}{q} \geq (n-1)(\frac{1}{2} - \frac{1}{r}), \quad \mu > s_e\right\}, \tag{1.12}$$

where

$$s_e = \frac{n+1}{2}(\frac{1}{2} - \frac{1}{r}). \tag{1.13}$$

We remark here that μ in the above sets is strictly greater than the optimal exponents s_w and s_e. This fact will imply a loss of regularity for high frequencies in the Strichartz estimates.

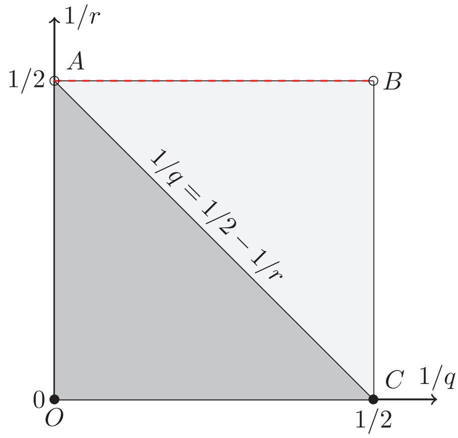

FIGURE 1. The range of (q, r) when $n = 3$. If $(q, r, \mu) \in \Lambda_w$, then (q, r) is in the triangle region ACO; while if $(q, r, \mu) \in \Lambda_e$, then (q, r) is in the region ABC.

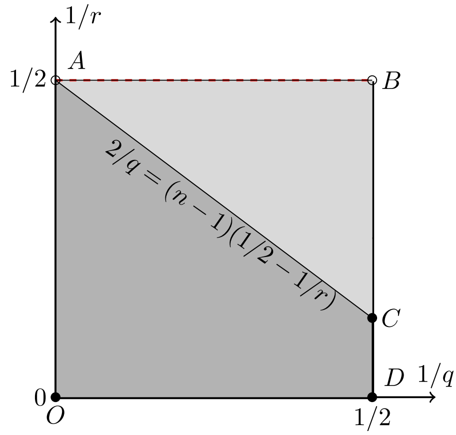

FIGURE 1. The range of (q, r) when $n \geq 4$. If $(q, r, \mu) \in \Lambda_w$, then (q, r) is in the region ACDO; while if $(q, r, \mu) \in \Lambda_e$, then (q, r) is in the triangle region ABC.

Our result about the homogeneous Strichartz estimate is the following.

Theorem 1.1 (Homogeneous Strichartz estimate)**.** *Let (M°, g) be any non-trapping asymptotically hyperbolic manifold of dimension $n \geq 2$ and let Δ_g be the Laplace-Beltrami operator on (M°, g) and $\rho^2 = (n-1)^2/4$. Assume Δ_g has no pure point eigenvalue and has no resonance at the bottom of the continuous spectrum ρ^2. Suppose that u is a solution to the Cauchy problem*

$$\begin{cases} \partial_t^2 u - \Delta_g u - \rho^2 u = 0, \quad (t, z) \in I \times M^\circ; \\ u(0) = u_0(z), \ \partial_t u(0) = u_1(z), \end{cases} \tag{1.14}$$

for some initial data $u_0 \in H^{\mu,0}(M^\circ), u_1 \in H^{\mu-1,-\epsilon}(M^\circ)$ defined in (1.9), and the time interval $I \subseteq \mathbb{R}$. Then

$$\|u(t, z)\|_{L_t^q(I; L_z^r(M^\circ))} \lesssim \|u_0\|_{H^{\mu,0}(M^\circ)} + \|u_1\|_{H^{\mu-1,-\epsilon}(M^\circ)}, \tag{1.15}$$

where $(q, r, \mu) \in \Lambda_w \cup \Lambda_e$ defined in (1.10) and (1.12), $0 < \epsilon \ll 1$.

Remark 1.1. The Strichartz estimate is global in time but with an arbitrary small loss in regularity of high frequency which is a bit weaker than the estimates in [1, 2, 23, 29] on the hyperbolic space $\mathbb{H}^n$. The loss comes from our techniques since we lack the (standard) Littlewood-Paley square function estimate due to the non-doubling property of the manifold (or a good representation of the fundamental solution as in [1, 2, 29]).

Remark 1.2. Compared with [1, 2, 23, 29], the general setting considered here may have conjugate points which can lead the usual dispersive estimate to fail. It is known that the sharp regularity Strichartz estimate in Euclidean space fails for admissible pairs (e.g. $q = 2$ and $r = \infty$ when $n = 3$), but we obtain the inequalities for the admissible pairs including $q = 2$.

Remark 1.3. We exclude the case $r = 2$. At the special point A, that is, $q = \infty, r = 2$, the usual Strichartz estimate holds. For example, in the Euclidean space, the Strichartz estimate holds at A if $\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} < \infty$; however one also can recover the estimate (1.15) at A but with $\epsilon = 1$ by using Proposition 3.1 below. In this sense, the result gains some regularity in low frequency.

Theorem 1.2 (Inhomogeneous Strichartz estimate). *Let Δ_g be as in Theorem 1.1 and suppose that u is a solution to the Cauchy problem*

$$\begin{cases} \partial_t^2 u - \Delta_g u - \rho^2 u = F(t,z), \quad (t,z) \in I \times M^\circ; \\ u(0) = 0, \ \partial_t u(0) = 0 \end{cases} \tag{1.16}$$

and the time interval $I \subseteq \mathbb{R}$. Then

$$\|u(t,z)\|_{L^q_t(I; L^r_z(M^\circ))} \lesssim \|F\|_{L^{\tilde{q}'}_t(I; H^{\mu+\tilde{\mu}-1,0}_{\tilde{r}'}(M^\circ))}, \tag{1.17}$$

where $(q, r, \mu), (\tilde{q}, \tilde{r}, \tilde{\mu}) \in \Lambda_w \cup \Lambda_e$.

The proof of the inhomogeneous Strichartz estimate will be divided into two cases. The first case $q > \tilde{q}'$ is proved using the TT^*-method and the Christ-Kiselev lemma [9]. The second case when $q = \tilde{q} = 2$ is more complicated to treat due to the failure of the Christ-Kiselev lemma, and the usual dispersive estimate fails due to the conjugate points. We overcome these difficulties following the idea of Hassell and the last author [13]. In this argument, we classify the microlocalized pseudo-differential operator via the wavefront set propagated along the bicharacteristic flow and parametrize the wavefront set off the diagonal case by a phase function with an unchanged sign. Finally we can show some dispersive estimate in some special cases; see Proposition 5.3 for details. Combining this with the TT^*-method again, we show the inhomogeneous Strichartz estimate when $q = \tilde{q} = 2$.

1.4. **The small data global existence and Strauss conjecture.** We now apply the previous estimates to the nonlinear wave equation with small data. We introduce the class of nonlinearities: let $F_p \in C^1$ behaving like $\pm|u|^p$ or $\pm|u|^{p-1}u$, hence such that

$$|F_p(u)| + |u||F_p'(u)| \leq C|u|^p$$

for some constant $C > 0$. Consider the family of nonlinear equations

$$\begin{cases} \partial_t^2 u(t,z) - \Delta_g u(t,z) + mu(t,z) = F_p(u), \\ u(0) = u_0(z), \ \partial_t u(0) = u_1(z), \end{cases} \tag{1.18}$$

where the constant satisfies

$$m \geq -\rho^2 := -(n-1)^2/4. \tag{1.19}$$

The problem under consideration belongs to the realm of the dichotomy between global existence vs. blow-up for the nonlinear equation (1.18) with $m = 0$ as investigated for the first time by F. John in [14] on the Euclidean space. John determined the critical power to be $p_S = 1 + \sqrt{2}$ for the problem when $n = 3$ by proving global existence results for $p > 1 + \sqrt{2}$ and blow-up results for $p < 1 + \sqrt{2}$. Later, Strauss [28] conjectured that the critical power $p_c(n) = p_S(n)$ (above which global existence for small data holds) for other dimensions $n \geq 2$ should be the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

See [32] and the references therein for a complete account on the state of the art. On the real hyperbolic space $\mathbb{H}^n$, Metcalfe-Taylor [23] gave a proof of small data global existence for (1.18) with $m = 0$ and $p \geq 5/3$ for dimension $n = 3$, and then Anker-Pierfelice [1] proved global existence for the problem (1.18) with $m > -\rho^2$ and $p \in (1, 1 + \frac{4}{n-1}]$ where $n \geq 2$. Metcalfe-Taylor [24] gave an alternative proof for $n = 3$. Notice that the spectrum of the Laplacian on $\mathbb{H}^n$ is contained in $[\rho^2, \infty)$; these results are more like a nonlinear Klein-Gordon equation instead of a nonlinear wave equation. For the limit case $m = -\rho^2$, i.e. the shifted wave equation, Fontaine [10] was the first one to provide small data global existence for $n = 2, 3$ and $p \geq 2$. Anker-Pierfelice-Vallarino [2] proved wider couples of Strichartz estimates and a stronger local well-posedness result for the nonlinear shifted wave equation. The Strichartz estimate established in [2] could be applied to show small data global existence for any $p \in (1, 1 + 4/(n-1)]$, even though such results have not been proved explicitly in [2]. Hence it illustrates that the critical power of global existence holds for the shifted wave equation with small data $p_c(n) = 1$. This result on $\mathbb{H}^n$ is explicitly stated and proved by the first three authors [26]. Tataru [29] actually proved dispersive estimates, which are strong enough to ensure global results, as pointed out in [26]. On Damek-Ricci spaces (which contain Riemannian symmetric spaces of rank one), Anker-Pierfelice-Vallarino [3] prove also global results. In [26], the authors also showed the small data global existence result for (1.18) on a manifold with variable curvature under the assumption that $\mathrm{Spec}(-\Delta_g + m) \subset (c, +\infty)$ with $c > 0$. The final remark of [26] raised a question about the small data global existence for (1.18) with $p > 1$ and $m = -\kappa\rho^2$ on a manifold with variable negative curvature with sectional curvatures $K \in [-\tilde{\kappa}, -\kappa]$ for some $\tilde{\kappa} \geq \kappa > 0$. Our second result partially answers this problem. More precisely, we prove

Theorem 1.3. *Let (M°, g) be a non-trapping asymptotically hyperbolic manifold of dimension $n \geq 2$ and let Δ_g be the Laplace-Beltrami operator on (M°, g) as in Theorem 1.1. Let $\rho^2 = (n-1)^2/4$ and let $p \in (1, 1 + \frac{4}{n-1})$. Then there exists a constant $\nu_1 > 0$ such that the Cauchy problem*

$$\begin{cases} \partial_t^2 u - \Delta_g u - \rho^2 u = F_p(u), & (t, z) \in I \times M^\circ; \\ u(0) = \nu u_0(z), \ \partial_t u(0) = \nu u_1(z) \end{cases} \tag{1.20}$$

has global solution where $\nu \in (0, \nu_1]$ and $u_0 \in H^{\mu,0}(M^\circ), u_1 \in H^{\mu-1,-\epsilon}(M^\circ)$ defined in (1.9) for ϵ very small and $\mu > \frac{n+1}{2}(\frac{1}{2} - \frac{1}{p+1})$.

Remark 1.4. The assumption on the regularity of the initial data is not sharp. The usual investigations for small data global existence require more care; see Wang [32].

Notice here that we do not reach the endpoint $p = 1 + \frac{4}{n-1}$. The reason is that there is a loss of derivatives in the inhomogeneous Strichartz estimates and so it is impossible to close the iteration in this latter case. In this case, one has to use another method, based on the strategy described in [4]. We postpone this issue to a later work since the techniques are very different.

Notation. We use $A \lesssim B$ to denote $A \leq CB$ for some large constant C which may vary from line to line and depend on various parameters, and similarly we use $A \ll B$ to denote $A \leq C^{-1}B$. We employ $A \sim B$ when $A \lesssim B \lesssim A$. If the constant C depends on a special parameter other than the above, we shall denote it explicitly by subscripts. For instance, C_ϵ should be understood as a positive constant not only depending on p, q, n, and M but also on ϵ. Throughout this paper, pairs of conjugate indices are written as p, p', where $\frac{1}{p} + \frac{1}{p'} = 1$ with $1 \leq p \leq \infty$.

Organization of this paper. Our paper is organized as follows. We recall the properties of the microlocalized spectral measure in Section 2. In Section 3, we define the microlocalized propagator and prove the energy estimate and the microlocalized dispersive estimate. We conclude this section by showing the microlocalized Strichartz estimate. We prove Theorem 1.1 in Section 4 and Theorem 1.2 in Section 5. Finally, we prove the global existence of Theorem 1.3 in Section 6.

2. The spectral measure

In this section, we briefly review the key elements of the microlocalized spectral measure, which was constructed and proved by Chen-Hassell [7, Theorem 1.3], [8]. This is an analogue of a result of Hassell and the fourth author [13, Proposition 1.5] for the non-trapping asymptotically conic manifold. The property not only gives the decay of spectral measure in frequency but also captures the oscillatory behaviour of the spectral measure.

Proposition 2.1. *Let (M°, g) and $H = -\Delta_g - (n-1)^2/4$ be as in Theorem 1.1. Then for low energy, i.e. $\lambda \leq 2$, the Schwartz kernel of the spectral measure $dE_{\sqrt{H}}(\lambda; z, z')$ satisfies*

$$dE_{\sqrt{H}}(\lambda; z, z') = \lambda \left((\rho_L\rho_R)^{\frac{n-1}{2}+i\lambda} a(\lambda; z, z') - (\rho_L\rho_R)^{\frac{n-1}{2}-i\lambda} a(-\lambda; z, z') \right), \tag{2.1}$$

where $a \in \mathcal{C}^\infty([-2,2]_\lambda \times M_0^2)$ and ρ_L and ρ_R are respectively the boundary defining functions for the left and right boundary in the double space M_0^2. Furthermore, there holds

$$|dE_{\sqrt{H}}(\lambda; z, z')| \leq C\lambda^2(1 + d(z, z'))e^{-(n-1)d(z,z')/2}. \tag{2.2}$$

For the high energy, i.e. $\lambda \geq 1/2$, there exists a finite pseudo-differential operator partition of the identity operator

$$\mathrm{Id} = \sum_{k=0}^{N} Q_k(\lambda), \tag{2.3}$$

where the Q_k are uniformly bounded as operators on L^2 and N is independent of λ, such that

$$(2.4)\quad (Q_k(\lambda)dE_{\sqrt{H}}(\lambda)Q_k^*(\lambda))(z,z') = \lambda^{n-1}\sum_{\pm} e^{\pm i\lambda d(z,z')}a_\pm(\lambda;z,z') + b(\lambda;z,z'),$$

where $d(\cdot,\cdot)$ is the Riemannian distance on M°, and for any α, there exists a constant C_α such that the $a_\pm$ satisfies

$$(2.5)\qquad |\partial_\lambda^\alpha a_\pm(\lambda;z,z')| \le \begin{cases} C_\alpha\lambda^{-\alpha}(1+\lambda d(z,z'))^{-\frac{n-1}{2}}, & d(z,z')\le 1,\\ C_\alpha\lambda^{-\frac{n-1}{2}-\alpha}e^{-(n-1)d(z,z')/2}, & d(z,z')\ge 1,\end{cases}$$

and b satisfies

$$(2.6)\qquad |\partial_\lambda^\alpha b(\lambda;z,z')| \le C_\alpha\lambda^{-K-\alpha}e^{-(n-1)d(z,z')/2}\quad \forall\alpha,\ K>0.$$

Moreover, if (M°,g) is in addition simply connected with non-positive sectional curvatures, then the estimates above are true for spectral measure without microlocalization; that is, in this case we can take $\{Q_k(\lambda)\}$ to be the trivial partition of unity.

Remark 2.1. For example, a Cartan-Hadamard manifold is a simply connected manifold with non-positive sectional curvatures; hence we have the estimates above without microlocalization. The non-positive sectional curvatures imply that the manifold is non-trapping and has no conjugate points.

Next we show an inequality for an integral operator which is similar to a result of Anker-Pierfelice-Vallarino [2] on $\mathbb{H}^n$. This is close to a non-Euclidean feature of hyperbolic space related to the Kunze-Stein phenomenon [15].

Lemma 2.1. *Let M° be the manifold as in Theorem 1.1 and let the kernel K satisfy the pointwise bound*

$$(2.7)\qquad |K(z,z')| \le e^{-\rho_\delta d(z,z')},\quad \rho_\delta = \rho-\delta = (n-1)/2-\delta.$$

Then for any $q\in(2,\infty]$, there exist a constant C and $0<\delta_0(q) := (n-1)(\frac12-\frac1q)$ such that

$$(2.8)\qquad \Big\|\int_{M^\circ} K(z,z')f(z')dg(z')\Big\|_{L^q(M^\circ)} \le C\|f\|_{L^{q'}(M^\circ)}$$

holds for all $0<\delta<\delta_0$.

Proof. The proof is a variant of the argument in [8, Section 4.2], where the estimates of the spectral measure are established. We show that there exists a constant C such that

$$(2.9)\qquad \Big|\int_{M^\circ\times M^\circ} K(z,z')f(z')h(z)dg(z')dg(z)\Big| \le C\|f\|_{L^{q'}}\|h\|_{L^{q'}}.$$

We split the left hand side into several pieces, restricting the kernel to different regions. Recall that M is the compactification of M° and M_0^2 is the blow-up space. Let O be a neighbourhood of the front face FF in M_0^2. We write

$$K(z,z') = K(z,z')\chi_O + K(z,z')\chi_{M_0^2\setminus O},$$

where χ is the usual bump function. We first consider (2.9) with the kernel $K(z,z')\chi_{M_0^2\setminus O}$. Since the other cases are similar, we only prove (2.9) when both

z, z' are near the boundary $\{x = 0\}$ of M where x is the boundary-defining function. Away from the front face, the distance $d(z, z')$ is comparable to $-\log(xx')$. Since $q > 2$ and $0 < \delta \ll 1$, we obtain

$$\begin{aligned} &\Big| \int_{\{x,x'\leq\eta\}\cap M_0^2\setminus O} K(z,z')f(z')h(z)dg(z')dg(z)\Big| \\ &\lesssim \int_{x,x'\leq\eta} (xx')^{q\rho_\delta} \frac{dx'}{x'^n}\frac{dx}{x^n}\|f\|_{L^{q'}}\|h\|_{L^{q'}} \leq C_\eta\|f\|_{L^{q'}}\|h\|_{L^{q'}}. \end{aligned} \tag{2.10}$$

Now consider the kernel $K(z, z')\chi_O$ near the front face. Further decompose the set O into subsets $O_i \subset M_0^2$,

$$O_i = \{(x, x', y, y') : x, x' \leq \eta; \quad d_h(y, y_i), d_h(y', y_i) \leq \eta\},$$

for some $y_i \in \partial M$ where the distance d_h is measured by the metric $h(0)$ on ∂M. Use the local coordinates (x, y) on M which is near $(0, y_i) \in \partial M$ to define a map ϕ_i such that

$$\phi_i : U_i \mapsto U_i' \subset \mathbb{H}^n, \tag{2.11}$$

where $U_i = \{(x, y) \in M : x \leq \eta, d_h(y, y_i) \leq \eta\}$ and U_i' is a neighbourhood of the origin $(0, 0)$ (using the upper half-space model) in the real hyperbolic space $\mathbb{H}^n$. The map ϕ_i induces a diffeomorphism Φ_i,

$$\Phi_i : O_i \mapsto O_i', \tag{2.12}$$

where O_i' is a subset of $(\mathbb{B}^n)_0^2$, the double space for $\mathbb{H}^n$. Let r be the geodesic distance on $\mathbb{H}^n$; then the kernel satisfies

$$|\phi_i \circ K(z, z')\chi_{O_i} \circ \phi_i^{-1}| \leq Ce^{-\rho_\delta r}. \tag{2.13}$$

We need the following lemma proved in [2, Lemma 5.1]:

Lemma 2.2. *Let $q \geq 2$,*

$$\|f * \kappa\|_{L^q(\mathbb{H}^n)} \leq C_q\|f\|_{L^{q'}(\mathbb{H}^n)} \left(\int_0^\infty (\sinh r)^{n-1}(1 + r)e^{-(n-1)r/2}|\kappa(r)|^{q/2}dr\right)^{2/q}. \tag{2.14}$$

Using this lemma with $\kappa(r) = e^{-\rho_\delta r}$ and the fact that

$$\begin{aligned} &\int_0^\infty (\sinh r)^{n-1}(1 + r)e^{-(n-1)r/2}|\kappa(r)|^{q/2}dr \\ &\leq \int_0^\infty (1 + r)e^{-\frac{n-1}{2}(\frac{q}{2}-1)r}e^{\frac{q\delta}{2}r}dr < \infty, \quad 0 < \delta < \delta_0, \end{aligned} \tag{2.15}$$

we obtain that the integral operator with kernel (2.13) is bounded from $L^{q'}(\mathbb{H}^n)$ into $L^q(\mathbb{H}^n)$. Therefore it shows that the integral operator with the kernel $K(z, z')\chi_O$ is bounded from $L^{q'}(U_i)$ to $L^q(U_i)$ since ϕ_i are bounded and invertible maps from $L^{q'}(U_i)$ to $L^{q'}(U_i')$. $\square$

3. Dispersive estimate and microlocalized Strichartz estimate

In this section, we define the microlocalized wave propagator and prove the microlocalized L^2-estimates and the dispersive estimates. As a final conclusion of this section, we prove microlocalized Strichartz estimates.

3.1. **Microlocalized wave propagator and L^2-estimates.** We first define the microlocalized wave propagator. Denote $U(t) = e^{it\sqrt{H}}$. For any $\sigma \in \mathbb{R}$, we define

$$ {}^{\sigma}U(t) = e^{it\sqrt{H}} H^{-\frac{\sigma}{2}}. \tag{3.1}$$

In the following application, in particular, we are interested in the cases $\sigma = 0$, $\sigma = 1/2$, and $\sigma = 1$. Choose $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(\lambda) = 1$ for $\lambda \le 1$ and vanishes when $\lambda \ge 2$. Then we define

$$\begin{aligned} {}^{\sigma}U^{\mathrm{low}}(t) = \int_0^\infty e^{it\lambda}\chi(\lambda)\lambda^{-\sigma} dE_{\sqrt{H}}(\lambda), \quad {}^{\sigma}U^{\mathrm{high}}(t) \\ = \int_0^\infty e^{it\lambda}(1-\chi)(\lambda)\lambda^{-\sigma} dE_{\sqrt{H}}(\lambda). \end{aligned} \tag{3.2}$$

Let $\varphi \in C_c^\infty([1/2, 2])$ and take values in $[0, 1]$ such that $1 = \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\lambda)$ for any $\lambda \neq 0$. We define

$$\begin{aligned} {}^{\sigma}U_j^{\mathrm{low}}(t) &= \int_0^\infty e^{it\lambda}\varphi(2^{-j}\lambda)\chi(\lambda)\lambda^{-\sigma} dE_{\sqrt{H}}(\lambda), \\ {}^{\sigma}U_j^{\mathrm{high}}(t) &= \int_0^\infty e^{it\lambda}\varphi(2^{-j}\lambda)(1-\chi)(\lambda)\lambda^{-\sigma} dE_{\sqrt{H}}(\lambda). \end{aligned} \tag{3.3}$$

For the high-energy operator partition of identity operator $Q_k(\lambda)$ in Proposition 2.1, we further define

$$ {}^{\sigma}U_{j,k}^{\mathrm{high}}(t) = \int_0^\infty e^{it\lambda}\varphi(2^{-j}\lambda)(1-\chi)(\lambda)\lambda^{-\sigma} Q_k(\lambda) dE_{\sqrt{H}}(\lambda),\ 0 \le k \le N. \tag{3.4}$$

The above definition of the operator is well-defined. Indeed, we have

Proposition 3.1 (L^2-estimates). *Let ${}^{\sigma}U_j^{\mathrm{low}}(t)$ and ${}^{\sigma}U_{j,k}^{\mathrm{high}}(t)$ be defined as in (3.3) and (3.4). Then there exists a constant C independent of t, j, k such that*

$$ \|{}^{\sigma}U_j^{\mathrm{low}}(t)\|_{L^2\to L^2} \le C2^{-\sigma j}, \quad \|{}^{\sigma}U_{j,k}^{\mathrm{high}}(t)\|_{L^2\to L^2} \le C2^{-\sigma j} \tag{3.5}$$

for all $k \ge 0, j \in \mathbb{Z}$.

Remark 3.1. The estimate of ${}^{\sigma}U_j^{\mathrm{low}}(t)$ will not be used in the following proofs. In the following argument, we only need estimates of ${}^{\sigma}U_{j,k}^{\mathrm{high}}(t)$ for the interpolation argument.

Proof. The proof essentially follows the argument in [13, 33] in which Hassell and the last author considered the cases of asymptotically conic manifolds. One also can find a modified version in [6] on the asymptotically hyperbolic setting. We here outline the proof for the convenience of the reader.

We first show that the above definition of the operator is well-defined. To this end, it suffices to show that the above integrals in the definitions are well-defined over any compact dyadic interval in $(0, \infty)$. Let $A(\lambda) = e^{it\lambda}\chi(\lambda)\varphi(2^{-j}\lambda)\lambda^{-\sigma}$ or $A(\lambda) = e^{it\lambda}\varphi(2^{-j}\lambda)(1-\chi)(\lambda)\lambda^{-\sigma}Q_k(\lambda)$. Then $A(\lambda)$ is compactly supported in $[a, b]$ with $a = 2^{j-1}$ and $b = 2^{j+1}$ and $\mathcal{C}^1$ in $\lambda \in (0, \infty)$. After integrating by parts, we see that the integral

$$\int_a^b A(\lambda) dE_{\sqrt{H}}(\lambda)$$

is given by

$$E_{\sqrt{H}}(b)A(b) - E_{\sqrt{H}}(a)A(a) - \int_a^b \frac{d}{d\lambda} A(\lambda) E_{\sqrt{H}}(\lambda)\, d\lambda. \tag{3.6}$$

From the construction of the pseudo-differential operator $Q_k(\lambda)$ in [8, Section 6.1], similarly as [13, Corollary 3.3], we can show that $Q_k(\lambda)$ and each operator $\lambda\partial_\lambda Q_k(\lambda)$ is bounded on $L^2(M^\circ)$ uniformly in λ. Then this means that the integrals are well-defined over any dyadic compact interval in $(0,\infty)$; hence the operators $U_j^{\text{low}}(t)$ and $U_{j,k}^{\text{high}}(t)$ are well-defined.

Next we show these operators are bounded on L^2. We only consider ${}^\sigma U_{j,k}^{\text{high}}(t)$ since the other is handled in the same way. We have by [13, Lemma 5.3],

$$\begin{aligned}
{}^\sigma U_{j,k}^{\text{high}}(t){}^\sigma U_{j,k}^{\text{high}}(t)^* &= \int (1-\chi)^2(\lambda)\varphi\big(\frac{\lambda}{2^j}\big)\varphi\big(\frac{\lambda}{2^j}\big)\lambda^{-2\sigma} Q_k(\lambda) dE_{\sqrt{H}}(\lambda) Q_k(\lambda)^* \\
&= -\int \frac{d}{d\lambda}\Big((1-\chi)^2(\lambda)\varphi\big(\frac{\lambda}{2^j}\big)\varphi\big(\frac{\lambda}{2^j}\big) Q_k(\lambda)\lambda^{-2\sigma}\Big) E_{\sqrt{H}}(\lambda) Q_k(\lambda)^* \\
&\quad - \int (1-\chi)^2(\lambda)\varphi\big(\frac{\lambda}{2^j}\big)\varphi\big(\frac{\lambda}{2^j}\big)\lambda^{-2\sigma} Q_k(\lambda) E_{\sqrt{H}}(\lambda) \frac{d}{d\lambda} Q_k(\lambda)^*.
\end{aligned} \tag{3.7}$$

On one hand, we note that this is independent of t and also recall that $Q_k(\lambda)$ and each operator $\lambda\partial_\lambda Q_k(\lambda)$ is bounded on $L^2(M^\circ)$ uniformly in λ. On the other hand, the integrand is a bounded operator on L^2, with an operator bound of the form $C\lambda^{-1-2\sigma}$ where C is uniform. By the support property of φ, the L^2 operator norm of the integral is uniformly bounded by $2^{-2j\sigma}$, as we are integrating over a dyadic interval in λ, and the proposition is proved. $\square$

3.2. **Dispersive estimates.** In this subsection, we prove the microlocalized dispersive estimates, which are the key estimates to derive the Strichartz estimates.

Proposition 3.2. *Let ${}^\sigma U_j^{\text{low}}(t)$ and ${}^\sigma U_{j,k}^{\text{high}}(t)$ be defined in (3.3) and (3.4). Let $\rho = (n-1)/2$. Then there exist constants C independent of t, j, k for all $j \in \mathbb{Z}$ such that*

- *For $j \geq 0$, $\sigma \geq 0$, and $|t-\tau| \leq 2$,*

$$\begin{aligned}
&\|{}^\sigma U_{j,k}^{\text{high}}(t)({}^\sigma U_{j,k}^{\text{high}}(\tau))^*\|_{L^1\to L^\infty} \\
&\qquad \leq C 2^{j[(n+1)/2-2\sigma]}(2^{-j} + |t-\tau|)^{-(n-1)/2}.
\end{aligned} \tag{3.8}$$

- *For $j \geq 0$, $\sigma \geq 0$ and $|t-\tau| \geq 2$,*

$$\begin{aligned}
&\|{}^\sigma U_{j,k}^{\text{high}}(t)({}^\sigma U_{j,k}^{\text{high}}(\tau))^*\|_{L^1\to L^\infty} \\
&\qquad \leq C 2^{j[(n+1)/2-2\sigma]}|t-\tau|^{-K} \quad \forall K \geq 0.
\end{aligned} \tag{3.9}$$

- *For $j \leq 0$, $0 \leq \sigma < 3/2$, and $0 \leq \epsilon \ll \min\{1, 3-2\sigma\}$,*

$$\|{}^\sigma U_j^{\text{low}}(t)({}^\sigma U_j^{\text{low}}(\tau))^*\|_{L^1\to L^\infty} \leq C 2^{\mp\epsilon j}(1+|t-\tau|)^{2\sigma-3\mp\epsilon}. \tag{3.10}$$

Proof. As before, we have by [13, Lemma 5.3]

$${}^\sigma U_{j,k}^{\text{high}}(t)({}^\sigma U_{j,k}^{\text{high}}(\tau))^* = \int_0^\infty e^{i(t-\tau)\lambda}(1-\chi)^2(\lambda)\varphi^2\big(\frac{\lambda}{2^j}\big)\lambda^{-2\sigma} Q_k(\lambda) dE_{\sqrt{H}}(\lambda) Q_k(\lambda)^* \tag{3.11}$$

and

$$({3.12})\qquad {}^{\sigma}U_j^{\mathrm{low}}(t)({}^{\sigma}U_j^{\mathrm{low}}(\tau))^* = \int_0^\infty e^{i(t-\tau)\lambda}\chi^2(\lambda)\varphi^2\big(\frac{\lambda}{2^j}\big)\lambda^{-2\sigma}dE_{\sqrt{H}}(\lambda).$$

Let $\phi(\lambda) = \varphi^2(\lambda)$. Then the proposition is a consequence of the following lemma about the microlocalized dispersive estimates.

Lemma 3.1 (Microlocalized dispersive estimates)**.** *Let $Q(\lambda)$ be the operator Q_k constructed as in Proposition 2.1 and suppose $\phi \in C_c^\infty([1/2,2])$ and takes value in $[0,1]$. Let $\rho = (n-1)/2$ and $0 < \delta \ll 1$. Then, for $j \geq 0$ and any $\sigma \geq 0$, there exist positive constant C independent of j and points $z, z' \in M^\circ$ such that*

- *when $|t| \leq 2$,*

(3.13)
$$\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)(1-\chi)^2(\lambda)\lambda^{-2\sigma}\big(Q(\lambda)E'_{\sqrt{H}}(\lambda)Q^*(\lambda)\big)(z,z')d\lambda\Big| \\ \leq C2^{j[(n+1)/2-2\sigma]}(2^{-j}+|t|)^{-(n-1)/2}e^{-(\rho-\delta)d(z,z')};$$

- *when $|t| \geq 2$,*

(3.14)
$$\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)(1-\chi)^2(\lambda)\lambda^{-2\sigma}\big(Q(\lambda)E'_{\sqrt{H}}(\lambda)Q^*(\lambda)\big)(z,z')d\lambda\Big| \\ \leq C2^{j[(n+1)/2-2\sigma]}|t|^{-K}e^{-(\rho-\delta)d(z,z')} \quad \forall K \geq 0;$$

and for $j \leq 0$, there exist constant C independent of j and points $z, z' \in M^\circ$ such that

(3.15)
$$\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\chi^2(\lambda)\lambda^{-2\sigma}E'_{\sqrt{H}}(\lambda;z,z')d\lambda\Big| \\ \leq C2^{\mp\epsilon j}(1+|t|)^{2\sigma-3\mp\epsilon}e^{-(\rho-\delta)d(z,z')},\ 0 \leq \sigma < 3/2,\ 0 \leq \epsilon \ll \min\{1, 3-2\sigma\}.$$

Note that $dE_{\sqrt{H}}(\lambda) = E'_{\sqrt{H}}(\lambda)d\lambda$; thus we have proved the result in Proposition 3.2 once we prove the lemma. □

Remark 3.2. In the proof of Proposition 3.2, the factor $e^{-(\rho-\delta)d(z,z')}$ is used as a bounded constant. This is enough to obtain the high-frequency estimate (3.31) in Proposition 3.3 below. However, the factor $e^{-(\rho-\delta)d(z,z')}$ is needed to obtain the low-frequency estimates (3.32) and (3.33).

The proof of Lemma 3.1. We shall rely on Proposition 2.1. We first prove (3.13) and (3.14), which are for the high frequencies. Using Proposition 2.1, it suffices to estimate

$$({3.16})\qquad \int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\lambda^{n-1-2\sigma}e^{\pm i\lambda d(z,z')}\tilde{a}_\pm(\lambda;z,z')d\lambda$$

and

$$({3.17})\qquad \int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\lambda^{-2\sigma}\tilde{b}(\lambda;z,z')d\lambda,$$

where $\tilde{a}_\pm = (1-\chi)^2(\lambda)a_\pm(\lambda;z,z')$ and $\tilde{b} = (1-\chi)^2(\lambda)b(\lambda;z,z')$ with $a_\pm$ and b satisfying (2.5) and (2.6). It is easy to verify that $\tilde{a}_\pm$ and $\tilde{b}$ have the same property as $a_\pm$ and b, respectively; that is, $\tilde{a}_\pm$ satisfies (2.5) and $\tilde{b}$ satisfies (2.6). Hence we briefly relabel $\tilde{a}_\pm$ to $a_\pm$ and $\tilde{b}$ to b without confusion from now on.

For any $K > 0$, we have by (2.6) in Proposition 2.1

$$\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\lambda^{-2\sigma}b(\lambda;z,z')d\lambda\Big| \le \int_0^\infty \phi(2^{-j}\lambda)\lambda^{-K-2\sigma}d\lambda\, e^{-(n-1)d(z,z')/2}$$
$$\le 2^{j(1-K-2\sigma)}e^{-(n-1)d(z,z')/2}.$$

We use (2.6) and N integrations by parts to obtain

$$\begin{aligned}
&\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\lambda^{-2\sigma}b(\lambda;z,z')d\lambda\Big|\\
&\le \Big|\int_0^\infty \big(\frac{1}{it}\frac{\partial}{\partial\lambda}\big)^N\big(e^{it\lambda}\big)\phi(2^{-j}\lambda)\lambda^{-2\sigma}b(\lambda;z,z')d\lambda\Big|\\
&\le C_N|t|^{-N}\int_{2^{j-1}}^{2^{j+1}}\lambda^{-K-N-2\sigma}d\lambda\, e^{-(n-1)d(z,z')/2}\\
&\le C_N|t|^{-N}2^{j(1-K-N-2\sigma)}e^{-(n-1)d(z,z')/2}.
\end{aligned}$$

Note that $j \ge 0$; therefore we obtain

$$\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\lambda^{-2\sigma}b(\lambda;z,z')d\lambda\Big| \le C_N(1+|t|)^{-N}2^{j(1-K-2\sigma)}e^{-(n-1)d(z,z')/2}, \tag{3.18}$$

which implies that (3.17) is bounded by the right hand side of (3.13) and (3.14).

Next we estimate (3.16). Due to the property of $a_\pm$, we divide it into two cases.

Case 1. $d(z,z') \le 1$. By using (2.5), we obtain

$$\begin{aligned}
&\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\lambda^{n-1-2\sigma}e^{\pm i\lambda d(z,z')}a_\pm(\lambda;z,z')d\lambda\Big|\\
&=\Big|\int_0^\infty \left(\frac{1}{i(t-d(z,z'))}\frac{\partial}{\partial\lambda}\right)^N\big(e^{i(t-d(z,z'))\lambda}\big)\phi(2^{-j}\lambda)\lambda^{n-1-2\sigma}a_\pm(\lambda;z,z')d\lambda\Big|\\
&\le C_N|t-d(z,z')|^{-N}\int_{2^{j-1}}^{2^{j+1}}\lambda^{n-1-2\sigma-N}(1+\lambda d(z,z'))^{-\frac{n-1}{2}}d\lambda\\
&\le C_N2^{j(n-2\sigma-N)}|t-d(z,z')|^{-N}(1+2^jd(z,z'))^{-(n-1)/2}.
\end{aligned}$$

Case 2. $d(z,z') \ge 1$. By using (2.5) again, we obtain

$$\begin{aligned}
&\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\lambda^{n-1-2\sigma}e^{\pm i\lambda d(z,z')}a_\pm(\lambda;z,z')d\lambda\Big|\\
&=\Big|\int_0^\infty \left(\frac{1}{i(t-d(z,z'))}\frac{\partial}{\partial\lambda}\right)^N\big(e^{i(t-d(z,z'))\lambda}\big)\phi(2^{-j}\lambda)\lambda^{n-1-2\sigma}a_\pm(\lambda;z,z')d\lambda\Big|\\
&\le C_N|t-d(z,z')|^{-N}e^{-(n-1)d(z,z')/2}\int_{2^{j-1}}^{2^{j+1}}\lambda^{n-1-2\sigma-N}\lambda^{-\frac{n-1}{2}}d\lambda\\
&\le C_N2^{j(n-2\sigma-N)}|t-d(z,z')|^{-N}2^{-j(n-1)/2}e^{-(n-1)d(z,z')/2}.
\end{aligned}$$

It follows that for $d(z,z')\le 1$,

$$\begin{aligned}&\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)(1-\chi)^2(\lambda)\lambda^{-2\sigma}\big(Q(\lambda)E'_{\sqrt{H}}(\lambda)Q^*(\lambda)\big)(z,z')d\lambda\Big|\\&\le C_N2^{j(n-2\sigma)}\big(1+2^j|t-d(z,z')|\big)^{-N}(1+2^jd(z,z'))^{-(n-1)/2}\end{aligned}\tag{3.19}$$

and for $d(z,z')\ge 1$,

$$\begin{aligned}&\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)(1-\chi)^2(\lambda)\lambda^{-2\sigma}\big(Q(\lambda)E'_{\sqrt{H}}(\lambda)Q^*(\lambda)\big)(z,z')d\lambda\Big|\\&\le C_N2^{j(n-2\sigma)}\big(1+2^j|t-d(z,z')|\big)^{-N}2^{-j(n-1)/2}e^{-(n-1)d(z,z')/2}.\end{aligned}\tag{3.20}$$

Consider the case $|t|\le 2$. We first consider the case, $d(z,z')\le 1$. If $|t|\sim d(z,z')$, it is clear to see (3.13). Otherwise, we have $|t-d(z,z')|\ge c|t|$ for some small constant c. Then choose $N=(n-1)/2$ to prove (3.13). For the second case, $d(z,z')\ge 1$, by using $j\ge 0$, it follows from the fact that $2^{-j}+|t|\lesssim 1$. Therefore we have proved (3.13).

Next we consider the case $|t|\ge 2$. We first consider the case $d(z,z')\le 1$. Since $|t|\ge 2$, we have $|t-d(z,z')|\ge \frac14|t|$. Then by (3.19) for any N,

$$\begin{aligned}\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)(1-\chi)^2(\lambda)\lambda^{-2\sigma}\big(Q(\lambda)E'_{\sqrt{H}}(\lambda)Q^*(\lambda)\big)(z,z')d\lambda\Big|&\\ \le C_N2^{j(n-2\sigma)}2^{-Nj}|t|^{-N}.&\end{aligned}$$

For the second case, $d(z,z')\ge 1$, if $|t|\sim d(z,z')$, it is clear to see for $0<\delta\ll 1$ that

$$\begin{aligned}&\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)(1-\chi)^2(\lambda)\lambda^{-2\sigma}\big(Q(\lambda)E'_{\sqrt{H}}(\lambda)Q^*(\lambda)\big)(z,z')d\lambda\Big|\\&\le C_N2^{j[(n+1)/2-2\sigma]}e^{-(n-1)d(z,z')/2}\le C_{N,\delta}2^{j[(n+1)/2-2\sigma]}|t|^{-N}e^{-(\rho-\delta)d(z,z')}.\end{aligned}$$

Otherwise, we have $|t-d(z,z')|\ge c|t|$ for some small constant c. Then by (3.20) for any N,

$$\begin{aligned}\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)(1-\chi)^2(\lambda)\lambda^{-2\sigma}\big(Q(\lambda)E'_{\sqrt{H}}(\lambda)Q^*(\lambda)\big)(z,z')d\lambda\Big|&\\ \le C_N2^{j[(n+1)/2-2\sigma]}2^{-Nj}|t|^{-N}e^{-(n-1)d(z,z')/2}.&\end{aligned}$$

By using the fact that $j\ge 0$, we have proved (3.14).

We next prove (3.15), which is for the low frequency, i.e. $j\le 0$, and for any $0\le\sigma<3/2$.

Case 1. $|t|\lesssim 1$. In this case, we know from (2.2) that

$$\begin{aligned}&\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\lambda^{-2\sigma}\chi^2(\lambda)E'_{\sqrt{H}}(\lambda;z,z')d\lambda\Big|\\&\qquad\le C\int_{2^{j-1}}^{2^{j+1}}\phi(2^{-j}\lambda)\lambda^{2-2\sigma}(1+d(z,z'))e^{-(n-1)d(z,z')/2}d\lambda\\&\qquad\le C2^{j(3-2\sigma)}(1+d(z,z'))e^{-(n-1)d(z,z')/2},\end{aligned}\tag{3.21}$$

which implies (3.15) when $|t|\lesssim 1$.

Case 2. $|t| \gg 1$. In this case, we further consider two subcases.

Subcase 1. $|t| \le 2d(z,z')$. In this subcase, arguing as above, we obtain

$$\begin{aligned}
&\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\lambda^{-2\sigma}\chi^2(\lambda)E'_{\sqrt{H}}(\lambda;z,z')d\lambda\Big| \\
&\qquad \le C\int_{2^{j-1}}^{2^{j+1}} \phi(2^{-j}\lambda)\lambda^{2-2\sigma}(1+d(z,z'))e^{-(n-1)d(z,z')/2}d\lambda \\
&\qquad \le C2^{j(3-2\sigma)}(1+d(z,z'))e^{-(n-1)d(z,z')/2} \\
&\qquad \le C2^{j(3-2\sigma)}|t|^{-N}e^{-(\rho-\delta)d(z,z')}
\end{aligned} \tag{3.22}$$

for any arbitrary large $N>0$ and $0<\delta \ll 1$.

Subcase 2. $|t| \ge 2d(z,z'), |t| \gg 1$. To show (3.15), it suffices to show that, for $0<\delta \ll 1$,

$$\Big|\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\lambda^{-2\sigma}\chi^2(\lambda)E'_{\sqrt{H}}(\lambda;z,z')d\lambda\Big| \lesssim 2^{\mp\epsilon j}|t|^{2\sigma-3\mp\epsilon}e^{-(\rho-\delta)d(z,z')}. \tag{3.23}$$

To this end, let $\bar\lambda = \lambda/t$ and recall that $\sum_k \varphi(2^{-k}\lambda)=1$. We write

$$\begin{aligned}
&\int_0^\infty e^{it\lambda}\phi(2^{-j}\lambda)\lambda^{-2\sigma}\chi^2(\lambda)E'_{\sqrt{H}}(\lambda;z,z')d\lambda \\
&= t^{2\sigma-1}\int_0^\infty e^{i\lambda}\lambda^{-2\sigma}\phi(2^{-j}\bar\lambda)\chi^2(\bar\lambda)E'_{\sqrt{H}}(\bar\lambda;z,z')d\lambda \\
&= t^{2\sigma-1}\sum_{k\in\mathbb{Z}}\int_0^\infty e^{i\lambda}\lambda^{-2\sigma}\varphi(2^{-k}\lambda)\phi(2^{-j}\bar\lambda)\chi^2(\bar\lambda)E'_{\sqrt{H}}(\bar\lambda;z,z')d\lambda.
\end{aligned} \tag{3.24}$$

Define

$$\begin{aligned}
I &= t^{2\sigma-1}\sum_{k\le 0}\int_0^\infty e^{i\lambda}\lambda^{-2\sigma}\varphi(2^{-k}\lambda)\phi(2^{-j}\bar\lambda)\chi^2(\bar\lambda)E'_{\sqrt{H}}(\bar\lambda;z,z')d\lambda; \\
II &= t^{2\sigma-1}\sum_{k\ge 1}\int_0^\infty e^{i\lambda}\lambda^{-2\sigma}\varphi(2^{-k}\lambda)\phi(2^{-j}\bar\lambda)\chi^2(\bar\lambda)E'_{\sqrt{H}}(\bar\lambda;z,z')d\lambda.
\end{aligned} \tag{3.25}$$

Recall that $\bar\lambda = \lambda/t$, by (2.2) and $\lambda/t \sim 2^j$. Then we have

$$\begin{aligned}
|I| &= |t|^{2\sigma-1}\left|\sum_{k\le 0}\int_0^\infty e^{i\lambda}\lambda^{-2\sigma}\varphi(2^{-k}\lambda)\phi(2^{-j}\bar\lambda)\chi^2(\bar\lambda)E'_{\sqrt{H}}(\bar\lambda;z,z')d\lambda\right| \\
&\le t^{2\sigma-1}2^{\mp j\epsilon}\sum_{k\le 0}\int_{2^k}^{2^{k+1}} \lambda^{-2\sigma}(t^{-1}\lambda)^{2\pm\epsilon}(1+d(z,z'))e^{-(n-1)d(z,z')/2}d\lambda \\
&\lesssim 2^{\mp j\epsilon}t^{2\sigma-3\mp\epsilon}(1+d(z,z'))e^{-(n-1)d(z,z')/2}, \quad 0\le\sigma<3/2, 0\le\epsilon\ll\min\{1,3-2\sigma\},
\end{aligned} \tag{3.26}$$

which gives (3.23). By (2.1) in Proposition 2.1, we have

$$
\begin{aligned}
II &= t^{2\sigma-1}\sum_{k\geq 1}\int_0^\infty e^{i\lambda}\lambda^{-2\sigma}\varphi(2^{-k}\lambda)\phi(2^{-j}\bar\lambda)\chi^2(\bar\lambda)E'_{\sqrt{H}}(\bar\lambda;z,z')d\lambda \\
&= 2^{\mp\epsilon j}t^{2\sigma-1}(\rho_L\rho_R)^{\frac{n-1}{2}}\sum_{k\geq 1}\int_0^\infty e^{i\lambda}(t^{-1}\lambda)^{1\pm\epsilon}\lambda^{-2\sigma}\varphi(2^{-k}\lambda)\phi(2^{-j}\bar\lambda)\chi^2(\bar\lambda) \\
&\quad\times\Big((\rho_L\rho_R)^{i\bar\lambda}a(\bar\lambda;z,z')-(\rho_L\rho_R)^{-i\bar\lambda}a(-\bar\lambda;z,z')\Big)d\lambda.
\end{aligned}
\tag{3.27}
$$

By integration by parts, we estimate that

$$
\begin{aligned}
II &\lesssim t^{2\sigma-2\mp\epsilon}2^{\mp j\epsilon}(\rho_L\rho_R)^{\frac{n-1}{2}}\sum_{k\geq 1}\int_0^\infty\big(\frac{d}{d\lambda}\big)^4\Big(\lambda^{1\pm\epsilon-2\sigma}\varphi(2^{-k}\lambda)\phi(2^{-j}\bar\lambda)\chi^2(\bar\lambda) \\
&\quad\times\Big((\rho_L\rho_R)^{i\bar\lambda}a(\bar\lambda;z,z')-(\rho_L\rho_R)^{-i\bar\lambda}a(-\bar\lambda;z,z')\Big)\Big)d\lambda.
\end{aligned}
\tag{3.28}
$$

If none of the derivatives hit the term $\Big((\rho_L\rho_R)^{i\bar\lambda}a(\bar\lambda;z,z')-(\rho_L\rho_R)^{-i\bar\lambda}a(-\bar\lambda;z,z')\Big)$, since $|\bar\lambda|=|\lambda/t|\leq 1$, then we use the smoothness of a at 0 to obtain

$$
\Big((\rho_L\rho_R)^{i\bar\lambda}a(\bar\lambda;z,z')-(\rho_L\rho_R)^{-i\bar\lambda}a(-\bar\lambda;z,z')\Big)\lesssim\bar\lambda\lesssim\lambda t^{-1}.
$$

If the derivatives hit the other terms we gain λ^{-4}. In this case, note that $0\leq\sigma<3/2$ and $0\leq\epsilon\ll 1$, and we show that

$$
|II_1|\lesssim t^{2\sigma-3\mp\epsilon}2^{\mp j\epsilon}(\rho_L\rho_R)^{\frac{n-1}{2}}\sum_{k\geq 1}\int_{2^k}^{2^{k+1}}\lambda^{-2\mp\epsilon}d\lambda\lesssim t^{2\sigma-3\mp\epsilon}2^{\mp j\epsilon}(\rho_L\rho_R)^{\frac{n-1}{2}}.
\tag{3.29}
$$

If at least one derivative hits the term $\Big((\rho_L\rho_R)^{i\bar\lambda}a(\bar\lambda;z,z')-(\rho_L\rho_R)^{-i\bar\lambda}a(-\bar\lambda;z,z')\Big)$, since $a\in\mathcal{C}^\infty$, we gain t^{-1} at least. Note that $\lambda/t\lesssim 1$, and we gain in total $\lambda^{-3}t^{-1}$; then

$$
\begin{aligned}
|II_2| &\lesssim t^{2\sigma-3\mp\epsilon}2^{\mp j\epsilon}(\rho_L\rho_R)^{\frac{n-1}{2}}(\ln(\rho_L\rho_R))^4\sum_{k\geq 1}\int_{2^k}^{2^{k+1}}\lambda^{-2\pm\epsilon}d\lambda \\
&\lesssim t^{2\sigma-3\mp\epsilon}2^{\mp j\epsilon}(\rho_L\rho_R)^{\frac{n-1}{2}}(\ln(\rho_L\rho_R))^4.
\end{aligned}
\tag{3.30}
$$

From [8, Proposition 3.4], we have

$$
(\rho_L\rho_R)^{\frac{n-1}{2}}(\ln(\rho_L\rho_R))^4\leq(1+d(z,z'))^4e^{-(n-1)d(z,z')/2}.
$$

Therefore we prove (3.23); hence we have (3.15). The proof of Lemma 3.1 is then complete. □

3.3. **Microlocalized Strichartz estimate.** In this subsection, we use the L^2-estimate and dispersive estimate for the microlocalized wave propagator to obtain the microlocalized Strichartz estimate.

Proposition 3.3. *Let ${}^\sigma U_j^{\mathrm{low}}(t)$ and ${}^\sigma U_{j,k}^{\mathrm{high}}(t)$ be defined in (3.3) and (3.4) and let $n \geq 3$. Then for every pair $(q,r) \in [2,\infty] \times (2,\infty]$, there exists a constant C depending only on n, q, and r such that:*

- *For $j \geq 0$ and $\sigma \geq 0$,*

$$\Big(\int_{\mathbb{R}} \| {}^\sigma U_{j,k}^{\mathrm{high}}(t) f\|_{L^r}^q dt \Big)^{\frac{1}{q}} \leq C(1+j) 2^{j(s-\sigma)} \|f\|_{L^2}, \tag{3.31}$$

where $s = s_e$ as in (1.13) when $2/q \geq (n-1)(1/2 - 1/r)$ and $s = s_w$ defined in (1.11) when $2/q \leq (n-1)(1/2-1/r)$.

- *For $j \leq 0$, if $0 \leq \sigma < 1$,*

$$\Big(\int_{\mathbb{R}} \| {}^\sigma U_{j}^{\mathrm{low}}(t) f\|_{L^r}^q dt \Big)^{\frac{1}{q}} \leq C 2^{\epsilon j} \|f\|_{L^2} \quad \forall 0 \leq \epsilon \ll 1, \quad \epsilon < 1 - \sigma. \tag{3.32}$$

- *For $j \leq 0$, $\sigma = 1$, and $q \geq 2$,*

$$\Big(\int_{\mathbb{R}} \| {}^\sigma U_{j}^{\mathrm{low}}(t) f\|_{L^r}^q dt \Big)^{\frac{1}{q}} \leq C 2^{-j\epsilon} \|f\|_{L^2} \quad \forall 0 < \epsilon \ll 1. \tag{3.33}$$

In addition if $q \neq 2$, one can choose $\epsilon = 0$.

Remark 3.3. The log regularity j in (3.31) appears on the line $\frac{2}{q} = (n-1)(\frac{1}{2} - \frac{1}{r})$. This loss can be removed using Keel-Tao's argument [16, Sections 3-7], but we do not pursue here sharp regularity.

Proof of Proposition 3.3. We closely follow Keel-Tao's argument [16, Sections 3-7]. By the TT^* argument, it suffices to show that

$$\begin{aligned} &\Big| \iint \langle ({}^\sigma U_{j,k}^{\mathrm{high}}(\tau))^* F(\tau), ({}^\sigma U_{j,k}^{\mathrm{high}}(t))^* G(t) \rangle d\tau dt \Big| \\ &\qquad \lesssim 2^{2j(s-\sigma)}(1+j)^2 \|F\|_{L_t^{q'} L^{r'}} \|G\|_{L_t^{q'} L^{r'}} \end{aligned} \tag{3.34}$$

and

$$\Big| \iint \langle ({}^\sigma U_{j}^{\mathrm{low}}(\tau))^* F(\tau), ({}^\sigma U_{j}^{\mathrm{low}}(t))^* G(t) \rangle d\tau dt \Big| \lesssim C \Lambda(j)^2 \|F\|_{L_t^{q'} L^{r'}} \|G\|_{L_t^{q'} L^{r'}}, \tag{3.35}$$

where $\Lambda(j) = 2^{\epsilon j}$ when $0 \leq \sigma < 1$ with $0 \leq \epsilon \ll 1$ and $\Lambda(j) = 2^{-\epsilon j}$ when $\sigma = 1$ with $0 < \epsilon \ll 1$. In particular, if $\sigma = 1$ and $q \neq 2$ one can choose $\epsilon = 0$.

To this end, we consider four cases.

Case 1. $j \geq 0$ and $|t - \tau| \leq 2$. By the interpolation of the bilinear form of (3.8) and the energy estimate in Proposition 3.1, we have

$$\begin{aligned} &\langle ({}^\sigma U_{j,k}^{\mathrm{high}}(\tau))^* F(\tau), ({}^\sigma U_{j,k}^{\mathrm{high}}(t))^* G(t) \rangle \\ &\leq C 2^{j[(n+1)(\frac{1}{2} - \frac{1}{r}) - 2\sigma]} (2^{-j} + |t-\tau|)^{-(n-1)(\frac{1}{2} - \frac{1}{r})} \|F\|_{L^{r'}} \|G\|_{L^{r'}}. \end{aligned}$$

Therefore we obtain by Hölder's and Young's inequalities

$$\begin{aligned} &\Big| \iint \langle ({}^\sigma U_{j,k}^{\mathrm{high}}(\tau))^* F(\tau), ({}^\sigma U_{j,k}^{\mathrm{high}}(t))^* G(t) \rangle d\tau dt \Big| \\ &\lesssim 2^{j[(n+1)(\frac{1}{2} - \frac{1}{r}) - 2\sigma]} \iint_{|t-\tau| \leq 2} (2^{-j} + |t-\tau|)^{-(n-1)(\frac{1}{2} - \frac{1}{r})} \|F(\tau)\|_{L^{r'}} \|G(t)\|_{L^{r'}} dt d\tau \\ &\lesssim 2^{2j(s_e - \sigma)} \max\{ 2^{j[(n-1)(\frac{1}{2} - \frac{1}{r}) - \frac{2}{q}]}, 1\} \|F\|_{L_t^{q'} L^{r'}} \|G\|_{L_t^{q'} L^{r'}} \end{aligned}$$

when $\frac{2}{q} \neq (n-1)(\frac{1}{2}-\frac{1}{r})$. If $\frac{2}{q} = (n-1)(\frac{1}{2}-\frac{1}{r})$, we similarly have

$$\begin{aligned}&\Big|\iint \langle ({}^{\sigma}U_{j,k}^{\mathrm{high}}(\tau))^* F(\tau), ({}^{\sigma}U_{j,k}^{\mathrm{high}}(t))^* G(t)\rangle d\tau dt\Big| \\ &\lesssim (1+j)2^{2j(s_e-\sigma)}\|F\|_{L_t^{q'}L^{r'}}\|G\|_{L_t^{q'}L^{r'}}.\end{aligned}$$

Case 2. $j \geq 0$ and $|t-\tau| \geq 2$. Similarly, by the interpolation of the bilinear form of (3.9) and the energy estimate in Proposition 3.1, we have

$$\begin{aligned}&\langle ({}^{\sigma}U_{j,k}^{\mathrm{high}}(\tau))^* F(\tau), ({}^{\sigma}U_{j,k}^{\mathrm{high}}(t))^* G(t)\rangle \\ &\qquad \leq C2^{j[(n+1)(\frac{1}{2}-\frac{1}{r})-2\sigma]}|t-\tau|^{-2N(\frac{1}{2}-\frac{1}{r})}\|F\|_{L^{r'}}\|G\|_{L^{r'}}.\end{aligned}$$

Therefore, by using Hölder's and Young's inequalities and choosing N enough, we obtain

$$\begin{aligned}&\Big|\iint \langle ({}^{\sigma}U_{j,k}^{\mathrm{high}}(\tau))^* F(\tau), ({}^{\sigma}U_{j,k}^{\mathrm{high}}(t))^* G(t)\rangle ds dt\Big| \\ &\lesssim 2^{j[(n+1)(\frac{1}{2}-\frac{1}{r})-2\sigma]}\iint_{|t-\tau|\geq 2}|t-\tau|^{-2N(\frac{1}{2}-\frac{1}{r})}\|F(\tau)\|_{L^{r'}}\|G(t)\|_{L^{r'}}dt d\tau \\ &\lesssim 2^{2j(s_e-\sigma)}\|F\|_{L_t^{q'}L^{r'}}\|G\|_{L_t^{q'}L^{r'}}.\end{aligned}$$

By the definition of s, we collect the two cases to prove (3.31).

Case 3. $j \leq 0$ and $0 \leq \sigma < 1$. By using (3.15) with positive sign and small δ satisfying $0 < \delta < \delta_0(r)$ as in Lemma 2.1, we use (2.8) to obtain

$$\begin{aligned}&\langle ({}^{\sigma}U_j^{\mathrm{low}}(\tau))^* F(\tau), ({}^{\sigma}U_j^{\mathrm{low}}(t))^* G(t)\rangle \\ &\quad \leq C\|{}^{\sigma}U_j^{\mathrm{low}}(t)({}^{\sigma}U_j^{\mathrm{low}}(\tau))^* F\|_{L^r}\|G(t)\|_{L^{r'}} \\ &\quad \leq C2^{2\epsilon j}(1+|t-\tau|)^{2\sigma-3+2\epsilon}\Big\|\int e^{-(\rho-\delta)d(z,z')}F dg(z')\Big\|_{L^r}\|G(t)\|_{L^{r'}} \\ &\quad \leq C2^{2\epsilon j}(1+|t-\tau|)^{2\sigma-3+2\epsilon}\|F(\tau)\|_{L^{r'}}\|G(t)\|_{L^{r'}}.\end{aligned}$$

Note that if $0 \leq \sigma < 1$, for $q \geq 2$, it gives $2/q < 3-2\sigma-2\epsilon$ when $0 \leq \epsilon \ll 1-\sigma$. Therefore, by using Hölder's and Young's inequalities, we obtain for $q \geq 2$:

$$\begin{aligned}&\Big|\iint \langle ({}^{\sigma}U_j^{\mathrm{low}}(\tau))^* F(\tau), ({}^{\sigma}U_j^{\mathrm{low}}(t))^* G(t)\rangle d\tau dt\Big| \\ &\lesssim 2^{2\epsilon j}\iint (1+|t-\tau|)^{2\sigma-3+2\epsilon}\|F(t)\|_{L^{r'}}\|G(\tau)\|_{L^{r'}}dt d\tau \\ &\lesssim 2^{2\epsilon j}\|F\|_{L_t^{q'}L^{r'}}\|G\|_{L_t^{q'}L^{r'}}.\end{aligned}$$

This proves (3.35).

Case 4. $j \leq 0$, $\sigma = 1$ and $q \geq 2$. By using (3.15) with negative sign and a similar argument as above, we have

$$\begin{aligned}&\langle ({}^{\sigma}U_j^{\mathrm{low}}(\tau))^* F(\tau), ({}^{\sigma}U_j^{\mathrm{low}}(t))^* G(t)\rangle \\ &\quad \leq C\|{}^{\sigma}U_j^{\mathrm{low}}(t)({}^{\sigma}U_j^{\mathrm{low}}(\tau))^* F\|_{L^r}\|G(t)\|_{L^{r'}} \\ &\quad \leq C2^{-2j\epsilon}(1+|t-\tau|)^{-1-2\epsilon}\Big\|\int e^{-(\rho-\delta)d(z,z')}F dg(z')\Big\|_{L^r}\|G(t)\|_{L^{r'}} \\ &\quad \leq C2^{-2j\epsilon}(1+|t-\tau|)^{-1-2\epsilon}\|F(\tau)\|_{L^{r'}}\|G(t)\|_{L^{r'}}.\end{aligned}$$

This proves (3.35). In particular if $q > 2$, it is clear that one can choose $\epsilon = 0$. $\square$

4. Homogeneous Strichartz estimates

In this section, we prove Theorem 1.1 by using the microlocalized Strichartz estimate in Proposition 3.3. Recall that $H = -\Delta_g - \rho^2$ and let u be the solution of

$$\partial_t^2 u + Hu = 0, \quad u(0) = u_0(z), \quad \partial_t u(0) = u_1(z). \tag{4.1}$$

Then we have

$$u(t,z) = \frac{U(t)+U(-t)}{2}u_0 + \frac{U(t)-U(-t)}{2i\sqrt{H}}u_1,$$

where $U(t) = e^{it\sqrt{H}}$. By recalling that ${}^\sigma U(t) = e^{it\sqrt{H}}H^{-\frac{\sigma}{2}}$ and using (3.2), we aim to estimate

$$\begin{aligned}&\|u\|_{L^q(\mathbb{R};L^r(M^\circ))}\\ &\lesssim \sum_{\pm}\sum_{\sigma\in\{0,1\}}\left(\|{}^\sigma U^{\mathrm{low}}(\pm t)u_\sigma\|_{L^q(\mathbb{R};L^r(M^\circ))} + \|{}^\sigma U^{\mathrm{high}}(\pm t)u_\sigma\|_{L^q(\mathbb{R};L^r(M^\circ))}\right).\end{aligned} \tag{4.2}$$

To prove (1.15) in Theorem 1.1, it is enough to prove that

$$\|{}^\alpha U^{\mathrm{high}}(t)f\|_{L^q_t(\mathbb{R}:L^r(M^\circ))} \lesssim \|f\|_{L^2(M^\circ)}, \tag{4.3}$$

with $\alpha = \mu$, and

$$\|{}^\beta U^{\mathrm{low}}(t)f\|_{L^q_t(\mathbb{R}:L^r(M^\circ))} \lesssim \|f\|_{L^2(M^\circ)}, \tag{4.4}$$

where β equals 0 or $1-\epsilon$ with $0<\epsilon\ll 1$. Recall ${}^\sigma U_j^{\mathrm{low}}(t)$ and ${}^\sigma U_{j,k}^{\mathrm{high}}(t)$ defined in (3.3) and (3.4). Then we have

$${}^\sigma U^{\mathrm{high}}(t)f = \sum_{j\geq 0}\sum_{k=0}^{N} {}^\sigma U_{j,k}^{\mathrm{high}}(t)f$$

and

$${}^\sigma U^{\mathrm{low}}(t)f = \sum_{j\leq 0} {}^\sigma U_j^{\mathrm{low}} f.$$

By using Proposition 3.3 with $\sigma = \alpha = \mu$, we obtain for $j\geq 0$, $0\leq k\leq N$,

$$\|{}^\alpha U_{j,k}^{\mathrm{high}}(t)f\|_{L^q_t(\mathbb{R}:L^r(M^\circ))} \lesssim 2^{j(s-\mu)}\|f\|_{L^2(M^\circ)}, \quad s = s_e,\ s_w.$$

Note that $\mu > s$, and by taking summation in $j\geq 0$ and finite k, we prove (4.3).

If $\beta = 0$, by using Proposition 3.3 with $\sigma = 0$, we obtain for $j\leq 0$,

$$\|{}^\beta U_j^{\mathrm{low}}(t)f\|_{L^q_t(\mathbb{R}:L^r(M^\circ))} \lesssim 2^{\epsilon j}\|f\|_{L^2(M^\circ)},$$

and if $\beta = 1-\epsilon$ with $0<\epsilon\ll 1$, choose $0<\tilde{\epsilon}<\epsilon = 1-\beta$. We use (3.32) in Proposition 3.3 with $\sigma = \beta = 1-\epsilon$ to obtain

$$\|{}^\beta U_j^{\mathrm{low}}(t)f\|_{L^q_t(\mathbb{R}:L^r(M^\circ))} \lesssim 2^{\tilde{\epsilon} j}\|f\|_{L^2(M^\circ)}.$$

By summing in $j\leq 0$, we obtain (4.4) with $\beta = 0$ and $1-\epsilon$. Hence we have proved (1.15) in Theorem 1.1.

5. Inhomogeneous Strichartz estimates

In this section, we prove the inhomogeneous Strichartz estimate in Theorem 1.2. To this purpose, we divide into two cases. The first case that $q > \tilde{q}'$ is much easier to prove due to the Christ-Kiselev lemma [9]. The second case when $q = \tilde{q} = 2$ is more complicated since the usual dispersive estimate fails due to the conjugate points. We call the inhomogeneous Strichartz estimate a double endpoint estimate when $q = \tilde{q} = 2$; otherwise we call it a non-double endpoint inhomogeneous Strichartz estimate.

5.1. **Inhomogeneous Strichartz estimates for non-double endpoint.**

In this subsection, we prove

Proposition 5.1. *Let $(q, r, \mu), (\tilde{q}, \tilde{r}, \tilde{\mu}) \in \Lambda_w \cup \Lambda_e$ and suppose at least one of $q, \tilde{q}$ does not equal 2. Then the following inequalities hold:*

- *Low-frequency estimate*

$$\Big\| \int_{\tau<t} \frac{\sin{(t-\tau)\sqrt{H}}}{\sqrt{H}} \chi(\sqrt{H}) F(\tau) d\tau \Big\|_{L^q_t L^r_z} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_z}; \tag{5.1}$$

- *High-frequency estimate*

$$\Big\| \int_{\tau<t} \frac{\sin{(t-\tau)\sqrt{H}}}{\sqrt{H}} (1-\chi)(\sqrt{H}) F(\tau) d\tau \Big\|_{L^q_t L^r_z} \lesssim \|H^{\frac{\mu+\tilde{\mu}-1}{2}} F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_z}, \tag{5.2}$$

where $\chi \in \mathcal{C}_c^\infty([0,\infty)$ such that $\chi(\lambda) = 1$ for $\lambda \leq 1$ and vanishes when $\lambda \geq 2$.

Remark 5.1. We can obtain a special inhomogeneous Strichartz estimate that we shall require in the next section. For $p \in (1, 1+4/(n-1))$, we have

$$\Big\| \int_{\tau<t} \frac{\sin{(t-\tau)\sqrt{H}}}{\sqrt{H}} F(\tau) d\tau \Big\|_{L^{p+1}_t L^{p+1}_z} \lesssim \|F\|_{L_t^{\frac{p+1}{p}} L_z^{\frac{p+1}{p}}}. \tag{5.3}$$

Indeed, the low-frequency part follows from (5.1). Choose $\mu = \tilde{\mu} = 1/2$. Then we can check that

$$(p+1, p+1, 1/2) \in \Lambda_e, \quad p+1 > 2,$$

when $p \in (1, 1+4/(n-1))$. Hence the high-frequency part follows from (5.2).

Proof of Proposition 5.1. We first prove (5.1). Recall that $U(t) = e^{it\sqrt{H}}$. Then

$$\begin{aligned} \frac{\sin{(t-\tau)\sqrt{H}}}{\sqrt{H}} \chi(\sqrt{H}) &= H^{-\frac{1}{2}} \chi(\sqrt{H}) (U(t)U(\tau)^* - U(-t)U(-\tau)^*)/2i \\ &= \frac{1}{2i}\big({}^\sigma U^{\mathrm{low}}(t)({}^\sigma U^{\mathrm{low}}(\tau))^* - {}^\sigma U^{\mathrm{low}}(-t)({}^\sigma U^{\mathrm{low}}(-\tau))^*\big), \ \sigma = 1/2, \end{aligned}$$

where

$${}^\sigma U^{\mathrm{low}}(t) = \int_0^\infty e^{it\lambda} \chi^{1/2}(\lambda) \lambda^{-\sigma} dE_{\sqrt{H}}(\lambda). \tag{5.4}$$

This is just the analogue of (3.2) with $\chi(\lambda)$ there replaced by $\chi^{1/2}(\lambda)$, which causes no problems. Since the other term can be treated similarly, it suffices to show that

$$\int_{\tau<t} {}^\sigma U^{\mathrm{low}}(t) ({}^\sigma U^{\mathrm{low}}(\tau))^* F(\tau) d\tau, \quad \sigma = 1/2, \tag{5.5}$$

satisfies the bounds in (5.1). As before, by using Proposition 3.3 with $\sigma = 1/2$, we obtain for $j \le 0$,

$$\|{}^{\sigma}U_j^{\text{low}}(t)f\|_{L_t^q(\mathbb{R}:L^r(M^\circ))} \lesssim 2^{\epsilon j}\|f\|_{L^2(M^\circ)}, \quad 0 \le \epsilon \ll 1,$$

and hence we further have

$$\|{}^{\sigma}U^{\text{low}}(t)f\|_{L_t^q(\mathbb{R}:L^r(M^\circ))} \lesssim \sum_{j\le 0}\|{}^{\sigma}U_j^{\text{low}}(t)f\|_{L_t^q(\mathbb{R}:L^r(M^\circ))} \lesssim \|f\|_{L^2(M^\circ)}.$$

By the duality, we have the following:

$$\Big\|\int_{\mathbb{R}} {}^{\sigma}U^{\text{low}}(t)({}^{\sigma}U^{\text{low}}(\tau))^*F(\tau)d\tau\Big\|_{L_t^q(\mathbb{R}:L^r(M^\circ))} \lesssim \|F\|_{L_t^{\tilde{q}'}(\mathbb{R}:L^{\tilde{r}'}(M^\circ))}.$$

Under the assumption that at least one of $q, \tilde{q}$ is not 2, we have $q > \tilde{q}'$. Hence by using the Christ-Kiselev lemma [9], we obtain

$$\Big\|\int_{\tau<t} {}^{\sigma}U^{\text{low}}(t)({}^{\sigma}U^{\text{low}}(\tau))^*F(\tau)d\tau\Big\|_{L_t^q(\mathbb{R}:L^r(M^\circ))} \lesssim \|F\|_{L_t^{\tilde{q}'}(\mathbb{R}:L^{\tilde{r}'}(M^\circ))}.$$

Therefore we have shown that (5.5) satisfies the bounds in (5.1), as desired.

Next we prove (5.2). Similarly as above, we write

$$\begin{aligned}
&\frac{\sin{(t-\tau)\sqrt{H}}}{\sqrt{H}}H^{-\frac{\mu+\tilde{\mu}-1}{2}}(1-\chi)(\sqrt{H})\\
&= H^{-\frac{\mu+\tilde{\mu}}{2}}(1-\chi)(\sqrt{H})(U(t)U(\tau)^* - U(-t)U(-\tau)^*)/2i\\
&= \frac{1}{2i}\big({}^{\mu}U^{\text{high}}(t)({}^{\tilde{\mu}}U^{\text{high}}(\tau))^* - {}^{\mu}U^{\text{high}}(-t)({}^{\tilde{\mu}}U^{\text{high}}(-\tau))^*\big),
\end{aligned}$$

where

$$ {}^{\sigma}U^{\text{high}}(t) = \int_0^\infty e^{it\lambda}(1-\chi)^{1/2}(\lambda)\lambda^{-\sigma}dE_{\sqrt{H}}(\lambda). \tag{5.6}$$

Here we have replaced $(1-\chi)(\lambda)$ in (3.2) by $(1-\chi)^{1/2}(\lambda)$, which is inconsequential. To prove (5.2), it suffices to show that

$$\Big\|\int_{\tau<t} {}^{\mu}U^{\text{high}}(t)({}^{\tilde{\mu}}U^{\text{high}}(\tau))^*F(\tau)d\tau\Big\|_{L_t^qL_z^r} \lesssim \|F\|_{L_t^{\tilde{q}'}L_z^{\tilde{r}'}}. \tag{5.7}$$

Applying Proposition 3.3 with $\sigma = \mu$ and its dual version with $\sigma = \tilde{\mu}$, we have for all $j \ge 0$ and $k = 0, \ldots, N$,

$$\big\|{}^{\mu}U_{j,k}^{\text{high}}(t)f\big\|_{L_t^q(\mathbb{R}:L^r(M^\circ))} \lesssim 2^{j(s-\mu)}\|f\|_{L^2(M^\circ)}, \quad s = s_e, s_w,$$

and

$$\Big\|\int_{\mathbb{R}}({}^{\tilde{\mu}}U_{j,k}^{\text{high}}(\tau))^*F(\tau)d\tau\Big\|_{L^2(M^\circ)} \lesssim 2^{j(s-\tilde{\mu})}\|F\|_{L_t^{\tilde{q}'}L_z^{\tilde{r}'}}, \quad s = s_e, s_w.$$

Therefore we obtain, for all $k, k' \in \{0, \ldots, N\}$ and $j, j' \ge 0$,

$$\Big\|\int_{\mathbb{R}} {}^{\mu}U_{j,k}^{\text{high}}(t)({}^{\tilde{\mu}}U_{j',k'}^{\text{high}}(\tau))^*F(\tau)d\tau\Big\|_{L_t^q(\mathbb{R}:L^r(M^\circ))} \lesssim 2^{j(s-\mu)}2^{j'(s-\tilde{\mu})}\|F\|_{L_t^{\tilde{q}'}(\mathbb{R}:L^{\tilde{r}'}(M^\circ))}.$$

Let

$$ {}^{\sigma}U_{\ge,k}^{\text{high}}(t) = \sum_{j\ge 0}{}^{\sigma}U_{j,k}^{\text{high}}(t), \tag{5.8}$$

since $\mu, \tilde{\mu} > s$. Then we sum over j and j' to show that

$$\Big\| \int_{\mathbb{R}} {}^{\mu}U^{\text{high}}_{\geq,k}(t)({}^{\tilde{\mu}}U^{\text{high}}_{\geq,k'}(\tau))^* F(\tau)d\tau \Big\|_{L^q_t(\mathbb{R}:L^r(M^\circ))} \lesssim \|F\|_{L^{\tilde{q}'}_t(\mathbb{R}:L^{\tilde{r}'}(M^\circ))}. \tag{5.9}$$

Further by taking the summation in k, k' which range over a finite set and using the Christ-Kiselev lemma with $q > \tilde{q}'$, we prove (5.7). □

5.2. Inhomogeneous Strichartz estimates on the double endpoint.

We prove the following result on the double endpoint inhomogeneous Strichartz estimate.

Proposition 5.2. *Let $(q, r, \mu), (\tilde{q}, \tilde{r}, \tilde{\mu}) \in \Lambda_w \cup \Lambda_e$ and let $q = \tilde{q} = 2$. The following inequalities hold:*

- *Low-frequency estimate*

$$\Big\| \int_{\tau<t} \frac{\sin{(t-\tau)\sqrt{H}}}{\sqrt{H}} \chi(\sqrt{H}) F(\tau) d\tau \Big\|_{L^2_t L^r_z} \lesssim \|F\|_{L^2_t L^{\tilde{r}'}_z}, \tag{5.10}$$

- *High-frequency estimate*

$$\Big\| \int_{\tau<t} \frac{\sin{(t-\tau)\sqrt{H}}}{\sqrt{H}} (1-\chi)(\sqrt{H}) F(\tau) d\tau \Big\|_{L^2_t L^r_z} \lesssim \|H^{\frac{\mu+\tilde{\mu}-1}{2}} F\|_{L^2_t L^{\tilde{r}'}_z}, \tag{5.11}$$

where $\chi \in \mathcal{C}^\infty_c([0,\infty)$ such that $\chi(\lambda) = 1$ for $\lambda \leq 1$ and vanishes when $\lambda \geq 2$.

Proof. The above argument breaks down here due to the failure of the Christ-Kiselev lemma. We follow the argument in Keel-Tao [16] to overcome this obstacle, but we need the usual dispersive estimates which are known to be false when there exist conjugate points on the manifold. However we can recover this by following the argument in [13].

We first prove (5.10). Recall ${}^{\sigma}U^{\text{low}}(t)$ in (5.4). As before, it suffices to show that

$$\Big\| \int_{\tau<t} {}^{\sigma}U^{\text{low}}(t)({}^{\sigma}U^{\text{low}}(\tau))^* F(\tau)d\tau \Big\|_{L^2_t(\mathbb{R}:L^r(M^\circ))} \lesssim \|F\|_{L^2_t(\mathbb{R}:L^{\tilde{r}'}(M^\circ))}, \quad \sigma = 1/2. \tag{5.12}$$

To show (5.12), it is enough to show the bilinear form estimate

$$|T(F,G)| \lesssim \|F\|_{L^2_t L^{\tilde{r}'}_z} \|G\|_{L^2_t L^{r'}_z}, \tag{5.13}$$

where $T(F,G)$ is the bilinear form

$$T(F,G) = \iint_{\tau<t} \langle {}^{\sigma}U^{\text{low}}(t)({}^{\sigma}U^{\text{low}}(\tau))^* F(\tau), G(t)\rangle_{L^2}\, d\tau dt. \tag{5.14}$$

Note that

$$\begin{aligned} {}^{\sigma}U^{\text{low}}(t)({}^{\sigma}U^{\text{low}}(\tau))^* &= \int_0^\infty e^{i(t-\tau)\lambda}\chi(\lambda)\lambda^{-2\sigma} dE_{\sqrt{H}}(\lambda) \\ &= \sum_{j\leq 0} \int_0^\infty e^{i(t-\tau)\lambda}\chi(\lambda)\varphi\big(2^{-j}\lambda\big)\lambda^{-2\sigma} dE_{\sqrt{H}}(\lambda). \end{aligned} \tag{5.15}$$

Note that the summation term is close to (3.12):

$$ {}^{\sigma}U^{\text{low}}_j(t)({}^{\sigma}U^{\text{low}}_j(\tau))^* = \int_0^\infty e^{i(t-\tau)\lambda}\chi^2(\lambda)\varphi^2\big(\frac{\lambda}{2^j}\big)\lambda^{-2\sigma} dE_{\sqrt{H}}(\lambda). \tag{5.16}$$

Therefore we can use the same argument to prove the same dispersive estimate (3.15). Using (3.15) with positive sign, we obtain

$$\begin{aligned}&\langle {}^{\sigma}U^{\mathrm{low}}(t)({}^{\sigma}U^{\mathrm{low}}(\tau))^*F(\tau),G(t)\rangle_{L^2}\\&\leq C\sum_{j\leq 0}2^{\epsilon j}(1+|t-\tau|)^{2\sigma-3+\epsilon}\Big\|\int e^{-(\rho-\delta)d(z,z')}F(\tau)dg(z')\Big\|_{L^r}\|G(t)\|_{L^{r'}}\\&\leq C(1+|t-\tau|)^{2\sigma-3+\epsilon}\|F(\tau)\|_{L^{\tilde r'}}\|G(t)\|_{L^{r'}}.\end{aligned}$$

By using Hölder's and Young's inequalities and the fact that $\sigma=1/2$ and $0<\epsilon\ll 1$, we obtain

$$\begin{aligned}|T(F,G)|&\lesssim\iint_{\tau<t}(1+|t-\tau|)^{2\sigma-3+\epsilon}\|F(\tau)\|_{L^{r'}}\|G(t)\|_{L^{r'}}dtd\tau\\&\lesssim\|F\|_{L^2_tL^{\tilde r'}}\|G\|_{L^2_tL^{r'}}.\end{aligned}$$

This proves (5.13) and hence (5.10).

We next prove (5.11). Recall ${}^{\sigma}U^{\mathrm{high}}(t)$ in (5.6). As before, it suffices to show that

$$\Big\|\int_{\tau<t}{}^{\mu}U^{\mathrm{high}}(t)({}^{\tilde\mu}U^{\mathrm{high}}(\tau))^*F(\tau)d\tau\Big\|_{L^2_tL^r_z}\lesssim\|F\|_{L^2_tL^{\tilde r'}_z}. \tag{5.17}$$

To show (5.17), it is enough to show the bilinear form estimate

$$|T(F,G)|\lesssim\|F\|_{L^2_tL^{\tilde r'}_z}\|G\|_{L^2_tL^{r'}_z}, \tag{5.18}$$

where $T(F,G)$ is the bilinear form

$$T(F,G)=\iint_{\tau<t}\langle{}^{\mu}U^{\mathrm{high}}(t)({}^{\tilde\mu}U^{\mathrm{high}}(\tau))^*F(\tau),G(t)\rangle_{L^2}\ d\tau dt. \tag{5.19}$$

Note that

$$\begin{aligned}&{}^{\mu}U^{\mathrm{high}}(t)({}^{\tilde\mu}U^{\mathrm{high}}(\tau))^*\\&=\sum_{k,k'=0}^{N}\int_0^\infty e^{i(t-\tau)\lambda}(1-\chi)(\lambda)\lambda^{-(\mu+\tilde\mu)}Q_k(\lambda)dE_{\sqrt H}(\lambda)Q_{k'}(\lambda)^*\\&=\sum_{j\geq 0}\sum_{k,k'=0}^{N}\int_0^\infty e^{i(t-\tau)\lambda}(1-\chi)(\lambda)\varphi(2^{-j}\lambda)\lambda^{-(\mu+\tilde\mu)}Q_k(\lambda)dE_{\sqrt H}(\lambda)Q_{k'}(\lambda)^*,\end{aligned} \tag{5.20}$$

in which the summation term is close to (3.11):

$${}^{\sigma}U^{\mathrm{high}}_{j,k}(t)({}^{\sigma}U^{\mathrm{high}}_{j,k}(\tau))^*=\int_0^\infty e^{i(t-\tau)\lambda}(1-\chi)^2(\lambda)\varphi^2\big(\frac{\lambda}{2^j}\big)\lambda^{-2\sigma}Q_k(\lambda)dE_{\sqrt H}(\lambda)Q_k(\lambda)^*.$$

The difference between the powers of functions $1-\chi$ and φ is harmless. From Lemma 5.2 below, the case "near-diagonal" (k is close to k') satisfies the same property of the case $k = k'$. Thus it also leads to (3.8) and (3.9); hence it proves (5.18). In the case "off diagonal" in which the conjugate points are not separated, we cannot prove the similar dispersive estimate like (3.8) and (3.9). However, we can prove the following, which also leads to (5.18).

Lemma 5.1. *Let ${}^{\sigma}U^{\mathrm{high}}_{\geq,k}(t)$ be defined as in (5.8). Then for each pair $(k,k') \in \{0,1,\dots,N\}^2$ there exists a constant C such that either*

$$\iint_{\tau<t} \langle {}^{\mu}U^{\mathrm{high}}_{\geq,k}(t)({}^{\tilde{\mu}}U^{\mathrm{high}}_{\geq,k'}(\tau))^* F(\tau), G(t)\rangle_{L^2}\, d\tau dt \leq C\|G\|_{L^2_\tau L^{r'}_z}\|F\|_{L^2_t L^{\tilde{r}'}_z} \tag{5.21}$$

or

$$\iint_{\tau>t} \langle {}^{\mu}U^{\mathrm{high}}_{\geq,k}(t)({}^{\tilde{\mu}}U^{\mathrm{high}}_{\geq,k'}(\tau))^* F(\tau), G(t)\rangle_{L^2}\, d\tau dt \leq C\|G\|_{L^2_\tau L^{r'}_z}\|F\|_{L^2_t L^{\tilde{r}'}_z}. \tag{5.22}$$

We postpone the proof for a moment. Now we see how Lemma 5.1 implies (5.18). On the one hand, for every pair (k,k'), we have by (5.9)

$$\iint \langle {}^{\mu}U^{\mathrm{high}}_{\geq,k}(t)({}^{\tilde{\mu}}U^{\mathrm{high}}_{\geq,k'}(\tau))^* F(\tau), G(t)\rangle_{L^2}\, d\tau dt \leq C\|G\|_{L^2_\tau L^{r'}_z}\|F\|_{L^2_t L^{\tilde{r}'}_z}. \tag{5.23}$$

Hence for every pair (k,k'), by (5.21) or subtracting (5.22) from (5.23), we obtain

$$\iint_{\tau<t} \langle {}^{\mu}U^{\mathrm{high}}_{\geq,k}(t)({}^{\tilde{\mu}}U^{\mathrm{high}}_{\geq,k'}(\tau))^* F(\tau), G(t)\rangle_{L^2}\, d\tau dt \leq C\|G\|_{L^2_\tau L^{r'}_z}\|F\|_{L^2_t L^{\tilde{r}'}_z}.$$

Finally by summing over all k and k', we obtain (5.18). Once we prove Lemma 5.1, we complete the proof of Proposition 5.2. □

To prove Lemma 5.1, we need a result about the dispersive estimates. To state and prove the dispersive estimates, we need to categorize all microlocalization pairs $\{Q_k, Q_{k'}\}_{k,k'=0}^N$ and the property of spectral measure.

Lemma 5.2. *The partition of the identity $Q_k(\lambda)$ can be chosen so that the pairs of indices (k,k'), $1 \leq k,k' \leq N$, can be divided into three classes,*

$$\{1,\dots,N\}^2 = J_{near} \cup J_{not-out} \cup J_{not-inc},$$

so that

- *if $(k,k') \in J_{near}$, then $Q_k(\lambda)dE_{\sqrt{H}}(\lambda)Q_{k'}(\lambda)^*$ satisfies the conclusions of Proposition 2.1;*
- *if $(k,k') \in J_{non-inc}$, then $Q_k(\lambda)$ is not incoming-related to $Q_{k'}(\lambda)$ in the sense that no point in the operator wavefront set (microlocal support) of $Q_k(\lambda)$ is related to a point in the operator wavefront set of $Q_{k'}(\lambda)$ by backward bicharacteristic flow;*
- *if $(k,k') \in J_{non-out}$, then $Q_k(\lambda)$ is not outgoing-related to $Q_{k'}(\lambda)$ in the sense that no point in the operator wavefront set of $Q_k(\lambda)$ is related to a point in the operator wavefront set of $Q_{k'}(\lambda)$ by forward bicharacteristic flow.*

Proof. This is an analogue of [13, Lemma 8.2], which is stated in the asymptotically conic manifold. The proof of the non-trapping asymptotically hyperbolic manifold is given in [6], which is essentially due to [11]. □

Using the not-incoming or not-outgoing property of $Q_k(\lambda)$ with respect to $Q_{k'}(\lambda)$, we obtain a similar lemma [13, Lemma 8.5] for spectral measure. We omit the details but we point out the key idea, which also was used in [6] considering the endpoint inhomogeneous Strichartz estimate for Schrödinger on the same setting considered here.

The essential key point is that the phase function in the oscillation expression of the Schwartz kernel of $Q_k(\lambda)dE_{\sqrt{H}}(\lambda)Q_{k'}(\lambda)^*$ has an unchanged sign when $(k,k')\in J_{non-inc}$ or $(k,k')\in J_{non-out}$. More precisely, there exists a small constant $c>0$ such that the phase function $\Phi\le -c$ when $(k,k')\in J_{non-out}$ and $\Phi\ge c$ when $(k,k')\in J_{non-inc}$. For simplicity, we take only one example to illustrate the idea. If Q_k is not incoming-related to $Q_{k'}$, we only consider

$$Q_k(\lambda)dE_{\sqrt{H}}(\lambda)Q_{k'}(\lambda)^* = \int_{\mathbb{R}^m} e^{i\lambda\Phi(z,z',v)}\lambda^{n-1+\frac{m}{2}}a(\lambda,z,z',v)dv,$$

where $\Phi(z,z',v)\ge c>0$ and $|(\lambda\partial_\lambda)^\alpha a|\le C_\alpha e^{-(\rho-\delta)d(z,z')}$, where $\rho=(n-1)/2$ and $0<\delta\ll 1$. Here the parameter $0\le m\le n-1$ is connected to the conjugate points, which is the degenerate rank of the projection from the phase space to the base. If we review the previous result in [13] and references therein, we will find that $m=0$ if there is no conjugate points in the manifold. Then the expression will be similar to the case $k=k'$ in which the conjugate points are separated. If $m>0$, then it causes a difficulty in showing the dispersive estimate when $\lambda\to\infty$. However, if we restrict to $\tau<t$, then the microlocalized wave propagator

$$\int_0^\infty e^{i(t-\tau)\lambda}\int_{\mathbb{R}^m} e^{i\lambda\Phi(z,z',v)}\lambda^{n-1+\frac{m}{2}}a(\lambda,z,z',v)dvd\lambda$$

has the phase function satisfying $(t-\tau)+\Phi\ge\max\{|t-\tau|,c\}$ due to the fact that Φ and $t-\tau$ have the same signs. Hence we can overcome the difficulties by integration by parts. More precisely, we shall prove that

Proposition 5.3. *Let $\rho=(n-1)/2$ and $0<\delta\ll 1$. There exists a constant C independent of t,z,z' for all $(k,k')\in\{0,1,\dots,N\}^2$, $j\ge 0$, such that the following pointwise estimates hold for any $K\ge 0$:*

- *If $k=0$ or $k'=0$ or $(k,k')\in J_{near}$, then for all $t\neq\tau$ we have*

$$\begin{aligned}&\Big|\int_0^\infty e^{i(t-\tau)\lambda}(1-\chi)(\lambda)\varphi(2^{-j}\lambda)\lambda^{-(\mu+\tilde{\mu})}Q_k(\lambda)dE_{\sqrt{H}}(\lambda)Q_{k'}(\lambda)^*\Big|\\ &\qquad\lesssim\begin{cases}2^{j[\frac{n+1}{2}-(\mu+\tilde{\mu})]}(2^{-j}+|t-\tau|)^{-\frac{n-1}{2}}e^{-(\rho-\delta)d(z,z')}, & |t-\tau|\le 2;\\ 2^{j[\frac{n+1}{2}-(\mu+\tilde{\mu})]}|t-\tau|^{-K}e^{-(\rho-\delta)d(z,z')}, & |t-\tau|\ge 2.\end{cases}\end{aligned}\tag{5.24}$$

- *If $(k,k')\in J_{non-out}$, that is, Q_k is not outgoing related to $Q_{k'}$ and $t<\tau$, then*

$$\begin{aligned}&\Big|\int_0^\infty e^{i(t-\tau)\lambda}(1-\chi)(\lambda)\varphi(2^{-j}\lambda)\lambda^{-(\mu+\tilde{\mu})}Q_k(\lambda)dE_{\sqrt{H}}(\lambda)Q_{k'}(\lambda)^*\Big|\\ &\qquad\lesssim\begin{cases}2^{j[\frac{n+1}{2}-(\mu+\tilde{\mu})]}(2^{-j}+|t-\tau|)^{-\frac{n-1}{2}}e^{-(\rho-\delta)d(z,z')}, & |t-\tau|\le 2;\\ 2^{j[\frac{n+1}{2}-(\mu+\tilde{\mu})]}|t-\tau|^{-K}e^{-(\rho-\delta)d(z,z')}, & |t-\tau|\ge 2.\end{cases}\end{aligned}\tag{5.25}$$

- *Similarly, if $(k,k') \in J_{non-inc}$, that is, Q_k is not incoming related to $Q_{k'}$ and $\tau < t$, then*

(5.26)
$$\Big|\int_0^\infty e^{i(t-\tau)\lambda}(1-\chi)(\lambda)\varphi(2^{-j}\lambda)\lambda^{-(\mu+\tilde{\mu})}Q_k(\lambda)dE_{\sqrt{H}}(\lambda)Q_{k'}(\lambda)^*\Big|$$
$$\lesssim \begin{cases} 2^{j[\frac{n+1}{2}-(\mu+\tilde{\mu})]}(2^{-j}+|t-\tau|)^{-\frac{n-1}{2}}e^{-(\rho-\delta)d(z,z')}, & |t-\tau|\leq 2; \\ 2^{j[\frac{n+1}{2}-(\mu+\tilde{\mu})]}|t-\tau|^{-K}e^{-(\rho-\delta)d(z,z')}, & |t-\tau|\geq 2. \end{cases}$$

Now we prove Lemma 5.1 assuming Proposition 5.3.

Proof of Lemma 5.1. The main argument is to repeat the argument in the proof of Proposition 3.3 with $j \geq 0$ due to [16] if we have the dispersive estimate. In the case that $(k,k') \in J_{near}$, we have the dispersive estimate (5.24). We repeat the argument in the proof of Proposition 3.3 and sum in $j \geq 0$ to obtain (5.21). We would like to remark that $\mu, \tilde{\mu} > s$ ensures that the summation in $j \geq 0$ converges. If $(k,k') \in J_{non-inc}$, we obtain (5.21) due to the dispersive estimate (5.26) when $\tau < t$. Finally, in the case that $(k,k') \in J_{non-out}$, we obtain (5.22) since we have the dispersive estimate (5.25) for $\tau > t$. □

Proof of Proposition 5.3. The proof is modified from the proof for the Schrödinger equation in [13, Lemma 8.6] adapted to the wave equation.

We first prove (5.24). If one of k, k' equals 0, we have the expression of microlocalized spectral measure in Proposition 2.1 since the support of Q_0 is far away from the boundary. From the above result, if $(k,k') \in J_{near}$, by Lemma 5.2, we also have the expression of microlocalized spectral measure in Proposition 2.1. Hence we can prove (5.24) by using the same argument used to prove (3.13) and (3.14) in Lemma 3.1. We omit the details here.

We only prove (5.26) since (5.25) follows from the same argument. Assume that Q_k is not incoming-related to $Q_{k'}$. In this case, for the sake of simplicity, we only consider

$$Q_k(\lambda)dE_{\sqrt{H}}(\lambda)Q_{k'}(\lambda)^* = \int_{\mathbb{R}^m} e^{i\lambda\Phi(z,z',v)}\lambda^{n-1+\frac{m}{2}}a(\lambda,z,z',v)dv,$$

where $\Phi(z,z',v) \geq \epsilon > 0$, $0 \leq m \leq n-1$, and a is a smooth function which is compactly supported in the v such that $|(\lambda\partial_\lambda)^\alpha a| \leq C_\alpha e^{-(\rho-\delta)d(z,z')}$. For example, see [13, (8-13), Lemma 8.5]. Then we need to show that for $\tau < t$ and $j \geq 0$,

(5.27)
$$\Big|\int_0^\infty e^{i(t-\tau)\lambda}(1-\chi)(\lambda)\varphi(2^{-j}\lambda)\lambda^{-(\mu+\tilde{\mu})}\int_{\mathbb{R}^m} e^{i\lambda\Phi(z,z',v)}\lambda^{n-1+\frac{m}{2}}a(\lambda,z,z',v)dvd\lambda\Big|$$
$$\lesssim \begin{cases} 2^{j[\frac{n+1}{2}-(\mu+\tilde{\mu})]}(2^{-j}+|t-\tau|)^{-\frac{n-1}{2}}e^{-(\rho-\delta)d(z,z')}, & |t-\tau|\leq 2; \\ 2^{j[\frac{n+1}{2}-(\mu+\tilde{\mu})]}|t-\tau|^{-K}e^{-(\rho-\delta)d(z,z')}, & |t-\tau|\geq 2. \end{cases}$$

Indeed, we can directly obtain by integration by parts

$$
\begin{aligned}
&\Big| \int_0^\infty e^{i(t-\tau)\lambda}(1-\chi)(\lambda)\varphi(2^{-j}\lambda)\lambda^{-(\mu+\tilde{\mu})} \int_{\mathbb{R}^m} e^{i\lambda\Phi(z,z',v)}\lambda^{n-1+\frac{m}{2}} a(\lambda,z,z',v)dvd\lambda\Big| \\
&\quad \lesssim \Big| \int_{\mathbb{R}^m}\int_0^\infty |(t-\tau)+\Phi(z,z',v)|^{-K} \\
&\qquad \times \partial_\lambda^K \Big((1-\chi)(\lambda)\varphi(2^{-j}\lambda)\lambda^{-(\mu+\tilde{\mu})}\lambda^{n-1+\frac{m}{2}} a(\lambda,z,z',v)\Big)\, d\lambda dv\Big|.
\end{aligned}
$$

Note that a is compactly supported in variable v, $t-\tau>0$, and $\Phi(z,z',v)\geq \epsilon>0$. Consequently,

$$
\begin{aligned}
&\Big| \int_0^\infty e^{i(t-\tau)\lambda}(1-\chi)(\lambda)\varphi(2^{-j}\lambda)\lambda^{-(\mu+\tilde{\mu})} \int_{\mathbb{R}^m} e^{i\lambda\Phi(z,z',v)}\lambda^{n-1+\frac{m}{2}} a(\lambda,z,z',v)dvd\lambda\Big| \\
&\lesssim |(t-\tau)+\epsilon|^{-K}\int_{2^{j-1}}^{2^{j+1}} \lambda^{n-1+\frac{m}{2}-\mu-\tilde{\mu}-K} d\lambda\, e^{-(\rho-\delta)d(z,z')} \\
&\lesssim 2^{j(n+\frac{m}{2}-\mu-\tilde{\mu})}(2^j(|t-\tau|+\epsilon))^{-K}e^{-(\rho-\delta)d(z,z')},
\end{aligned}
$$

which implies (5.27) by choosing K large enough. □

6. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3 by using the Strichartz estimates in Theorems 1.1 and 1.2.

Let $p>1$. The proof is standard and based on a contraction mapping argument in the Banach space $L^{p+1}(\mathbb{R}^+\times M^\circ)$. Define the map $\mathcal{T}$ by $v=\mathcal{T}u$ where v solves, given $u\in L^{p+1}(\mathbb{R}^+\times M^\circ)$,

$$
\begin{cases} \partial_t^2 v - \Delta_g v - \rho^2 v = F_p(u), \quad (t,z)\in I\times M^\circ; \\ u(0)=\nu u_0(z),\ \partial_t u(0)=\nu u_1(z). \end{cases} \tag{6.1}
$$

Notice first that all the Strichartz estimates are global in time so one has $I=\mathbb{R}^+$. Choose $q=r=\tilde{q}=\tilde{r}=p+1>2$. Then we can verify, for any $p\in(1,1+\frac{4}{n-1})$,

$$
(p+1,p+1,\mu)\in\Lambda_e,\quad s_e<\mu,
$$

and

$$
(p+1,p+1,1/2)\in\Lambda_e,\quad s_e<1/2.
$$

Therefore, for fixing $0<\epsilon\ll 1$, we can apply Theorem 1.1 with $(p+1,p+1,\mu_0)\in\Lambda_e$ and Theorem 1.2 with $(p+1,p+1,1/2)\in\Lambda_e$ (or directly (5.3)) to obtain

$$
\begin{aligned}
&\|v(t,z)\|_{L^{p+1}(I\times M^\circ)} \\
&\quad \lesssim \nu\Big(\|u_0\|_{H^{\mu,0}(M^\circ)}+\|u_1\|_{H^{\mu-1,-\epsilon}(M^\circ)}\Big)+\||u(t,z)|^p\|_{L_t^{\frac{p+1}{p}}(I;L^{\frac{p+1}{p}}(M^\circ))}.
\end{aligned}
$$

Thus this gives

$$
\|v(t,z)\|_{L^{p+1}(I\times M^\circ)}\lesssim \nu\Big(\|u_0\|_{H^{\mu,0}(M^\circ)}+\|u_1\|_{H^{\mu-1,-\epsilon}(M^\circ)}\Big)+\|u\|^p_{L_t^{p+1}(I;L^{p+1}(M^\circ))}.
$$

Therefore the operator $\mathcal{T}$ maps $L^{p+1}(\mathbb{R}^+\times M^\circ)$ into itself. Furthermore, a standard computation shows that if ν is small enough, $\mathcal{T}$ maps a ball of $L^{p+1}(\mathbb{R}^+\times M^\circ)$ into itself and is actually a contraction. Hence by the Banach fixed point theorem this leads to the desired result (see for instance [26]).

Acknowledgments

We thank Jean-Marc Bouclet and Andrew Hassell for helpful discussions.

References

[1] Jean-Philippe Anker and Vittoria Pierfelice, *Wave and Klein-Gordon equations on hyperbolic spaces*, Anal. PDE **7** (2014), no. 4, 953–995, DOI 10.2140/apde.2014.7.953. MR3254350

[2] Jean-Philippe Anker, Vittoria Pierfelice, and Maria Vallarino, *The wave equation on hyperbolic spaces*, J. Differential Equations **252** (2012), no. 10, 5613–5661, DOI 10.1016/j.jde.2012.01.031. MR2902129

[3] Jean-Philippe Anker, Vittoria Pierfelice, and Maria Vallarino, *The wave equation on Damek-Ricci spaces*, Ann. Mat. Pura Appl. (4) **194** (2015), no. 3, 731–758, DOI 10.1007/s10231-013-0395-x. MR3345662

[4] M. Blair, Y. Sire, C. Sogge, Quasimode, eigenfunction and spectral projection bounds for Schrödinger operators on manifolds with critically singular potentials, J. Geom. Anal. (2019), doi.org/10.1007/s12220-019-00287-z.

[5] Jean-Marc Bouclet, *Littlewood-Paley decompositions on manifolds with ends* (English, with English and French summaries), Bull. Soc. Math. France **138** (2010), no. 1, 1–37, DOI 10.24033/bsmf.2584. MR2638890

[6] Xi Chen, *Resolvent and spectral measure on non-trapping asymptotically hyperbolic manifolds III: Global-in-time Strichartz estimates without loss* (English, with English and French summaries), Ann. Inst. H. Poincaré Anal. Non Linéaire **35** (2018), no. 3, 803–829, DOI 10.1016/j.anihpc.2017.08.003. MR3778653

[7] Xi Chen and Andrew Hassell, *Resolvent and spectral measure on non-trapping asymptotically hyperbolic manifolds I: Resolvent construction at high energy*, Comm. Partial Differential Equations **41** (2016), no. 3, 515–578, DOI 10.1080/03605302.2015.1116561. MR3473907

[8] Xi Chen and Andrew Hassell, *Resolvent and spectral measure on non-trapping asymptotically hyperbolic manifolds II: Spectral measure, restriction theorem, spectral multipliers* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **68** (2018), no. 3, 1011–1075. MR3805767

[9] Michael Christ and Alexander Kiselev, *Maximal functions associated to filtrations*, J. Funct. Anal. **179** (2001), no. 2, 409–425, DOI 10.1006/jfan.2000.3687. MR1809116

[10] Jean Fontaine, *A semilinear wave equation on hyperbolic spaces*, Comm. Partial Differential Equations **22** (1997), no. 3-4, 633–659, DOI 10.1080/03605309708821277. MR1443052

[11] Colin Guillarmou and Andrew Hassell, *Uniform Sobolev estimates for non-trapping metrics*, J. Inst. Math. Jussieu **13** (2014), no. 3, 599–632, DOI 10.1017/S1474748013000273. MR3211800

[12] J. Ginibre and G. Velo, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal. **133** (1995), no. 1, 50–68, DOI 10.1006/jfan.1995.1119. MR1351643

[13] Andrew Hassell and Junyong Zhang, *Global-in-time Strichartz estimates on nontrapping, asymptotically conic manifolds*, Anal. PDE **9** (2016), no. 1, 151–192, DOI 10.2140/apde.2016.9.151. MR3461304

[14] Fritz John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979), no. 1-3, 235–268, DOI 10.1007/BF01647974. MR535704

[15] R. A. Kunze and E. M. Stein, *Uniformly bounded representations and harmonic analysis of the* 2×2 *real unimodular group*, Amer. J. Math. **82** (1960), 1–62, DOI 10.2307/2372876. MR163988

[16] Markus Keel and Terence Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), no. 5, 955–980. MR1646048

[17] Andrew Lawrie, Sung-Jin Oh, and Sohrab Shahshahani, *Profile decompositions for wave equations on hyperbolic space with applications*, Math. Ann. **365** (2016), no. 1-2, 707–803, DOI 10.1007/s00208-015-1305-x. MR3498926

[18] A. Lawrie, J. Lührmann, S.-J. Oh, and S. Shahshahani, *Local smoothing estimates for Schrödinger equations on hyperbolic space*, arXiv:1808.04777, 2018.

[19] A. Lawrie, J. Lührmann, S.-J. Oh, and S. Shahshahani, *Asymptotic stability of harmonic maps on the hyperbolic plane under the Schrödinger maps evolution*, arXiv:1909.06899, 2019.

[20] Noël Lohoué, *Estimation des fonctions de Littlewood-Paley-Stein sur les variétés riemanniennes à courbure non positive* (French), Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 4, 505–544. MR932796

[21] Rafe Mazzeo, *The Hodge cohomology of a conformally compact metric*, J. Differential Geom. **28** (1988), no. 2, 309–339. MR961517

[22] Rafe R. Mazzeo and Richard B. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987), no. 2, 260–310, DOI 10.1016/0022-1236(87)90097-8. MR916753

[23] Jason Metcalfe and Michael Taylor, *Nonlinear waves on 3D hyperbolic space*, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3489–3529, DOI 10.1090/S0002-9947-2011-05122-6. MR2775816

[24] Jason Metcalfe and Michael Taylor, *Dispersive wave estimates on 3D hyperbolic space*, Proc. Amer. Math. Soc. **140** (2012), no. 11, 3861–3866, DOI 10.1090/S0002-9939-2012-11534-5. MR2944727

[25] Robert S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), no. 3, 705–714. MR512086

[26] Yannick Sire, Christopher D. Sogge, and Chengbo Wang, *The Strauss conjecture on negatively curved backgrounds*, Discrete Contin. Dyn. Syst. **39** (2019), no. 12, 7081–7099, DOI 10.3934/dcds.2019296. MR4026182

[27] Elias M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory.*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1970. MR0252961

[28] Walter A. Strauss, *Nonlinear scattering theory at low energy*, J. Functional Analysis **41** (1981), no. 1, 110–133, DOI 10.1016/0022-1236(81)90063-X. MR614228

[29] Daniel Tataru, *Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation*, Trans. Amer. Math. Soc. **353** (2001), no. 2, 795–807, DOI 10.1090/S0002-9947-00-02750-1. MR1804518

[30] Michael Taylor, *Hardy spaces and BMO on manifolds with bounded geometry*, J. Geom. Anal. **19** (2009), no. 1, 137–190, DOI 10.1007/s12220-008-9054-7. MR2465300

[31] Terence Tao, *Nonlinear dispersive equations*: *Local and global analysis*, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. MR2233925

[32] C. Wang, *Recent progress on the Strauss conjecture and related problems*, Mathematica **48** (2018), no. 1, 111–130.

[33] Junyong Zhang, *Strichartz estimates and nonlinear wave equation on nontrapping asymptotically conic manifolds*, Adv. Math. **271** (2015), 91–111, DOI 10.1016/j.aim.2014.11.013. MR3291858

Department of Mathematics, Johns Hopkins University, Baltimore, Maryland 21218
Email address: sire@math.jhu.edu

Department of Mathematics, Johns Hopkins University, Baltimore, Maryland 21218
Email address: sogge@math.jhu.edu

School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, People's Republic of China
Email address: wangcbo@zju.edu.cn

Department of Mathematics, Beijing Institute of Technology, Beijing 100081, People's Republic of China; Cardiff University, Cardiff CF10 3AT, United Kingdom
Email address: zhang_junyong@bit.edu.cn, ZhangJ107@cardiff.ac.uk