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Joint Pricing and Rebalancing of Autonomous Mobility-on-Demand Systems

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Abstract— **This paper studies optimal pricing and rebalancing policies for Autonomous Mobility-on-Demand (AMoD) systems. We adopt a macroscopic planning perspective to tackle a profit maximization problem while ensuring that the system is load-balanced. We describe the system using a dynamic fluid model to show the existence and stability of an equilibrium (i.e., load balance) through pricing policies. We then develop an optimization framework that allows us to find optimal policies in terms of both pricing and rebalancing. We first maximize profit by only using pricing policies, then incorporate rebalancing, and finally we consider whether the solution is found sequentially or jointly. We apply each approach to a data-driven case study using real taxi data from New York City. Depending on which benchmarking solution we use, the joint problem (i.e., pricing and rebalancing) increases profits by 7% to 40%.**

I. INTRODUCTION

With the rise of Mobility-on-Demand (MoD) services (e.g. Uber, Lyft, DiDi) and the rapid technological evolution of self-driving vehicles, we are closer to having Autonomous Mobility-on-Demand (AMoD) systems. A crucial step in the proper functioning of such a service is to define pricing, rebalancing and routing policies for the operator's fleet. This paper focuses on the first two issues, while the interested reader is directed to [1] for a discussion on routing and rebalancing.

Pricing policies play an important role as they modulate the inflow of customers traveling between regions in the network. As a result, the controller has the ability to choose prices such that the induced demand ensures a balanced load of customers and vehicles arriving at each location. In addition, the selection of prices enables the operator to modulate demand such that the system can operate with smaller or larger fleet sizes. If we restrict a pricing policy to require balancing the load in every node, we expect the solution to concentrate on balancing the network rather than choosing the prices to maximize profit. To give the pricing policy more flexibility, AMoD systems can leverage rebalancing policies, i.e., send empty vehicles from regions with excess supply of vehicles to regions with excess demand with the objective of achieving higher profits.

Related Literature: Researchers have tackled the pricing problem using two main settings: one-sided, or two-sided markets depending on whether the MoD controller has full or limited control over the supply. One-sided markets assume full control over the vehicles [2], [3], whereas two-sided markets consider self-interested suppliers (drivers) [2], [4]. To the best of our knowledge, all these optimal pricing policies, except [3], do not rebalance externally. Rather, they incentivize the supply (human drivers) to reallocate by the use of compensations. Our model differs from [3], which uses a microscopic model and Reinforcement Learning techniques, by the level of abstraction performed. As an alternative to a microscopic model we employ a macroscopic (planning) model to assess the benefits of *jointly* solving the pricing and rebalancing problem over other approaches.

In contrast to pricing, the rebalancing of AMoD systems (without pricing) has been studied using simulation [5]–[7], queuing-theoretical [8], [9], and network-flow [10], [11] models and it has also been tackled jointly with routing schemes [1], [12]. In [5], the rebalancing problem is addressed using a data-driven parametric controller suited for real-time implementation. Alternatively, [10] uses a steady-state fluid model which serves as a basis for our results.

Key contributions: In this work we provide a theoretical framework to design optimal pricing policies for an AMoD provider. We analyze the system in the spirit of [10], converting the problem into profit maximization rather than an operational cost minimization. Different from the existing methods in pricing, we consider the destination of a customer when designing the pricing policy. This allows the fleet controller to modulate demand in such a way that the system is balanced by solely adjusting prices. Additionally, we incorporate the rebalancing policy optimization framework in [10] and formulate a *joint* optimization model. We compare this joint strategy with four different methodologies. First by only finding optimal prices, second by only rebalancing the fleet, third by sequentially solving the rebalancing and then pricing of the system, and fourth by jointly estimating pricing and rebalancing with a unique surge price by origin. We apply each approach to two case studies; one, with simulated data; and another, with real taxi data from New York City.

Organization: The paper is organized as follows. In Section II we introduce the fluid model consisting of queues of customers and vehicles at every region. In Section III, we show that the system is well-posed and establish the existence of a load balance equilibrium through the selection of prices. We also obtain local stability results. In Section IV, we state the problems of optimal pricing, optimal rebalancing and the *joint* formulation of these two. Then, we present case studies to assess the performance of the *joint* formulation in Section V. Finally, in Section VI we conclude.

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II. Model

In this section we present a steady-state deterministic fluid model to find optimal prices in an AMoD system while ensuring service to customers. This model is intended to serve as a relaxation of the corresponding stochastic queueing model where customers arrive according to a Poisson process and travel times are non-deterministic (usually exponentially distributed). The reason for making this relaxation is the flexibility it provides to perform analysis of the system.

Consider a fully-connected network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ where $\mathcal{N}$ is the set of nodes (regions) $\mathcal{N} = \{1, ..., N\}$ and $\mathcal{A} = \{(i, j) : i, j \in \mathcal{N} \times \mathcal{N}\}$ is the set of arcs. A customer requests a ride in region i, receives a transportation service from the AMoD platform, and is charged a price composed of the product of a *base* and a *surge* price. The total price is $p_{ij} = p^0_{ij} u_{ij}$ where p^0_{ij}, u_{ij} are the base and surge prices, respectively, for traveling from node i to j. Throughout the paper, we will use the surge price u_{ij} as our control variable, and we assume that $u_{ij} \geq 1$ as the platform is not willing to charge less than its base price.

We further assume that customers' arrival rate is a function of the surge price, namely $\lambda_{ij}(u_{ij}) : \mathbb{R}_{\geq 1} \mapsto \mathbb{R}_{\geq 0}$ for a customer travelling from i to j. This function is known as the *willigness-to-pay* or the *demand* function. Let the *base demand* be $\lambda^0_{ij} = \lambda_{ij}(1)$, i.e., the demand rate of customers when the surge price is at its minimum.

As in [10], we use a queueing model for this system with two queues per region. We let $c_i(t) \in \mathbb{R}_{\geq 0}$ be the number of customers at region i waiting to be assigned to a vehicle; and denote with $v_i(t) \in \mathbb{R}_{\geq 0}$ the number of available vehicles waiting in region i at time t. Moreover, the AMoD provider assigns vehicles to customers located in the same region at a service rate μ_i. We assume that $\mu_i > \sum_j \lambda^0_{ij}$, meaning that the platform assigns vehicles to customers faster than the rate at which customers arrive. This assumption is required to avoid building large customer queues. For the purpose of this paper, we consider the rate vectors $\boldsymbol{\lambda} = (\lambda_{ij};\ \forall i, j \in \mathcal{N})$ and $\boldsymbol{\mu} = (\mu_i;\ \forall i \in \mathcal{N})$ to be invariant (we use bold notation to represent a vector containing all the variables sharing the same symbol). This allows us to analyze the steady-state solution of the system. Finally, we let $T_{ij} \in \mathbb{R}_{\geq 0}$ be the travel time for a passenger to go from i to j, which we assume to be fixed and not dependent on the routing decisions of the AMoD system (see Fig. 1). To continue with our analysis, we make the following assumptions:

Assumption 1. The function $\lambda_{ij}(\cdot)$ is monotonically decreasing $\forall i, j \in \mathcal{N}$, i.e., as price increases, the demand rate decreases.

Assumption 2. There exists a surge price $u^{\max}_{ij}$ for which $\lambda_{ij}(u^{\max}_{ij}) = 0, \quad \forall i, j \in \mathcal{N}$.

Customer Dynamics: Consider a customer queue $c_i(t)$ for each region $i \in \mathcal{N}$ in the network. The queue dynamics are:

$$\dot{c}_i = \begin{cases} \sum_j \lambda_{ij}(u_{ij}), & \text{if } v_i = 0, \\ 0, & \text{if } v_i \geq 0 \text{ and } c_i = 0, \\ \sum_j \lambda_{ij}(u_{ij}) - \mu_i, & \text{if } v_i \geq 0 \text{ and } c_i \geq 0. \end{cases}$$

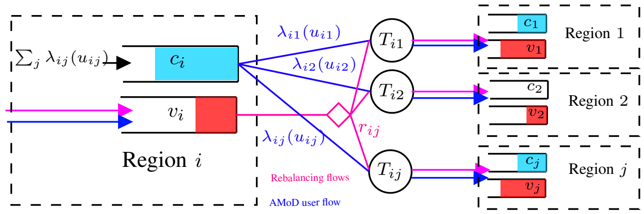


Fig. 1: Customer traveling from i to j arrive to region i at rate $\lambda_{ij}(u_{ij})$ and it takes T_{ij} units of time to reach j. The AMoD provider plans a pricing policy $\mathbf{u}$ and a rebalancing policy of empty vehicles r_{ij} to serve their customers such that its profit is maximized. Note this is a fluid model as opposed to a discrete event system.

In order to express the customer dynamics with shorter notation we let $H(x) = \mathbb{1}_{x>0}$ be an indicator function for positive values of x, and we use the following shorthand notation:

$$\lambda_{ij} := \lambda_{ij}(u_{ij}), \quad \lambda_i := \sum_j \lambda_{ij}, \quad v_i := v_i(t),$$
$$c_i := c_i(t), \quad v^i_j := v_j(t - T_{ji}), \quad c^i_j := c_j(t - T_{ji})$$

where λ_i is the total endogenous outgoing flow from node i; and c^i_j, v^i_j are the customer and vehicle queue levels in region j at time $t - T_{ij}$, respectively. Then, we rewrite the customer dynamics in compact form as follows:

$$\dot{c}_i = \lambda_i(1 - H(v_i)) + (\lambda_i - \mu_i)H(c_i)H(v_i).$$

Note that as a result of using a fluid model, the variables denoting the number of customers in a region are real-valued.

Vehicle Dynamics: The outflow rate corresponding to vehicles departing station i is given by:

$$\dot{v}^-_i = \begin{cases} -\lambda_i, & \text{if } v_i \geq 0 \text{ and } c_i = 0, \\ 0, & \text{if } v_i = 0, \\ -\mu_i, & \text{if } v_i \geq 0 \text{ and } c_i \geq 0. \end{cases}$$

which, by using the $H(x)$ notation above, can be written as $\dot{v}^-_i = -\lambda_i H(v_i) + (\lambda_i - \mu_i)H(v_i)H(c_i)$. In addition, the rate at which customer-carrying vehicles arrive at station i is given by: $\dot{v}^+_i = \sum_j (\lambda_{ji} H(v^i_j) - (\lambda_{ji} - \mu_j)H(v^i_j)H(c^i_j))$. Hence, the vehicle dynamics is $\dot{v}_i = \dot{v}^-_i + \dot{v}^+_i$, which lead to

$$\dot{v}_i = -\lambda_i H(v_i) + (\lambda_i - \mu_i)H(c_i)H(v_i) + \sum_j (\lambda_{ji} H(v^i_j) - (\lambda_{ji} - \mu_j)H(c^i_j)H(v^i_j)).$$

Then, the global system dynamics are expressed by the following differential equations

$$\dot{c}_i = \lambda_i(1 - H(v_i)) + (\lambda_i - \mu_i)H(c_i)H(v_i), \tag{1a}$$
$$\dot{v}_i = -\lambda_i H(v_i) + (\lambda_i - \mu_i)H(c_i)H(v_i) + \sum_j (\lambda_{ji} H(v^i_j) - (\lambda_{ji} - \mu_j)H(c^i_j)H(v^i_j)). \tag{1b}$$

which describe a non-linear, time-delayed, time-invariant, right-hand discontinuous system.

III. Well posedness, Equilibrium and Stability

Similar to [10], we say that the system (1) is *well posed* if two conditions are satisfied: *(i)* for any initial condition, there exists a solution of the differential equations in (1), and *(ii)*, the number of vehicles in the system remain invariant over time. In order to analyze the model, we use the framework of Filippov solutions [13]. Let us now give a proposition for the well-posedness of the system:

Proposition 1 (Well-posedness of the fluid model)**.**

1) *For every initial condition in the fluid model represented in* (1)*, there exist continuous functions* $c_i(t) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ *and* $v_i(t) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}, \forall i \in \mathcal{N}$*, satisfying the system of equations in the Fillipov sense.*
2) *For all* $t > 0$*, the total number of vehicles is invariant and equal to* $m = \sum_{i\in\mathcal{N}} v_i(0)$.

Proof: For 1), we use the framework in [14]. In particular, we check that all assumptions and conditions of [14, Thm II-1] are satisfied. This theorem, ensures the existence of Fillipov solutions to the time-delayed differential equations with discontinuous right-hand sides.

To prove the second claim, we separate the vehicle dynamics in two parts: vehicles in transit $v_{ij}(t)$, and vehicles at a specific region $v_i(t)$. For the vehicles queued at i we know their dynamics are as in (1b). For the vehicles in transit, we let the total be

$$v_{ij}(t) = \int_{t-T_{ij}}^{t} \lambda_{ij} H(v_i(\tau)) + (\lambda_{ij} - \mu_i) H(c_i(\tau)) H(v_i(\tau)) \; d\tau,$$

and their dynamics are

$$\begin{aligned}\dot{v}_{ij}(t) = \lambda_{ij} H(v_i) + (\lambda_{ij} - \mu_i) H(c_i) H(v_i) \\ -(\lambda_{ij} H(v_i^j) + (\lambda_{ij} - \mu_i) H(c_i^j) H(v_i^j)).\end{aligned}$$

Hence, we let the total number of vehicles in the system be $m(t) = \sum_i v_i(t) + \sum_{ij} v_{ij}(t)$ with dynamics:

$$\dot{m}(t) = \sum\nolimits_i \dot{v}_i(t) + \sum\nolimits_{ij} \dot{v}_{ij}(t), \tag{2a}$$

$$= \sum\nolimits_i \Big(-\lambda_i H(v_i) + (\lambda_i - \mu_i) H(c_i) H(v_i) \tag{2b}$$

$$+ \sum\nolimits_j \lambda_{ji} H(v_j^i) - (\lambda_{ji} - \mu_j) H(c_j^i) H(v_j^i) \Big) + \sum\nolimits_{ij} \dot{v}_{ij},$$

$$= \sum\nolimits_{ij} -\lambda_{ij} H(v_i) + (\lambda_{ij} - \mu_i) H(c_i) H(v_i) \tag{2c}$$

$$+ \sum\nolimits_{ij} \lambda_{ji} H(v_j^i) - (\lambda_{ji} - \mu_j) H(c_j^i) H(v_j^i) + \sum\nolimits_{ij} \dot{v}_{ij},$$

$$= 0. \tag{2d}$$

Note this result is obtained by expanding the first term in (2a) using (1b), rearranged terms and found that $-\sum_i \dot{v}_i(t) = \sum_{ij} \dot{v}_{ij}(t) \implies \dot{m} = 0$, which implies that the fleet size remains invariant over time. ☐

Equilibria: We say that the system is in equilibrium if customer queues (and therefore, waiting times) do not grow to infinity. We show the existence of an equilibrium in the fluid model (1) when we control the prices of every origin-destination pair. Additionally, we show that by having the ability to control the prices, one can have find multiple equilibria for a desired fleet size, giving the flexibility to AMoD managers to operate the system at different demand levels. Most remaining proofs are omitted due to space limitations, but can be found in [15].

Theorem 1 (Existence of equilibria)**.** *Let* $\mathcal{U}$ *be a set of prices* $\mathbf{u}$*, such that when* $\mathbf{u} \in \mathcal{U}$ *we have*

$$\sum\nolimits_j \lambda_{ij}(u_{ij}) - \lambda_{ji}(u_{ji}) = 0, \quad \forall i \in \mathcal{N}, \tag{3}$$

and let $m_{\mathbf{u}} := \sum_{ij} T_{ij} \lambda_{ij}(u_{ij})$*. Then, if* $\mathbf{u} \in \mathcal{U}$*, and* $m > m_{\mathbf{u}}$*, an equilibrium exists with* $\mathbf{c} = 0$ *and* $\mathbf{v} > 0$*. Otherwise no equilibrium exists.*

Lemma 1 (Existence of an equilibrium)**.** *The set* $\mathcal{U}$ *is never empty, hence, at least one equilibrium exists.*

Proof: We use the fact that there exists a price $u_{ij}^{\max}$ for which $\lambda_{ij}(u_{ij}^{\max}) = 0$ for all $i, j \in \mathcal{N}$. Then, setting $\mathbf{u} = \mathbf{u}^{\max}$, implies that an equilibrium exists. This strategy means that we are not providing service to any request, nevertheless the equilibrium exists. ☐

Lemma 2 (Infinite number of equilibria)**.** *If there is a positive demand tour in the graph, then there exists an infinite number of price vectors* $\mathbf{u}$ *which can steer the system to an equilibrium point.*

Proof: Assume that there exists at least one Eulerian tour (or `cycle`) in the graph for which $\lambda_{ij}^0 > 0$ for all $(i,j) \in \texttt{cycle}$. Then, let $\boldsymbol{\lambda}^{\texttt{cycle}} = \{\lambda_{ij}^0 \mid (i,j) \in \texttt{cycle}\}$ and the minimum rate on that tour be $\lambda_{\min}^{\texttt{cycle}} = \min\{\lambda_{ij}\}_{(i,j)\in\texttt{cycle}}$. Then by setting $u_{ij} = u_{ij}^{\max}$ for all $(i,j) \notin \texttt{cycle}$, we can express the equilibrium condition as

$$\sum_{j:(i,j)\in\texttt{cycle}} \lambda_{ij}(u_{ij}) - \lambda_{ji}(u_{ji}) = 0, \quad \forall i : (i,j) \in \texttt{cycle}. \tag{4}$$

Now, we use the fact that $\lambda_{ij}(u_{ij})$ is a monotonically decreasing function and we focus on $(i,j) \in \texttt{cycle}$. Hence for all $\lambda_{ij}(u_{ij}) > \lambda_{\min}^{\texttt{cycle}}$ we can find a u_{ij} such that $\lambda_{ij}(u_{ij}) = \lambda_{\min}^{\texttt{cycle}}$. Then, extending this for higher prices on $\lambda_{\min}^{\texttt{cycle}}$ and using the same argument as before, we show that there exists a pricing strategy $\mathbf{u}$ for which we can obtain an equilibrium with a tour demand rate with any value in the range $(0, \lambda_{\min}^{\texttt{cycle}})$. ☐

These two lemmata imply that by incorporating an origin-destination pricing strategy, we can operate a MoD service at equilibrium for any demand rate and with any fleet size.

Corollary 1 (Minimum number of vehicles in equilibria)**.** *The minimum number of vehicles to operate in an equilibrium induced by policy* $\mathbf{u}$ *is at least* $m > \underline{m} := \min_{\mathbf{u}} m_{\mathbf{u}}$ *where* $m_{\mathbf{u}} := \sum_{ij} T_{ij} \lambda_{ij}(u_{ij})$.

Stability: In this section we study local stability of the equilibria presented in the previous subsection. As an example, we look at cases when a disruptive change happens to the system, either because of an increase in customers or a decrease in the availability of vehicles. Let $\mathbf{u} \in \mathcal{U}$ and assume $m_{\mathbf{u}} > \underline{m}$. Then, we define the set of equilibria as

$$\Upsilon_{\mathbf{u}} := \{(\mathbf{c}, \mathbf{v}) \in \mathbb{R}^{2N} \mid c_i = 0, v_i > 0, \quad \forall i \in \mathcal{N}, \text{ and} \sum\nolimits_i v_i = m - m_{\mathbf{u}}\}. \tag{5}$$

Definition 1 (Locally asymptotically stable)**.** *A set of equilibria* $\Upsilon_{\mathbf{u}}$ *is locally asymptotically stable if for an equilibrium*

$(\underline{\mathbf{c}}, \underline{\mathbf{v}}) \in \Upsilon_{\mathbf{u}}$, *there exists a neighborhood* $\mathcal{B}_{\mathbf{u}}^{\delta}(\underline{\mathbf{c}}, \underline{\mathbf{v}})$ *such that every evolution of* (1) *starting at* $(\mathbf{c}(\tau), \mathbf{v}(\tau)) = (\underline{\mathbf{c}}, \underline{\mathbf{v}})$, *and with* $(\mathbf{c}(0), \mathbf{v}(0)) \in \mathcal{B}_{\mathbf{u}}^{\delta}(\underline{\mathbf{c}}, \underline{\mathbf{v}})$ *has a limit which belongs to the equilibrium set* $\Upsilon_{\mathbf{u}}$ *i.e.,* $(\lim_{t\to+\infty} \mathbf{c}(t), \lim_{t\to+\infty} \mathbf{v}(t)) \in \Upsilon_{\mathbf{u}}$, *where* $\tau \in [-\max_{i,j} T_{ij}, 0)$ *and*

$$\mathcal{B}_{\mathbf{u}}^{\delta}(\underline{\mathbf{c}}, \underline{\mathbf{v}}) := \{(\mathbf{c}, \mathbf{v}) \in \mathbb{R}^{2N} \mid c_i > 0, v_i = \underline{v_i},\ \forall i \in \mathcal{N}, \text{ and } ||(\mathbf{c} - \underline{\mathbf{c}}, 0)|| < \delta\}. \quad (6)$$

Theorem 2 (Stability of the equilibria)**.** *Let* $\mathbf{u} \in \mathcal{U}$ *and* $m_{\mathbf{u}} > \underline{m}$*; then, the set of equilibria* $\Upsilon_{\mathbf{u}}$ *is locally asymptotically stable.*

IV. Optimal Strategies

In this section, we present an optimization framework to find optimal prices given endogenous demand rates. The model aims to maximize the revenue of an AMoD provider while ensuring load balancing of vehicles. We then present a formulation which uses rebalancing (without prices) to ensure load balancing. Finally, we combine these two ideas into a single joint model.

Optimal Pricing: We are looking for the best pricing policy that ensures the existence of an equilibrium (3). Hence, we define the feasible set of the pricing problem to be $\mathcal{F} = \{\mathbf{u} : \sum_i (\lambda_{ij}(u_{ij}) - \lambda_{ji}(u_{ji})) = 0,\ \forall j \in \mathcal{N},\ \mathbf{u} \in [1, \mathbf{u}^{\max}]\}$ and the profit maximization problem as

$$\max_{\mathbf{u}\in\mathcal{F}} \quad \sum_{ij} \lambda_{ij}(u_{ij}) u_{ij} p_{ij}^0 - c_{ij}^o \lambda_{ij}(u_{ij}) - c^c(\lambda_{ij}^0 - \lambda_{ij}(u_{ij})), \quad (7)$$

where $\lambda_{ij}(u_{ij}) u_{ij} p_{ij}^0$ and c_{ij}^o are the total revenue and the operational cost of request i to j, respectively; and c^c is an additional penalty that the AMoD service incurs when a costumer exists the platform because of a high price.

Note that if the functions $J_{ij}(u_{ij}) := \lambda_{ij}(u_{ij}) u_{ij} p_{ij}^0 - c_{ij}^o \lambda_{ij}(u_{ij}) - c^c(\lambda_{ij}^0(u_{ij}) - \lambda_{ij}(u_{ij}))$ are concave in the range of $[1, \mathbf{u}^{\max}]$, then the optimization problem is tractable (we maximize over a concave function with linear equality constraints). To ensure the concavity of the cost function J_{ij} we need its second derivative to satisfy

$$\ddot{J_{ij}} \leq 0 \implies \ddot{\lambda}_{ij}(u_{ij}) \leq -\frac{2}{u_{ij} p_{ij}^0 - c_{ij}^o - c^c} \dot{\lambda}_{ij}(u_{ij}). \quad (8)$$

Recall that by Assumption 1 (λ_{ij} is monotonically decreasing) $\dot{\lambda}_{ij} < 0$. Hence, for any linear demand function, the problem becomes tractable.

Optimal Rebalancing: We use the planning rebalancing model developed in [10]. In this setting, we aim to find a static rebalancing policy that reaches an equilibrium. Let the rebalancing flow be r_{ij}, that is, the rate at which empty vehicles flow from i to j. To solve the problem we use the following Linear Program (LP) that minimizes the empty travel time and seeks to equate the inflow and outflow of vehicles at each region by using N^2 variables

$$\min_{\mathbf{r}\geq 0} \quad \sum_{ij} T_{ij} r_{ij} \quad (9a)$$

$$\text{s.t.} \sum_i \lambda_{ij}^0 + r_{ij} - \lambda_{ji}^0 - r_{ji} = 0,\ \forall j \in \mathcal{N}. \quad (9b)$$

TABLE I: Different policies evaluated to plan the operation of an AMoD system.

Policy	Type	Formulation
$\mathcal{P}_{ij}$	Individual	(7)
$\mathcal{R}_{ij}$	Individual	(9)
$\mathcal{R}_{ij} \to \mathcal{P}_{ij}$	Sequential	(9) then (7)
$\mathcal{P}_i + \mathcal{R}_{ij}$	Joint with fixed price by origin	(10) with $u_{ij} = u_{ik}$ $\forall i, j, k \in \mathcal{N}$
$\mathcal{P}_{ij} + \mathcal{R}_{ij}$	Joint	(10)

Notice that in this case we use λ_{ij}^0 instead of $\lambda_{ij}(u_{ij})$ as we do not consider the possibility of decreasing the demand by adjusting prices. This LP is always feasible as one can always choose $r_{ij} = \lambda_{ji}^0 > 0$ for all $i, j \in \mathcal{N}$ which satisfies the set of constraints (9b). All the results presented in Section III hold for this problem as well and are studied in [10].

Joint Pricing and Rebalancing: We are interested in choosing the best policy which leverages different decisions that the MoD providers face. In particular, we would like to optimize the pricing, re-balancing and sizing problem. Then, we can write the planning optimization problem as,

$$\max_{\mathbf{u},\mathbf{r},m} \quad \sum_{ij} \lambda_{ij}(u_{ij}) u_{ij} p_{ij}^0 - c_{ij}^o \lambda_{ij}(u_{ij}) - c^c(\lambda_{ij}^0(u_{ij}) - \lambda_{ij}(u_{ij})) - c^r(r_{ij} T_{ij}) - c^f m \quad (10a)$$

$$\text{s.t.} \quad \sum_i \lambda_{ij}(u_{ij}) + r_{ij} - \lambda_{ji}(u_{ji}) - r_{ji} = 0, \quad \forall j \in \mathcal{N} \quad (10b)$$

$$\sum_{ij} T_{ij}(\lambda_{ij}(u_{ij}) + r_{ij}) \leq m, \quad (10c)$$

$$\mathbf{u} \in [1, \mathbf{u}^{\max}], \quad (10d)$$

where c^r and c^f are the cost of rebalancing and the cost of owning and maintaining a vehicle per unit time, respectively. Note that to ensure that solving (10) reaches a global maximum, we must validate that (8) holds for $\mathbf{u} \in [1, \mathbf{u}^{\max}]$.

This problem, if solvable, yields a solution with higher profits than the individual formulations of pricing (7) and (9), or the sequential approach of solving first the rebalancing problem and then adjusting prices. This happens given that the problem is *jointly* solving for $\mathbf{u}$ and $\mathbf{r}$ considering simultaneously the full objective of the profit maximization problem (10a).

V. Experiments

We carry out two case studies to assess the benefits of solving the joint problem over other approaches. Our first experiment uses a fictitious transportation network to analyze sensitivities with respect to the network size. The second one consists of a data-driven case study using historical data from New York City. We report empirical results of the achievable profit improvement of the AMoD system when solving the problem using the different methodologies presented in Table I.

We begin with the individual the policies $\mathcal{P}_{ij}$ and $\mathcal{R}_{ij}$ to see the equilibrium under a pricing policy or rebalancing strategy. We then turn to a *sequential* approach $\mathcal{R}_{ij} \to \mathcal{P}_{ij}$ which solves the problem by finding a rebalancing policy and, once the system is in equilibrium, select the prices. Our motivation for this methodology comes from the fact that many companies tend to separate their pricing and rebalancing processes, which would result in solving the problem sequentially. Note that the sequential policy $\mathcal{P}_{ij} \rightarrow \mathcal{R}_{ij}$ is not included because once the pricing problem is solved, the system is at equilibrium and the rebalancing problem becomes trivial (i.e., $\mathbf{r} = 0$). Finally, the *joint with fixed prices by origin* policy $\mathcal{P}_i + \mathcal{R}_{ij}$ is motivated by the fact that current MoD services only use the origin (not the destination) when setting surge prices [16], [17].

Note that in order to have a tractable solution for formulations (7) and (10) we require a function satisfying (8). To achieve this, we assume a linear demand (willingness-to-pay) function, specifically we let

$$\lambda_{ij}(u_{ij}) = \frac{\lambda_{ij}^0}{u_{ij}^{\max} - 1}(u_{ij}^{\max} - u_{ij}), \quad (11)$$

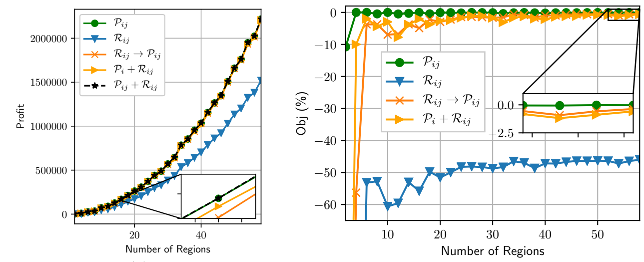
where we set $u_{ij}^{\max} = 4$ as suggested in [17]. Hence, by using this linear demand function, we get a tractable Quadratic Program (QP) with linear constraints. Arguably, linear demand functions may not be as accurate as desired for realistic implementations of this model. However, using linear functions allows us to recover a global maximum solution to the problem and assess the potential benefits that *joint* policies may achieve compared to other strategies.

For both experiments we let the *base* price be proportional to the travel time using $p_{ij}^0 = 0.5T_{ij}$, where \$0.5 is the average price a user pays in dollars per minute of taxi ride reported in [18]. Additionally, we set the operation and rebalancing cost per kilometer be $c^o = c^r = \$0.72$ as in [19]; the lost customer cost c^c is equal to \$5, and the reguralizer parameter on the fleet size to be $c^f = \$1 \times 10^{-10}$.

Uniform Demand: We compare the solution of the different methods for a network with random uniform demands. For each strategy, we let the *base* demand be $\lambda_{ij}^0 \sim U(0, 4)$ and travel time between regions be $T_{ij} \sim U(0, 40)$ for all i, j. Then, we solve the problem for networks with a number of regions ranging between 0 and 60.

Figure 2a shows the value of the cost function (10a) for each methodology. Moreover, in Figure 2b we observe the relative deviation in profits for the solution of each strategy against the joint pricing and rebalancing solution. We see that as the number of regions increases, the deviation converges to a stable value. To explain this phenomenon, we define the *potential* of region i to be the load balance deviation when no pricing or rebalancing policy is applied, namely, $\zeta_i = \sum_j \lambda_{ij}^0 - \lambda_{ji}^0$. Then, since we draw samples from the same uniform distribution to assign all $\lambda_{ij}^0\ \forall i, j \in \mathcal{N}$, the expected value of ζ_i is equal to zero for all i. Hence, this convergence behavior is simply a direct implication of the law of large numbers. Note that, for the same reason, the individual policy $\mathcal{P}_{ij}$ converges to zero.

New York City Case Study: We perform a case study of New York city using the data available at [20]. Specifically, we analyze the data set of *High Volume For-Hire Vehicle Trip Records* of November 2019 [20]. To analyze stable distributions of trips in the network, we filter the data to consider only working days (Monday to Friday). Then, we focus on four time slots: Morning Peak (AM) from 7:00-10:00 hrs, Noon (MD) from 12:00-15:00 hrs, Afternoon Peak



(a) (b)

Fig. 2: Objective function value under different number of regions (zones) and AMoD strategies, (a) Shows absolute values of (10a) when different policies are implemented while (b) plots the relative difference between the joint solution and the others.

TABLE II: Relative deviation in percentage of each policy compared to the joint strategy $\mathcal{P}_{ij} + \mathcal{R}_{ij}$ for different time slots

Policy	**AM**	**MD**	**PM**	**NT**
$\mathcal{P}_{ij}$	-29.83	-8.77	-6.64	-26.00
$\mathcal{R}_{ij}$	-33.33	-28.74	-29.20	-40.67
$\mathcal{R}_{ij} \rightarrow \mathcal{P}_{ij}$	-13.72	-9.38	-10.89	-15.75
$\mathcal{P}_i + \mathcal{R}_{ij}$	-5.3	-5.3	-5.1	-7.0

(PM) from 17:00-20:00 hrs and Night (NT) from 00:00-3:00 hrs. For every time window in November 2019, we collect data on origin-destination pairs and travel times of every trip. Then, we compute the average hourly demand and travel times, and we use these values to preform our analysis and test the different solutions.

Table II shows the deviation in profits (in percentage) between the different approaches and the joint formulation. As a reminder, Table I summarizes all policy definitions. We observe that the joint method outperforms all the other methods in the range from 5% to 40%, highlighting the benefit of solving this problem using a joint strategy. In particular, we observe that each of the individual strategies performs on average worse than strategies that optimize both pricing and rebalancing. Also, it is relevant to stress the 5% deviation of the policy with *fixed surge price by origin*, as it matches our expectations of the relevance of considering the destination when pricing.

To better understand the different approaches, we generated plots of the pricing distribution and trend. Figure 3 shows histograms comparing the value of the solution $\mathbf{u}$ for the individual pricing policy and the joint strategy. As expected, we observe the distribution of the individual approach to have higher variance than the joint. This happens given the hard constraint to reach an equilibrium. When no rebalancing is considered as in $\mathcal{P}_{ij}$ the policy chooses prices to ensure $\mathbf{u} \in \mathcal{F}$. In contrast, when solving the joint problem, the solution leverages rebalancing and pricing and gives the pricing decision more flexibility to concentrate to select values that maximize profits.

Finally, we quantify how relevant the pricing is relative to the rebalancing component when balancing the load of the system. Letting $\mathbf{r}^*$ and $\mathbf{u}^*$ be the solution of (10), we define a load dispersion metric as follows $\bar{\zeta}_0 = \frac{1}{N}\sum_i |(\sum_j \lambda_{ij}^0 - \lambda_{ji}^0)|$ when nothing is applied, $\bar{\zeta}_{\mathbf{r}} = \frac{1}{N}\sum_i |(\sum_j \lambda_{ij}^0 + r_{ij} - \lambda_{ji}^0 - r_{ij})|$ when the rebalancing component is applied, and $\bar{\zeta}_{\mathbf{u}} = \frac{1}{N}\sum_i |(\sum_j \lambda_{ij}(u_{ij}) - \lambda_{ji}(u_{ji}))|$ when the pricing component (but no rebalancing) is applied. Note that we do not define

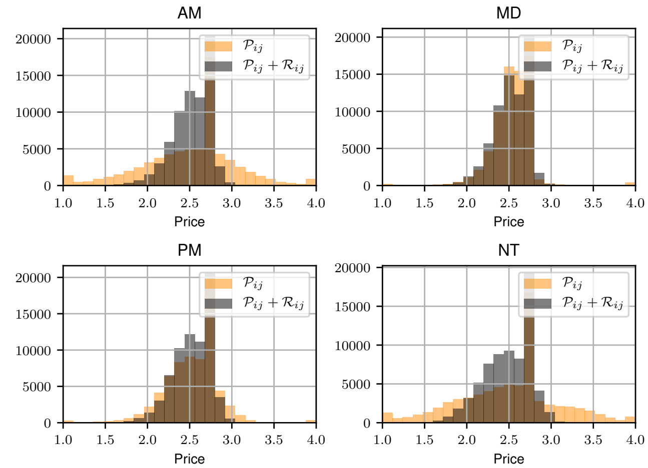


Fig. 3: Distribution of prices $\mathbf{u}^*$ for different policies at different time slots

TABLE III: Dispersion on the average absolute value of potentials when components of the joint policy $\mathbf{u}^*$ and $\mathbf{r}^*$ are applied.

	AM	MD	PM	NT
ζ_0	57.03	16.62	34.77	17.64
$\bar{\zeta}_{\mathbf{u}^*}$	20.44	4.10	6.80	6.24
$\bar{\zeta}_{\mathbf{r}^*}$	36.71	13.23	28.36	11.49

$\bar{\zeta}_{\mathbf{u},\mathbf{r}}$ as the result will be zero given that the system is at equilibrium by (10b). Table III shows this dispersion metric for the different time slots considered. Interestingly, we see that the pricing component of the policy reduces this metric in all cases, showing its relevance for load balancing the system while also maximizing profit.

VI. Conclusion

In this paper we studied how a pricing policy which considers origin-destinations can stabilize the system and reach an equilibrium in terms of balancing the load of customer and vehicles. In addition, we formulate a profit maximization optimization model which considers selecting pricing and rebalancing policies jointly. Moreover, we quantify the achievable benefits of solving the problem jointly compared to other methodologies using a data-driven case study of the New York City transportation network. Our results suggest that solving the problem jointly increases the profits of the AMoD provider by up to 40% when comparing it to individual strategies, 15% when comparing it to sequential strategies, and 7% when comparing it to a policy that restricts to a unique *surge* price per origin.

Future Work: This work can be extended as follows. First, we would like to provide a framework capable of handling more realistic nonlinear demand functions. Second, we would like to complement this model with real-time strategies through the use of a stochastic fluid model [21], as well as a discrete event system [22] with the aim to provide stochastic and microscopic results of the joint policy. Third, we are interested in coupling this joint solution with the routing problem in [1] in order to give an overall optimization framework to operate AMoD systems. Finally, we would like to solve the problem from a welfare maximization perspective rather than from the profit maximization and compare its results.

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