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Truncation of unitary operads

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ABSTRACT

We introduce truncation ideals of a $\Bbbk$-linear unitary symmetric operad and use them to study ideal structure, growth property and to classify operads of low Gelfand-Kirillov dimension.

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0. Introduction

Operad theory originates from work of Boardman-Vogt [4] and May [29] in homotopy theory in 1970s. Since then many applications of both topological and algebraic operads have been discovered in algebra, category theory, combinatorics, geometry, mathematical physics and topology [9,10,26,28]. In this paper we study operads from the algebraic viewpoint.

Following tradition, let $\mathcal{A}ss$ denote the associative algebra operad that encodes the category of unital associative algebras. (In the book [26], it is denoted by uAs.) Given an operadic ideal $\mathcal{I}$ of $\mathcal{A}ss$, one can define the quotient operad $\mathcal{A}ss/\mathcal{I}$. Quotient operads of $\mathcal{A}ss$ relate to polynomial identity algebras (PI-algebras) closely. In fact, a PI-algebra is equivalent to an algebra over $\mathcal{A}ss/\mathcal{I}$ for some nonzero operadic ideal of $\mathcal{A}ss$ [3,18]. It is worth mentioning that an operadic ideal is essentially equivalent to so-called T-ideal. For an introduction to PI-algebras and T-ideals, we refer to [30, Chapter 13].

We are mainly interested in those operads that have some common properties with $\mathcal{A}ss/\mathcal{I}$. Let $\Bbbk$ be a base field. Let $\mathcal{P} := (\mathcal{P}(n))_{n\geq 0}$ denote a $\Bbbk$-linear operad. Recall that $\mathcal{P}$ is *unitary* if $\mathcal{P}(0) = \Bbbk\mathbb{1}_0$ with a basis element $\mathbb{1}_0$ (called a 0-*unit*), see [9, Section 2.2]. Denote by Op_+ the category of unitary operads, in which a morphism preserves the 0-unit. Operads in this paper are usually unitary. We say $\mathcal{P}$ is *2-unitary*, if $\mathcal{P}$ is a unitary operad equipped a morphism $\mathcal{M}ag \to \mathcal{P}$ in Op_+, where $\mathcal{M}ag$ is the unital magmatic operad (see Subsection 8.4 or [25, Section 4.1.10]), or equivalently, there is an element $\mathbb{1}_2 \in \mathcal{P}(2)$ (call a *2-unit*) such that

$$\mathbb{1}_2 \circ (\mathbb{1}_0, \mathbb{1}) = \mathbb{1} = \mathbb{1}_2 \circ (\mathbb{1}, \mathbb{1}_0), \tag{E0.0.1}$$

where $\mathbb{1} \in \mathcal{P}(1)$ is the identity of the operad $\mathcal{P}$ and $\circ$ is composition in $\mathcal{P}$. An operad $\mathcal{P}$ is called *2a-unitary* if $\mathcal{P}$ is a unitary operad equipped with a morphism $\mathcal{A}ss \to \mathcal{P}$ in Op_+, or equivalently, $\mathcal{P}$ is 2-unitary with a 2-unit $\mathbb{1}_2$ satisfying

$$\mathbb{1}_2 \circ (\mathbb{1}_2, \mathbb{1}) = \mathbb{1}_2 \circ (\mathbb{1}, \mathbb{1}_2).$$

Note that every quotient operad $\mathcal{A}ss/\mathcal{I}$ is 2a-unitary and that there are many other interesting 2-unitary (respectively, 2a-unitary) operads [Example 2.2 and Lemma 2.3].

All operads in this paper are $\Bbbk$-linear. An operad usually means a *symmetric* operad and the word symmetric could be omitted. Plain operads are used in a few places.

0.1. Definition of truncations

Given a unitary operad $\mathcal{P}$, one can define restriction operators [9, Section 2.2.1] as follows. We are using different notation from [9]. Some explanations concerning the restriction operators are given in [9, Section 2.2]. Let $[n]$ be the set $\{1, \cdots, n\}$ and I be a subset of $[n]$. Let χ_I be the characteristic function of I, i.e. $\chi_I(x) = 1$ for $x \in I$ and

$\chi_I(x) = 0$ otherwise. Then one defines the *restriction operator* $\pi^I : \mathcal{P}(n) \to \mathcal{P}(s)$, where $s = |I|$, by

$$\pi^I(\theta) = \theta \circ (\mathbb{1}_{\chi_I(1)}, \cdots, \mathbb{1}_{\chi_I(n)})$$

for all $\theta \in \mathcal{P}(n)$. The restriction operator also appeared in many other papers, see for example, [24]. For $k \geq 1$, the *k-th truncation* of $\mathcal{P}$, denoted by ${}^k\varUpsilon$, is defined by

$${}^k\varUpsilon(n) = \begin{cases} \bigcap\limits_{I \subset [n],\, |I|=k-1} \operatorname{Ker} \pi^I, & \text{if } n \geq k; \\ 0, & \text{if } n < k. \end{cases} \tag{E0.0.2}$$

By convention, ${}^0\varUpsilon = \mathcal{P}$. The *truncation* $\{{}^k\varUpsilon\}_{k\geq 1}$ of $\mathcal{P}$ is a sequence of ideals that are naturally associated to $\mathcal{P}$. In the case of $\mathcal{P} = \mathcal{A}ss$,

$${}^1\varUpsilon = {}^2\varUpsilon = \operatorname{Ker}(\mathcal{A}ss \to \mathcal{C}om)$$

where $\mathcal{C}om$ is the commutative algebra operad defined by $\mathcal{C}om(n) = \Bbbk$ for all $n \geq 0$. We will use the truncation to study the growth of operads, as well as their ideal structure and classification of operads of low growth.

0.2. Truncations and Gelfand-Kirillov dimension

The first application of the truncations concerns the growth property. The growth of a T-ideal (in the theory of PI algebras) has been studied by many authors, see for instance [20,11–14]. This paper deals with a similar question in the framework of operad theory. Next we define the Gelfand-Kirillov dimension of an operad. For the definition of Gelfand-Kirillov dimension of an algebra, we refer to [21]. The *Gelfand-Kirillov dimension* (or *GKdimension* for short) of an operad $\mathcal{P}$ is defined to be

$$\operatorname{GKdim} \mathcal{P} := \limsup_{n\to\infty} \left(\log_n \Big(\sum_{i=0}^{n} \dim_{\Bbbk} \mathcal{P}(i)\Big) \right). \tag{E0.0.3}$$

The *exponent* of $\mathcal{P}$ is defined to be

$$\exp(\mathcal{P}) := \limsup_{n\to\infty} (\dim \mathcal{P}(n))^{\frac{1}{n}}. \tag{E0.0.4}$$

When we talk about the GKdimension or the exponent of an operad $\mathcal{P}$, we usually implicitly assume that $\mathcal{P}$ is *locally finite*, namely, $\dim_{\Bbbk} \mathcal{P}(n) < \infty$ for all $n \geq 0$. We say $\mathcal{P}$ *has polynomial growth* if $\operatorname{GKdim} \mathcal{P} < \infty$. It is easy to see that $\operatorname{GKdim} \mathcal{A}ss = \infty$ and $\operatorname{GKdim} \mathcal{C}om = 1$. The *generating series* or *Hilbert series* of $\mathcal{P}$ is defined to be

$$G_{\mathcal{P}}(t) = \sum_{n=0}^{\infty} \dim_{\Bbbk} \mathcal{P}(n) t^n \in \mathbb{Z}[[t]].$$

Also see Definition 4.1.[1]

Our first result is to give a characterization of operads that have finite GKdimension.

Theorem 0.1 *(Theorem 5.3). Let $\mathcal{P}$ be a 2-unitary operad.*

(1) *If $\mathcal{P}$ has polynomial growth, then the generating series $G_{\mathcal{P}}(t)$ is rational. As a consequence,* $\operatorname{GKdim} \mathcal{P} \in \mathbb{N}$.
(2) *$\mathcal{P}$ has polynomial growth if and only if there is an integer k such that ${}^k\Upsilon = 0$. And*

$$\operatorname{GKdim} \mathcal{P} = \max\{k \mid {}^k\Upsilon \neq 0\} + 1 = \min\{k \mid {}^k\Upsilon = 0\}.$$

Theorem 0.1(1) answers an open question (or rather fulfills an expectation) of Khoroshkin-Piontkovski [19, Expectation 3] for 2-unitary symmetric operads. When $\mathcal{P}$ has finite Gröbner basis [19], Theorem 0.1(2) is a consequence of a more general result [19, Theorem 0.1.5]. Our proof is not dependent on the Gröbner basis. It follows from Corollary 6.12 that the GKdimension of a unitary operad can be a non-integer. There are some other results concerning the *exponent* of an operad, see for example Theorem 0.8(2). In the next corollary, let $\{{}^k\Upsilon\}_{k\geq 0}$ be the truncation of $\mathcal{A}ss$.

Corollary 0.2 *(Corollary 5.4). Let $\mathcal{I}$ be an operadic ideal of $\mathcal{A}ss$ and $\mathcal{P}$ be the quotient operad $\mathcal{A}ss/\mathcal{I}$. Let k be a positive integer. Then* $\operatorname{GKdim} \mathcal{P} \leq k$ *if and only if $\mathcal{I} \supseteq {}^k\Upsilon$. In particular,*

$$\operatorname{GKdim}(\mathcal{A}ss/{}^k\Upsilon) = \begin{cases} 1, & k = 1, 2, \\ k, & k \geq 3. \end{cases}$$

0.3. Chain conditions on ideals of an operad

The second application of the truncations concerns the ideal structure of operads. We say an operad $\mathcal{P}$ is *artinian* (respectively, *noetherian*) if the set of ideals of $\mathcal{P}$ satisfies the descending chain condition (respectively, ascending chain condition).

Theorem 0.3 *(Theorem 5.6). Let $\mathcal{P}$ be a 2-unitary operad that is locally finite.*

(1) *If* $\operatorname{GKdim} \mathcal{P} < \infty$, *then $\mathcal{P}$ is noetherian.*
(2) $\operatorname{GKdim} \mathcal{P} < \infty$ *if and only if $\mathcal{P}$ is artinian.*

[1] In [26, Section 5.1.10, p. 128], the *generating series* of $\mathcal{P}$ is defined to be $E_{\mathcal{P}}(x) := \sum_{n\geq 0} \frac{\dim_{\Bbbk} \mathcal{P}(n)}{n!} x^n$, which is also called the *Hilbert-Poincaré series* of $\mathcal{P}$.

(3) [An operadic version of Hopkins' Theorem] *If $\mathcal{P}$ is artinian, then it is noetherian.*

We have a version of Artin-Wedderburn Theorem for operads. Similar to the definition given before Theorem 0.3, we can define left or right artinian operads [Definition 1.9 (2, 3)]. We say an operad $\mathcal{P}$ is *semiprime*, if it does not contain an ideal $\mathcal{N} \neq 0$ such that $\mathcal{N}^2 = 0$ [Definition 1.11 (4)]. An operad $\mathcal{P}$ is called *bounded above* if $\mathcal{P}(n) = 0$ for all $n \gg 0$. The next result contains Theorems 3.6 and 6.5.

Theorem 0.4 *(Operadic versions of Artin-Wedderburn Theorem). Suppose $\mathcal{P}$ is semiprime. In parts (1) and (2), $\mathcal{P}$ is either a plain operad or a symmetric operad. In part (3), $\mathcal{P}$ is a symmetric operad.*

(1) *If $\mathcal{P}$ is reduced and left or right artinian, then*

$$\mathcal{P}(n) = \begin{cases} 0, & n \neq 1, \\ \Lambda, & n = 1, \end{cases}$$

where Λ is a semisimple algebra.

(2) *If $\mathcal{P}$ is unitary, bounded above, and left or right artinian, then*

$$\mathcal{P}(n) = \begin{cases} 0, & n \neq 0, 1, \\ \Bbbk, & n = 0, \\ \Lambda, & n = 1, \end{cases}$$

where Λ is an augmented semisimple algebra.

(3) *If $\mathcal{P}$ is 2-unitary and left or right artinian, then $\mathcal{P}$ is as in Example 2.4(1) and $\mathcal{P}(1)$ is an augmented semisimple algebra.*
If, further, $\mathcal{P}(1)$ is finite dimensional over $\Bbbk$, then $\mathcal{P}$ is locally finite, $\operatorname{GKdim} \mathcal{P} = 2$ or $\operatorname{GKdim} \mathcal{P} = 1$ (and hence $\mathcal{P} = \mathcal{C}om$), and $\mathcal{P}(1)$ is a finite dimensional augmented semisimple algebra.

Note that there are unitary and left (or right) artinian operads that are not bounded above. Such examples are given in Example 2.4(2).

0.4. Classifications of operads of low Gelfand-Kirillov dimension

The third application of truncations concerns classifications of 2-unitary operads. The classification of 2-unitary operads of GKdimension 1 is easy.

Proposition 0.5 *(Proposition 2.12). Let $\mathcal{P}$ be a (symmetric or plain) 2-unitary operad. If $\operatorname{GKdim}(\mathcal{P}) < 2$, then $\mathcal{P} \cong \mathcal{C}om$.*

A 2-unitary operad consists of a triple $(\mathcal{P}, \mathbb{1}_0, \mathbb{1}_2)$ satisfying (E0.0.1). A morphism between two 2-unitary operads means a morphism of operads that preserves $\mathbb{1}_0$ and $\mathbb{1}_2$. All 2-unitary operads form a category with morphisms being defined as above.

Theorem 0.6 *(Theorem 6.3). There are natural equivalences between*

(a) *the category of finite dimensional, not necessarily unital, $\Bbbk$-algebras;*
(b) *the category of 2-unitary operads of GKdimension ≤ 2;*
(c) *the category of 2a-unitary operads of GKdimension ≤ 2.*

At this point we have not found any 2-unitary plain operad of GKdimension two that is not a symmetric operad. It would be nice to show that every 2-unitary plain operad of GKdimension 2 admits a natural $\mathbb{S}$-module structure making it a symmetric operad.

Note that the category in Theorem 0.6(1) is equivalent to the category of finite dimensional unital augmented $\Bbbk$-algebras. The description of operads in the above theorem is given in Example 2.4(1).

For quotient operads of $\mathcal{A}ss$, we can classify a few more operads with small GKdimension.

Theorem 0.7 *(Theorem 6.6). Suppose* $\operatorname{char} \Bbbk = 0$*. Let $\mathcal{P}$ be a quotient operad of $\mathcal{A}ss$ and* $\operatorname{GKdim} \mathcal{P} = n$*. Let ${}^k\Upsilon$ be the k-th truncation of $\mathcal{A}ss$.*

(1) [Proposition 0.5] *If $n = 1$, $\mathcal{P} = \mathcal{A}ss/{}^1\Upsilon \cong \mathcal{C}om$.*
(2) [Gap Theorem] $\operatorname{GKdim} \mathcal{P}$ *can not be* 2, (*so can not be strictly between* 1 *and* 3).
(3) *If $n = 3$, then $\mathcal{P} = \mathcal{A}ss/{}^3\Upsilon$.*
(4) *If $n = 4$, then $\mathcal{P} = \mathcal{A}ss/{}^4\Upsilon$.*
(5) *There are at least two non-isomorphic quotient operads $\mathcal{P}$ such that* $\operatorname{GKdim} \mathcal{P} = 5$.

0.5. Other results related to truncations

We list two other results related to the truncations indirectly. In Theorem 0.9, operads $\mathcal{P}$ need not be unitary.

Using the Hilbert series of an operad $\mathcal{P}$, one can define another numerical invariant, *signature* of $\mathcal{P}$, denoted by $\mathcal{S}(\mathcal{P})$ [Definition 6.1]. Let $\mathcal{C}om \downarrow \mathsf{Op}_+$ denote the category of operads with a morphism $\mathcal{C}om \to \mathcal{P}$. (More precisely, an object in $\mathcal{C}om \downarrow \mathsf{Op}_+$ is a unitary operad with a morphism $\mathcal{C}om \to \mathcal{P}$ in Op_+ and morphisms are the commutative triangles.) Every operad in $\mathcal{C}om \downarrow \mathsf{Op}_+$ is canonically 2a-unitary, inherited from $\mathcal{C}om$. We prove the following

Theorem 0.8 *(Theorem 6.11). Let $\mathcal{C}om \downarrow \mathsf{Op}_+$ be defined as above.*

(1) *For every sequence of non-negative integers* $\mathbf{d} := \{d_1, d_2, \cdots\}$, *there is an operad* $\mathcal{P}$ *in* $\mathcal{C}om{\downarrow}\mathsf{Op}_+$ *such that* $\dim {}^k\Upsilon(k) = d_k$ *for all* $k \geq 1$.
(2) *Exponent* exp *of* (E0.0.4) *is a surjective map from* $\mathcal{C}om \downarrow \mathsf{Op}_+$ (*or the category of 2-unitary operads*) *to* $\{1\} \cup [2, \infty]$.

For a 2-unitary operad $\mathcal{P}$ with infinite GKdimension, we can show that $\exp(\mathcal{P}) \geq 2$. This implies that there are no 2-unitary operads that have subexponential growth [Definition 4.1(5)]. On the other hand, there are many unitary operads having subexponential growth [Example 2.2(3)]. Theorem 0.8(2) says that exp of an 2-unitary operad can be any real number larger than 2. However, for 2-unitary Hopf operads, we don't have any example that has non-integer exp.

The next result is a connection between the GKdimension of an operad and the GKdimension of finitely generated algebras over it.

Theorem 0.9 *(Theorem 5.9). Let* $\mathcal{P}$ *be an operad and* A *be an algebra over* $\mathcal{P}$. *Suppose* A *is generated by* g *elements as an algebra over* $\Bbbk$. *Then*

$$\operatorname{GKdim} A \leq g - 1 + \operatorname{GKdim} \mathcal{P}.$$

When $\mathcal{P}$ is the commutative algebra operad $\mathcal{C}om$, then the above theorem gives rise to a well-known fact that the GKdimension of a commutative algebra A is bounded by the number of generators of A [Example 5.10]. Note that every finitely generated PI-algebra has finite GKdimension, see for instance [20,8].

The theory of operads provides a unified approach to several different topics. Operads are also closely related to *clones* in universal algebra [34,7] and *species* in combinatorics [17,1,2]. Some ideas presented in this paper can be adapted to study both clones and species.

The paper is organized as follows. We recall some basic concepts in Section 1. In Section 2, we study basic properties of 2-unitary operads, and prove some lemmas that are needed in later sections. One of the main examples is given in Example 2.4. Proposition 0.5 is proved in Section 2. The main object of this paper, the sequence of truncation ideals, is defined in Section 3. As an application of truncations, a basis theorem is proved in Section 4. Binomial transform of generating series is defined in Section 5. Theorems 0.1, 0.9, 0.3 and Corollary 0.2 are proved in Section 5. In Section 6, we study the signature of an operad. Theorems 0.6, 0.7 and 0.8 are proved in Section 6. Theorem 0.4 is proved in Sections 3 and 6. In Section 7 we introduce the notion of a truncatified operad. Some basic material is reviewed in Section 8 (Appendix).

1. Preliminaries

Throughout let $\Bbbk$ be a fixed base field, and all unadorned $\otimes$ will be $\otimes_{\Bbbk}$. In this section, we recall some basic facts about operads from standard books such as [26] and [9,10]. Also see Section 8 for some extra material.

1.1. Operads

An algebraic structure of a certain type is usually defined by generating operations and relations, see for instance, the definition for associative algebras, commutative algebras, Lie algebras and so on. Given a type of algebras, the set of operations generated by the ones defining this algebra structure will form an operad, and an algebra of this type is exactly given by a set (or a vector space) together with an action of the operad on it. Roughly speaking, an operad can be viewed as a set of operations, each of which has a fixed number of inputs and one output, satisfying a set of compatibility laws.

In this paper we consider operads over $\Bbbk$-vector spaces. We now recall the *classical definition* of an operad. Usually the word "symmetric" is omitted in this paper.

Definition 1.1. Most of the following definitions are copied from [26, Chapter 5].

(1) A *plain operad* (sometimes called a *non-*Σ or *non-symmetric* operad) consists of the following data:
  (i) a sequence $(\mathcal{P}(n))_{n\geq 0}$ of sets, whose elements are called n*-ary operations*,
  (ii) an element $\mathbb{1} \in \mathcal{P}(1)$ called the *identity*,
  (iii) for all integers $n \geq 1$, $k_1, \cdots, k_n \geq 0$, a *composition map*

$$\begin{gathered} \circ \colon \mathcal{P}(n) \times \mathcal{P}(k_1) \times \cdots \times \mathcal{P}(k_n) \longrightarrow \mathcal{P}(k_1 + \cdots + k_n) \\ (\theta, \theta_1, \cdots, \theta_n) \mapsto \theta \circ (\theta_1, \cdots, \theta_n), \end{gathered}$$

satisfying the following coherence axioms:
(OP1) (Identity)

$$\theta \circ (\mathbb{1}, \mathbb{1}, \cdots, \mathbb{1}) = \theta = \mathbb{1} \circ \theta;$$

(OP2) (Associativity)

$$\begin{aligned} &\theta \circ (\theta_1 \circ (\theta_{1,1}, \cdots, \theta_{1,k_1}), \cdots, \theta_n \circ (\theta_{n,1}, \cdots, \theta_{n,k_n})) \\ &= (\theta \circ (\theta_1, \cdots, \theta_n)) \circ (\theta_{1,1}, \cdots, \theta_{1,k_1}, \cdots, \theta_{n,1}, \cdots, \theta_{n,k_n}), \end{aligned}$$

where in the left hand side, $\theta_i \circ (\theta_{i,1}, \cdots, \theta_{i,k_i}) = \theta_i$ in case $k_i = 0$.

(2) A plain operad $\mathcal{P}$ is called an *operad* (or a *symmetric operad*), if there exists a right action $*$ of the symmetric group $\mathbb{S}_n$ on $\mathcal{P}(n)$ for each n, satisfying the following compatibility condition:
(OP3) (Equivariance)

$$\begin{aligned} &(\theta * \sigma) \circ (\theta_1 * \sigma_1, \cdots, \theta_n * \sigma_n) \\ =&(\theta \circ (\theta_{\sigma^{-1}(1)}, \cdots, \theta_{\sigma^{-1}(n)})) * \vartheta_{n;k_1,\cdots,k_n}(\sigma, \sigma_1, \cdots, \sigma_n), \end{aligned}$$

where $\vartheta_{n;k_1,\cdots,k_n}$ is defined in Section 8.

(3) An operad (respectively, a plain operad) is said to be $\Bbbk$-*linear* if $\mathcal{P}(n)$ is a $\Bbbk\mathbb{S}_n$-module (respectively, a $\Bbbk$-module) for each n and all composition maps are $\Bbbk$-multilinear.
(4) A $\Bbbk$-linear operad is called *unitary* if $\mathcal{P}(0) = \Bbbk\mathbb{1}_0 \cong \Bbbk$, which is the unit object in the symmetric monoidal category $\mathrm{Vect}_{\Bbbk}$. Here $\mathbb{1}_0$ is a basis for $\mathcal{P}(0)$ and is called a 0-*unit* of $\mathcal{P}$.
(5) If $\mathcal{P}(0) = 0$, $\mathcal{P}$ is called *reduced*.
(6) If $\mathcal{P}(1) = \Bbbk$, $\mathcal{P}$ is called *connected*.

Unless otherwise stated, all operads considered here will be $\Bbbk$-linear. In some occasions, it will be more convenient to use another definition, called the *partial definition* of an operad.

Definition 1.2 *([9, Section 2.1], [26, Section 5.3.4]).* An *operad* consists of the following data:

(i) a sequence $(\mathcal{P}(n))_{n\geq 0}$ of right $\Bbbk\mathbb{S}_n$-modules, whose elements are called *n-ary operations*,
(ii) an element $\mathbb{1} \in \mathcal{P}(1)$ called the *identity*,
(iii) for all integers $m \geq 1$, $n \geq 0$, and $1 \leq i \leq m$, a *partial composition map*

$$- \underset{i}{\circ} - \colon \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1) \quad (1 \leq i \leq m),$$

satisfying the following axioms:

(OP1′) (Identity)
for $\theta \in \mathcal{P}(n)$ and $1 \leq i \leq n$,

$$\theta \underset{i}{\circ} \mathbb{1} = \theta = \mathbb{1} \underset{1}{\circ} \theta;$$

(OP2′) (Associativity)
for $\lambda \in \mathcal{P}(l)$, $\mu \in \mathcal{P}(m)$ and $\nu \in \mathcal{P}(n)$,

$$\begin{cases} (\lambda \underset{i}{\circ} \mu) \underset{i-1+j}{\circ} \nu = \lambda \underset{i}{\circ} (\mu \underset{j}{\circ} \nu), & 1 \leq i \leq l, 1 \leq j \leq m, \\ (\lambda \underset{i}{\circ} \mu) \underset{k-1+m}{\circ} \nu = (\lambda \underset{k}{\circ} \nu) \underset{i}{\circ} \mu, & 1 \leq i < k \leq l; \end{cases}$$

(OP3′) (Equivariance)
for $\mu \in \mathcal{P}(m)$, $\phi \in \mathbb{S}_m$, $\nu \in \mathcal{P}(n)$ and $\sigma \in \mathbb{S}_n$,

$$\begin{cases} \mu \underset{i}{\circ} (\nu * \sigma) = & (\mu \underset{i}{\circ} \nu) * \sigma', \\ (\mu * \phi) \underset{i}{\circ} \nu = & (\mu \underset{\phi(i)}{\circ} \nu) * \phi'', \end{cases}$$

where

$$\begin{aligned} \sigma' &= \vartheta_{m;1,\cdots,1,\underset{i}{n},1,\cdots,1}(1_m, 1_1, \cdots, 1_1, \underset{i}{\sigma}, 1_1, \cdots, 1_1), \\ \phi'' &= \vartheta_{m;1,\cdots,1,\underset{i}{n},1,\cdots,1}(\phi, 1_1, \cdots, 1_1, \underset{i}{1_n}, 1_1 \cdots, 1_1) \end{aligned} \tag{E1.2.1}$$

(see (E8.1.3) for the definition of $\vartheta_{m;1,\cdots,1,\underset{i}{n},1,\cdots,1}$).

Remark 1.3. The above two definitions for operads are equivalent by [26, Proposition 5.3.4]. Let $\mathcal{P}$ be an operad in the sense of Definition 1.1. Then the partial compositions

$$- \underset{i}{\circ} -: \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1) \ \ (1 \le i \le m)$$

associated to $\mathcal{P}$ are defined by

$$\mu \underset{i}{\circ} \nu = \mu \circ (\mathbb{1}_1, \cdots, \mathbb{1}_1, \underset{i}{\nu}, \mathbb{1}_1, \cdots, \mathbb{1}_1).$$

Conversely, let $\mathcal{P}$ be an operad in the sense of Definition 1.2, then one can define composition maps by

$$\theta \circ (\theta_1, \cdots, \theta_n) = (\cdots((\theta \underset{n}{\circ} \theta_n) \underset{n-1}{\circ} \theta_{n-1}) \underset{n-2}{\circ} \theta_{n-2} \cdots) \underset{1}{\circ} \theta_1.$$

One can show that the axioms (OP1)-(OP3) are equivalent to the axioms (OP1′)-(OP3′) respectively.

We will use the partial definition in several examples in later sections.

Example 1.4. [26, Section 5.2.11] For every $\Bbbk$-vector space V, the sequence $(\mathcal{E}nd_V(n))_{n\ge 0}$ together with the composition map defined as in (E8.1.6) gives rise to an operad, which is denoted by $\mathcal{E}nd_V$. We call $\mathcal{E}nd_V$ the *endomorphism operad* of V. It is easy to see that $\mathcal{E}nd_V$ is not unitary unless $V = \Bbbk$.

If $\mathcal{T}$ is a $\Bbbk$-linear symmetric monoidal category with internal hom-bifunctor

$$\mathrm{Hom}_{\mathcal{T}}(-,-) : \mathcal{T}^{op} \times \mathcal{T} \to \mathcal{T},$$

then endomorphism operad $\mathcal{E}nd_V$ can be defined for any object $V \in \mathcal{T}$. Some results in this paper can be extended from $\mathrm{Vect}_{\Bbbk}$ to $\mathcal{T}$.

Let $\mathcal{P}, \mathcal{P}'$ be ($\Bbbk$-linear) operads. A *morphism* from $\mathcal{P}$ to $\mathcal{P}'$ is a sequence of $\mathbb{S}_n$-morphism $\gamma = (\gamma_n \colon \mathcal{P}(n) \to \mathcal{P}'(n))_{n\ge 0}$, satisfying

$$\gamma(\mathbb{1}) = \mathbb{1}'$$

where $\mathbb{1}$ and $\mathbb{1}'$ are identities of $\mathcal{P}$ and $\mathcal{P}'$, respectively, and

$$\gamma(\theta \circ (\theta_1, \cdots, \theta_n)) = \gamma(\theta) \circ (\gamma(\theta_1), \cdots, \gamma(\theta_n))$$

for all $\theta, \theta_1, \cdots \theta_n$.

We denote by Op the category of operads. The category of unitary operads is denoted by Op_+, in which morphisms are operadic morphisms preserving 0-units.

Recall that $\mathcal{M}ag$ and $\mathcal{A}ss$ are the operads governing the unital magmatic and unital associative algebra, respectively. See Sections 8.3 and 8.4 for details.

Definition 1.5. Retain the above notation.

(1) A *2-unitary operad* $\mathcal{P}$ is a unitary operad $\mathcal{P}$ equipped with a morphism $\mathcal{M}ag \to \mathcal{P}$ in Op_+.
(2) A *2a-unitary operad* $\mathcal{P}$ is a unitary operad $\mathcal{P}$ equipped with a morphism $\mathcal{A}ss \to \mathcal{P}$ in Op_+.

Let $\mathcal{P}$ and $\mathcal{P}'$ be 2-unitary operads. A morphism of 2-unitary operads is a morphism $\gamma\colon \mathcal{P} \to \mathcal{P}'$ in Op_+ satisfying the following commutative diagram

$$\begin{tikzcd} & \mathcal{M}ag \arrow[dl] \arrow[dr] & \\ \mathcal{P} \arrow[rr, "\gamma"] & & \mathcal{P}' \end{tikzcd}$$

The category of 2-unitary operads is denoted by $\mathcal{M}ag \downarrow \mathsf{Op}_+$. Similarly, one can define the category of 2a-unitary operads, denoted by $\mathcal{A}ss \downarrow \mathsf{Op}_+$.

1.2. Algebras and free algebras over an operad

Given a type of algebras, there is a notion of “free” algebras, which can be constructed by using the associated operad.

Definition 1.6. [26, Sections 5.2.1 and 5.2.3] An *algebra over* $\mathcal{P}$, or a $\mathcal{P}$*-algebra* for short, is a $\Bbbk$-vector space A equipped with a morphism $\gamma\colon \mathcal{P} \to \mathcal{E}nd_A$. Also see [9, Proposition 1.1.15].

Let $\mathcal{P}$ be an operad and V a $\Bbbk$-vector space. Set

$$\mathcal{P}(V)_n = \mathcal{P}(n) \otimes_{\Bbbk\mathbb{S}_n} V^{\otimes n}, \qquad \mathcal{P}(V) = \bigoplus_{n\geq 0} \mathcal{P}(V)_n$$

where a pure tensor $\theta \otimes x_1 \otimes \cdots \otimes x_n$ in $\mathcal{P}(n) \otimes_{\Bbbk\mathbb{S}_n} V^{\otimes n}$ is denoted by $[\theta, x_1, \cdots, x_n]$. Then we have

$$\mathcal{P}(V)^{\otimes n} = \bigoplus_{m \geq 0} \bigoplus_{k_1 + \cdots + k_n = m} \mathcal{P}(V)_{k_1} \otimes \cdots \otimes \mathcal{P}(V)_{k_n}.$$

The composition in $\mathcal{P}$ gives a linear map

$$\gamma_n \colon \mathcal{P}(n) \to \mathrm{Hom}_{\Bbbk}\left(\bigoplus_{k_1 + \cdots + k_n = m} \mathcal{P}(V)_{k_1} \otimes \cdots \otimes \mathcal{P}(V)_{k_n}, \mathcal{P}(V)_m \right)$$

$$\begin{aligned}\gamma_n(\theta)([\theta_1, x_{1,1}, \cdots, x_{1,k_1}] \otimes \cdots \otimes [\theta_n, x_{n,1}, \cdots, x_{n,k_n}])& \\ = [\theta \circ (\theta_1, \cdots, \theta_n), x_{1,1}, \cdots, x_{1,k_1}, \cdots, x_{n,1}, \cdots, x_{n,k_n}]&,\end{aligned}$$

which extends to a linear map $\gamma_n \colon \mathcal{P}(n) \to \mathcal{E}nd_{\mathcal{P}(V)}(n)$. One can check that γ_n is well defined and the sequence $\gamma = (\gamma_n)_{n \geq 0}$ is a morphism of operads, i.e., $\mathcal{P}(V)$ is a $\mathcal{P}$-algebra. We mention that $\mathcal{P}(V)$ is a *free $\mathcal{P}$-algebra* in the following sense.

Proposition 1.7. *[26, Proposition 5.2.1] Let A be a $\mathcal{P}$-algebra, and V a $\Bbbk$-vector space. Then every linear map $f \colon V \to A$ extends uniquely to a morphism $f \colon \mathcal{P}(V) \to A$ of $\mathcal{P}$-algebras.*

Remark 1.8. The above proposition can be restated as follows. Given an operad $\mathcal{P}$, the functor $V \mapsto \mathcal{P}(V)$ is a left adjoint to the forgetful functor from the category of $\mathcal{P}$-algebras to the category $\mathrm{Vect}_{\Bbbk}$ of $\Bbbk$-vector spaces.

1.3. Operadic ideals and quotient operads

We denote by $\mathbb{S}$ the disjoint union of all $\mathbb{S}_n$, $n \geq 0$. We call a family

$$\mathcal{M} = (\mathcal{M}(0), \mathcal{M}(1), \cdots, \mathcal{M}(n), \cdots)$$

of right $\Bbbk\mathbb{S}_n$-modules $\mathcal{M}(n)$ a (right) $\mathbb{S}$*-module over* $\Bbbk$. Thus a $\Bbbk$-linear operad is an $\mathbb{S}$-module over $\Bbbk$ equipped with a family of suitable composition maps.

An $\mathbb{S}$-submodule $\mathcal{N}$ of $\mathcal{M}$ is a sequence $\mathcal{N} = (\mathcal{N}(n))_{n \geq 0}$, where each $\mathcal{N}(n)$ is an $\mathbb{S}_n$-submodule of $\mathcal{M}(n)$. Given $\mathcal{M}, \mathcal{N}$, one defines the quotient $\mathbb{S}$-module $\mathcal{M}/\mathcal{N}$ by setting $(\mathcal{M}/\mathcal{N})(n) = \mathcal{M}(n)/\mathcal{N}(n)$.

Definition 1.9. Let $\mathcal{P}$ be an operad and $\mathcal{I}$ is a $\mathbb{S}$-submodule of $\mathcal{P}$.

(1) [26, Section 5.2.14]. We call $\mathcal{I}$ an *operadic ideal* (or simply *ideal*) of $\mathcal{P}$ if the operad structure on $\mathcal{P}$ passes to $\mathcal{P}/\mathcal{I}$. In this case, $\mathcal{P}/\mathcal{I}$ is called a *quotient operad* of $\mathcal{P}$. More explicitly, $\mathcal{I}$ is an ideal if and only if

$$\mathcal{I}(n) \circ (\mathcal{P}(k_1), \cdots, \mathcal{P}(k_n)) \subseteq \mathcal{I}(k_1 + \cdots + k_n)$$

and

$$\mathcal{P}(n) \circ (\mathcal{P}(k_1), \cdots, \mathcal{P}(k_{s-1}), \mathcal{I}(k_s), \mathcal{P}(k_{s+1}), \cdots, \mathcal{P}(k_n)) \subseteq \mathcal{I}(k_1 + \cdots + k_n)$$

for all $n > 0, k_1, \cdots, k_n \geq 0$. In other words, for any family of operations $\theta, \theta_1, \cdots, \theta_n$, if one of them is in $\mathcal{I}$, then so is $\theta \circ (\theta_1, \cdots, \theta_n)$.

(2) An $\mathbb{S}$-submodule $\mathcal{I}$ of $\mathcal{P}$ is called a *right ideal* of $\mathcal{P}$, if for every $\lambda \in \mathcal{I}(m)$ and $\mu \in \mathcal{P}(n)$, $\lambda \underset{i}{\circ} \mu \in \mathcal{I}(m+n-1)$ for every $1 \leq i \leq m$. We say $\mathcal{P}$ is *right artinian* if the set of right ideals of $\mathcal{P}$ satisfies the descending chain condition.

(3) An $\mathbb{S}$-submodule $\mathcal{I}$ of $\mathcal{P}$ is called a *left ideal* of $\mathcal{P}$, if for every $\lambda \in \mathcal{P}(m)$ and $\mu \in \mathcal{I}(n)$, $\lambda \underset{i}{\circ} \mu \in \mathcal{I}(m+n-1)$ for every $1 \leq i \leq m$. We say $\mathcal{P}$ is *left artinian* if the set of left ideals of $\mathcal{P}$ satisfies the descending chain condition.

It is easy to see that $\mathcal{I}$ is an ideal if and only if it is both a left and a right ideal.

Let $\{\mathcal{I}^j\}_{j\in J}$ be a family of ideals of $\mathcal{P}$. Let $\sum_{j\in J} \mathcal{I}^j$ and $\bigcap_{j\in J} \mathcal{I}^j$ be the $\mathbb{S}$-modules given by

$$(\sum\nolimits_{j\in J} \mathcal{I}^j)(n) = \sum\nolimits_{j\in J} \mathcal{I}^j(n), \qquad (\bigcap\nolimits_{j\in J} \mathcal{I}^j)(n) = \bigcap\nolimits_{j\in J} \mathcal{I}^j(n)$$

for all $n \geq 0$. The following lemmas are easy and their proofs are omitted.

Lemma 1.10. *Let $\{\mathcal{I}^j\}_{j\in J}$ be a family of ideals (respectively, left or right ideals) of an operad $\mathcal{P}$. Then both $\sum_{j\in J} \mathcal{I}^j$ and $\bigcap_{j\in J} \mathcal{I}^j$ are ideals (respectively, left or right ideals) of $\mathcal{P}$.*

Let $\mathcal{I}$ and $\mathcal{J}$ be $\mathbb{S}$-submodules (or ideals) of $\mathcal{P}$. The product $\mathcal{I}\mathcal{J}$ is defined to be the $\mathbb{S}$-submodule of $\mathcal{P}$ generated by elements of the form $\mu \underset{i}{\circ} \nu$ for all possible $\mu \in \mathcal{I}(m)$, $\nu \in \mathcal{J}(n)$ and $1 \leq i \leq m$.

Definition 1.11. Let $\mathcal{P}$ be an operad.

(1) Let X be a property that is defined on operads (or a class of operads). We define X-radical of $\mathcal{P}$ to be

$$X\,\mathrm{rad}(\mathcal{P}) := \bigcap \{\mathcal{I} \mid \mathcal{P}/\mathcal{I} \text{ has property } X\}.$$

(2) For example, if $(GK \leq k)$ denotes the property that the GKdim of $\mathcal{P}$ is no more than k, then

$$(GK \leq k)\,\mathrm{rad}(\mathcal{P}) := \bigcap \{\mathcal{I} \mid \mathrm{GKdim}(\mathcal{P}/\mathcal{I}) \leq k\}.$$

(3) We say $\mathcal{P}$ is *semiprime* if $\mathcal{P}$ does not contain an ideal $\mathcal{N} \neq 0$ such that $\mathcal{N}^2 = 0$.

(4) If p denotes the property of $\mathcal{P}$ being semiprime, then

$$p.\operatorname{rad}(\mathcal{P}) := \bigcap\{\mathcal{I} \mid \mathcal{P}/\mathcal{I} \text{ does not contain an ideal } \mathcal{N} \neq 0 \text{ such that } \mathcal{N}^2 = 0\}.$$

Lemma 1.12. *Let $\mathcal{I}$ and $\mathcal{J}$ be $\mathbb{S}$-submodules of an operad $\mathcal{P}$.*

(1) *If $\mathcal{I}$ and $\mathcal{J}$ are right ideals of $\mathcal{P}$, then so is $\mathcal{I}\mathcal{J}$.*
(2) *If $\mathcal{I}$ is a left ideal of $\mathcal{P}$, then so is $\mathcal{I}\mathcal{J}$.*
(3) *If $\mathcal{I}$ is an ideal of $\mathcal{P}$ and $\mathcal{J}$ is a right ideal of $\mathcal{P}$, then $\mathcal{I}\mathcal{J}$ is an ideal of $\mathcal{P}$.*

We conclude this section with the following fact. Recall that $\mathcal{C}om$ denotes the operad that encodes the category of unital commutative algebras, namely, $\mathcal{C}om(n) = \Bbbk$ for all $n \geq 0$. Let $\mathcal{U}ni$ be the trivial unitary operad defined by

$$\mathcal{U}ni(n) = \begin{cases} \Bbbk \mathbb{1}_0 \cong \Bbbk, & n = 0, \\ \Bbbk \mathbb{1}_1 \cong \Bbbk, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

Lemma 1.13.

(1) *[9, Proposition 2.2.21] The operad $\mathcal{C}om$ is the terminal object in the category of unitary operads.*
(2) *The operad $\mathcal{U}ni$ is the initial object in the category of unitary operads.*

2. Unitary and 2-unitary operads

Let $\mathcal{P}$ be a unitary operad with a fixed 0-unit $\mathbb{1}_0 \in \mathcal{P}(0)$. An element $\mathbb{1}_2 \in \mathcal{P}(2)$ is called a *right 2-unit* if

$$\mathbb{1}_2 \circ (\mathbb{1}, \mathbb{1}_0) = \mathbb{1}. \tag{E2.0.1}$$

An element $\mathbb{1}_2 \in \mathcal{P}(2)$ is called a *left 2-unit* if

$$\mathbb{1}_2 \circ (\mathbb{1}_0, \mathbb{1}) = \mathbb{1}. \tag{E2.0.2}$$

If both (E2.0.1) and (E2.0.2) hold for the same $\mathbb{1}_2$, then it is called a *2-unit.*

Recall from Definition 1.5 that a 2-unitary operad is a unitary operad $\mathcal{P}$ equipped with a morphism $\varphi\colon \mathcal{M}ag \to \mathcal{P}$ in Op_+, where $\mathcal{M}ag$ is the unital magmatic algebra operad.

Lemma 2.1. *Let $\mathcal{P}$ be a unitary operad with a 0-unit $\mathbb{1}_0$.*

(1) *$\mathcal{P}$ is 2-unitary if and only if it has a 2-unit $\mathbb{1}_2$.*

(2) *$\mathcal{P}$ is a 2a-unitary if and only if it has a 2-unit $\mathbb{1}_2$ satisfying*

$$\mathbb{1}_2 \underset{1}{\circ} \mathbb{1}_2 = \mathbb{1}_2 \underset{2}{\circ} \mathbb{1}_2. \tag{E2.1.1}$$

Proof. (1) Let $\mathcal{P}$ be a 2-unitary operad with operadic morphism $\varphi\colon \mathcal{M}ag \to \mathcal{P}$, where, by the convention in Subsection 8.4, $\mathcal{M}ag = (\Bbbk u, \Bbbk \mathbb{1}, \Bbbk \nu, \cdots)$ is the unital magmatic algebra operad. Denote

$$\mathbb{1}_0 = \varphi_0(u) \in \mathcal{P}(0), \quad \mathbb{1}_2 = \varphi_2(\nu) \in \mathcal{P}(2).$$

Since $\nu \underset{i}{\circ} u = \mathbb{1}$ for $i = 1, 2$, we have

$$\mathbb{1}_2 \underset{i}{\circ} \mathbb{1}_0 = \varphi_1(\nu \underset{i}{\circ} u) = \varphi(\mathbb{1}) = \mathbb{1}_{\mathcal{P}}.$$

Therefore, $\mathbb{1}_2$ is a 2-unit of $\mathcal{P}$.

Conversely, if $\mathcal{P}$ has a 2-unit $\mathbb{1}_2$, then one can define a morphism $\varphi\colon \mathcal{M}ag \to \mathcal{P}$ in Op_+ by $\varphi_0(u) = \mathbb{1}_0$ and $\varphi_2(\nu) = \mathbb{1}_2 \neq 0$.

(2) The proof is similar to part (1) and we omit it. □

Note that a 2-unit may not be unique. For example, if $\mathbb{1}_2$ is a 2-unit, then so is $\mathbb{1}_2 * (12)$, where (12) is the non-identity element in $\mathbb{S}_2$.

Suggested by (E2.0.1)-(E2.0.2), sometimes we denote $\mathbb{1}$ by $\mathbb{1}_1$. It is easy to see that (E2.0.1) implies that

$$\mathbb{1}_2 \circ (\theta, \mathbb{1}_0) = \theta \tag{E2.1.2}$$

for all $\theta \in \mathcal{P}(n)$ and that (E2.0.2) implies that

$$\mathbb{1}_2 \circ (\mathbb{1}_0, \theta) = \theta \tag{E2.1.3}$$

for all $\theta \in \mathcal{P}(n)$.

2.1. Examples of 2-unitary operads

Example 2.2. Parts (1) and (2) are examples of 2-unitary operads and part (3) is an example of unitary operad.

(1) There are many commonly-used 2-unitary operads from textbooks, such as the unital magmatic algebra operad $\mathcal{M}ag$, unitary operads $\mathcal{A}ss$ and $\mathcal{C}om$, the unitary A_∞-algebra operad (denoted by $\mathcal{A}_\infty$), the Poisson operad (denoted by $\mathcal{P}ois$), the operad governing unital dg associative algebras.
(2) One can easily show that every quotient operad of a 2-unitary operad is again 2-unitary.

(3) Let $\mathcal{M}$ be an $\mathbb{S}$-module with $\mathcal{M}(0) = 0$. Then $\mathcal{U}ni \oplus \mathcal{M}$ is an unitary operad with partial composition defined by

$$\begin{aligned} \mathbb{1}_1 \underset{1}{\circ} \theta = \theta = \theta \underset{i}{\circ} \mathbb{1}_1, & \qquad \forall \theta \in \mathcal{M}, \\ \theta \underset{i}{\circ} \mathbb{1}_0 = 0, & \qquad \forall \theta \in \mathcal{M}, \\ \theta_1 \underset{i}{\circ} \theta_2 = 0, & \qquad \forall \theta_1, \theta_2 \in \mathcal{M}. \end{aligned}$$

One can use the partial definition to check that this operad is unitary, but not 2-unitary.

Of course, any non-unitary operads can not be 2-unitary. In the rest of this subsection we give some examples of 2-unitary operads different from ones in Example 2.2. The following lemma is easy to prove.

Lemma 2.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be unitary operads.*

(1) *If $\mathcal{P}$ and $\mathcal{Q}$ are 2-unitary, then so is the Hadamard product [26, Section 5.3.2] (also called Segre product or white product) of $\mathcal{P}$ and $\mathcal{Q}$. In fact, the 2-unit in $\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{Q}$ is just $\mathbb{1}_2^{\mathcal{P}} \otimes \mathbb{1}_2^{\mathcal{Q}}$, where $\mathbb{1}_2^{\mathcal{P}}$ and $\mathbb{1}_2^{\mathcal{Q}}$ are 2-units in $\mathcal{P}$ and $\mathcal{Q}$, respectively.*
(2) *Suppose $\mathcal{P}$ is 2-unitary with 2-unit $\mathbb{1}_2^{\mathcal{P}}$ and $f : \mathcal{P} \to \mathcal{Q}$ is a morphism of unitary operads. Then $\mathcal{Q}$ is 2-unitary with 2-unit $f(\mathbb{1}_2^{\mathcal{P}})$.*

The next example will be used in the classification of 2-unitary operads of GKdimension two.

Example 2.4. Let $\Lambda = \mathbb{k}\mathbb{1}_1 \oplus \bar{\Lambda}$ be an augmented algebra with augmentation ideal $\bar{\Lambda}$. We consider the plain operad $\mathcal{D}_\Lambda$ generated by the sequence $(\mathbb{k}\mathbb{1}_0, \Lambda, \mathbb{k}\mathbb{1}_2, 0, 0, \cdots)$ of vector spaces and subject to the following relations

$$\begin{aligned} & \delta \circ \mathbb{1}_0 = 0, \text{ for all } \delta \in \bar{\Lambda}, \\ & \mathbb{1}_2 \underset{i}{\circ} \mathbb{1}_0 = \mathbb{1}_1, \text{ for } i = 1, 2, \\ & \delta \circ \delta' = \delta\delta', \text{ for all } \delta, \delta' \in \Lambda, \\ & \mathbb{1}_2 \underset{1}{\circ} \mathbb{1}_2 = \mathbb{1}_2 \underset{2}{\circ} \mathbb{1}_2, \\ & \mathbb{1}_2 \circ (\delta, \delta') = 0, \text{ for all } \delta, \delta' \in \bar{\Lambda}, \\ & \delta \circ \mathbb{1}_2 = \mathbb{1}_2 \underset{1}{\circ} \delta + \mathbb{1}_2 \underset{2}{\circ} \delta, \text{ for all } \delta \in \bar{\Lambda}, \end{aligned}$$

where $\delta\delta'$ is the product of δ and δ' in Λ.

(1) Next we give an explicit description of $\mathcal{D}_\Lambda$. For this, we choose an arbitrary basis $\{\delta_i \mid i \in T\}$ for $\bar{\Lambda}$ where T is an index set. Suppose that $\{\Omega_{ij}^k \mid i,j,k \in T\}$ are the corresponding structural constants, namely,

$$\delta_i \delta_j = \sum_{k \in T} \Omega_{ij}^k \delta_k \tag{E2.4.1}$$

for all $i, j \in T$. We assume that 0 is not in T. Then we have

$$\mathcal{D}_\Lambda(0) = \Bbbk \mathbb{1}_0 \cong \Bbbk, \ \mathcal{D}_\Lambda(1) = \Lambda = \Bbbk \mathbb{1}_1 \oplus \bar{\Lambda},$$

and

$$\mathcal{D}_\Lambda(n) = \Bbbk \mathbb{1}_n \oplus \bigoplus_{i \in [n], j \in T} \Bbbk \delta_{(i)j}^n \cong \Bbbk \mathbb{1}_n \oplus \bar{\Lambda}^{\oplus n} \tag{E2.4.2}$$

for $n \geq 2$. For consistency of notations, we set $\delta_{(1)j}^1 = \delta_j$ for each $j \in T$, and $\delta_{(i)0}^n = \mathbb{1}_n$ for all $i \in [n]$.

We use the partial definition of an operad [Definition 1.2]. The partial composition

$$- \underset{i}{\circ} - \colon \mathcal{D}_\Lambda(m) \otimes \mathcal{D}_\Lambda(n) \to \mathcal{D}_\Lambda(m+n-1) \quad (i \in [m])$$

is defined by

$$\delta_{(s)t}^m \underset{i}{\circ} \delta_{(k)l}^n = \begin{cases} \delta_{(k+i-1)l}^{m+n-1}, & t = 0, l \geq 0, \\ \delta_{(s)t}^{m+n-1}, & t \geq 1, l = 0, 1 \leq s \leq i-1, \\ \sum\limits_{h=i}^{i+n-1} \delta_{(h)t}^{m+n-1}, & t \geq 1, l = 0, s = i, \\ \delta_{(s+n-1)t}^{m+n-1}, & t \geq 1, l = 0, i < s \leq m, \\ \sum\limits_{v \in T} \Omega_{tl}^v \delta_{(i+k-1)v}^{m+n-1}, & t \geq 1, l \geq 1, s = i, \\ 0, & t \geq 1, l \geq 1, s \neq i \end{cases} \tag{E2.4.3}$$

for all $n \geq 1$, and $\mathbb{1}_1 \underset{1}{\circ} \mathbb{1}_0 = \mathbb{1}_0$, $\delta_j \underset{1}{\circ} \mathbb{1}_0 = 0$ for all $j \in T$. If we separate $\mathbb{1}_m$ from elements of the form $\delta_{(k)l}^m$ for $k \in [m]$ and $0 \neq l \in T$, it is easy to see that (E2.4.3) is equivalent to

$$\mathbb{1}_m \underset{i}{\circ} \mathbb{1}_n = \mathbb{1}_{m+n-1},$$

$$\mathbb{1}_m \underset{i}{\circ} \delta_{(k)l}^n = \delta_{(k+i-1)l}^{m+n-1},$$

$$\delta^m_{(s)t} \underset{i}{\circ} \mathbb{1}_n = \begin{cases} \delta^{m+n-1}_{(s)t}, & 1 \le s \le i-1, \\ \sum\limits_{h=i}^{i+n-1} \delta^{m+n-1}_{(h)t}, & s = i, \\ \delta^{m+n-1}_{(s+n-1)t}, & i < s \le m, \end{cases}$$

$$\delta^m_{(s)t} \underset{i}{\circ} \delta^n_{(k)l} = \begin{cases} \sum\limits_{v\in T} \Omega^v_{tl} \delta^{m+n-1}_{(i+k-1)v}, & s = i, \\ 0, & s \ne i. \end{cases}$$

It is easy to see that the above defining equations are independent of the choices of the basis $\{\delta_i \mid i \in T\}$. Note that $- \underset{1}{\circ} -$ in A is just the associative multiplication of Λ. By the second relation on the above list, we obtain

$$\delta^n_{(i)j} = \mathbb{1}_n \underset{i}{\circ} \delta_j \tag{E2.4.4}$$

for all $i \in [n], j \in T$.

One can now directly check via a tedious computation that $\mathcal{D}_\Lambda$ is a 2-unitary plain operad with the partial composition defined above. In fact, it is easily seen that $\mathcal{D}_\Lambda(0) = \Bbbk\mathbb{1}_0$, $\mathcal{D}_\Lambda(1) = \Lambda$ and for every $n \ge 2$, $\mathcal{D}_\Lambda(n) = \Bbbk\mathbb{1}_n \oplus \bar{\Lambda}^n$, where $\bar{\Lambda}^n = \{\mathbb{1}_n \underset{i}{\circ} \delta \mid \delta \in \bar{\Lambda}, 1 \le i \le n\}$ is isomorphic to n copies of $\bar{\Lambda}$ as a vector space.

Observe that there is a natural right action of $\mathbb{S}_n$ on $\mathcal{D}_\Lambda(n)$ given by

$$\mathbb{1}_n * \sigma = \mathbb{1}_n \text{ and } (\mathbb{1}_n \underset{i}{\circ} \delta) * \sigma = \mathbb{1}_n \underset{\sigma^{-1}(i)}{\circ} \delta$$

for all $\sigma \in \mathbb{S}_n$ and all n. It is easily checked that $\mathcal{D}_\Lambda$ is a symmetric operad under the above $\mathbb{S}$-action. Furthermore, this action is uniquely determined. In fact, by $\mathbb{1}_2 * (12) = \mathbb{1}_2$ and $\mathbb{1}_n = \mathbb{1}_2 \underset{1}{\circ} \mathbb{1}_{n-1} = \mathbb{1}_2 \underset{2}{\circ} \mathbb{1}_{n-1}$, we have inductively that $\mathbb{1}_n * \sigma = \mathbb{1}_n$, and by (OP3′) we have

$$\mathbb{1}_n \underset{\sigma^{-1}(i)}{\circ} \delta = (\mathbb{1}_n * \sigma) \underset{\sigma^{-1}(i)}{\circ} \delta = (\mathbb{1}_n \underset{i}{\circ} \delta) * \sigma$$

for all $\sigma \in \mathbb{S}_n$ and all $n \ge 2$.

A $\Bbbk$-linear basis of $\mathcal{D}_\Lambda$ is explicitly given in (E2.4.2). When T is a finite set with d elements, the generating function of $\mathcal{D}_\Lambda$ is

$$G_{\mathcal{D}_\Lambda}(t) = \sum_{n=0}^{\infty} (1 + dn)t^n = \frac{1}{1-t} + \frac{dt}{(1-t)^2}.$$

As a consequence, $\mathcal{D}_\Lambda$ has GKdimension two. We will see later that every 2-unitary operad of GKdimension two is of this form.

An algebra A over $\mathcal{D}_\Lambda$ means a unital commutative associative algebra together with a set of derivations $\{\delta_i\}_{i\in T}$ satisfying

(i) $\delta_i(x)\delta_j(y) = 0$ for all $i, j \in T$ and all $x, y \in A$, and
(ii) (E2.4.1): $\delta_i\delta_j = \sum_{k\in T} \Omega_{ij}^k \delta_k$.
Note that a $\mathcal{D}_\Lambda$-algebra is a special kind of commutative differential $\Bbbk$-algebra. Similar algebras have been studied by Goodearl in [15, Section 1].

(2) Let $\mathcal{I} := \{I_n\}_{n\geq 2}$ be a descending chain of ideals of Λ inside $\bar{\Lambda}$ such that $I_m I_n \subseteq I_{m+n-1}$ for all m and n. Denote $I_1 = \Lambda$. We define a unitary operad, denoted by $\mathcal{D}_\Lambda^{\mathcal{I}}$, associated to $\mathcal{I}$. For the sake of using $\Bbbk$-linear bases, suppose we can choose a descending chain of subsets $\{T_n\}$ of T such that $\{\delta_i \mid i \in T_n\}$ is a $\Bbbk$-linear basis of I_n (this is not essential). Define

$$\mathcal{D}_\Lambda^{\mathcal{I}}(n) = \begin{cases} \Bbbk \mathbb{1}_0, & n = 0, \\ \Lambda = \Bbbk \mathbb{1}_1 \oplus \bigoplus_{j\in T} \Bbbk \delta_j, & n = 1, \\ \bigoplus_{i\in[n], j\in T_n} \Bbbk \delta_{(i)j}^n, & n \geq 2. \end{cases}$$

Alternatively, denote by $\delta_{(i)}^n$ the n-ary operation $\mathbb{1}_n \underset{i}{\circ} \delta \in \mathcal{D}_\Lambda(n)$ for all $\delta \in \bar{\Lambda}$. Then $\mathcal{D}_\Lambda^{\mathcal{I}}(0) = \Bbbk \mathbb{1}_0$ and $\mathcal{D}_\Lambda^{\mathcal{I}}(1) = \Lambda$, and for $n \geq 2$,

$$\mathcal{D}_\Lambda^{\mathcal{I}}(n) = \{\delta_{(i)}^n \mid \delta \in I_n, 1 \leq i \leq n\}.$$

One can check that the suboperad $\mathcal{D}_\Lambda^{\mathcal{I}}$ of $\mathcal{D}_\Lambda$ is a unitary, but not 2-unitary.
An algebra over $\mathcal{D}_\Lambda^{\mathcal{I}}$ is a $\Bbbk$-vector space A with a fixed element $e \in A$ and a collection of n-ary operations $\delta_{(i)}^n : A^{\otimes n} \to A$ for all $\delta \in I_n$ and $1 \leq i \leq n$, $n \geq 1$, satisfying
(a) $\delta(e) = 0$ for all $\delta \in \bar{\Lambda}$.
(b)

$$\delta_{(i)}^n(x_1, \cdots, x_{k-1}, e, x_{k+1}, \cdots, x_n) = \begin{cases} \delta_{(i-1)}^{n-1}(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n), & \text{if } k < i, \\ 0, & \text{if } k = i, \\ \delta_{(i)}^{n-1}(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n), & \text{if } k > i; \end{cases}$$

for all $\delta \in I_n$, $1 \leq i \leq n$, $n \geq 2$ and all $x_s \in A$, $s = 1, \cdots, n$, $s \neq k$.
(c)

$$\delta_{(i)}^m(x_1, \cdots, x_{k-1}, (\delta')_{(j)}^n(x_k, \cdots, x_{k+n-1}), x_{k+n}, \cdots, x_{m+n-1}) = \begin{cases} (\delta\delta')_{(i+j-1)}^{m+n-1}(x_1, \cdots, x_{m+n-1}), & \text{if } k = i \\ 0, & \text{if } k \neq i; \end{cases}$$

for all $1 \leq i, k \leq m, n \geq 1$, $\delta \in I_m, \delta' \in I_n$, and for all $x_s \in A$ where $s = 1, \cdots, m + n + 1$.

To see that, let A be a $\mathcal{D}_\Lambda^{\mathcal{I}}$-algebra that is given by $\gamma = (\gamma_n : \mathcal{D}_\Lambda^{\mathcal{I}}(n) \to \mathcal{E}nd_A(n))_{n\geq 0}$. Let $e = \gamma_0(\mathbb{1}_0) \in A$. Since $D_\Lambda^{\mathcal{I}}$ is a suboperad of D_Λ, relation (a) follows from the equation $\delta \circ \mathbb{1}_0 = 0$ for all $\delta \in \bar{\Lambda}$. Similarly, relations (b) and (c) are deduced from relations of $D_\Lambda^{\mathcal{I}}$ such as (E2.4.3).

A special case is when $I_n = I$ for all $n \geq 2$. In this case, the above defined operad is denoted by $\mathcal{D}_\Lambda^I$. Suppose T' is a subset of T such that $\{\delta_i \mid i \in T'\}$ is a $\Bbbk$-linear basis of I. Then

$$\mathcal{D}_\Lambda^I(n) = \begin{cases} \Bbbk\mathbb{1}_0, & n = 0, \\ \Lambda, & n = 1, \\ \bigoplus_{i\in[n], j\in T'} \Bbbk\delta_{(i)j}^n, & n \geq 2. \end{cases}$$

2.2. Some elementary operators on 2-unitary operads

Let s be an integer no more than n, and $I \subseteq [n]$ a subset consisting of s elements. Clearly, there exists a unique 1-1 correspondence from $[s]$ to I that preserves the ordering. Choosing $I \subseteq [n]$ is equivalent to giving an order preserving map

$$\overrightarrow{I} : [s] \longrightarrow I \subseteq [n].$$

Let χ_I be the characteristic function of I, i.e. $\chi_I(x) = 1$ for $x \in I$ and $\chi_I(x) = 0$ otherwise.

We recall the following useful operators. Let $\mathcal{P}$ be a (2-)unitary operad. Consider the following *restriction operator* [9, Section 2.2.1]

$$\pi^I : \mathcal{P}(n) \to \mathcal{P}(s), \qquad \pi^I(\theta) = \theta \circ (\mathbb{1}_{\chi_I(1)}, \cdots, \mathbb{1}_{\chi_I(n)}) \tag{E2.4.5}$$

for all $\theta \in \mathcal{P}(n)$. The *contraction operator* is defined by $\Gamma^I = \pi^{\hat{I}}$ where $\hat{I}$ is the complement of I in $[n]$, or

$$\Gamma^I : \mathcal{P}(n) \to \mathcal{P}(n-s), \qquad \Gamma^I(\theta) = \theta \circ (\mathbb{1}_{\chi_{\hat{I}}(1)}, \cdots, \mathbb{1}_{\chi_{\hat{I}}(n)}) \tag{E2.4.6}$$

for all $\theta \in \mathcal{P}(n)$.

Recall that $\bullet$ denotes the usual composition of two functions that is omitted sometimes.

Lemma 2.5. *[9, Lemma 2.2.4(1)] Retain the above notation.*

(1) *Let $I \subseteq [n]$ with $|I| = s$ and $J \subseteq [s]$. Let $\widetilde{J} := \overrightarrow{I}(J)$ be the image of J under $\overrightarrow{I}$. Then $\pi^{\widetilde{J}} = \pi^J \bullet \pi^I$.*
(2) *For each $W \subseteq \hat{I}$, $\pi^I = \Gamma^{\hat{I}} = \Gamma^{W'} \bullet \Gamma^W$ for some subset W' of $[n - |W|]$ with $|W'| + |W| = n - |I|$.*

(3) *If $k \notin I$, then $\pi^I = \pi^{I'} \bullet \Gamma^k$ for some $I' \subseteq [n-1]$ with $|I'| = |I|$.*

Proof. (1) This is [9, Lemma 2.2.4(1)]. It follows from (OP2).

(2, 3) Easy consequences of part (1). □

If $\mathcal{P}$ is 2-unitary, we can define another operator as follows. The *extension operator* $\Delta_I : \mathcal{P}(n) \to \mathcal{P}(n+s)$ is defined by

$$\Delta_I(\theta) = \theta \circ (\mathbb{1}_{\chi_I(1)+1}, \cdots, \mathbb{1}_{\chi_I(n)+1})$$

for all $\theta \in \mathcal{P}(n)$. If $I = \{i_1, \cdots, i_s\}$ with $i_1 < i_2 < \cdots < i_s$, then we also write π^I, Γ^I and Δ_I as $\pi^{i_1,\cdots,i_s}$, $\Gamma^{i_1,\cdots,i_s}$ and $\Delta_{i_1,\cdots,i_s}$ respectively.

Assume that $\mathcal{P}$ is 2-unitary. For every $n \geq 3$, we define inductively that

$$\mathbb{1}_n = \mathbb{1}_2 \circ (\mathbb{1}_{n-1}, \mathbb{1}_1). \tag{E2.5.1}$$

Note that one might also define inductively

$$\mathbb{1}'_n = \mathbb{1}_2 \circ (\mathbb{1}_1, \mathbb{1}'_{n-1}) \tag{E2.5.2}$$

for all $n \geq 3$. By convention, $\mathbb{1}_n = \mathbb{1}'_n$ for $n = 0, 1, 2$. Unless $\mathcal{P}$ is 2a-unitary, it is not automatic that $\mathbb{1}'_n = \mathbb{1}_n$ for any $n \geq 3$. In fact, $\mathbb{1}_3 = \mathbb{1}'_3$ means that the binary operation given by $\mathbb{1}_2$ is associative.

Definition 2.6. Let $\mathcal{P}$ and $\mathcal{Q}$ be operads.

(1) Let $\mathcal{Q}$ be a (unitary) operad. We call $\mathcal{P}$ *$\mathcal{Q}$-augmented* if there are morphisms of operads $f : \mathcal{Q} \to \mathcal{P}$ and $g : \mathcal{P} \to \mathcal{Q}$ such that $gf = \mathrm{Id}_{\mathcal{Q}}$.
(2) $\mathcal{P}$ is called *$\mathcal{C}om$-augmented* if there is a morphism $\mathcal{C}om \to \mathcal{P}$, or equivalently, there exists a 2-unit $\mathbb{1}_2 \in \mathcal{P}(2)$ satisfying (E2.1.1) and $\mathbb{1}_2 * (12) = \mathbb{1}_2$. In this case it is automatic that the morphism $\mathcal{C}om \to \mathcal{P}$ has the unique left inverse $\mathcal{P} \to \mathcal{C}om$.

It is easy to see that $\mathcal{C}om$-augmented operads are 2a-unitary. Observe that the 2a-unitary property of a 2-unitary operad may be dependent on choices of $\mathbb{1}_2$. For example, if $(\mathbb{1}_0, \mathbb{1}_1, \mathbb{1}_2) = (1_0, 1_1, 1_2)$ as elements in $\mathbb{S}_n$ for $n = 0, 1, 2$, then $(\mathcal{A}ss, \mathbb{1}_0, \mathbb{1}_1, \mathbb{1}_2)$ is a 2a-unitary operad. Suppose char $\Bbbk \neq 2$. If we set $(\mathbb{1}_0, \mathbb{1}_1, \mathbb{1}_2) = (1_0, 1_1, \frac{1}{2}(1_2 + 1_2 * (12)))$, $(\mathcal{A}ss, \mathbb{1}_0, \mathbb{1}_1, \mathbb{1}_2)$ is only 2-unitary, but not 2a-unitary.

Lemma 2.7. *Let $\mathcal{P}$ be a 2a-unitary operad, namely, $\mathbb{1}_3 = \mathbb{1}'_3$. Then the following hold.*

(1) *For every $n \geq 3$, $\mathbb{1}_n = \mathbb{1}'_n$.*
(2) *For every $n \geq 1$ and $k_1, \cdots, k_n \geq 0$, $\mathbb{1}_n \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_n}) = \mathbb{1}_{k_1+\cdots+k_n}$.*

Proof. (1) Use induction on n. Assume that $\mathbb{1}_k = \mathbb{1}'_k$ for all $3 \le k \le n-1$. Then

$$\begin{aligned}\mathbb{1}_n =&\mathbb{1}_2 \circ (\mathbb{1}_{n-1}, \mathbb{1}_1) = \mathbb{1}_2 \circ (\mathbb{1}'_{n-1}, \mathbb{1}_1) = \mathbb{1}_2 \circ (\mathbb{1}_2 \circ (\mathbb{1}_1, \mathbb{1}'_{n-2}), \mathbb{1}_1 \circ \mathbb{1}_1)\\ =&(\mathbb{1}_2 \circ (\mathbb{1}_2, \mathbb{1}_1)) \circ (\mathbb{1}_1, \mathbb{1}'_{n-2}, \mathbb{1}_1) = (\mathbb{1}_2 \circ (\mathbb{1}_1, \mathbb{1}_2)) \circ (\mathbb{1}_1, \mathbb{1}_{n-2}, \mathbb{1}_1)\\ =&\mathbb{1}_2 \circ (\mathbb{1}_1, \mathbb{1}_2 \circ (\mathbb{1}_{n-2}, \mathbb{1}_1)) = \mathbb{1}_2 \circ (\mathbb{1}_1, \mathbb{1}_{n-1}) = \mathbb{1}_2 \circ (\mathbb{1}_1, \mathbb{1}'_{n-1})\\ =&\mathbb{1}'_n.\end{aligned}$$

(2) This follows from induction. $\square$

By definition, $\mathcal{M}ag$ and $\mathcal{A}ss$ are the initial objects in the category of 2-unitary and $2a$-unitary operads, respectively. It is easy to see that Lemma 2.3 holds for $2a$-unitary operads.

For any $l, r \ge 0$, we define the function $\iota^l_r \colon \mathcal{P}(n) \to \mathcal{P}(l+n+r)$ by

$$\iota^l_r(\theta) = \mathbb{1}_3 \circ (\mathbb{1}_l, \theta, \mathbb{1}_r).$$

We simply write $\iota_r = \iota^0_r$ and $\iota^l = \iota^l_0$.

Lemma 2.8. *Retain the above notation. Let $\mathcal{P}$ be a 2-unitary operad and let $\theta \in \mathcal{P}(n)$.*

(1) $\pi^I(\mathbb{1}_n) = \mathbb{1}_{|I|}$ *for all* $I \subseteq [n]$.
(2) $\iota_r(\theta) = \mathbb{1}_2 \circ (\theta, \mathbb{1}_r)$.
(3) $\iota^l(\theta) = \mathbb{1}_2 \circ (\mathbb{1}_l, \theta)$.
(4) $\iota^l_r = \iota_r \bullet \iota^l$. *Moreover,* $\iota^l_r = \iota^l \bullet \iota_r$ *for all* $l, r \ge 0$ *if and only if* $\mathcal{P}$ *is* $2a$*-unitary.*

Proof. (1) This follows by induction on n.

(2) We compute

$$\begin{aligned}\iota_r(\theta) &= \mathbb{1}_3 \circ (\mathbb{1}_0, \theta, \mathbb{1}_r) = (\mathbb{1}_2 \circ (\mathbb{1}_2, \mathbb{1}_1)) \circ (\mathbb{1}_0, \theta, \mathbb{1}_r)\\ &= \mathbb{1}_2 \circ (\mathbb{1}_2 \circ (\mathbb{1}_0, \theta), \mathbb{1}_1 \circ \mathbb{1}_r)\\ &= \mathbb{1}_2 \circ (\theta, \mathbb{1}_r).\end{aligned}$$

(3) We compute

$$\begin{aligned}\iota^l(\theta) &= \mathbb{1}_3 \circ (\mathbb{1}_l, \theta, \mathbb{1}_0) = (\mathbb{1}_2 \circ (\mathbb{1}_2, \mathbb{1}_1)) \circ (\mathbb{1}_l, \theta, \mathbb{1}_0)\\ &= \mathbb{1}_2 \circ (\mathbb{1}_2 \circ (\mathbb{1}_l, \theta), \mathbb{1}_1 \circ \mathbb{1}_0)\\ &= \mathbb{1}_2 \circ (\mathbb{1}_2 \circ (\mathbb{1}_l, \theta), \mathbb{1}_0)\\ &= \mathbb{1}_2 \circ (\mathbb{1}_l, \theta).\end{aligned}$$

(4) Using parts (2) and (3), we compute

$$\begin{aligned}\iota_r^l(\theta) &= \mathbb{1}_3 \circ (\mathbb{1}_l, \theta, \mathbb{1}_r) = (\mathbb{1}_2 \circ (\mathbb{1}_2, \mathbb{1}_1)) \circ (\mathbb{1}_l, \theta, \mathbb{1}_r) \\ &= \mathbb{1}_2 \circ (\mathbb{1}_2 \circ (\mathbb{1}_l, \theta), \mathbb{1}_1 \circ \mathbb{1}_r) \\ &= \mathbb{1}_2 \circ (\iota^l(\theta), \mathbb{1}_r) \\ &= \iota_r \bullet \iota^l(\theta).\end{aligned}$$

If $\iota^l \bullet \iota_r = \iota_r^l$, taking $r = l = 1$, then

$$\mathbb{1}_3 = \iota_1(\iota^1(\mathbb{1}_1)) = \iota^1(\iota_1(\mathbb{1}_1)) = \mathbb{1}_2 \circ (\mathbb{1}_1, \mathbb{1}_2) = \mathbb{1}_3'.$$

Conversely, if $\mathbb{1}_3 = \mathbb{1}_3'$ (equivalently, if $\mathcal{P}$ is $2a$-unitary), then we have

$$\iota^l(\iota_r(\theta)) = \mathbb{1}_2 \circ (\mathbb{1}_l, \mathbb{1}_2 \circ (\theta, \mathbb{1}_r)) = (\mathbb{1}_2 \circ (\mathbb{1}_1, \mathbb{1}_2)) \circ (\mathbb{1}_l, \theta, \mathbb{1}_r) = \mathbb{1}_3 \circ (\mathbb{1}_l, \theta, \mathbb{1}_r) = \iota_r^l(\theta)$$

for all θ. □

Example 2.9. Let $\mathcal{P} = \mathcal{A}ss$. Assume $n = 5$, $I = \{2, 4\}$ and $\sigma = (14)(235)$. Then $\Gamma^I(\sigma) = (123) \in \mathbb{S}_3$, $\pi^I(\sigma) = (12) \in \mathbb{S}_2$, $\Delta_I(\sigma) = (1624735) \in \mathbb{S}_7$ and $\iota_2^1(\sigma) = (25)(346) \in \mathbb{S}_8$. Following the convention introduced in Section 8.1 the sequences corresponded to

$$\sigma, \qquad \Gamma^I(\sigma), \qquad \pi^I(\sigma), \qquad \Delta_I(\sigma) \qquad \text{and} \qquad \iota_2^1(\sigma)$$

are given by

$$(4,5,2,1,3),\ (3,1,2),\ (2,1),\ (5,6,7,2,3,1,4) \text{ and } (1,5,6,3,2,4,7,8),$$

respectively.

By an easy calculation, we have the following useful lemmas.

Lemma 2.10. *Let $\mathcal{P}$ be 2-unitary. Let $n, l, r \geq 0$ be integers and $i, j, i_1, \cdots, i_s \in [n]$. Then the following hold.*

(1) *Assume that $i_1 < \cdots < i_s$, then*

$$\Delta_{i_1,\cdots,i_s} = \Delta_{i_s+s-1} \bullet \cdots \bullet \Delta_{i_2+1} \bullet \Delta_{i_1} = \Delta_{i_1} \bullet \cdots \bullet \Delta_{i_s},$$

and

$$\Gamma^{i_1,\cdots,i_s} = \Gamma^{i_s-s+1} \bullet \cdots \bullet \Gamma^{i_2-1} \bullet \Gamma^{i_1} = \Gamma^{i_1} \bullet \cdots \bullet \Gamma^{i_s}.$$

(2) $\Gamma^{i+1} \bullet \Delta_i = \mathrm{id}$.
(3) $\Gamma^i \bullet \Delta_i = \mathrm{id}$.

(4) $\Delta_j \bullet \Gamma^i = \begin{cases} \Gamma^i \bullet \Delta_{j+1}, & i \leq j; \\ \Gamma^{i+1} \bullet \Delta_j, & i > j. \end{cases}$

(5) $\Gamma^{l+i} \bullet \iota_r^l = \iota_r^l \bullet \Gamma^i$, *and* $\Delta_{l+i} \bullet \iota_r^l = \iota_r^l \bullet \Delta_i$.

(6) $\Gamma^1 \bullet \iota^1 = \mathrm{id}$, *and* $\Gamma^{n+1} \bullet \iota_1|_{\mathcal{P}(n)} = \mathrm{id}_{\mathcal{P}(n)}$.

Proof. This follows from easy computations and (OP2). □

Lemma 2.11. *Let $\mathcal{P}$ be 2-unitary. Let $n, k_1, \cdots, k_n \geq 0$ be integers. Then, for each $\theta \in \mathcal{P}(n)$,*

$$\theta \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_n}) = ((\Delta_1)^{k_1-1} \bullet \cdots \bullet (\Delta_n)^{k_n-1})(\theta),$$

where, by convention, $(\Delta_i)^{-1}$ means Γ^i in case $k_i = 0$.

Proof. We use (OP2) in the following computation. If $k_s \geq 2$, we have

$$\begin{aligned} &\theta \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_s}, \underbrace{\mathbb{1}_1, \cdots, \mathbb{1}_1}_{t}) \\ &= \theta \circ (\mathbb{1}_1 \circ \mathbb{1}_{k_1}, \cdots, \mathbb{1}_1 \circ \mathbb{1}_{k_{s-1}}, \mathbb{1}_2 \circ (\mathbb{1}_{k_s-1}, \mathbb{1}_1), \underbrace{\mathbb{1}_1 \circ \mathbb{1}_1, \cdots, \mathbb{1}_1 \circ \mathbb{1}_1}_{t}) \\ &= \Delta_s(\theta) \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_{s-1}}, \mathbb{1}_{k_s-1}, \underbrace{\mathbb{1}_1, \mathbb{1}_1, \cdots, \mathbb{1}_1}_{t+1}). \end{aligned}$$

If $k_s = 0$, then

$$\begin{aligned} &\theta \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_s}, \underbrace{\mathbb{1}_1, \cdots, \mathbb{1}_1}_{t}) \\ &= \theta \circ (\mathbb{1}_1 \circ \mathbb{1}_{k_1}, \cdots, \mathbb{1}_1 \circ \mathbb{1}_{k_{s-1}}, \mathbb{1}_0 \circ (), \underbrace{\mathbb{1}_1 \circ \mathbb{1}_1, \cdots, \mathbb{1}_1 \circ \mathbb{1}_1}_{t}) \\ &= \Gamma^s(\theta) \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_{s-1}}, \underbrace{\mathbb{1}_1, \mathbb{1}_1, \cdots, \mathbb{1}_1}_{t}). \end{aligned}$$

Combining the above, we have

$$\begin{aligned} &\theta \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_s}, \underbrace{\mathbb{1}_1, \cdots, \mathbb{1}_1}_{t}) \\ &= \begin{cases} \Delta_s(\theta) \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_{s-1}}, \mathbb{1}_{k_s-1}, \underbrace{\mathbb{1}_1, \mathbb{1}_1, \cdots, \mathbb{1}_1}_{t+1}) & \text{if } k_s \geq 2; \\ \Gamma^s(\theta) \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_{s-1}}, \underbrace{\mathbb{1}_1, \cdots, \mathbb{1}_1}_{t}) & \text{if } k_s = 0. \end{cases} \end{aligned} \tag{E2.11.1}$$

The lemma follows by applying the formula (E2.11.1) iteratively. □

Note that Lemmas 2.10 and 2.11 hold for plain operads. Now we have the following classification result.

Proposition 2.12. *Let $\mathcal{P}$ be a (symmetric or plain) 2-unitary operad. If* $\operatorname{GKdim}(\mathcal{P}) < 2$, *then $\mathcal{P} \cong \mathcal{C}om$.*

Proof. Assume that $\mathcal{P}$ is not $\mathcal{C}om$. Let $n = \min\{m \mid \mathcal{P}(m) \neq \Bbbk \mathbb{1}_m\}$. Since $\mathcal{P}$ is unitary, $n \geq 1$. Since $\mathcal{P}(n-1) = \Bbbk \mathbb{1}_{n-1}$, there is a nonzero element $\theta \in \mathcal{P}(n)$ such that $\pi^I(\theta) = 0$ where $I = [n-1]$. For every $J \subseteq [n]$ such that $|J| = n-1$,

$$\pi^{\emptyset} \bullet \pi^J(\theta) = \pi^{\emptyset}(\theta) = \pi^{\emptyset} \bullet \pi^I(\theta) = 0.$$

Firstly since $\pi^{\emptyset} : \mathcal{P}(n-1) \to \mathcal{P}(0)$ is an isomorphism,

$$\pi^J(\theta) = 0 \tag{E2.12.1}$$

for all $J \subseteq [n]$ with $|J| = n-1$. For every $w \geq n+1$ and $0 \leq i \leq w-n$, let $\theta_i^w = \iota_{w-i-n}^i(\theta)$. We claim that $\{\theta_0^w, \theta_1^w \cdots, \theta_{w-n}^w\}$ are linearly independent. We prove this by induction on w. The initial case is when $w = n+1$. Suppose

$$a\theta_0^w + b\theta_1^w = 0. \tag{E2.12.2}$$

By (E2.12.1), we have $\Gamma^1(\theta_0^w) = 0$ and $\Gamma^1(\theta_1^w) = \theta$. Thus $b\theta = 0$ after applying Γ^1 to (E2.12.2). Hence $b = 0$. Applying Γ^w to (E2.12.2), we obtain that $a = 0$. Therefore the claim holds for $w = n+1$. Now suppose the claim holds for w, and we consider the equation

$$\sum_{s=0}^{w-n+1} a_s \theta_s^{w+1} = 0. \tag{E2.12.3}$$

Since $\Gamma^{w+1}(\theta_s^{w+1}) = \begin{cases} \theta_s^w, & s < w-n+1, \\ 0, & s = w-n+1, \end{cases}$ we obtain that $\sum_{s=0}^{w} a_s \theta_s^w = 0$ after applying Γ^{w+1} to (E2.12.3). By induction hypothesis, $a_s = 0$ for all $s = 0, \cdots, w-n$. Using Γ^1 instead of Γ^{w+1}, we obtain that $a_s = 0$ for all $s = 1, \cdots, w-n+1$. Therefore we proved the claim by induction.

By the claim $\dim \mathcal{P}(w) \geq w - n$ for all w, which implies that $\operatorname{GKdim} \mathcal{P} \geq 2$, a contradiction. □

Recall that $*$ denote the right action of $\mathbb{S}_n$ on $\mathcal{P}(n)$. The following lemma is easy.

Lemma 2.13. *Let $\mathcal{P}$ be a unitary operad.*

(1) *Let n be a positive integer and I a subset of $[n]$. Then, for all $\theta \in \mathcal{P}(n), \sigma \in \mathbb{S}_n$,*

$$\pi^I(\theta * \sigma) = \pi^{\sigma I}(\theta) * \pi^I(\sigma), \tag{E2.13.1}$$

where $\sigma I = \{\sigma(i) \mid i \in I\} \subseteq [n]$.

(2) *Let $\mu \in \mathcal{P}(m)$, $\nu \in \mathcal{P}(n)$ and $1 \le i \le m$. Then*

$$\pi^I(\mu \underset{i}{\circ} \nu) = \pi^J(\mu) \underset{j}{\circ} \pi^{I'}(\nu) \tag{E2.13.2}$$

where

$$\begin{aligned} J &= (I \cap [i-1]) \cup \{i\} \cup ((I \cap \{i+n, \cdots, m+n-1\}) - (n-1)), \\ I' &= (I \cap \{i, \cdots, i+n-1\}) - (i-1), \\ j &= |I \cap [i-1]| + 1. \end{aligned}$$

Proof. (1) First we recall (OP3). For all $k_1 \cdots, k_n \ge 0$, $\theta_i \in \mathcal{P}(k_i), \sigma_i \in \mathbb{S}_{k_i}, 1 \le i \le n$ and $\theta \in \mathcal{P}(n), \sigma \in \mathbb{S}_n$,

$$(\theta * \sigma) \circ (\theta_1 * \sigma_1, \cdots, \theta_n * \sigma_n) = (\theta \circ (\theta_{\sigma^{-1}(1)}, \cdots, \theta_{\sigma^{-1}(n)})) * (\sigma \circ (\sigma_1, \cdots, \sigma_n)). \tag{E2.13.3}$$

Let $k = |I|$. Take $\theta_i = \mathbb{1}_1 \in \mathcal{P}(1)$ and $\sigma_i = \mathbb{1}_1 \in \mathbb{S}_1$ for $i \in I$ and $\theta_i = \mathbb{1}_0 \in \mathcal{P}(0)$ and $\sigma_i = \mathbb{1}_0 \in \mathbb{S}_0$ otherwise. By (E2.13.3), we obtain

$$\begin{aligned} \pi^I(\theta * \sigma) =& (\theta * \sigma) \circ (\theta_1 * \sigma_1, \cdots, \theta_n * \sigma_n) \\ =& (\theta \circ (\theta_{\sigma^{-1}(1)}, \cdots, \theta_{\sigma^{-1}(n)})) * (\sigma \circ (\sigma_1, \cdots, \sigma_n)) \\ =& \pi^{\sigma I}(\theta) * \pi^I(\sigma). \end{aligned}$$

(2) This follows from the definition of π and (OP2). $\square$

2.3. Some basic lemmas

We show the following properties of ideals of $\mathcal{P}$.

Lemma 2.14. *Let $\mathcal{P}$ be 2-unitary operads. Let $\mathcal{I}$ and $\mathcal{I}'$ be ideals of $\mathcal{P}$.*

(1) *For each integer $n \ge 0$ and each subset $I \subseteq [n]$, $\pi^I : \mathcal{I}(n) \to \mathcal{I}(|I|)$ is surjective.*
(2) *If $\mathcal{I}(n) = \mathcal{I}'(n)$ for some n, then $\mathcal{I}(s) = \mathcal{I}'(s)$ for all $s \le n$.*

Proof. (1) Without loss of generality, we may assume that the complement $\hat{I}$ of I is $\{i_1, \cdots, i_s\}$ with $1 \le i_1 < \cdots < i_s \le n$. Since $\pi^I = \Gamma^{i_1, \cdots, i_s} = \Gamma^{i_1} \bullet \cdots \bullet \Gamma^{i_s}$ [Lemma 2.10(1)], it suffices to prove that each $\Gamma^{i_t} : \mathcal{I}(n+t-s) \to \mathcal{I}(n+t-s-1)$ is surjective, which follows from the fact $\Gamma^{i_t} \bullet \Delta_{i_t} = \mathrm{id}$ [Lemma 2.10(3)] or $\Gamma^{i_t} \bullet \Delta_{i_t - 1} = \mathrm{id}$ [Lemma 2.10(2)]. The proof is completed.

(2) This is an easy consequence of part (1). □

Let X be a subset of an operad $\mathcal{P}$. The operadic ideal of $\mathcal{P}$ generated by X is the unique minimal ideal of $\mathcal{P}$ that contains X. We denote by $\langle X \rangle$ the ideal generated by X. An ideal is said to be *finitely generated* if it can be generated by a finite subset.

Lemma 2.15. *Let $\Bbbk$ be a field of characteristic zero, and $\mathcal{I}$ an ideal of 2-unitary operad $\mathcal{P}$.*

(1) *If $\mathcal{I}$ is finitely generated, then $\mathcal{I}$ is generated by a subset $X \subseteq \mathcal{P}(n)$ for some n.*
(2) *Suppose $\mathcal{P}$ is a quotient operad of $\mathcal{A}ss$. Then $\mathcal{I}$ is finitely generated if and only if there exists some $n \geq 0$ and some $x \in \mathcal{I}(n)$, such that $\mathcal{I} = \langle x \rangle$.*

Proof. (1) Let X be a finite generating set of $\mathcal{I}$. Then there exists some n such that $X \subset \bigcup_{0 \leq i \leq n} \mathcal{I}(n)$. Therefore we can take $X \subseteq \langle \mathcal{I}(n) \rangle$ by Lemma 2.14(1).

(2) It suffices to show the "only if" part. Without loss of generality, we suppose $\mathcal{P} = \mathcal{A}ss$. By part (1), we know $\mathcal{I}$ can be generated by $\mathcal{I}(n)$ for some n. Then $\mathcal{I}(n)$ is a right submodule and hence a direct summand of $\Bbbk\mathbb{S}_n$ by $\operatorname{char} \Bbbk = 0$. Since a direct summand of a cyclic module is always cyclic, we have $\mathcal{I}(n) = x \cdot \Bbbk\mathbb{S}_n$ for some $x \in \mathcal{I}(n)$. Clearly, x is the desired generator of $\mathcal{I}$, which completes the proof. □

Theorem 2.16. *Let $\mathcal{P}$ be $\mathcal{A}ss/\mathcal{I}$ for some ideal $\mathcal{I} \subseteq \mathcal{A}ss$. Let $k \geq 0$ be an integer and M a submodule of the $\Bbbk\mathbb{S}_k$-module $\mathcal{P}(k)$.*

(1) *As an $\mathbb{S}$-module, $\langle M \rangle$ is generated by elements of the form $\iota_r^l \bullet \Delta_{i_1} \bullet \cdots \bullet \Delta_{i_s} \bullet \pi^I(x)$, $x \in M$. More explicitly, for every $n \geq 0$, $\langle M \rangle(n)$ is a $\Bbbk\mathbb{S}_n$-submodule generated by*

$$X_n = \left\{ (\iota_r^l \bullet \Delta_{i_1} \bullet \cdots \bullet \Delta_{i_s} \bullet \pi^I)(x) \;\middle|\; \substack{x \in M, I \subseteq [k], l, r \geq 0, l+r+s+|I|=n, \\ 1 \leq i_t \leq |I|+t-1, t=1,\cdots,s} \right\}.$$

(2) *If $\Gamma^i(M) = 0$ for all $1 \leq i \leq k$, then $\langle M \rangle(k) = M$ and $\langle M \rangle(n) = 0$ for all $n < k$.*

Proof. (1) Let $\mathcal{I}(n)$ denote the $\Bbbk\mathbb{S}_n$-submodule of $\mathcal{P}(n)$ generated by the subset X_n and let X be $\bigcup_n X_n$. We claim that $\mathcal{I} = (\mathcal{I}(n))_{n \geq 0}$ is an ideal of $\mathcal{P}$.

By definition we need to show that $\theta \circ (\theta_1, \cdots, \theta_n) \in \mathcal{I}$ provided that one of θ, $\theta_1, \cdots, \theta_n$ is in $\mathcal{I}$. By (OP2) or (E2.0.2), it suffices to show that

$$\mathbb{1}_s \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_{t-1}}, \theta, \mathbb{1}_{k_{t+1}}, \cdots, \mathbb{1}_{k_s}) \in X, \quad \text{and} \quad \theta \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_n}) \in X$$

if $\theta \in X$. Since $\mathcal{P}$ is $2a$-unitary, we have $\mathbb{1}_3 = \mathbb{1}_3'$ and $\iota_r \bullet \iota^l = \iota^l \bullet \iota_r$ [Lemma 2.8(4)]. It follows that

$$\mathbb{1}_s \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_{t-1}}, \theta, \mathbb{1}_{k_{t+1}}, \cdots, \mathbb{1}_{k_s}) = \iota^{k_1+\cdots+k_{t-1}}_{k_{t+1}+\cdots+k_s}(\theta)$$

and therefore the former holds. For the latter, Lemma 2.11 shows that $\theta \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_n})$ is obtained by applying Γ^i's and Δ_j's on θ iteratively. The commutativity relations in Lemma 2.10(2-4) together with Lemma 2.11 imply that

$$\theta \circ (\mathbb{1}_{k_1}, \cdots, \mathbb{1}_{k_n}) \in X_{k_1+\cdots+k_n}.$$

Clearly $M \subseteq X_k \subseteq \mathcal{I}(k)$, and $\mathcal{I} \subseteq \langle M \rangle$. By definition $\langle M \rangle$ is the minimal ideal containing M, which forces that $\langle M \rangle \subseteq \mathcal{I}$ and hence $\langle M \rangle = \mathcal{I}$.

(2) If $\Gamma^i(M) = 0$ for all $1 \le i \le k$, then $\pi^I(M) = 0$ for any $I \subset [k]$ with $|I| < k$, and the statement follows. $\square$

3. Truncation ideals

3.1. The truncation ideals ${}^k\Upsilon$

We first recall the definition of truncation ideals (E0.0.2) from the introduction. Let $\mathcal{P}$ be a unitary operad (or a unitary plain operad). For integers $k, n \ge 0$, we use ${}^k\Upsilon_{\mathcal{P}}(n)$ to denote the subspace of $\mathcal{P}(n)$ defined by

$$ {}^k\Upsilon_{\mathcal{P}}(n) = \bigcap_{I \subset [n],\, |I| \le k-1} \operatorname{Ker} \pi^I = \begin{cases} \bigcap\limits_{I \subset [n],\, |I| = k-1} \operatorname{Ker} \pi^I, & \text{if } n \ge k; \\ 0, & \text{otherwise.} \end{cases} \tag{E3.0.1}$$

By convention, we denote ${}^0\Upsilon_{\mathcal{P}} = \mathcal{P}$. It is easily deduced from Lemma 2.13 that ${}^k\Upsilon_{\mathcal{P}}(n)$ is an $\mathbb{S}_n$-submodule of $\mathcal{P}(n)$. Therefore we obtain an $\mathbb{S}$-submodule ${}^k\Upsilon_{\mathcal{P}} = ({}^k\Upsilon_{\mathcal{P}}(n))_{n \ge 0}$ of $\mathcal{P}$. If no confusion, we write ${}^k\Upsilon = {}^k\Upsilon_{\mathcal{P}}$ for brevity. For two ideals $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{P}$, let $\mathcal{I}\mathcal{J}$ denote the $\mathbb{S}$-module generated by all elements of the form $\mu \underset{i}{\circ} \nu$ for all $\mu \in \mathcal{I}(m)$ and $\nu \in \mathcal{J}(n)$ and all i. It is easy to see that $\mathcal{I}\mathcal{J}$ is also an ideal of $\mathcal{P}$.

Proposition 3.1. *Let $\mathcal{P}$ be a unitary operad (respectively, a unitary plain operad).*

(1) *${}^k\Upsilon$ is an ideal of $\mathcal{P}$ for any $k \ge 1$.*
(2) *If $m, n \ge 1$, then ${}^m\Upsilon{}^n\Upsilon \subseteq {}^{m+n-1}\Upsilon$, and if $m = 0$ or $n = 0$, then ${}^m\Upsilon{}^n\Upsilon \subseteq {}^{m+n}\Upsilon$.*

Proof. (1) Let $n > 0$, $k_1, \cdots, k_n \ge 0$ be integers, and $\theta \in \mathcal{P}(n)$, $\theta_i \in \mathcal{P}(k_i)$ for $i = 1, \cdots, n$. We need to show that if either $\theta \in {}^k\Upsilon(n)$ or $\theta_i \in {}^k\Upsilon(k_i)$ for some $i \in [n]$, then $\theta \circ (\theta_1, \cdots, \theta_n) \in {}^k\Upsilon(m)$, where $m = k_1 + \cdots + k_n$.

Let I be an arbitrary subset of $[m]$ with $|I| = k - 1$. Then we have

$$\pi^I(\theta \circ (\theta_1, \cdots, \theta_n)) = (\theta \circ (\theta_1, \cdots, \theta_n)) \circ (\mathbb{1}_{\chi_I(1)}, \cdots, \mathbb{1}_{\chi_I(m)})$$

$$\begin{aligned}&= \theta \circ (\theta_1 \circ (\mathbb{1}_{\chi(1)}, \cdots, \mathbb{1}_{\chi_I(k_1)}), \cdots, \theta_n \circ (\mathbb{1}_{\chi_I(k_1+\cdots+k_{n-1}+1)}, \cdots, \mathbb{1}_{\chi_I(m)}))\\ &= \theta \circ (\pi^{I_1}(\theta_1), \cdots, \pi^{I_n}(\theta_n))\end{aligned}$$

where in the last equality, each I_i is a subset of $[k_i]$ determined by I, with $|I_i| \leq k_i$ and $\sum_{i=1}^{n} |I_i| = |I| = k-1$.

If $\theta_i \in {}^k\Upsilon(k_i)$ for some $i \in [n]$, then $\pi^{I_i}(\theta_i) = 0$ by Lemma 2.5 and the fact that $|I_i| \leq k-1$. So $\pi^I(\theta \circ (\theta_1, \cdots, \theta_n)) = 0$.

We are left to show that if $\theta \in {}^k\Upsilon(n)$, then $\pi^I(\theta \circ (\theta_1, \cdots, \theta_n)) = 0$. Set $J = \{i \in [n] \mid I_i \neq \emptyset\}$. By $\sum_{i=1}^{n} |I_i| = k-1$, we have $s := |J| \leq k-1$. Consequently, $\pi^J(\theta) = 0$. Observe that if $I_i = \emptyset$ and $\mathcal{P}(0) = \Bbbk\mathbb{1}_0$, then

$$\pi^{I_i}(\theta_i) = \theta_i \circ (\mathbb{1}_0, \cdots, \mathbb{1}_0) = \lambda_i \mathbb{1}_0$$

for some $\lambda_i \in \Bbbk$. Therefore, we have

$$\begin{aligned}\pi^I(\theta \circ (\theta_1, \cdots, \theta_n)) &= \theta \circ (\pi^{I_1}(\theta_1), \cdots, \pi^{I_n}(\theta_n))\\ &= (\prod_{i \notin J} \lambda_i)(\pi^J(\theta)) \circ (\pi^{I_{j_1}}(\theta_{j_1}), \cdots, \pi^{I_{j_s}}(\theta_{j_s}))\\ &= 0\end{aligned}$$

where $J = \{j_1, \cdots, j_s\}$ and $1 \leq j_1 < j_2 < \cdots < j_s \leq n$.

(2) If $m = 0$ or $n = 0$, the assertion follows from part (1). For the rest of the proof, we assume that $mn > 0$.

Let $\mu \in {}^m\Upsilon(m_0)$ and $\nu \in {}^n\Upsilon(n_0)$ and let $i \leq m_0$. It suffices to show that

$$\mu \underset{i}{\circ} \nu \in {}^{m+n-1}\Upsilon(m_0 + n_0 - 1)$$

for all i. Let $I \subseteq [m_0 + n_0 - 1]$ such that $|I| \leq m + n - 2$. By Lemma 2.13(2), we have

$$\pi^I(\mu \underset{i}{\circ} \nu) = \pi^J(\mu) \underset{j}{\circ} \pi^{I'}(\nu) \tag{E3.1.1}$$

where

$$\begin{aligned}&J = (I \cap [i-1]) \cup \{i\} \cup ((I \cap \{i + n_0, i + n_0 + 1, \cdots, m_0 + n_0 - 1\}) - (n_0 - 1)),\\ &I' = I \cap \{i, i+1, \ldots, i + n_0 - 1\} - (i-1),\\ &j = |I \cap [i-1]| + 1.\end{aligned}$$

If $|I'| \leq n-1$, then $\pi^{I'}(\nu) = 0$, whence $\pi^I(\mu \underset{i}{\circ} \nu) = 0$ by (E3.1.1). Otherwise, $|I'| \geq n$ and then

$$|J| = |I| + 1 - |I'| \le m + n - 2 + 1 - n = m - 1.$$

In this case $\pi^J(\mu) = 0$, whence $\pi^I(\mu \underset{i}{\circ} \nu) = 0$ by (E3.1.1). Therefore $\mu \underset{i}{\circ} \nu \in {}^{m+n-1}\Upsilon(m_0 + n_0 - 1)$ as required. $\square$

Note that for any operad $\mathcal{P}$, $\mathcal{P}(1)$ is always a unital associative algebra; and for a unitary operad $\mathcal{P}$, $\mathcal{P}(1)$ is an augmented algebra and ${}^1\Upsilon(1)$ is the augmented ideal of $\mathcal{P}(1)$.

In later sections we will also use a modification of truncation ideals that we define now. Let M be an $\mathbb{S}_k$-submodule of ${}^k\Upsilon(k)$. We consider the following two conditions

(E3.1.2) $\nu \circ m \in M$ for all $\nu \in \mathcal{P}(1)$ and $m \in M$.
(E3.1.3) $m \underset{i}{\circ} \nu \in M$ for all $\nu \in \mathcal{P}(1)$, $m \in M$ and $1 \le i \le k$.

Define ${}^k\Upsilon^M$ by

$${}^k\Upsilon^M(n) = \{\mu \in {}^k\Upsilon(n) \mid \pi^I(\mu) \in M, \ \forall\, I \subseteq [n], |I| = k\}.$$

We have the following proposition similar to Proposition 3.1.

Proposition 3.2. *Let $\mathcal{P}$ be a unitary operad. Let M (respectively, N) be an $\mathbb{S}_m$ (respectively, $\mathbb{S}_n$)-submodule of ${}^m\Upsilon(m)$ (respectively, ${}^n\Upsilon(n)$).*

(1) *If* (E3.1.2) *holds, then ${}^m\Upsilon^M$ is a left ideal of $\mathcal{P}$.*
(2) *If* (E3.1.3) *holds, then ${}^m\Upsilon^M$ is a right ideal of $\mathcal{P}$.*
(3) *$({}^m\Upsilon^M)({}^n\Upsilon^N) \subseteq {}^{m+n-1}\Upsilon^{MN}$, where MN is an $\mathbb{S}_{m+n-1}$-submodule generated by elements of the form $\mu \underset{i}{\circ} \nu$ for all $\mu \in M$ and $\nu \in N$ and $1 \le i \le m$.*

Proof. (1) By Lemma 2.13, ${}^m\Upsilon^M$ is an $\mathbb{S}$-module. Next we show that ${}^m\Upsilon^M$ is a left ideal.

For $\nu \in \mathcal{P}(m_0)$ and $\mu \in {}^m\Upsilon^M(n_0)$, $I \subseteq [m_0 + n_0 - 1]$ with $|I| = m$, by Lemma 2.13 (2),

$$\pi^I(\nu \underset{i}{\circ} \mu) = \pi^J(\nu) \underset{j}{\circ} \pi^{I'}(\mu)$$

where J, I' and j are given as after (E3.1.1). If $|I'| \le m - 1$, then $\pi^{I'}(\mu) = 0$, and $\pi^I(\nu \underset{i}{\circ} \mu) = 0$. Otherwise $|I'| = m$ (which is maximum possible) and $I \subset \{i, i+1, \cdots, i + n_0 - 1\}$, then $j = 1$, $|J| = 1$, and

$$\pi^J(\nu) \underset{1}{\circ} \pi^I(\mu) \in M$$

by assumption of M. Thus $\pi^I(\nu \underset{i}{\circ} \mu) \in M$.

(2) The proof is similar to the proof of part (1). For $\mu \in {}^m\Upsilon^M(m_0)$ and $\nu \in \mathcal{P}(n_0)$, $I \subseteq [m_0+n_0-1]$ with $|I| = m$, by Lemma 2.13(2),

$$\pi^I(\mu \underset{i}{\circ} \nu) = \pi^J(\mu) \underset{j}{\circ} \pi^{I'}(\nu)$$

where J, I' and j are given as after (E3.1.1). If $|J| \leq m-1$, then $\pi^J(\mu) = 0$, and $\pi^I(\nu \underset{i}{\circ} \mu) = 0$. If $|J| = m$, then $\pi^J(\mu) \in M$ and $\pi^{I'}(\nu) \in \mathcal{P}(1)$, and by the assumption on M, we obtain that $\pi^J(\mu) \underset{j}{\circ} \pi^{I'}(\nu) \in M$. If $|J| = m+1$ (maximal possible), then $I' = \emptyset$ and $\pi^{I'}(\nu) \in \mathcal{P}(0)$. Then

$$\pi^J(\mu) \underset{j}{\circ} \pi^{I'}(\nu) = \pi^J(\mu) \underset{j}{\circ} \pi^{\emptyset}(\nu)\mathbb{1}_0 = \pi^{J\setminus\{j'\}}(\mu)\pi^{\emptyset}(\nu) \in M$$

for some j'. Combining these cases, we have $\pi^I(\mu \underset{i}{\circ} \nu) \in M$. Therefore ${}^m\Upsilon^M$ is a right ideal.

(3) Let $\mu \in {}^m\Upsilon^M(m_0)$ and $\nu \in {}^n\Upsilon^N(n_0)$ and let $i \leq m_0$. It suffices to show that

$$\mu \underset{i}{\circ} \nu \in {}^{m+n-1}\Upsilon^{MN}(m_0+n_0-1)$$

for all i. By Proposition 3.1(2), $\mu \underset{i}{\circ} \nu \in {}^{m+n-1}\Upsilon(m_0+n_0-1)$.

Let $I \subseteq [m_0+n_0-1]$ such that $|I| = m+n-1$. It suffices to show that $\pi^I(\mu \underset{i}{\circ} \nu) \in MN$. By Lemma 2.13(2),

$$\pi^I(\mu \underset{i}{\circ} \nu) = \pi^J(\mu) \underset{j}{\circ} \pi^{I'}(\nu)$$

where

$$\begin{aligned}
&J = (I \cap [i-1]) \cup \{i\} \cup ((I \cap \{i+n_0, i+n_0+1, \cdots, m_0+n_0-1\}) - (n_0-1)),\\
&I' = (I \cap \{i, i+1, \ldots, i+n_0-1\}) - (i-1), \text{ and}\\
&j = |I \cap [i-1]| + 1.
\end{aligned}$$

In particular, $|I'| + |J| = m+n$. If $|I'| \leq n-1$ or $|J| \leq m-1$, then $\pi^{I'}(\nu) = 0$ or $\pi^J(\mu) = 0$. Hence $\pi^I(\mu \underset{i}{\circ} \nu) = 0 \in M$. The remaining case is when $|I'| = n$ and $|J| = m$. Then, in this case, $\pi^J(\mu) \in M$ and $\pi^{I'}(\nu) \in N$. Hence $\pi^J(\mu) \underset{j}{\circ} \pi^{I'}(\nu) \in MN$ by definition. Combining all cases, $\pi^I(\mu \underset{i}{\circ} \nu) \in MN$ as required. □

A version of Proposition 3.2 holds for plain operads. The following lemmas are clear.

Lemma 3.3. *Let $f : \mathcal{P} \to \mathcal{Q}$ be a morphism of unitary operads in Op_+. Then, for every $I \subseteq [n]$ with $|I| = k-1$, we have a commutative diagram*

$$\begin{tikzcd}
\mathcal{P}(n) \arrow[r] \arrow[d, "\pi^I"'] & \mathcal{Q}(n) \arrow[d, "\pi^I"] \\
\mathcal{P}(k-1) \arrow[r] & \mathcal{Q}(k-1).
\end{tikzcd}$$

As a consequence, f maps from ${}^k\Upsilon_{\mathcal{P}}$ to ${}^k\Upsilon_{\mathcal{Q}}$ for all $k \geq 0$.

Recall from Definition 1.1(6) that an operad $\mathcal{P}$ is called *connected* if $\mathcal{P}(1) = \Bbbk \cdot \mathbb{1}_1 \cong \Bbbk$.

Lemma 3.4. *Let $\mathcal{P}$ be a connected unitary operad. Then ${}^1\Upsilon = {}^2\Upsilon$.*

Proof. In this case, $\pi^\emptyset : \mathcal{P}(1) \to \mathcal{P}(0)$ is an isomorphism. Then

$$\mathrm{Ker}(\pi^i : \mathcal{P}(n) \to \mathcal{P}(1)) = \mathrm{Ker}(\pi^\emptyset : \mathcal{P}(n) \to \mathcal{P}(0))$$

for all $i \leq n$. Therefore ${}^1\Upsilon = {}^2\Upsilon$. $\square$

Recall that operads $\mathcal{C}om$ and $\mathcal{U}ni$ are defined before Lemma 1.13.

Lemma 3.5. *Let $\mathcal{P}$ be a unitary operad.*

(1) *${}^1\Upsilon$ is the maximal ideal of $\mathcal{P}$ and $\mathcal{P}/{}^1\Upsilon$ is isomorphic to either $\mathcal{C}om$ or $\mathcal{U}ni$.*
(2) *If $\mathcal{P}/{}^1\Upsilon \cong \mathcal{C}om$ and $\mathcal{P}$ is connected, then $\mathcal{P}$ is 2-unitary.*
(3) *$\mathcal{U}ni \oplus {}^1\Upsilon$ is a suboperad of $\mathcal{P}$ and it is unitary, but not 2-unitary.*
(4) *If $\mathcal{P}/{}^1\Upsilon \cong \mathcal{U}ni$, then $\mathcal{P} = \mathcal{U}ni \oplus {}^1\Upsilon$.*

Proof. (1) Since $\mathcal{P}$ is unitary, $\mathcal{P}(0) = \Bbbk\mathbb{1}_0$. By definition,

$${}^1\Upsilon(n) = \mathrm{Ker}(\pi^\emptyset : \mathcal{P}(n) \to \mathcal{P}(0)).$$

Then $\dim(\mathcal{P}/{}^1\Upsilon)(n)$ is either 0 or 1 for each n. If $(\mathcal{P}/{}^1\Upsilon)(2) = 0$, then one can check that $(\mathcal{P}/{}^1\Upsilon)(n) = 0$ for all $n \geq 2$. Consequently, $\mathcal{P}/{}^1\Upsilon = \mathcal{U}ni$. If $(\mathcal{P}/{}^1\Upsilon)(2) \neq 0$, then one can check that $\mathcal{P}/{}^1\Upsilon$ is 2a-unitary and $(\mathcal{P}/{}^1\Upsilon)(n) = \Bbbk\mathbb{1}_n$ for all n. Consequently, $\mathcal{P}/{}^1\Upsilon = \mathcal{C}om$.

(2) Since $\mathcal{P}$ is connected, ${}^1\Upsilon = {}^2\Upsilon$ by Lemma 3.4. Since $\mathcal{P}(2) \neq {}^1\Upsilon(2)$, there is an $f \in \mathcal{P}(2)$ such that $\pi^1(f) = \mathbb{1}_1$. Since $\pi^\emptyset \bullet \pi^2(f) = \pi^\emptyset(f) = \pi^\emptyset \bullet \pi^1(f) = \mathbb{1}_0$, we obtain that $\pi^2(f) = \mathbb{1}_1$. Thus f is a 2-unit by definition.

(3) This follows from the fact that ${}^1\Upsilon$ is an ideal of $\mathcal{P}$.

(4) This follows from part (3). $\square$

Let $\mathcal{P}$ be a (plain or symmetric) operad. For each $n \geq 0$, we denote an $\mathbb{S}$-submodule $\mathcal{P}_{\geq n}$ of $\mathcal{P}$ as follows:

$$\mathcal{P}_{\geq n}(i) = \begin{cases} 0, & 0 \leq i < n, \\ \mathcal{P}(n), & i \geq n. \end{cases}$$

Now we are ready to prove the following Artin-Wedderburn Theorem for reduced operads and unitary operads.

Theorem 3.6. *Let $\mathcal{P}$ be a semiprime plain or symmetric operad.*

(1) *If $\mathcal{P}$ is reduced and left or right artinian, then*

$$\mathcal{P}(n) = \begin{cases} 0, & n \neq 1, \\ \Lambda, & n = 1, \end{cases}$$

where Λ is a semisimple algebra.

(2) *If $\mathcal{P}$ is unitary, bounded above, and left or right artinian, then*

$$\mathcal{P}(n) = \begin{cases} 0, & n \neq 0, 1, \\ \mathbb{k}, & n = 0, \\ \Lambda, & n = 1, \end{cases}$$

where Λ is an augmented semisimple algebra.

Proof. We only prove the results for symmetric operads. The proofs for plain operads are similar.

(1) Since $\mathcal{P}$ is reduced, $\mathcal{P}_{\geq n}$ is an ideal for every n. Since $\mathcal{P}$ is artinian and $\{\mathcal{P}_{\geq n}\}_{n=0}^{\infty}$ is a descending chain of ideals, $\mathcal{P}_{\geq n} = 0$ for some n. Let n be the largest integer such that $\mathcal{P}(n) \neq 0$. If $n \geq 2$, then $\mathcal{P}$ being reduced implies that $\mathcal{P}(n)$ is an ideal such that $\mathcal{P}(n)^2 = 0$. This contradicts the hypothesis that $\mathcal{P}$ is semiprime. Therefore $\mathcal{P}(n) = 0$ for all $n \geq 2$. Let $\mathcal{P}(1) = \Lambda$. In this case the left (or right) ideals of $\mathcal{P}$ coincide with the left (or right) ideals of Λ. Thus Λ is left or right artinian and semiprime. This implies that Λ is semisimple as desired.

(2) In the proof of part (2), we need to use truncation ideals ${}^{k}\Upsilon$ of $\mathcal{P}$. By definition, $\bigcap_{k\geq 1} {}^{k}\Upsilon = 0$. Since $\mathcal{P}$ is left or right artinian, ${}^{k}\Upsilon = 0$ for some k. Let n be the largest integer such that ${}^{n}\Upsilon \neq 0$. If $n \geq 2$, by Proposition 3.1(2), $({}^{n}\Upsilon)^2 \subseteq {}^{2n-1}\Upsilon = 0$. This contradicts the hypothesis that $\mathcal{P}$ is semiprime. Therefore ${}^{2}\Upsilon = 0$. Let $\mathcal{P}(1) = \Lambda$. By Proposition 3.2 (1, 2), if Λ is not left (respectively, right) artinian, then $\mathcal{P}$ is not left (respectively, right) artinian. Since $\mathcal{P}$ is left or right artinian, so is Λ. Let N be an ideal of Λ such that $N^2 = 0$. By Proposition 3.2 (1, 2), ${}^{1}\Upsilon^{N}$ is an ideal of $\mathcal{P}$. By Proposition 3.2 (3),

$$({}^{1}\Upsilon^{N})^2 \subseteq {}^{1}\Upsilon^{N^2} = {}^{1}\Upsilon^{0} = {}^{2}\Upsilon = 0.$$

Since $\mathcal{P}$ is semiprime, ${}^1\Upsilon^N = 0$, consequently, $N = 0$. Thus Λ is semiprime. Since Λ is left artinian or right artinian, Λ is semisimple. It remains to show that $\mathcal{P}(n) = 0$ for all $n \geq 2$. If not, let $n \geq 2$ be the largest integer such that $\mathcal{P}(n) \neq 0$ (such n exists since $\mathcal{P}$ is bounded above). For every element $0 \neq \mu \in \mathcal{P}(n)$, $x := \pi^i(\mu) \neq 0$ for some i as ${}^2\Upsilon = 0$. Let I be the ideal of Λ generated by x. For every element $f \in I$, f can be written as $\sum_{s=1}^{w} a_s x b_s$ with $a_s, b_s \in \Lambda$. Let

$$g = \sum_{s=1}^{w} a_s \underset{1}{\circ} (\mu \underset{i}{\circ} b_s) \in \mathcal{P}(n).$$

Then $f = \pi^i(g)$. Since I is a nonzero ideal of a semisimple ring, $I = e\Lambda = \Lambda e$ for some idempotent $e \in I$. Hence we may assume that $f = e$ is a nonzero idempotent. Let $\nu = \pi^{i,j}(g)$. Then $f = \pi^1(\nu)$ or $f = \pi^2(\nu)$. By symmetry, we assume that $f = \pi^1(\nu)$. Let $h = g \underset{i}{\circ} \nu \in \mathcal{P}(n+1)$. Then

$$\pi^i(h) = \pi^i(g) \circ \pi^1(\nu) = f \circ f = f \neq 0$$

which contradicts the fact that $\mathcal{P}(n+1) = 0$. Therefore $\mathcal{P}(n) = 0$ for all $n \geq 2$ as required. □

Lemma 3.7. *Let $\mathcal{P}$ be a 2-unitary operad and $\mathcal{I}$ an ideal of $\mathcal{P}$. Then for each $k \geq 1$, $\mathcal{I}(k-1) = 0$ if and only if $\mathcal{I} \subset {}^k\Upsilon$.*

Proof. ($\Leftarrow$) is obvious. Next we show the other implication ($\Rightarrow$). Suppose $\mathcal{I}(k-1) = 0$ for some $k \geq 1$.

If $n \geq k-1$, then we have

$$\pi^I(\theta) \in \mathcal{I}(k-1) = 0$$

for any $\theta \in \mathcal{I}(n)$ and any $I \subseteq [n]$ with $|I| = k-1$, and hence $\theta \in {}^k\Upsilon(n)$.

If $n < k-1$, for every $\theta \in \mathcal{I}(n)$, we have

$$(\Delta_{i_1} \bullet \cdots \bullet \Delta_{i_{k-1-n}})(\theta) \in \mathcal{I}(k-1) = 0,$$

for all possible $i_1, \cdots, i_{k-1-n}$. Since $\mathcal{P}$ is 2-unitary, each Δ_i is injective by Lemma 2.10 (2) (or (3)). It follows that $\theta = 0$ and hence $\mathcal{I} \subset {}^k\Upsilon$. □

3.2. The unique maximal ideal of a quotient operad of $\mathcal{A}ss$

In this subsection we assume that $\mathcal{P} = \mathcal{A}ss/\mathcal{W}$ for some ideal $\mathcal{W}$. We use Φ_n to denote the alternating sum $\sum_{\sigma \in \mathbb{S}_n} \mathrm{sgn}(\sigma)\sigma$, where $\mathrm{sgn}(\sigma) = 1$ if σ is an even permutation, and $\mathrm{sgn}(\sigma) = -1$ if σ is an odd permutation. When applied to an associative algebra, the operator Φ_2 gives exactly the usual commutator.

Lemma 3.8. *As an ideal of $\mathcal{P}$, ${}^1\Upsilon = {}^2\Upsilon = \langle \Phi_2 \rangle$.*

Proof. We only consider the case $\mathcal{P} = \mathcal{A}ss$. By Lemma 3.4, ${}^1\Upsilon = {}^2\Upsilon$. Clearly, we have ${}^2\Upsilon \supseteq \langle \Phi_2 \rangle$ since $\Phi_2 \in {}^2\Upsilon(2)$. It suffices to show that ${}^2\Upsilon \subseteq \langle \Phi_2 \rangle$. By definition $\pi^i(\sigma) = 1_1$ for all $n \geq 1$, $\sigma \in \mathbb{S}_n$, and $1 \leq i \leq n$. Thus we have

$$ {}^2\Upsilon(n) = \left\{ \sum_{\sigma \in \mathbb{S}_n} \lambda_\sigma \sigma \mid \sum_{\sigma \in \mathbb{S}_n} \lambda_\sigma = 0, \lambda_\sigma \in \Bbbk \right\}. $$

It is well-known that ${}^2\Upsilon(n)$ is generated by the set $\{1_n - \sigma \mid \sigma \in \mathbb{S}_n, \sigma \neq 1_n\}$. (It may not be a basis unless $\mathcal{P} = \mathcal{A}ss$). For any $\sigma_1, \cdots, \sigma_s \in \mathbb{S}_n$, we write

$$ \begin{aligned} 1_n - \sigma_1 \cdots \sigma_s =& (1_n - \sigma_s) + (\sigma_s - \sigma_{s-1}\sigma_s) + \cdots + (\sigma_2 \cdots \sigma_s - \sigma_1 \cdots \sigma_s) \\ =& (1_n - \sigma_s) + (1_n - \sigma_{s-1})\sigma_s + \cdots + (1_n - \sigma_1)\sigma_2 \cdots \sigma_s. \end{aligned} $$

Since $\{(12), (23), \cdots, (n-1, n)\}$ generates the group $\mathbb{S}_n$, the above equality implies that ${}^2\Upsilon(n)$ is generated by the elements $1_n - (12), 1_n - (23), \cdots, 1_n - (n-1, n)$ as a right $\mathbb{S}_n$-module. For each $i \geq 1$, we have $\mathbb{1}_n - (i, i+1) = \iota^{i-1}_{n-i-1}(\Phi_2)$, and hence ${}^2\Upsilon(n) \subseteq \langle \Phi_2 \rangle(n)$. The assertion follows. $\square$

Lemma 3.9. *Let $\mathcal{I} \subsetneq \mathcal{P}$ be an ideal. Then either $\mathcal{I} = {}^2\Upsilon$ or $\mathcal{I} \subseteq {}^3\Upsilon$.*

Proof. First we claim that $\mathcal{I} \subseteq {}^2\Upsilon$. Otherwise, there exist $n \geq 1$ and $\theta \in \mathcal{I}(n)$ such that $\pi^i(\theta) \neq 0$ for some $1 \leq i \leq n$. It follows that $\mathbb{1}_1 \in \mathcal{I}$, and hence $\mathcal{P} \subseteq \mathcal{I}$, which leads to a contradiction.

Next assume that $\mathcal{I} \nsubseteq {}^3\Upsilon$. Then there exist $n \geq 2$ and $\theta \in \mathcal{I}(n)$, such that $\pi^{i,j}(\theta) \neq 0$ for some $1 \leq i < j \leq n$. Note that $\pi^{i,j}(\theta) \in {}^2\Upsilon(2)$ and hence $\pi^{i,j}(\theta) = \lambda \Phi_2$ for some $\lambda \neq 0$. Now Lemma 3.8 implies that $\mathcal{I} = {}^2\Upsilon$. $\square$

3.3. A descending chain of ideals

In this subsection we assume that $\mathcal{P} = \mathcal{A}ss$. By definition and Lemma 2.5 (1), ${}^{k+1}\Upsilon \subseteq {}^k\Upsilon$ for all $k \geq 0$. Thus we obtain a descending chain of ideals

$$ {}^1\Upsilon = {}^2\Upsilon \supseteq {}^3\Upsilon \supseteq {}^4\Upsilon \supseteq \cdots $$

of $\mathcal{A}ss$. Then for any ideal $\mathcal{I}$ of $\mathcal{A}ss$, after taking intersections with ${}^k\Upsilon$'s, we also obtain a descending chain of subideals

$$ \mathcal{I} \cap {}^2\Upsilon \supseteq \mathcal{I} \cap {}^3\Upsilon \supseteq \mathcal{I} \cap {}^4\Upsilon \supseteq \cdots. $$

Before continuing, we introduce a useful lemma.

Lemma 3.10. *Let $n \geq k \geq 0$ be integers, and θ be in $\mathcal{A}ss(n)$. Then*

(1) $\Phi_2 \circ (\theta, 1_1) \in {}^{k+1}\Upsilon(n+1)$ *if and only if* $\theta \in {}^{k}\Upsilon(n)$.
(1′) $\Phi_2 \circ (1_1, \theta) \in {}^{k+1}\Upsilon(n+1)$ *if and only if* $\theta \in {}^{k}\Upsilon(n)$.
(2) $1_2 \circ (\theta, \Phi_2) \in {}^{k+2}\Upsilon(n+2)$ *if and only if* $\theta \in {}^{k}\Upsilon(n)$.
(2′) $1_2 \circ (\Phi_2, \theta) \in {}^{k+2}\Upsilon(n+2)$ *if and only if* $\theta \in {}^{k}\Upsilon(n)$.

Proof. To avoid confusion, we use τ to denote the 2-cycle $(12) \in \mathbb{S}_2$. Thus $\Phi_2 = 1_2 - \tau$. We only prove (1) and (2), and the argument for (1′) and (2′) are the same.

(1) ($\Leftarrow$) First we assume that $\theta \in {}^{k}\Upsilon(n)$. Take any subset I of $[n+1]$ with $|I| = k$. We claim that $\pi^I(\Phi_2 \circ (\theta, 1_1)) = 0$. By definition,

$$\begin{aligned}\pi^I(\Phi_2 \circ (\theta, 1_1)) &= (\Phi_2 \circ (\theta, 1_1)) \circ (1_{\chi_I(1)}, \cdots, 1_{\chi_I(n)}, 1_{\chi_I(n+1)}) \\ &= \Phi_2 \circ (\pi^{I_1}(\theta), 1_{\chi_I(n+1)}),\end{aligned} \tag{E3.10.1}$$

where $I_1 = I \cap [n]$. There are two cases: $n+1 \in I$ or $n+1 \notin I$. If $n+1 \in I$, then $|I_1| = k-1$, and hence $\pi^{I_1}(\theta) = 0$ by assumption. The claim follows in this case. Now we assume that $n+1 \notin I$, i.e., $I = I_1 \subseteq [n]$. Obviously one has $\tau \circ (\theta', 1_0) = \theta'$, and hence $\Phi_2 \circ (\theta', 1_0) = 0$ for any θ'. Now in both cases, we have $\pi^I(\Phi_2 \circ (\theta, 1_1)) = 0$. Therefore the claim holds and the "if" part follows.

($\Rightarrow$) Next we prove the "only if" part. Assume that $\Phi_2 \circ (\theta, 1_1) \in {}^{k+1}\Upsilon(n+1)$. We need only to show that $\pi^I(\theta) = 0$ for every subset $I \subseteq [n]$ with $|I| = k-1$. Set $I' = I \cup \{n+1\}$. Clearly I' is a subset of $[n+1]$ with $|I'| = k$. By (E3.10.1), we have

$$0 = \pi^{I'}(\Phi_2 \circ (\theta, 1_1)) = \Phi_2 \circ (\pi^I(\theta), 1_1) = 1_2 \circ (\pi^I(\theta), 1_1) - \tau \circ (\pi^I(\theta), 1_1).$$

Note that

$$\{1_2 \circ (\sigma, 1_1) \mid \sigma \in \mathbb{S}_{k-1}\} \cup \{\tau \circ (\sigma, 1_1) \mid \sigma \in \mathbb{S}_{k-1}\}$$

are linearly independent in $\Bbbk\mathbb{S}_k$. It follows that $1_2 \circ (\pi^I(\theta), 1_1) = 0$ and hence $\pi^I(\theta) = 0$.

(2) For every $I \subset [n]$, denote by $\tilde{I}$ the set $I \cup \{n+1, n+2\}$. Then we obtain a 1-1 correspondence between subsets of $[n]$ and the ones of $[n+2]$ containing both $n+1$ and $n+2$.

($\Leftarrow$) Assume that $\theta \in {}^{k}\Upsilon(n)$. Then for every $J \subseteq [n+2]$ with $|J| = k+1$, we claim that

$$\pi^J(1_2 \circ (\theta, 1_2 - \tau)) = 0.$$

Easy calculations show that

$$\Gamma^{n+1}(1_2 \circ (\theta, 1_2 - \tau)) = \Gamma^{n+2}(1_2 \circ (\theta, 1_2 - \tau)) = 0.$$

Thus, if $\{n+1, n+2\} \nsubseteq J$, then $\pi^J(1_2 \circ (\theta, 1_2 - \tau)) = 0$ since π^J will factor through Γ^{n+1} or Γ^{n+2} in this case. Now we may assume that $J = \tilde{I}$ for some $I \subseteq [n]$. Then

$$\pi^{\tilde{I}}(1_2 \circ (\theta, 1_2 - \tau)) = 1_2 \circ (\pi^I(\theta), 1_2 - \tau) = 0. \qquad \text{(E3.10.2)}$$

The "if" part follows.

($\Rightarrow$) For the "only if" part, again we use (E3.10.2) and the fact that $1_2 \circ (\pi^I(\theta), 1_2 - \tau)) = 0$ if and only if $\pi^I(\theta) = 0$. $\square$

The main result of this subsection is the following separability property of the ideals ${}^k\Upsilon$ of $\mathcal{A}ss$.

Proposition 3.11.

(1) *Let $\mathcal{I}$ be a nonzero ideal of $\mathcal{A}ss$. Then $\mathcal{I} \cap {}^k\Upsilon \neq \mathcal{I} \cap {}^{k+1}\Upsilon$ for all $k \gg 0$.*
(2) *${}^k\Upsilon \neq {}^{k+1}\Upsilon$ for every $k \geq 2$.*

Proof. (1) Note that $\bigcap_{k\geq 0} {}^k\Upsilon = 0$ since ${}^k\Upsilon(k-1) = 0$ for all $k \geq 1$. By Lemma 3.9 and the assumption $\mathcal{I} \neq 0$, we have $I \cap {}^2\Upsilon \neq 0$. Thus $\mathcal{I} \cap {}^{k_0}\Upsilon \neq \mathcal{I} \cap {}^{k_0+1}\Upsilon$ for some $k_0 \geq 1$. There exist some $k_0 \geq 1$, $n \geq k_0$, and $\theta \in \mathcal{I}(n)$ such that $\theta \in {}^{k_0}\Upsilon(n)$ while $\theta \notin {}^{k_0+1}\Upsilon(n)$. By Lemma 3.10, $\Phi_2 \circ (\theta, 1_1) \in {}^{k_0+1}\Upsilon(n+1)$, and $\Phi_2 \circ (\theta, 1_1) \notin {}^{k_0+2}\Upsilon(n+1)$, which implies that $\mathcal{I} \cap {}^{k_0+1}\Upsilon \neq \mathcal{I} \cap {}^{k_0+2}\Upsilon$. By induction we may show that $\mathcal{I} \cap {}^k\Upsilon \neq \mathcal{I} \cap {}^{k+1}\Upsilon$ for all $k \geq k_0$.

(2) The statement follows from the above proof and the fact that $\Phi_2 \in {}^2\Upsilon$ and $\Phi_2 \notin {}^3\Upsilon$. $\square$

Remark 3.12. Recall that the descending chain condition (DCC, for short) for an object C means that any descending chain

$$C \supseteq C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$$

of subobjects of C is stable, that is, $C_k = C_{k+1} = \cdots$ for sufficiently large k. The proposition says that the DCC does NOT hold for any nonzero ideal of $\mathcal{A}ss$ and $\mathcal{A}ss$ is not artinian.

Let $\mathcal{P}$ be a unitary operad and let ${}^n\Upsilon$ be ${}^n\Upsilon_{\mathcal{P}}$. Let Υ denote the $\mathbb{S}$-submodule of $\mathcal{P}$ given by

$$\Upsilon(n) = {}^n\Upsilon(n), \quad (n = 0, 1, \cdots). \qquad \text{(E3.12.1)}$$

Proposition 3.13. *Let $\mathcal{P}$ be a unitary operad.*

(1) *Υ is closed under partial compositions.*

(2) $\Bbbk \mathbb{1}_1 \oplus \Upsilon$ *is a unitary operad.*
(3) $\Bbbk \mathbb{1}_1 \oplus \Upsilon$ *is* $\mathcal{U}ni$*-augmented.*

Proof. (1) By the proof of Proposition 3.1 (2), ${}^m\Upsilon(m){}^n\Upsilon(n) \subseteq {}^{m+n-1}\Upsilon(m+n-1)$ for all m, n. The assertion follows.

(2, 3) These follow from part (1). $\square$

4. Dimension computation, basis theorem and categorification

4.1. Definitions of growth properties

We collect some definitions.

Definition 4.1. Let $\mathcal{M} = (\mathcal{M}(n))_{n\geq 0}$ be an $\mathbb{S}$-module (or a $\Bbbk$-linear operad).

(1) The sequence $(\dim \mathcal{M}(0), \dim \mathcal{M}(1), \cdots)$ is called the *dimension sequence* (or simply *dimension*) of $\mathcal{M}$. We call $\mathcal{M}$ *locally finite* if $\dim_{\Bbbk} \mathcal{M}(n) < \infty$ for all n.
(2) The *generating series* of $\mathcal{M}$ is defined to be
$$G_{\mathcal{M}}(t) = \sum_{n=0}^{\infty} \dim \mathcal{M}(n) t^n \in \mathbb{Z}[[t]].$$
The *exponential generating series* of $\mathcal{M}$ is defined to be
$$E_{\mathcal{M}}(t) = \sum_{n=0}^{\infty} \frac{\dim \mathcal{M}(n)}{n!} t^n \in \mathbb{Q}[[t]].$$
(3) The *exponent* of $\mathcal{M}$ is defined to be
$$\exp(\mathcal{M}) := \limsup_{n\to\infty} (\dim \mathcal{M}(n))^{\frac{1}{n}}.$$
We say $\mathcal{M}$ has *exponential growth* if $\exp(\mathcal{M}) > 1$. We say $\mathcal{M}$ has *finite exponent* if $\exp(\mathcal{M}) < \infty$.
(4) We say that $\mathcal{M}$ has *polynomial growth* if there are $0 < C, k < \infty$ such that $\dim \mathcal{M}(n) < Cn^k$ for all $n > 0$. The infimum of such k is called the *order of polynomial growth* and denoted by $o(\mathcal{M})$.
(5) We say $\mathcal{M}$ has *sub-exponential growth* if $\exp(\mathcal{M}) \leq 1$ and if $\mathcal{M}$ does not have polynomial growth.
(6) The *Gelfand-Kirillov dimension* (*GKdimension* for short) of $\mathcal{M}$ is defined to be
$$\operatorname{GKdim}(\mathcal{M}) = \limsup_{n\to\infty} \log_n \left(\sum_{i=0}^{n} \dim_{\Bbbk} \mathcal{M}(i) \right)$$
which is the same as (E0.0.3).

When we talk about the growth of an operad $\mathcal{P}$, we implicitly assume that $\mathcal{P}$ is locally finite. It is easy to see that $\exp(\mathcal{A}ss) = \infty$, so $\mathcal{A}ss$ has (infinite) exponential growth. And $\operatorname{GKdim}(\mathcal{C}om) = 1$, so $\mathcal{C}om$ has polynomial growth. We will see that for every integer $k \geq 1$, there exists a quotient operad $\mathcal{P}/{}^k\Upsilon$ has polynomial growth of order (no more than) k. First we state a lemma for arbitrary unitary operads.

Lemma 4.2. *Let $\mathcal{P}$ be a $\Bbbk$-linear (symmetric or plain) unitary operad. If ${}^k\Upsilon = 0$ for some k, then $\operatorname{GKdim} \mathcal{P} \leq k$.*

As usual

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.$$

Proof. Consider the restriction operator $\pi^I : \mathcal{P}(n) \to \mathcal{P}(k-1)$ for all $n \geq k-1$, which induces an injective map

$$(\pi^I)' : \quad \mathcal{P}(n)/\operatorname{Ker}\pi^I \to \mathcal{P}(k-1)$$

where $I \subseteq [n]$ with $|I| = k-1$. By hypothesis and definition,

$$0 = {}^k\Upsilon(n) = \bigcap_{I \subset [n],\, |I|=k-1} \operatorname{Ker}\pi^I.$$

Hence we have an injective map

$$\mathcal{P}(n) \xrightarrow{\cong} \frac{\mathcal{P}(n)}{(\bigcap_{I\subset[n],\,|I|=k-1} \operatorname{Ker}\pi^I)} \to \bigoplus_{I\subset[n],\,|I|=k-1} \frac{\mathcal{P}(n)}{\operatorname{Ker}\pi^I} \to \bigoplus_{I\subset[n],\,|I|=k-1} \mathcal{P}(k-1),$$

which implies that

$$\dim \mathcal{P}(n) \leq \dim \mathcal{P}(k-1)\binom{n}{k-1}$$

for all $n \geq k-1$. The assertion follows. $\square$

Let $\mathcal{P}$ be a unitary operad and $\mathcal{I}$ an ideal of $\mathcal{P}$. Let $d^k_{\mathcal{I}}(n)$ denote the codimension of $({}^k\Upsilon \cap \mathcal{I})(n)$ in $\mathcal{I}(n)$, that is,

$$d^k_{\mathcal{I}}(n) = \dim_{\Bbbk} \frac{\mathcal{I}}{{}^k\Upsilon \cap \mathcal{I}}(n) = \dim_{\Bbbk} \mathcal{I}(n) - \dim_{\Bbbk}({}^k\Upsilon \cap \mathcal{I})(n), \tag{E4.2.1}$$

where the second equality holds if $\mathcal{P}$ is locally finite. If $\mathcal{I} = \mathcal{P}$, we have

$$d^k(n) = \dim_{\Bbbk} \frac{\mathcal{P}}{{}^k\Upsilon}(n) = \dim_{\Bbbk} \mathcal{P}(n) - \dim_{\Bbbk} {}^k\Upsilon(n),$$

where the second equation holds if $\mathcal{P}$ is locally finite.

We do not assume that $d^k_{\mathcal{I}}(n)$ is finite. When $\mathcal{P}$ is locally finite, we will give a recursive formula for $d^k_{\mathcal{I}}(n)$. The key idea is to find a basis for the quotient module $\frac{{}^k\Upsilon \cap \mathcal{I}}{{}^{k+1}\Upsilon \cap \mathcal{I}}(n)$, so we can calculate dim $\frac{{}^k\Upsilon \cap I}{{}^{k+1}\Upsilon \cap \mathcal{I}}(n)$ for all n.

For every subset $I \subseteq [n]$, we use c_I to denote the element in $\mathbb{S}_n$ which corresponds to the permutation

$$c_I := (1, \cdots, i_1 - 1, i_1 + 1, \cdots, i_s - 1, i_s + 1, \cdots, n, i_1, \cdots, i_s), \tag{E4.2.2}$$

where $I = \{i_1, \cdots, i_s\}$ with $i_1 < \cdots < i_s$. Let $\mathcal{P}$ be a 2-unitary operad. By an easy calculation we have

$$\Gamma^I((\mathbb{1}_2 \circ (\mathbb{1}_{n-s}, \mathbb{1}_s)) * c_I) = \mathbb{1}_{n-s}.$$

In fact, we have a more general result.

Lemma 4.3. *Let $\mathcal{P}$ be a 2-unitary operad. Let $n \geq k$ be integers and set $s = n - k$.*

(1) *Let $I \subseteq [n]$ be a subset with $|I| = s$. Then $\Gamma^I((\mathbb{1}_2 \circ (\theta, \mathbb{1}_s)) * c_I) = \theta$ for all $\theta \in \mathcal{P}(k)$.*
(2) *Let $J \subseteq [n]$ be a subset with $|J| = k$. Then for every $\theta \in {}^k\Upsilon(k)$ and every $\sigma \in \mathbb{S}_n$,*

$$\pi^J((\mathbb{1}_2 \circ (\theta, \mathbb{1}_s)) * \sigma) = 0$$

unless $J = \{\sigma^{-1}(1), \cdots, \sigma^{-1}(k)\}$.

Proof. (1) To avoid possible confusion, we use 1_n for $\mathbb{1}_{\mathbb{S}_n} \in \mathbb{S}_n$ for all $n \geq 0$. Applying (OP3) and using the fact that $\theta * 1_k = \theta$ for all $\theta \in \mathcal{P}(k)$, we have

$$\begin{aligned}
\Gamma^I(\mathbb{1}_2 \circ (\theta, \mathbb{1}_s) * c_I) =& ((\mathbb{1}_2 \circ (\theta, \mathbb{1}_s)) * c_I) \circ (\mathbb{1}_{\chi_{\hat{I}}(1)}, \cdots, \mathbb{1}_{\chi_{\hat{I}}(n)}) \\
=& ((\mathbb{1}_2 \circ (\theta, \mathbb{1}_s)) * c_I) \circ (\mathbb{1}_{\chi_{\hat{I}}(1)} * 1_{\chi_{\hat{I}}(1)}, \cdots, \mathbb{1}_{\chi_{\hat{I}}(n)} * 1_{\chi_{\hat{I}}(n)}) \\
=& [(\mathbb{1}_2 \circ (\theta, \mathbb{1}_s)) \circ (\mathbb{1}_{\chi_{\hat{I}}(c_I^{-1}(1))}, \cdots, \mathbb{1}_{\chi_{\hat{I}}(c_I^{-1}(n))})] \\
& * [c_I \circ (1_{\chi_{\hat{I}}(1)}, \cdots, 1_{\chi_{\hat{I}}(n)})] \\
=& [(\mathbb{1}_2 \circ (\theta, \mathbb{1}_s)) \circ (\underbrace{\mathbb{1}_1, \cdots, \mathbb{1}_1}_{k}, \underbrace{\mathbb{1}_0, \cdots, \mathbb{1}_0}_{s})] * 1_k \\
=& \mathbb{1}_2 \circ (\theta \circ (\underbrace{\mathbb{1}_1, \cdots, \mathbb{1}_1}_{k}), \mathbb{1}_s \circ (\underbrace{\mathbb{1}_0, \cdots, \mathbb{1}_0}_{s})) \\
=& \mathbb{1}_2 \circ (\theta, \mathbb{1}_0) = \theta,
\end{aligned}$$

where the second to last equality is Lemma 2.8(1) and the last equality uses the hypothesis that $\mathcal{P}$ is 2-unitary.

(2) We will consider the special case $\sigma = 1_n \in \mathbb{S}_n$, and the general case follows from Lemma 2.13. If there exists some $r \in [k]$ such that $r \notin J$, then $\pi^J = \pi^{J'} \bullet \Gamma^r$ for some $J' \subseteq [n-1]$ by Lemma 2.5(3). We have

$$\Gamma^r(\mathbb{1}_2 \circ (\theta, \mathbb{1}_{n-k})) = \mathbb{1}_2 \circ (\Gamma^r(\theta), \mathbb{1}_{n-k}) = 0,$$

as $\theta \in {}^k\Upsilon(k)$. Hence $\pi^J(\mathbb{1}_2 \circ (\theta, \mathbb{1}_{n-k})) = 0$ as desired. $\square$

As an immediate consequence of the above lemma we have the following.

Corollary 4.4. *Let $I, I' \subseteq [n]$ be subsets with $|I| = |I'| = n - k =: s$. For $\theta \in {}^k\Upsilon(k)$, we have*

$$\Gamma^{I'}((\mathbb{1}_2 \circ (\theta, \mathbb{1}_s)) * c_I) = \begin{cases} \theta, & \text{if } I' = I; \\ 0, & \text{otherwise.} \end{cases} \tag{E4.4.1}$$

We are now in a position to give a recursive formula to compute the dimension of ${}^k\Upsilon \cap \mathcal{I}$. By convention, ${}^0\Upsilon = \mathcal{P}$. Let

$$G_{\mathcal{I}}^k(t) = \sum_{n=0}^{\infty} d_{\mathcal{I}}^k(n) t^n$$

and let

$$f_{\mathcal{I}}(k) = d_{\mathcal{I}}^{k+1}(k) - d_{\mathcal{I}}^k(k)$$

for all $k \geq 0$. Clearly, it follows from (E4.2.1) that

$$f_{\mathcal{I}}(k) = \dim_{\Bbbk}({}^k\Upsilon \cap \mathcal{I})(k), \quad \text{and} \quad f_{\mathcal{P}}(k) = \dim_{\Bbbk} {}^k\Upsilon(k).$$

Note that if $f_{\mathcal{I}}(k)$ is not finite, then it denotes a cardinal.

Theorem 4.5. *Let $\mathcal{P}$ be 2-unitary and $\mathcal{I}$ an ideal of $\mathcal{P}$. Let $n \geq k \geq 0$ be integers.*

(1) *Let $\{\theta_i \mid 1 \leq i \leq f_{\mathcal{I}}(k)\}$ be a basis of $({}^k\Upsilon \cap \mathcal{I})(k)$. Then*

$$\{\mathbb{1}_2 \circ (\theta_i, \mathbb{1}_{n-k}) * c_I \mid 1 \leq i \leq f_{\mathcal{I}}(k), I \subseteq [n], |I| = n - k\}$$

forms a basis of $(({}^k\Upsilon \cap \mathcal{I})/({}^{k+1}\Upsilon \cap \mathcal{I}))(n)$. Consequently,

$$\dim_{\Bbbk} \frac{{}^k\Upsilon \cap \mathcal{I}}{{}^{k+1}\Upsilon \cap \mathcal{I}}(n) = f_{\mathcal{I}}(k)\tbinom{n}{k}.$$

(2)
$$d_{\mathcal{I}}^{k+1}(n) = d_{\mathcal{I}}^{k}(n) + f_{\mathcal{I}}(k)\tbinom{n}{k}.$$

Equivalently,

$$G_{\mathcal{I}}^{k+1}(t) - G_{\mathcal{I}}^{k}(t) = f_{\mathcal{I}}(k)\, \frac{t^k}{(1-t)^{k+1}}.$$

(3) *If $\mathcal{I} = \mathcal{P}$, then*

$$d^{k+1}(n) = d^k(n) + (d^{k+1}(k) - d^k(k))\tbinom{n}{k},$$

for all n.

Proof. (1) Let $I, I' \subseteq [n]$ be subsets with $|I| = |I'| = n-k$. By Corollary 4.4, we have

$$\Gamma^{I'}(\mathbb{1}_2 \circ (\Gamma^I(\theta), \mathbb{1}_{n-k}) * c_I) = \begin{cases} \Gamma^I(\theta), & \text{if } I' = I, \\ 0, & \text{otherwise,} \end{cases} \tag{E4.5.1}$$

for all $\theta \in ({}^k\Upsilon \cap \mathcal{I})(n)$, because $\Gamma^I(\theta) \in ({}^k\Upsilon \cap \mathcal{I})(k)$. For each $\theta \in ({}^k\Upsilon \cap \mathcal{I})(n)$, we set

$$\theta' = \theta - \sum_{\substack{I \subseteq [n] \\ |I| = n-k}} \mathbb{1}_2 \circ (\Gamma^I(\theta), \mathbb{1}_{n-k}) * c_I. \tag{E4.5.2}$$

Then (E4.5.1) implies that $\theta' \in ({}^{k+1}\Upsilon \cap \mathcal{I})(n)$, and hence the image of the elements of the form

$$\mathbb{1}_2 \circ (\theta_i, \mathbb{1}_{n-k}) * c_I$$

span $\dfrac{{}^k\Upsilon \cap \mathcal{I}}{{}^{k+1}\Upsilon \cap \mathcal{I}}(n)$.

Next we show the linear independency. Assume that

$$\sum_{\substack{1 \le i \le f_{\mathcal{I}}(k) \\ I \subseteq [n], |I| = n-k}} \lambda_{i,I} \mathbb{1}_2 \circ (\theta_i, \mathbb{1}_{n-k}) * c_I \in ({}^{k+1}\Upsilon \cap \mathcal{I})(n)$$

for some $\lambda_{i,I} \in \Bbbk$. Then, for each I, by applying Γ^I we obtain that

$$\sum_{1 \le i \le f_{\mathcal{I}}(k)} \lambda_{i,I} \theta_i = 0,$$

again we use Corollary 4.4 here. It follows that all $\lambda_{i,I}$'s must be zero.

(2,3) Easy consequences of part (1). $\square$

4.2. Basis theorem

As a consequence of Theorem 4.5(1), we have the following result concerning a $\Bbbk$-linear basis of $\mathcal{P}$. In theorem below, if z_k is not finite, then it denotes a cardinal.

Recall that an operad $\mathcal{P}$ is *finitely generated* if there is a finite dimensional subspace X such that every element in $\mathcal{P}$ is generated by X by using operad composition and $\mathbb{S}_n$-actions for $n \geq 0$.

Theorem 4.6. *Suppose $\mathcal{P}$ is a 2-unitary operad.*

(1) [Basis theorem] *For each $k \geq 0$, let*

$$\Theta^k := \{\theta_1^k, \cdots, \theta_{z_k}^k\}$$

be a $\Bbbk$-linear basis for ${}^k\Upsilon(k)$. Let $\mathbf{B}_k(n)$ be the set

$$\{\mathbb{1}_2 \circ (\theta_i^k, \mathbb{1}_{n-k}) * c_I \mid 1 \leq i \leq z_k, I \subseteq [n], |I| = n-k\}.$$

Then $\mathcal{P}(n)$ has a $\Bbbk$-linear basis

$$\bigcup_{0\leq k\leq n} \mathbf{B}_k(n) = \{\mathbb{1}_n\} \cup \bigcup_{1\leq k\leq n} \mathbf{B}_k(n),$$

and, for every $k \geq 1$, ${}^k\Upsilon(n)$ has a $\Bbbk$-linear basis $\bigcup_{k\leq i\leq n} \mathbf{B}_i(n)$.

(2) *$\mathcal{P}$ is generated by $\{\mathbb{1}_0, \mathbb{1}_1, \mathbb{1}_2\} \cup \{{}^k\Upsilon(k) \mid k \geq 1\}$.*

(3) *If $\mathcal{P}$ is locally finite and ${}^n\Upsilon = 0$ for some n, then it is finitely generated.*

Proof. (1) For each $n \geq 0$, $\mathcal{P}(n)$ admits a decreasing filtration $\{{}^k\Upsilon(n)\}_{k=0}^{\infty}$. As a vector space, we have

$$\mathcal{P}(n) \cong \bigoplus_{k=0}^{\infty} {}^k\Upsilon(n)/{}^{k+1}\Upsilon(n) \cong \Bbbk\mathbb{1}_n \oplus \bigoplus_{k=1}^{\infty} {}^k\Upsilon(n)/{}^{k+1}\Upsilon(n).$$

By Theorem 4.5 (1), $\mathbf{B}_k(n)$ is a $\Bbbk$-linear basis of ${}^k\Upsilon(n)/{}^{k+1}\Upsilon(n)$. Note that $\mathbf{B}_k(n)$ is empty if $k \geq n+1$. The first assertion follows. The proof of the second assertion is similar.

(2) This follows from part (1).

(3) This follows from part (2) and the fact that ${}^k\Upsilon(k) = 0$ for all $k \geq n$. $\square$

As a consequence of the above basis theorem, we have the following corollary. A morphism f of operads is called a morphism of 2-unitary operads if f preserves $\mathbb{1}_i$ for $i = 0, 1, 2$. Before we prove the corollary, we need the following lemma.

Lemma 4.7. *Let $\mathcal{P}$ be a 2-unitary operad and $\mathcal{I}$ be an ideal of $\mathcal{P}$. Then ${}^k\Upsilon_{\mathcal{P}/\mathcal{I}} \cong {}^k\Upsilon_{\mathcal{P}}/({}^k\Upsilon_{\mathcal{P}} \cap \mathcal{I})$.*

Proof. Let $\mathcal{Q} = \mathcal{P}/\mathcal{I}$. The canonical morphism $\varphi : \mathcal{P} \to \mathcal{Q}$ induces a natural map $f : {}^k\Upsilon_{\mathcal{P}} \to {}^k\Upsilon_{\mathcal{Q}}$ by Lemma 3.3. Since $\mathcal{I}$ is the kernel of φ, f induces a natural injective morphism

$$g : {}^k\Upsilon_{\mathcal{P}}/({}^k\Upsilon_{\mathcal{P}} \cap \mathcal{I}) \to {}^k\Upsilon_{\mathcal{Q}}.$$

It remains to show that g is surjective, equivalently, to show that, for each n,

$$\phi : ({}^k\Upsilon_{\mathcal{P}}(n) + \mathcal{I}(n))/\mathcal{I}(n) \to {}^k\Upsilon_{\mathcal{Q}}(n)$$

is surjective. For every $x \in {}^k\Upsilon_{\mathcal{Q}}(n)$, let $\theta \in \mathcal{P}(n)$ such that $\varphi(\theta) = x$. Suppose $\theta \in {}^i\Upsilon_{\mathcal{P}}(n)$ for some i. We will use induction to show that $i \geq k$ for some choice of θ. There is nothing to be proved if $i \geq k$. Assume now that $i < k$. Then $\Gamma^J(\theta) \in {}^i\Upsilon_{\mathcal{P}}(i)$ when $|J| = n - i$. Let

$$\theta' = \theta - \sum_{\substack{J \subseteq [n] \\ |J| = n-i}} \mathbb{1}_2 \circ (\Gamma^J(\theta), \mathbb{1}_{n-i}) * c_J$$

which is similar to the element given in (E4.5.2). By Corollary 4.4 or the proof of Theorem 4.5 (1), $\Gamma^J(\theta') = 0$ for all $J \subseteq [n]$ with $|J| = n - i$. This means that $\theta' \in {}^{i+1}\Upsilon_{\mathcal{P}}(n)$. For each J as above, we have

$$\varphi(\Gamma^J(\theta)) = \Gamma^J(\varphi(\theta)) = \Gamma^J(x) = 0$$

as $x \in {}^k\Upsilon_{\mathcal{Q}}(n)$ and $k > i$. Thus $\Gamma^J(\theta) \in \mathcal{I}(i)$ for all J. Consequently,

$$\Omega := \sum_{\substack{J \subseteq [n] \\ |J| = n-i}} \mathbb{1}_2 \circ (\Gamma^J(\theta), \mathbb{1}_{n-i}) * c_J \in \mathcal{I}(n).$$

Hence $\phi(\theta') = \phi(\theta) = x$. Replacing θ by θ' we move i to $i + 1$. The assertion follows by induction. □

Recall from (E3.12.1) $\Upsilon_{\mathcal{P}}$ denote the $\mathbb{S}$-submodule $({}^n\Upsilon_{\mathcal{P}}(n))_{n\geq 0}$.

Corollary 4.8. *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are 2-unitary operads. Let $f : \mathcal{P} \to \mathcal{Q}$ be a morphism of 2-unitary operads.*

(1) *f is uniquely determined by $f\,|_{\Upsilon_{\mathcal{P}}}$.*
(2) *f is injective if and only if $f\,|_{\Upsilon_{\mathcal{P}}}$ is.*
(3) *f is surjective if and only if $f\,|_{\Upsilon_{\mathcal{P}}}$ is.*
(4) *f is an isomorphism if and only if $f\,|_{\Upsilon_{\mathcal{P}}}$ is.*

Proof. Since f is a morphism of operads, it follows from Lemma 3.3 that f maps ${}^k\Upsilon_{\mathcal{P}}$ to ${}^k\Upsilon_{\mathcal{Q}}$ for every k. Consequently, f maps ${}^k\Upsilon_{\mathcal{P}}(k)$ to ${}^k\Upsilon_{\mathcal{Q}}(k)$ for every k. This defines a map $f : \Upsilon_{\mathcal{P}} \to \Upsilon_{\mathcal{Q}}$. Since f preserves $\mathbb{1}_i$ for $i = 0, 1, 2$, it preserves $\mathbb{1}_n$ for all n. Therefore f maps $\mathbb{1}_2 \circ (\theta, \mathbb{1}_{n-k}) * c_I$ to $\mathbb{1}_2 \circ (f(\theta), \mathbb{1}_{n-k}) * c_I$ for all $\theta \in {}^k\Upsilon_{\mathcal{P}}(k)$.

(1) Since f is a morphism of 2-unitary operads, $\mathcal{P}$ is generated by elements in Θ^k for $k \geq 0$ by Theorem 4.6 (1). The assertion follows.

(2) Suppose f is not injective. Let $\mathcal{I}$ be the nonzero kernel Ker f. Then $\mathcal{I}$ is an ideal of $\mathcal{P}$. Since $\mathcal{I} \neq 0$, $\mathcal{I} \cap {}^k\Upsilon_{\mathcal{P}} \neq \mathcal{I} \cap {}^{k+1}\Upsilon_{\mathcal{P}}$ for some k. Let $x \in (\mathcal{I} \cap {}^k\Upsilon_{\mathcal{P}})(n) \setminus (\mathcal{I} \cap {}^{k+1}\Upsilon_{\mathcal{P}})(n)$ for some $n > k$. Then there is a subset I of $[n]$ with $|I| = k$ such that $0 \neq \pi^I(x) \in (\mathcal{I} \cap {}^k\Upsilon_{\mathcal{P}})(k)$. So $f\,|_{\Upsilon_{\mathcal{P}}}$ is not injective. The converse is easy.

(3) Suppose $f\,|_{\Upsilon_{\mathcal{P}}}$ is surjective. Since $\mathcal{Q}$ is generated by $\{{}^k\Upsilon_{\mathcal{Q}}(k)\}_{k\geq 1}$ by Theorem 4.6 (2), f is surjective.

Conversely, suppose that f is surjective. Then $\mathcal{Q}$ is a quotient operad of $\mathcal{P}$. By Lemma 4.7, f maps surjectively from ${}^k\Upsilon_{\mathcal{P}}(k)$ to ${}^k\Upsilon_{\mathcal{Q}}(k)$ for each k. The assertion follows.

(4) This is a consequence of parts (2) and (3). $\square$

4.3. Categorification of binomial coefficients

Following the basis theorem [Theorem 4.6 (1)], for each $I \subseteq [n]$ with $|I| = n - k$, we define a $\Bbbk$-linear map

$$\Lambda_I^n : {}^k\Upsilon(k) \to {}^k\Upsilon(n) \tag{E4.8.1}$$

by

$$\Lambda_I^n(\theta) = \mathbb{1}_2 \circ (\theta, \mathbb{1}_{n-k}) * c_I. \tag{E4.8.2}$$

Lemma 4.9. *Retain the above notation. For every $n \geq k$ and every $\sigma \in \mathbb{S}_n$, the following diagram is commutative in the quotient space ${}^k\Upsilon/{}^{k+1}\Upsilon$*

$$\begin{array}{ccc} {}^k\Upsilon(k) & \xrightarrow{\Lambda_I^n} & {}^k\Upsilon(n) \\ {\scriptstyle *\Gamma^{\sigma^{-1}(I)}(\sigma)}\big\downarrow & & \big\downarrow{\scriptstyle *\sigma} \\ {}^k\Upsilon(k) & \xrightarrow[\Lambda_{\sigma^{-1}(I)}^n]{} & {}^k\Upsilon(n). \end{array}$$

As a consequence, if ${}^{k+1}\Upsilon = 0$, then

$$\Lambda_I^n(\theta) * \sigma = \Lambda_{\sigma^{-1}(I)}^n(\theta * \Gamma^{\sigma^{-1}(I)}(\sigma)).$$

Proof. Let θ be an element in ${}^k\Upsilon(k)$. For every $I' \subseteq [n]$ with $|I'| = n-k$, by Lemma 2.13 and Corollary 4.4,

$$\Gamma^{I'}(\Lambda_I^n(\theta)*\sigma)=\Gamma^{\sigma(I')}(\Lambda_I^n(\theta))*\Gamma^{I'}(\sigma)=\begin{cases}\theta*\Gamma^{I'}(\sigma), & \sigma(I')=I,\\ 0, & \sigma(I')\neq I\end{cases}$$

and

$$\Gamma^{I'}(\Lambda^n_{\sigma^{-1}(I)}(\theta*\Gamma^{\sigma^{-1}(I)}(\sigma)))=\begin{cases}\theta*\Gamma^{\sigma^{-1}(I)}(\sigma), & \sigma(I')=I,\\ 0, & \sigma(I')\neq I.\end{cases}$$

Thus $\Gamma^{I'}(\Lambda_I^n(\theta)*\sigma)=\Gamma^{I'}(\Lambda^n_{\sigma^{-1}(I)}(\theta*\Gamma^{\sigma^{-1}(I)}(\sigma)))$ for all I'. Therefore

$$\Lambda_I^n(\theta)*\sigma=\Lambda^n_{\sigma^{-1}(I)}(\theta*\Gamma^{\sigma^{-1}(I)}(\sigma))$$

modulo ${}^{k+1}\Upsilon$. The assertion follows. □

Let Mod-$\mathbb{S}_n$ denote the category of right $\Bbbk\mathbb{S}_n$-modules. Suggested by Lemma 4.9, we define the following functor

$$\mathcal{C}_k^n : \text{Mod-}\mathbb{S}_k \to \text{Mod-}\mathbb{S}_n$$

for $n \geq k$ as follows. Let T_k^n be the set $\{I \subset [n] \mid |I| = n-k\}$. Let M be a right $\mathbb{S}_k$-module. Then $\mathcal{C}_k^n(M)$ is a right $\mathbb{S}_n$-module such that

(i) as a vector space, $\mathcal{C}_k^n(M)=\bigoplus_{I\in T_k^n} M$, elements in $\mathcal{C}_k^n(M)$ are linear combinations of (m,I) for $m\in M$ and $I\in T_k^n$;
(ii) the $\mathbb{S}_n$-action on $\mathcal{C}_k^n(M)$ is determined by

$$(m,I)*\sigma := (m*\Gamma^{\sigma^{-1}(I)}(\sigma),\sigma^{-1}(I))$$

for all $(m,I)\in\mathcal{C}_k^n(M)$ and all $\sigma\in\mathbb{S}_n$.

Lemma 4.10. *Retain the notation as above.*

(1) *If M is a right $\mathbb{S}_k$-module, then $\mathcal{C}_k^n(M)$ is a right $\mathbb{S}_n$-module.*
(2) *Let A be an algebra. If M is an $(A,\mathbb{S}_k)$-bimodule, then $\mathcal{C}_k^n(M)$ is an $(A,\mathbb{S}_n)$-bimodule.*
(3) *The functor $\mathcal{C}_k^n(-)$ is equivalent to the tensor functor $-\otimes_{\mathbb{S}_k}\mathcal{C}_k^n(\mathbb{S}_k)$.*

Proof. (1) For $\sigma, \tau \in \mathbb{S}_n$, and $(m, I) \in \mathcal{C}_k^n(M)$,

$$\begin{aligned}((m, I) * \sigma) * \tau &= (m * \Gamma^{\sigma^{-1}(I)}(\sigma), \sigma^{-1}(I)) * \tau \\ &= ((m * \Gamma^{\sigma^{-1}(I)}(\sigma)) * \Gamma^{\tau^{-1}\sigma^{-1}(I)}(\tau), \tau^{-1}(\sigma^{-1}(I))) \\ &= ((m * \Gamma^{\sigma^{-1}(I)}(\sigma)) * \Gamma^{(\sigma\tau)^{-1}(I)}(\tau), (\sigma\tau)^{-1}(I)) \\ &= (m * (\Gamma^{\tau(\sigma\tau)^{-1}(I)}(\sigma) * \Gamma^{(\sigma\tau)^{-1}(I)}(\tau)), (\sigma\tau)^{-1}(I)) \\ &= (m * (\Gamma^{(\sigma\tau)^{-1}(I)}(\sigma * \tau)), (\sigma\tau)^{-1}(I)) \\ &= (m * (\Gamma^{(\sigma\tau)^{-1}(I)}(\sigma\tau)), (\sigma\tau)^{-1}(I)) \\ &= (m, I) * (\sigma\tau).\end{aligned}$$

(2) This follows from the definition and part (1).

(3) This follows from the Watts Theorem and the fact that $\mathcal{C}_k^n$ is exact. □

5. Binomial transform of generating series

In this section we study 2-unitary operads of finite Gelfand-Kirillov dimension. One tool is binomial transform [22,31,32] of generating series that is closely related to truncation ideals of 2-unitary operads.

5.1. Binomial transform

First of all, there are at least two versions of binomial transforms, we will use the following version. We also list some facts without proofs.

Let $a := \{a_0, a_1, a_2, \cdots\}$ be a sequence of numbers. Its generating series is denoted by

$$G_a(t) = \sum_{i=0}^{\infty} a_i t^i$$

and its exponential generating series is

$$E_a(t) = \sum_{i=0}^{\infty} \frac{a_i}{i!} t^i.$$

The *binomial transform* of a is a sequence $b := \{b_0, b_1, b_2, \cdots, \}$ defined by

$$b_i = \sum_{k=0}^{i} a_k (-1)^{i-k} \binom{i}{k} \tag{E5.0.1}$$

for all $i \geq 0$. It is well-known (see [31]) that

$$a_i = \sum_{k=0}^{i} b_k \binom{i}{k} \tag{E5.0.2}$$

for all $i \geq 0$, and

$$G_a(t) = \frac{1}{1-t}\, G_b(\frac{t}{1-t}), \qquad G_b(t) = \frac{1}{1+t}\, G_a(\frac{t}{1+t}) \tag{E5.0.3}$$

and

$$E_a(t) = e^t E_b(t), \qquad E_b(t) = e^{-t} E_a(t). \tag{E5.0.4}$$

Note that (E5.0.3) is equivalent to

$$\sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} b_k \, \frac{t^k}{(1-t)^{k+1}}. \tag{E5.0.5}$$

We also write

$$\mathcal{T}(\{a_i\}) = \{b_i\}, \quad \text{and} \quad \mathcal{T}^{-1}(\{b_i\}) = \{a_i\},$$

or

$$\mathcal{T}(\sum_{k=0}^{\infty} a_k t^k) = \sum_{k=0}^{\infty} b_k t^k, \quad \text{and} \quad \mathcal{T}^{-1}(\sum_{k=0}^{\infty} b_k t^k) = \sum_{k=0}^{\infty} a_k t^k,$$

where $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ are determined by each other via (E5.0.1)-(E5.0.2), and in this case we call $a = \{a_i\}$ the *inverse binomial transform* of $b = \{b_i\}$. For a sequence of non-negative numbers (called *a non-negative sequence*) $a = \{a_i\}$, define the *exponent* of a to be

$$\exp(a) := \limsup_{n\to\infty} a_n^{\frac{1}{n}}. \tag{E5.0.6}$$

When $\{a_n\}$ is a sequence of non-negative integers with infinitely many nonzero a_n's, then by [33, Lemma 1.1(1)],

$$\exp(a) = \limsup_{n\to\infty} \left(\sum_{i=0}^{n} a_i\right)^{\frac{1}{n}}. \tag{E5.0.7}$$

Lemma 5.1. *Let $b := \{b_n\}$ be a non-negative sequence with $b_0 = 1$ and $a := \{a_n\} = \mathcal{T}^{-1}(\{b_n\})$.*

(1) $\exp(a) = \exp(b) + 1$.

(2) *If* $b_n = 0$ *for* $n \gg 0$*, then* $\exp(a) = 1$*.*
(3) *For every real number* $r \geq 2$*, let* $b = \{\lfloor (r-1)^n \rfloor\}$*, then* $\exp(a) = r$*.*

Proof. First of all $\exp(a) \geq 1$ since $a_n \geq 1$ for each n. From calculus, the radius of convergence of the power series $G_a(t)$ is $r_a := \exp(a)^{-1}$. The same is true for b.

(1) By (E5.0.3), $r_a^{-1} = r_b^{-1} + 1$. The assertion follows.
(2) Since $\exp(b) = 0$, this is a special case of (1).
(3) Clearly $\exp(b) = r - 1$. The assertion follows from part (1). □

Next we apply binomial transform to operads. Let $\mathcal{P}$ be a 2-unitary operad and let $\mathcal{I}$ be an ideal of $\mathcal{P}$ or $\mathcal{I} = \mathcal{P}$. Let ${}^n\Upsilon$ be defined as (E3.0.1) and let ${}^0\Upsilon = \mathcal{P}$. Let

$$G_{\mathcal{I}}^w(t) = \sum_{n=0}^{\infty} \dim_{\Bbbk}(\frac{\mathcal{I}}{{}^w\Upsilon \cap \mathcal{I}}(n))t^n = \sum_{n=0}^{\infty} d_{\mathcal{I}}^w(n)t^n \tag{E5.1.1}$$

and

$$G_{\mathcal{I}}(t) = \sum_{n=0}^{\infty} \dim_{\Bbbk}(\mathcal{I}(n))t^n. \tag{E5.1.2}$$

Lemma 5.2. *Let* $\mathcal{P}$ *be a 2-unitary operad and let* $\mathcal{I}$ *be an ideal of* $\mathcal{P}$ *or* $\mathcal{I} = \mathcal{P}$*. Then* $G_{\mathcal{I}}^w(t)$ *and* $G_{\mathcal{I}}(t)$ *are*

$$G_{\mathcal{I}}^w(t) = \sum_{k=0}^{w-1} f_{\mathcal{I}}(k)\, \frac{t^k}{(1-t)^{k+1}} \tag{E5.2.1}$$

for all w *and*

$$G_{\mathcal{I}}(t) = \sum_{k=0}^{\infty} f_{\mathcal{I}}(k)\, \frac{t^k}{(1-t)^{k+1}} \tag{E5.2.2}$$

where $f_{\mathcal{I}}(k) = d_{\mathcal{I}}^{k+1}(k) - d_{\mathcal{I}}^k(k)$ *for all* k*.*

Proof. Since $G_{\mathcal{I}}(t) = \lim_{w\to\infty} G_{\mathcal{I}}^w(t)$, (E5.2.2) is a consequence of (E5.2.1). So we only prove (E5.2.1).

By Theorem 4.5(2), we have

$$G_{\mathcal{I}}^w(t) = G_{\mathcal{I}}^{w-1}(t) + f_{\mathcal{I}}(w-1)\frac{t^{w-1}}{(1-t)^w}$$

for all $w \geq 1$. When $w = 1$, the above equation becomes $0 = 0 + 0$ where $\mathcal{I} \neq \mathcal{P}$, or $\sum_{i=0}^{\infty} t^i = 0 + \frac{1}{1-t}$ where $\mathcal{I} = \mathcal{P}$, both of which hold clearly. We have

$$G_{\mathcal{I}}^{w}(t)=\sum_{k=1}^{w}(G_{\mathcal{I}}^{k}(t)-G_{\mathcal{I}}^{k-1}(t))=\sum_{k=0}^{w-1}f_{\mathcal{I}}(k)\frac{t^k}{(1-t)^{k+1}}.\quad\square$$

Lemma 5.2 tells us that the sequence $\{f_{\mathcal{I}}(n)\}$ is the binomial transform of $\{\dim_{\mathcal{I}}(n)\}$. By Definition 4.1(6) and (E5.0.5), we immediately get

$$\operatorname{GKdim}\mathcal{I}=\max\{k\mid f_{\mathcal{I}}(k)\neq 0\}+1. \tag{E5.2.3}$$

5.2. Operads with finite GKdimension

Using the truncation ideals, we give a characterization of 2-unitary operads with finite GKdimension.

Theorem 5.3. *Let $\mathcal{P}$ be a 2-unitary operad.*

(1) *If $\mathcal{P}$ has polynomial growth, then the generating series $G_{\mathcal{P}}(t)$ is rational. As a consequence,* $\operatorname{GKdim}\mathcal{P}\in\mathbb{N}$.
(2) *$\mathcal{P}$ has polynomial growth if and only if there is an integer k such that ${}^{k}\Upsilon=0$. And*

$$\operatorname{GKdim}\mathcal{P}=\max\{k\mid {}^{k}\Upsilon\neq 0\}+1=\min\{k\mid {}^{k}\Upsilon=0\}.$$

Proof. If ${}^{k}\Upsilon=0$, then $\mathcal{P}$ has finite GKdimension by Lemma 4.2. Conversely, we assume that $\operatorname{GKdim}\mathcal{P}<\infty$. By Lemma 5.2,

$$G_{\mathcal{P}}(t)=\sum_{n=0}^{\infty}f_{\mathcal{P}}(n)\,\frac{t^n}{(1-t)^{n+1}}. \tag{E5.3.1}$$

By definition, $f_{\mathcal{P}}(n)\geq 0$ for all n. Since $\operatorname{GKdim}\mathcal{P}<\infty$, there is an $N\in\mathbb{N}$ such that $f_{\mathcal{P}}(n)=0$ for all $n\geq N$ where $f_{\mathcal{P}}(n)=d_{\mathcal{P}}^{n+1}(n)-d_{\mathcal{P}}^{n}(n)=\dim{}^{n}\Upsilon(n)$. This implies that ${}^{n}\Upsilon(n)=0$ for all $n\geq N$. By Theorem 4.6(1), ${}^{n}\Upsilon=0$ for all $n\geq N$. In this case,

$$G_{\mathcal{P}}(t)=\sum_{n=0}^{N-1}f_{\mathcal{P}}(n)\,\frac{t^n}{(1-t)^{n+1}}$$

which is rational. It is clear and follows from (E5.2.3) that

$$\operatorname{GKdim}\mathcal{P}=\max\{n\mid f_{\mathcal{P}}(n)\neq 0\}+1=\max\{n\mid {}^{n}\Upsilon\neq 0\}+1. \tag{E5.3.2}$$

Therefore assertions in parts (1) and (2) follow. $\square$

Corollary 5.4. *Let $\mathcal{I}$ be an operadic ideal of $\mathcal{A}ss$ and $\mathcal{P}$ be the quotient operad $\mathcal{A}ss/\mathcal{I}$. Let k be a positive integer. Then $\operatorname{GKdim}\mathcal{P}\leq k$ if and only if $\mathcal{I}\supseteq{}^{k}\Upsilon$. In particular,*

$$\operatorname{GKdim}(\mathcal{A}ss/{}^k\Upsilon) = \begin{cases} 1, & k = 1, 2, \\ k, & k \geq 3. \end{cases}$$

Proof. By Theorem 5.3 (2) and Lemma 4.7, $\operatorname{GKdim} \mathcal{P} \leq k$ if and only if ${}^k\Upsilon_{\mathcal{P}} = 0$ if and only if $\mathcal{I} \supseteq {}^k\Upsilon_{\mathcal{A}ss}$.

By definition,

$$G_{\mathcal{A}ss}(t) = \sum_{n=0}^{\infty} f_{\mathcal{A}ss}(n) \frac{t^n}{(1-t)^{n+1}},$$

and

$$G_{\mathcal{A}ss/{}^k\Upsilon}(t) = G^k_{\mathcal{A}ss}(t) = \sum_{n=0}^{k-1} f_{\mathcal{A}ss}(n) \frac{t^n}{(1-t)^{n+1}}. \tag{E5.4.1}$$

Since $G_{\mathcal{A}ss}(t) = \sum_{k=0}^{\infty} k! t^k$, by (E5.0.1),

$$f_{\mathcal{A}ss}(n) = \sum_{s=0}^{n} (-1)^{n-s} s! \binom{n}{s}.$$

It is easy to check that $f_{\mathcal{A}ss}(n) \neq 0$ for all $n \neq 1$. The assertion concerning the GKdimension of $\mathcal{A}ss/{}^k\Upsilon$ follows from (E5.4.1). $\square$

Let $\sum_k a_k t^k$ and $\sum_k b_k t^k$ be two power series. If $a_k \leq b_k$ for all k, then we write $\sum_k a_k t^k \leq \sum_k b_k t^k$.

Lemma 5.5. *Let $\mathcal{I}$ be an ideal of $\mathcal{P}$. Then $\mathcal{T}(G_{\mathcal{I}}(t)) \leq \mathcal{T}(G_{\mathcal{P}}(t))$. As a consequence, if ${}^n\Upsilon = 0$ for some n, the set $\{G_{\mathcal{I}}(t) \mid \mathcal{I} \subseteq \mathcal{P}\}$ is finite.*

Proof. By Theorem 4.5(1) and Lemma 5.2,

$$G_{\mathcal{I}}(t) = \sum_{k=0}^{\infty} f_{\mathcal{I}}(k) \, \frac{t^k}{(1-t)^{k+1}}$$

where $f_{\mathcal{I}}(k) = \dim({}^k\Upsilon \cap \mathcal{I})(k)$. Since

$$0 \leq f_{\mathcal{I}}(k) = \dim({}^k\Upsilon \cap \mathcal{I})(k) \leq \dim {}^k\Upsilon(k) = f_{\mathcal{P}}(k)$$

for all k, we have

$$0 \leq \mathcal{T}(G_{\mathcal{I}}(t)) = \sum_{k=0}^{\infty} f_{\mathcal{I}}(k) t^k \leq \sum_{k=0}^{\infty} f_{\mathcal{P}}(k) t^k = \mathcal{T}(G_{\mathcal{P}}(t)).$$

If ${}^{n}\Upsilon = 0$, then there are only finitely many nonzero $f_{\mathcal{P}}(k)$. Therefore there are only finitely many possible choices $\{f_{\mathcal{I}}(k)\}_{k\geq 0}$. The assertion follows. $\square$

The classical Hopkins (or Hopkins-Levitzki) Theorem states that any right artinian ring with identity element is right noetherian. Using the truncation ideals, we show that similar phenomenon occurs in 2-unitary operads.

Theorem 5.6. *Let $\mathcal{P}$ be a locally finite 2-unitary operad.*

(1) *If* $\operatorname{GKdim}\mathcal{P} < \infty$*, then $\mathcal{P}$ is noetherian.*
(2) $\operatorname{GKdim}\mathcal{P} < \infty$ *if and only if $\mathcal{P}$ is artinian.*
(3) [An operadic version of Hopkins' Theorem] *If $\mathcal{P}$ is artinian, then it is noetherian.*

Proof. (1) Let

$$\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_n \subseteq \cdots$$

be an ascending chain of ideals of $\mathcal{P}$. Then we have

$$G_{\mathcal{I}_1}(t) \leq G_{\mathcal{I}_2}(t) \leq \cdots \leq G_{\mathcal{I}_n}(t) \leq \cdots$$

Since $\operatorname{GKdim}\mathcal{P} < \infty$, we have ${}^{k}\Upsilon = 0$ for some k. By Lemma 5.5, $\{G_{\mathcal{I}_i} \mid i = 1, 2, \cdots\}$ is finite and therefore the sequence $\{G_{\mathcal{I}_i}(t)\}_{i\geq 1}$ stabilizes. This implies that the sequence of ideals $\{\mathcal{I}_i\}_{i\geq 1}$ stabilizes.

(2) ($\Rightarrow$) The proof is similar to the proof of part (2) above. Let

$$\mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots \supseteq \mathcal{I}_n \supseteq \cdots$$

be a descending chain of ideals of $\mathcal{P}$. Since $\operatorname{GKdim}\mathcal{P} < \infty$, we have ${}^{k}\Upsilon = 0$ for some k. By Lemma 5.5, $\{G_{\mathcal{I}_i}(t) \mid i = 1, 2, \cdots\}$ is finite and therefore the sequence $\{G_{\mathcal{I}_i}(t)\}_{i\geq 1}$ stabilizes. This implies that the sequence of ideals $\{\mathcal{I}_i\}_{i\geq 1}$ stabilizes.

($\Leftarrow$) By Proposition 3.1 and Lemma 2.5(1), we have a descending chain of ideals

$${}^{0}\Upsilon \supseteq {}^{1}\Upsilon \supseteq {}^{2}\Upsilon \supseteq \cdots \supseteq {}^{k}\Upsilon \supseteq \cdots$$

of $\mathcal{P}$. If $\mathcal{P}$ is artinian, then this chain is stable. On the other hand, we have $\cap_{k\geq 0}{}^{k}\Upsilon = 0$ since ${}^{k}\Upsilon(k'-1) = 0$ for all $k \geq k' \geq 0$. It follows that ${}^{k}\Upsilon = 0$ for some sufficiently large k and hence $\mathcal{P}$ has finite GKdimension.

(3) This is a consequence of parts (1) and (2). $\square$

The Gelfand-Kirillov dimension (or GKdimension) is an important tool in the study of noncommutative algebra [30,21]. Similar to associative algebras, we introduce the notion of the GKdimension for algebras over any operad.

Definition 5.7. Let $\mathcal{P}$ be an operad and A a $\mathcal{P}$-algebra.

(1) Let X be a subset of A. We say that X is a *set of generators* of A if $A = \sum_{n\geq 0} \gamma_n(X)$, where $\gamma_n(X)$ denotes the image $\mathcal{P}(n) \otimes (\Bbbk X)^{\otimes n} \to A$.
(2) We say A is *finitely generated* if it has a set of generators which is finite.
(3) The *GKdimension* of A is defined to be

$$\operatorname{GKdim}(A) = \sup_V \left\{ \limsup_{n\to\infty} \log_n \left(\dim \sum_{i=0}^{n} \gamma_i(V) \right) \right\},$$

where the sup is taken over all finite dimensional subspaces $V \subseteq A$.

Remark 5.8. If A is an associative algebra, then the above defined notions of generators and GKdimension coincide with the standard ones in [30,21].

The next result connects the GKdimension of an operad $\mathcal{P}$ and the ones of finitely generated $\mathcal{P}$-algebras. We stress that $\mathcal{P}$ is not assumed to be 2-unitary here.

Theorem 5.9. *Let $\mathcal{P}$ be an operad with order of the growth $o(\mathcal{P})$ (see Definition 4.1(4)) and A an algebra over $\mathcal{P}$ with a finite set X of generators. Then A has finite GKdimension, precisely*

$$\operatorname{GKdim}(A) \leq o(\mathcal{P}) + r,$$

where $r = |X|$ is the cardinality of X. If, the generating series of $\mathcal{P}$ is rational, then

$$\operatorname{GKdim}(A) \leq \operatorname{GKdim}(\mathcal{P}) - 1 + r.$$

Proof. First we claim that

$$\dim(\mathcal{P}(n) \otimes_{\mathbb{S}_n} (\Bbbk X)^{\otimes n}) \leq \dim \mathcal{P}(n) \cdot \tbinom{n+r-1}{r-1}.$$

In fact, assume $X = \{x_1, \cdots, x_r\}$. We define a total ordering on X by $x_1 < x_2 < \cdots < x_r$. For any $x_{i_1}, \cdots, x_{i_n}$ with $1 \leq i_1, \cdots, i_n \leq r$, there exists some $\sigma \in \mathbb{S}_n$ such that $i_{\sigma(1)} \leq \cdots \leq i_{\sigma(n)}$, thus

$$\theta \otimes (x_{i_1} \otimes \cdots \otimes x_{i_n}) = (\theta * \sigma) \otimes \sigma^{-1} * (x_{i_1} \otimes \cdots \otimes x_{i_n}) = (\theta * \sigma) \otimes (x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(n)}}).$$

By a standard argument we obtain the desired inequality.

Consequently, we have

$$\dim(\gamma_n(\Bbbk X)) \leq \dim \mathcal{P}(n) \cdot \tbinom{n+r-1}{r-1}. \tag{E5.9.1}$$

By Definition 4.1(4), for any arbitrary small positive number ϵ, there is a positive number C such that

$$\dim \mathcal{P}(n) \leq C n^{o(\mathcal{P})+\epsilon} \tag{E5.9.2}$$

for all $n \geq 1$. Now let V be any finite dimensional subspace of A. Since X is a generating set of A, $V \subseteq \sum_{i=0}^{m} \gamma_i(\Bbbk X)$ for some integer $m \geq 1$. Then we have

$$\begin{aligned}
\gamma_n(V) &\subseteq \sum_{0 \leq i_1, \cdots, i_n \leq m} \gamma(\mathcal{P}(n) \otimes ((\gamma_{i_1}(\Bbbk X)^{i_1}) \otimes \cdots (\gamma_{i_n}(\Bbbk X)^{i_n}))) \\
&= \sum_{0 \leq i_1, \cdots, i_n \leq m} \gamma((\mathcal{P}(n) \circ (\mathcal{P}(i_1), \cdots, \mathcal{P}(i_n))) \otimes (\Bbbk X)^{i_1 + \cdots + i_n}) \\
&\subseteq \sum_{0 \leq i_1, \cdots, i_n \leq m} \gamma_{i_1 + \cdots + i_n}(\Bbbk X) \subseteq \sum_{i=0}^{mn} \gamma_i(\Bbbk X),
\end{aligned}$$

and hence $\sum_{i=0}^{n} \gamma_i(V) \subseteq \sum_{i=0}^{mn} \gamma_i(\Bbbk X)$. Combining with (E5.9.1) and (E5.9.2), we have

$$\begin{aligned}
\dim(\sum_{i=0}^{n} \gamma_i(V)) \leq \dim(\sum_{i=0}^{mn} \gamma_i(\Bbbk X)) &\leq \sum_{i=0}^{mn} \dim \mathcal{P}(i) \binom{i+r-1}{r-1} \\
&\leq \sum_{i=0}^{mn} C i^{o(\mathcal{P})+\epsilon} \binom{i+r-1}{r-1} \leq \sum_{i=0}^{mn} C_1 i^{o(\mathcal{P})+\epsilon+r-1} \\
&\leq C_2 n^{o(\mathcal{P})+\epsilon+r}
\end{aligned}$$

for some constants C_1 and C_2. Thus $\operatorname{GKdim}(A) \leq o(\mathcal{P}) + r$ when taking ϵ arbitrary small.

When $\mathcal{P}$ has rational generating series, one can easily check that $\operatorname{GKdim} \mathcal{P} = o(\mathcal{P}) + 1$. Thus $\operatorname{GKdim}(A) \leq \operatorname{GKdim}(\mathcal{P}) - 1 + r$. □

Example 5.10. Consider the operad $\mathcal{C}om$. It is easy to check that $\operatorname{GKdim} \mathcal{C}om = 1$, and consequently, any commutative algebra generated in n-elements has GKdimension no greater than n. Notice that the free algebra generated in n elements over $\mathcal{C}om$ is the polynomial algebra in n variables, and has GKdimension n.

Recall that $(GK \leq k) \operatorname{rad}(\mathcal{P})$ is defined in Definition 1.11(2).

Proposition 5.11. *Let $\mathcal{P}$ be a 2-unitary operad. Then $(GK \leq k) \operatorname{rad}(\mathcal{P}) = {}^k\Upsilon$.*

Proof. Let $\mathcal{I}$ be an ideal of $\mathcal{P}$ and let $\mathcal{P}' = \mathcal{P}/\mathcal{I}$. If $\operatorname{GKdim} \mathcal{P}' \leq k$, then ${}^k\Upsilon_{\mathcal{P}'} = 0$ by Theorem 5.3(2). By Lemma 4.7, ${}^k\Upsilon_{\mathcal{P}} \subseteq \mathcal{I}$. By definition, ${}^k\Upsilon_{\mathcal{P}} \subseteq (GK \leq k) \operatorname{rad}(\mathcal{P})$. For the other inclusion, note that,

$$\mathrm{GKdim}\,\mathcal{P}/{}^{k}\Upsilon \leq k$$

by Lemmas 4.2 and 4.7. Hence ${}^{k}\Upsilon \supseteq (GK \leq k)\,\mathrm{rad}(\mathcal{P})$. □

6. Signature of a 2-unitary operad

In this section we introduce the notion of the signature of an unitary operad, and classify some 2-unitary operads of low GKdimension. Note that we do not usually assume that $\mathcal{P}$ is locally finite.

6.1. Definition of the signature

Definition 6.1. Let $\mathcal{P}$ be a unitary operad. The *signature* of $\mathcal{P}$ is defined to be the sequence

$$\mathcal{S}(\mathcal{P}) := \{d_1, d_2, d_3, \cdots\}$$

where

$$d_k = \dim_{\Bbbk} {}^{k}\Upsilon(k)$$

for all $k \geq 1$. We leave out $d_0 = \dim_{\Bbbk} {}^{0}\Upsilon(0)$ because it is always 1.

We borrow the word "signature" from a paper of Brown-Gilmartin [6, Definition 5.3(1)]. There are some similarities between the signature of a connected Hopf algebra in the sense of [6] and the signature of a 2-unitary operad defined above.

The signature of $\mathcal{C}om$ is $\{0, 0, 0, \cdots\}$. Let $\mathcal{P}$ be a 2-unitary operad of GKdimension k. By (E5.3.2), we have the signature of $\mathcal{P}$ is of form

$$\{f_{\mathcal{P}}(1), \cdots, f_{\mathcal{P}}(k-1), 0, 0, \cdots\}$$

where $f_{\mathcal{P}}(k-1) \neq 0$, and

$$\dim \mathcal{P}(n) = \sum_{i=0}^{k-1} f_{\mathcal{P}}(i)\binom{n}{i},$$

where $\binom{n}{i} = 0$ if $n < i$. Thus the signature of $\mathcal{P}$ is uniquely determined by the Hilbert series of $\mathcal{P}$, and vice versa.

6.2. 2-Unitary operads of low GKdimension

We start with the following lemma.

Lemma 6.2. *Let $\mathcal{P}$ be a 2-unitary operad or a 2-unitary plain operad. Suppose that ${}^2\Upsilon = 0$. Then*

(1) *$\mathcal{P}$ is 2a-unitary.*
(2) *$\mathcal{P}$ is $\mathcal{C}om$-augmented, namely, there is a morphism from $\mathcal{C}om \to \mathcal{P}$.*

Proof. Note that ${}^2\Upsilon = 0$ means that, for each $\theta \in \mathcal{P}(n)$, if $\pi^i(\theta) = 0$ for all $i \in [n]$, then $\theta = 0$.

(1) It is easily seen that $\pi^i(\mathbb{1}_3) = \mathbb{1}_1 = \pi^i(\mathbb{1}'_3)$ for $i = 1, 2, 3$. So $\mathbb{1}_3 - \mathbb{1}'_3 \in \operatorname{Ker} \pi^i$ and $\mathbb{1}_3 - \mathbb{1}'_3 \in {}^2\Upsilon(3) = 0$. The assertion follows.

(2) We claim that $\mathbb{1}_2 * \tau = \mathbb{1}_2$ where $\tau = (12) \in \mathbb{S}_2$. The proof is similar to the proof of part (1) by using the fact that $\pi^i(\mathbb{1}_2 * \tau) = \mathbb{1}_1 = \pi^i(\mathbb{1}_2)$ for $i = 1, 2$. It follows by induction on n that, for every $n \geq 1$, $\mathbb{1}_n * \sigma = \mathbb{1}_n$ for all $\sigma \in \mathbb{S}_n$. Thus there is an operad morphism from $\mathcal{C}om$ to $\mathcal{P}$ by sending $\mathbb{1}_n \in \mathcal{C}om$ to $\mathbb{1}_n \in \mathcal{P}$. □

First we have the following classification of all 2-unitary operads of GKdimension 2.

Theorem 6.3. *There are natural equivalences between*

(a) *the category of finite dimensional, not necessarily unital, $\Bbbk$-algebras;*
(b) *the category of 2-unitary operads of GKdimension ≤ 2;*
(c) *the category of 2a-unitary operads of GKdimension ≤ 2.*

Proof. If $\mathcal{P}$ is a 2-unitary operad of GKdim ≤ 2, then, by Theorem 5.3(2), ${}^2\Upsilon = 0$. Hence Lemma 6.2 can be applied. In particular, two categories in parts (2) and (3) are the same. We now show that two categories in parts (1) and (2) are equivalent.

Suppose Λ is a finite dimensional augmented algebra. By Example 2.4(1), one can construct an operad, denoted by $\mathcal{D}_\Lambda$. Recall that $\mathcal{D}_\Lambda(0) = \Bbbk\mathbb{1}_0$, $\mathcal{D}_\Lambda(1) = \Lambda$, and for $n \geq 2$, $\mathcal{D}_\Lambda(n) = \Bbbk\mathbb{1}_n \oplus \bar{\Lambda}^n$, where $\mathbb{1}_n * \sigma = \mathbb{1}_n$ and $\bar{\Lambda}^n = \{\mathbb{1}_n \underset{i}{\circ} \delta \mid \delta \in \bar{\Lambda}, 1 \leq i \leq n\}$ is the $\Bbbk\mathbb{S}_n$-module with the action $(\mathbb{1}_n \underset{i}{\circ} \delta) * \sigma = \mathbb{1}_n \underset{\sigma^{-1}(i)}{\circ} \delta$ for all $\sigma \in \mathbb{S}_n$. Let $f \colon \Lambda \to \Lambda'$ be a homomorphism of algebras. Then f extends to a unique morphism $\widetilde{f} \colon \mathcal{D}_\Lambda \to \mathcal{D}_{\Lambda'}$ in $\mathcal{M}ag \downarrow \mathsf{Op}_+$ given by $\widetilde{f}_m(\mathbb{1}_m) = \mathbb{1}'_m$ for all $m \geq 0$ and $\widetilde{f}_n(\mathbb{1}_n \underset{i}{\circ} \delta) = \mathbb{1}'_n \underset{i}{\circ} f(\delta)$ for all $\delta \in \bar{\Lambda}$ and $n \geq 1$. Thus the assignment $(\Lambda \mapsto D_\Lambda, f \mapsto \widetilde{f})$ define a functor F from the category of finite dimensional augmented algebras to the category of 2-unitary operads of GKdim ≤ 2.

Let $\mathcal{D}$ be a 2-unitary operad of GKdimension ≤ 2. Observe that $\mathcal{D}(1)$ is an associative $\Bbbk$-algebra with identity $\mathbb{1}_1$. The map $\pi^\emptyset \colon \mathcal{D}(1) \to \mathcal{D}(0) = \Bbbk$ shows that $\mathcal{D}(1)$ is augmented. The restriction $G : \mathcal{D} \to \mathcal{D}(1)$ defines a functor from the category of 2-unitary operads of GKdim ≤ 2 to the category of finite dimensional augmented algebras. It is clear that $G \bullet F \cong \mathrm{Id}$. It remains to show that $F \bullet G \cong \mathrm{Id}$.

If $\operatorname{GKdim}\mathcal{D} = 1$, then $\mathcal{D} = \mathcal{C}om$ by Proposition 2.12. In this case we have $F \bullet G(\mathcal{D}) = \mathcal{D}$. For the rest of the proof we assume that $\operatorname{GKdim}\mathcal{D} = 2$. By Theorem 5.3(2), we have that $f_{\mathcal{D}}(1) \neq 0$ (or ${}^1\Upsilon_{\mathcal{D}} \neq 0$) and $f_{\mathcal{D}}(n) = 0$ (or ${}^n\Upsilon_{\mathcal{D}} = 0$) for all $n \geq 2$. Recall that $f_{\mathcal{D}}(0) = \dim {}^0\Upsilon_{\mathcal{D}}(0) = \dim \mathcal{D}(0) = 1$. Suppose $f_{\mathcal{D}}(1) = \dim {}^1\Upsilon_{\mathcal{D}}(1) = \dim \mathcal{D}(1) - 1 = d > 0$. Then by Lemma 5.2, we know that the generating series of $\mathcal{D}$ is

$$G_{\mathcal{D}}(t) = f_{\mathcal{D}}(0)\frac{1}{1-t} + f_{\mathcal{D}}(1)\frac{t}{(1-t)^2} = \sum_{n=0}^{\infty}(1+nd)t^n.$$

Since ${}^1\Upsilon(1)$ is the kernel of the $\Bbbk$-linear map $\pi^{\emptyset}\colon \mathcal{D}(1) \to \mathcal{D}(0)$ (sending $\theta \mapsto \theta \circ \mathbb{1}_0$), we can choose a $\Bbbk$-basis $\mathbb{1}_1, \delta_1, \cdots, \delta_d$ for $\mathcal{D}(1)$ with $\delta_i \circ \mathbb{1}_0 = 0$ for all $i = 1, \cdots, d$.

The claim that $F \bullet G \cong \mathrm{Id}$ is equivalent to the claim that $\mathcal{D}$ is naturally isomorphic to the operad introduced in Example 2.4(1). We separate the proof into several steps.

Step 1: Denote $\delta^n_{(i)j} \colon = \mathbb{1}_n \underset{i}{\circ} \delta_j$. We claim that $\{\mathbb{1}_n, \delta^n_{(i)j} \mid i \in [n], j \in [d]\}$ is a basis for $\mathcal{D}(n)$. In fact, since $\dim \mathcal{D}(n) = 1 + nd$, we only need show $\{\mathbb{1}_n, \delta^n_{(i)j} \mid i \in [n], j \in [d]\}$ are linearly independent. Assume that there exist $\{\lambda_0, \lambda_{ij} \in \Bbbk \mid i \in [n], j \in [d]\}$ such that $\lambda_0 \mathbb{1}_n + \sum\limits_{i,j} \lambda_{ij}\delta^n_{(i)j} = 0$. Then we have

$$0 = \pi^k(\lambda_0 \mathbb{1}_n + \sum_{i,j} \lambda_{ij}\delta^n_{(i)j}) = \lambda_0 \mathbb{1}_1 + \sum_j \lambda_{kj}\delta_j$$

since $\pi^k(\delta^n_{(i)j}) = \begin{cases} \delta_j, & i = k \\ 0, & i \neq k. \end{cases}$ It follows that $\lambda_0 = 0$ and $\lambda_{ij} = 0$ for all i, j. Therefore we proved our claim.

Step 2: For consistency, we set $\delta_0 = \mathbb{1}_1$, and $\delta^n_{(i)0} = \mathbb{1}_n$ for any $n \geq 1$ and any $i \in [n]$. For other $i \in [n], 0 \leq j \leq d, n \geq 1$, we have $\delta^n_{(i)j} = \mathbb{1}_n \underset{i}{\circ} \delta_j$ by definition. Next, we compute $\delta^m_{(s)t} \underset{i}{\circ} \delta^n_{(k)l}$ for all possible m, s, t, i, n, k, l.

Case 1: $t \geq 1$ and $l = 0$. We consider the special case $m = 1$. Suppose that $\delta_t \circ \mathbb{1}_n = \lambda^t_0 \mathbb{1}_n + \sum\limits_{\substack{1\leq i\leq n\\1\leq j\leq d}} \lambda^t_{ij}\delta^n_{(i)j}$. Then for any $k \in [n]$, we have

$$\begin{aligned}
\delta_t =&(\delta_t \circ \mathbb{1}_n) \circ (\mathbb{1}_0, \cdots, \mathbb{1}_0, \underset{k}{\mathbb{1}_1}, \mathbb{1}_0, \cdots, \mathbb{1}_0) \\
=&(\lambda^t_0 \mathbb{1}_n + \sum_{\substack{1\leq i\leq n\\1\leq j\leq d}} \lambda^t_{ij}\delta^n_{(i)j}) \circ (\mathbb{1}_0, \cdots, \mathbb{1}_0, \underset{k}{\mathbb{1}_1}, \mathbb{1}_0, \cdots, \mathbb{1}_0) \\
=&\lambda^t_0 \mathbb{1}_1 + \sum_{\substack{1\leq i\leq n\\1\leq j\leq d}} \lambda^t_{ij}(\mathbb{1}_n \circ (\mathbb{1}_1, \cdots, \mathbb{1}_1, \underset{i}{\delta_j}, \mathbb{1}_1, \cdots, \mathbb{1}_1)) \circ (\mathbb{1}_0, \cdots, \mathbb{1}_0, \underset{k}{\mathbb{1}_1}, \mathbb{1}_0, \cdots, \mathbb{1}_0) \\
=&\lambda^t_0 \mathbb{1}_1 + \sum_j \lambda^t_{kj}\delta_j.
\end{aligned}$$

It follows that $\lambda_0^t = 0$ and $\lambda_{ij}^t = \begin{cases} 1, & j = t, i \in [n], \\ 0, & \text{otherwise}. \end{cases}$ Therefore,

$$\delta_t \circ \mathbb{1}_n = \sum_{1 \le i \le n} \delta^n_{(i)t}. \tag{E6.3.1}$$

In general,

$$\begin{aligned} \delta^m_{(s)t} \underset{i}{\circ} \mathbb{1}_n =& (\mathbb{1}_m \underset{s}{\circ} \delta_t) \underset{i}{\circ} \mathbb{1}_n \\ =& \begin{cases} (\mathbb{1}_m \underset{i}{\circ} \mathbb{1}_n) \underset{s}{\circ} \delta_t, & i > s, \\ \mathbb{1}_m \underset{s}{\circ} (\delta_t \underset{1}{\circ} \mathbb{1}_n), & i = s, \\ (\mathbb{1}_m \underset{i}{\circ} \mathbb{1}_n) \underset{s+n-1}{\circ} \delta_t, & i < s \end{cases} \\ =& \begin{cases} \delta^{m+n-1}_{(s)t}, & i > s, \\ \sum\limits_{k=1}^{n} \delta^{m+n-1}_{(s+k-1)t}, & i = s, \\ \delta^{m+n-1}_{(s+n-1)t}, & i < s. \end{cases} \end{aligned}$$

Case 2: $t = 0$ and $l \ge 1$.

$$\mathbb{1}_m \underset{i}{\circ} \delta^n_{(k)l} = \mathbb{1}_m \underset{i}{\circ} (\mathbb{1}_n \underset{k}{\circ} \delta_l) = (\mathbb{1}_m \underset{i}{\circ} \mathbb{1}_n) \underset{i+k-1}{\circ} \delta_l = \mathbb{1}_{m+n-1} \underset{i+k-1}{\circ} \delta_l = \delta^{m+n-1}_{(i+k-1)l}.$$

Case 3: $t \ge 1$, $l \ge 1$ and $n = 1$.

For any $1 \le i < i' \le m$, $1 \le j, j' \le d$, we have

$$\pi^k(\mathbb{1}_m \circ (\mathbb{1}_1, \cdots, \mathbb{1}_1, \underset{i}{\delta_j}, \mathbb{1}_1, \cdots, \mathbb{1}_1, \underset{i'}{\delta_{j'}}, \mathbb{1}_1, \cdots, \mathbb{1}_1)) = 0$$

for any $k \in [m]$. So $\mathbb{1}_m \circ (\mathbb{1}_1, \cdots, \mathbb{1}_1, \underset{i}{\delta_j}, \mathbb{1}_1, \cdots, \mathbb{1}_1, \underset{i'}{\delta_{j'}}, \mathbb{1}_1, \cdots, \mathbb{1}_1) = 0$. It follows that

$$\delta^m_{(s)t} \underset{i}{\circ} \delta_l = 0 \tag{E6.3.2}$$

for any $1 \le s \ne i \le m$.

Step 3: Next we consider the multiplication of $\mathcal{D}(1)$. Suppose that $\delta_j \circ \delta_{j'} = \Omega^0_{jj'} \mathbb{1}_1 + \sum\limits_{k=1}^{d} \Omega^k_{jj'} \delta_k$, where $\Omega^k_{jj'}$ $(k = 0, 1, \cdots, d)$ are the structure constants of the associative algebra $\mathcal{D}(1)$ associated to the basis $\{\mathbb{1}_1, \delta_1, \cdots, \delta_d\}$. By (E6.3.1), we have $\delta_j \circ \mathbb{1}_2 = \mathbb{1}_2 \circ (\delta_j, \mathbb{1}_1) + \mathbb{1}_2 \circ (\mathbb{1}_1, \delta_j)$, and by (E6.3.2), we have $\mathbb{1}_2 \circ (\delta_j, \delta_{j'}) = 0$ for any $1 \le j, j' \le d$. It follows that

$$(\delta_j \circ \delta_{j'}) \circ \mathbb{1}_2 = \mathbb{1}_2 \circ (\delta_j \circ \delta_{j'}, \mathbb{1}_1) + \mathbb{1}_2 \circ (\mathbb{1}_1, \delta_j \circ \delta_{j'})$$

and hence $\Omega^0_{jj'} = 0$, which means that $\mathcal{D}(1) = \Bbbk \mathbb{1}_1 \oplus \overline{\mathcal{D}(1)}$ is an augmented algebra with $\overline{\mathcal{D}(1)} = \bigoplus_{j=1}^{d} \Bbbk \delta_j$.

Step 4: We now consider general $\delta^m_{(s)t} \underset{i}{\circ} \delta^n_{(k)l}$ for $t, l \geq 1$.

By (E6.3.2), we have $\delta^m_{(s)t} \underset{i}{\circ} \delta^n_{(k)l} = 0$ for any $i \neq s$. If $s = i$, we have

$$\begin{aligned}
\delta^m_{(s)t} \underset{i}{\circ} \delta^n_{(k)l} =& (\mathbb{1}_m \underset{s}{\circ} \delta_t) \underset{i}{\circ} (\mathbb{1}_n \underset{k}{\circ} \delta_l) = \mathbb{1}_m \underset{s}{\circ} (\delta_t \underset{1}{\circ} (\mathbb{1}_n \underset{k}{\circ} \delta_l)) \\
=& \mathbb{1}_m \underset{s}{\circ} ((\delta_t \underset{1}{\circ} \mathbb{1}_n) \underset{k}{\circ} \delta_l) \\
=& \mathbb{1}_m \underset{s}{\circ} ((\sum_{u=1}^{n} \delta^n_{(u)t}) \underset{k}{\circ} \delta_l) \qquad \text{by (E6.3.1)} \\
=& \mathbb{1}_m \underset{s}{\circ} (\delta^n_{(k)t} \underset{k}{\circ} \delta_l) \qquad \text{by (E6.3.2)} \\
=& \mathbb{1}_m \underset{s}{\circ} (\mathbb{1}_n \underset{k}{\circ} (\delta_t \underset{1}{\circ} \delta_l)) \\
=& (\mathbb{1}_m \underset{s}{\circ} \mathbb{1}_n) \underset{k+s-1}{\circ} (\delta_t \underset{1}{\circ} \delta_l) \\
=& \sum_{v=1}^{d} \Omega^v_{tl} \delta^{m+n-1}_{(k+s-1)v}.
\end{aligned}$$

The first 4 steps show that (E2.4.3) holds.

Step 5: Finally it follows from ${}^2\Upsilon_{\mathcal{D}} = 0$ that $\delta^n_{(i)j} * \sigma = \delta^n_{(\sigma^{-1}(i))j}$ for all $\sigma \in \mathbb{S}_n$.

As above, we have shown that a 2-unitary operad $\mathcal{D}$ is isomorphic to an operad introduced in Example 2.4(1) with $\Lambda = \mathcal{D}(1)$ and that $\mathcal{D}$ is uniquely determined by an augmented algebra $\mathcal{D}(1)$. This implies that $F \bullet G \cong \mathrm{Id}$, as required. $\square$

We make a remark.

Remark 6.4. The above proof works for non-locally finite operads when the use of the generating function is replaced by the basis Theorem 4.6 (1). Therefore, for non-locally finite 2-unitary operads, there are natural equivalences between the following categories:

(aI) the category of $\Bbbk$-algebras not necessarily having unit;
(aI') the category of unital augmented $\Bbbk$-algebras;
(bI) the category of 2-unitary operads with ${}^2\Upsilon = 0$;
(cI) the category of $2a$-unitary operads with ${}^2\Upsilon = 0$.

Every operad in one of the above categories is isomorphic to one given in Example 2.4(1).

Consequently, we have the following Artin-Wedderburn Theorem for 2-unitary operads.

Theorem 6.5. *Let $\mathcal{P}$ be a semiprime symmetric operad. If $\mathcal{P}$ is 2-unitary and left or right artinian, then $\mathcal{P}$ is as in Example 2.4(1) and $\mathcal{P}(1)$ is an augmented semisimple algebra.*

If, further, $\mathcal{P}(1)$ is finite dimensional over $\Bbbk$, then $\mathcal{P}$ is locally finite, $\operatorname{GKdim}\mathcal{P} = 2$ *or* $\operatorname{GKdim}\mathcal{P} = 1$ *(and hence $\mathcal{P} = \mathcal{C}om$), and $\mathcal{P}(1)$ is a finite dimensional augmented semisimple algebra.*

Proof. The proof of this part is similar to the proof of Theorem 3.6(2).

Let ${}^k\Upsilon$ be the truncation ideals of $\mathcal{P}$. By definition, $\bigcap_{k\geq 1} {}^k\Upsilon = 0$. Since $\mathcal{P}$ is left or right artinian, ${}^k\Upsilon = 0$ for some k. Let n be the largest integer such that ${}^n\Upsilon \neq 0$. If $n \geq 2$, by Proposition 3.1 (2), $({}^n\Upsilon)^2 \subseteq {}^{2n-1}\Upsilon = 0$. This contradicts the hypothesis that $\mathcal{P}$ is semiprime. Therefore ${}^2\Upsilon = 0$.

Let $A = \mathcal{P}(1)$. By Proposition 3.2(1,2), if A is not left (respectively, right) artinian, then $\mathcal{P}$ is not left (respectively, right) artinian. Since $\mathcal{P}$ is left or right artinian, so is A. Let N be an ideal of A such that $N^2 = 0$. By Proposition 3.2(1,2), ${}^1\Upsilon^N$ is an ideal of $\mathcal{P}$. By Proposition 3.2(3),

$$({}^1\Upsilon^N)^2 \subseteq {}^1\Upsilon^{N^2} = {}^1\Upsilon^0 = {}^2\Upsilon = 0.$$

Since $\mathcal{P}$ is semiprime, ${}^1\Upsilon^N = 0$, consequently, $N = 0$. Thus A is semiprime. Since A is left artinian or right artinian, A is semisimple.

By Remark 6.4, the operad $\mathcal{P}$ is given as in Example 2.4(1).

If, further, $\mathcal{P}(1)$ is finite dimensional, then by Theorem 4.6(1) $\mathcal{P}$ is locally finite. Since ${}^2\Upsilon = 0$, $\operatorname{GKdim}\mathcal{P} \leq 2$. If $\operatorname{GKdim}\mathcal{P} = 1$, then $\mathcal{P} = \mathcal{C}om$ by Proposition 2.12. Otherwise $\operatorname{GKdim}\mathcal{P} = 2$. The rest of assertion follows. $\square$

Next we consider the quotient operads of $\mathcal{A}ss$ of low GKdimension.

Theorem 6.6. *Suppose* $\operatorname{char}\Bbbk = 0$. *Let $\mathcal{P}$ be a quotient operad of $\mathcal{A}ss$ and n be* $\operatorname{GKdim}\mathcal{P}$. *Let ${}^k\Upsilon$ be the truncations of $\mathcal{A}ss$.*

(1) *If $n = 1$, then $\mathcal{P} = \mathcal{A}ss/{}^1\Upsilon \cong \mathcal{C}om$.*
(2) [Gap Theorem] $\operatorname{GKdim}\mathcal{P}$ *can not be* 2, (*so can not be strictly between* 1 *and* 3).
(3) *If $n = 3$, then $\mathcal{P} = \mathcal{A}ss/{}^3\Upsilon$.*
(4) *If $n = 4$, then $\mathcal{P} = \mathcal{A}ss/{}^4\Upsilon$.*
(5) *There are at least two non-isomorphic quotient operads $\mathcal{P}$ such that* $\operatorname{GKdim}\mathcal{P} = 5$.

Proof. Let $\mathcal{P} = \mathcal{A}ss/\mathcal{I}$ be a quotient operad of $\mathcal{A}ss$ of GKdimension n. Let ${}^k\Upsilon_{\mathcal{P}}$ and ${}^k\Upsilon$ be the truncations of $\mathcal{P}$ and $\mathcal{A}ss$, respectively. Since $\mathcal{P}$ is unitary, $f_{\mathcal{P}}(0) = \dim {}^0\Upsilon_{\mathcal{P}}(0) = \dim \mathcal{P}(0) = 1$.

(1) This is Proposition 2.12.

(2) Since $\mathcal{P}$ is a quotient of $\mathcal{A}ss$, $f_{\mathcal{P}}(1) = \dim {}^1\Upsilon_{\mathcal{P}}(1) = 0$. By (E5.2.3), $\operatorname{GKdim}\mathcal{P}$ is either 1 or at least 3.

(3) $\operatorname{GKdim}\mathcal{P} = 3$. From Corollary 5.4 and Lemma 3.9, it immediately follows that $\mathcal{I} = {}^3\Upsilon$ and $\mathcal{P} = \mathcal{A}ss/{}^3\Upsilon$.

(4) $\operatorname{GKdim}\mathcal{P} = 4$. Then $\dim \mathcal{P}(0) = \dim \mathcal{P}(1) = 1$. By Lemma 3.9, $\mathcal{I} \subseteq {}^3\Upsilon$. Hence $\dim \mathcal{P}(2) = 2$, and consequently by (E5.3.1), $\dim f_{\mathcal{P}}(1) = 0$ and $\dim f_{\mathcal{P}}(2) = 1$. Hence we have

$$G_{\mathcal{P}}(t) = \sum_{n=0}^{\infty} (1 + \tbinom{n}{2} + f_{\mathcal{P}}(3)\tbinom{n}{3})t^n.$$

Observe that $\mathcal{I}(3)$ must be a $\Bbbk\mathbb{S}_3$-submodule of

$${}^3\Upsilon(3) := \Bbbk((1,2,3)-(2,1,3)-(3,1,2)+(3,2,1)) + \Bbbk((1,3,2)-(2,1,3)-(3,1,2)+(2,3,1)),$$

where the permutations are written by the convention introduced in Appendix 8.1. Since ${}^3\Upsilon(3)$ above is a simple $\Bbbk\mathbb{S}_3$-module, we have either $\mathcal{I}(3) = 0$ or $\mathcal{I}(3) = {}^3\Upsilon(3)$.

If $\mathcal{I}(3) = {}^3\Upsilon(3)$, then $f_{\mathcal{P}}(3) = 0$, which is impossible. The only possibility is $\mathcal{I}(3) = 0$. In this case, $\dim \mathcal{P}(3) = 6$ and $f_{\mathcal{P}}(3) = 2$. So we have

$$\dim \mathcal{P}(n) = 1 + \tbinom{n}{2} + 2\tbinom{n}{3} = \dim(\mathcal{A}ss/{}^4\Upsilon)(n),$$

and consequently,

$$\dim \mathcal{I}(n) = \dim {}^4\Upsilon(n).$$

On the other hand, we have ${}^4\Upsilon \subseteq \mathcal{I}$. Therefore, we have $\mathcal{I}(n) = {}^4\Upsilon(n)$ for all $n \geq 4$. It follows that $\mathcal{I} = {}^4\Upsilon$ and $\mathcal{P} = \mathcal{A}ss/{}^4\Upsilon$.

(5) It is easy to see that $\dim \mathcal{A}ss/{}^4\Upsilon(4) = 15$ (for example, by the proof of part (4)). Hence $\dim {}^4\Upsilon(4) = 4! - 15 = 9$. Thus there is a nonzero $\Bbbk\mathbb{S}_4$-submodule $M \subsetneq {}^4\Upsilon(4)$. Since $\mathcal{A}ss(1) = \Bbbk$, both (E3.1.2) and (E3.1.3) hold trivially for M. By Proposition 3.2(1,2), ${}^4\Upsilon^M$ is an ideal of $\mathcal{A}ss$. By the choice of M, we have

$${}^5\Upsilon \subsetneq {}^4\Upsilon^M \subsetneq {}^4\Upsilon$$

which implies that

$$\operatorname{GKdim}\mathcal{A}ss/{}^5\Upsilon = 5 = \operatorname{GKdim}\mathcal{A}ss/{}^4\Upsilon^M.$$

Since the Hilbert series of $\mathcal{A}ss/{}^5\Upsilon$ and $\mathcal{A}ss/{}^4\Upsilon^M$ are different, these two operads are non-isomorphic. $\square$

6.3. $\mathcal{C}om$-augmented operad with a given signature

For the rest of this section we consider $\mathcal{C}om$-augmented operads. Let $\mathcal{C}om \downarrow \mathsf{Op}_+$ denote the category of $\mathcal{C}om$-augmented operads. For every $\mathcal{P}$ in $\mathcal{C}om \downarrow \mathsf{Op}_+$, there is a natural decomposition

$$\mathcal{P} = \mathcal{C}om \oplus {}^1\Upsilon$$

of $\mathbb{S}$-module, where ${}^1\Upsilon = \mathrm{Ker}(\mathcal{P} \to \mathcal{C}om)$.

Definition 6.7. Let $\{\mathcal{P}_i\}_{i\in I}$ be a family of operads in $\mathcal{C}om \downarrow \mathsf{Op}_+$. The *$\mathcal{C}om$-augmented sum* of $\{\mathcal{P}_i\}_{i\in I}$ is defined to be

$$\bigoplus_{i\in I} \mathcal{P}_i := \mathcal{C}om \oplus \bigoplus_{i\in I} {}^1\Upsilon_{\mathcal{P}_i} \tag{E6.7.1}$$

with relations, for all homogeneous element θ_k in $\mathcal{C}om \cup \bigcup_{i\in I} {}^1\Upsilon_{\mathcal{P}_i}$,

$$\theta_0 \circ (\theta_1, \cdots, \theta_n) = 0 \tag{E6.7.2}$$

whenever at least two of $\theta_0, \cdots, \theta_n$ are in different ${}^1\Upsilon_{\mathcal{P}_j}$. If all θ_k's are in the same $\mathcal{P}_j$, then the composition in $\bigoplus_{i\in I} \mathcal{P}_i$ agrees with the composition in $\mathcal{P}_j$.

Lemma 6.8. *Let $\{\mathcal{P}_i\}_{i\in I}$ be a family of operads in $\mathcal{C}om \downarrow \mathsf{Op}_+$.*

(1) *$\mathcal{P} := \bigoplus_{i\in I} \mathcal{P}_i$ is an operad in $\mathcal{C}om \downarrow \mathsf{Op}_+$.*
(2) *${}^k\Upsilon_{\mathcal{P}} = \bigoplus_{i\in I} {}^k\Upsilon_{\mathcal{P}_i}$ for all $k \geq 1$.*
(3) *$\mathcal{S}(\mathcal{P}) = \sum_{i\in I} \mathcal{S}(\mathcal{P}_i)$.*
(4) *For each subset $I' \subseteq I$, $\bigoplus_{i\in I'} {}^1\Upsilon_{\mathcal{P}_i}$ is an ideal of $\mathcal{P}$. As a consequence, if there are infinitely many i such that $\Upsilon_{\mathcal{P}_i} \neq 0$, then $\mathcal{P}$ is neither artinian nor noetherian.*

Proof. (1) We need to show (OP1), (OP2), (OP3) in Definition 1.1. Since all maps are linear or multilinear, we only need to consider elements in $\mathcal{C}om$, $\Upsilon_{\mathcal{P}_i}$, for $i \in I$. Using the relations in (E6.7.1), it amounts to verify (OP1), (OP2) and (OP3) for elements in $\mathcal{C}om \cup {}^1\Upsilon_{\mathcal{P}_i}$ for one i. In this setting (OP1), (OP2), (OP3) hold since $\mathcal{P}_i$ is an operad. Therefore $\bigoplus_{i\in I} \mathcal{P}_i$ is an operad. It is clear from (E6.7.1) that we can define a morphism from $\mathcal{C}om \to \bigoplus_{i\in I} \mathcal{P}_i$. So the assertion follows.

(2) Let $\mathcal{P}$ be $\bigoplus_{i\in I} \mathcal{P}_i$. It is clear from the definition that

$${}^1\Upsilon_{\mathcal{P}} = \bigoplus_{i\in I} {}^1\Upsilon_{\mathcal{P}_i}.$$

Inside this ideal, we have $\pi^I_{\mathcal{P}} = \bigoplus_{i\in I} \pi^I_{\mathcal{P}_i}$ for restriction maps defined in (E2.4.5). The assertion follows easily from this fact.

(3) This is an consequence of part (2).

(4) It is easy to show and the proof is omitted. $\square$

Next we will show the existence of operads with any given signature. We begin with a special case.

Example 6.9. Fix $w \geq 1$ and $d \geq 1$. In this example, we construct a $\mathcal{C}om$-augmented operad of signature $\{0, \cdots, 0, d, 0, \cdots\}$, where d is in wth position.

Let V be an $\mathbb{S}_w$-module of dimension d and let $\{\delta_1, \cdots, \delta_d\}$ be a $\Bbbk$-linear basis of V. If $w = 1$, we further assume that the multiplication $\delta_i \delta_j = 0$ for all i, j. Let $\mathcal{C}_w^n$ be defined as before Lemma 4.10. Define

$$\mathcal{P}(n) = \begin{cases} \Bbbk \mathbb{1}_n, & n < w, \\ \Bbbk \mathbb{1}_w \bigoplus V & n = w, \\ \Bbbk \mathbb{1}_n \bigoplus \mathcal{C}_w^n(V) & n > w. \end{cases} \tag{E6.9.1}$$

We recall the following notation. For $n = w + s$, where $s > 0$, and for every $I \subseteq [n]$ such that $|I| = s$ and for $j \in [d]$, let

$$(\delta, I) := \mathbb{1}_2 \circ (\delta, \mathbb{1}_s) * c_I$$

for all $\delta \in V$. As a vector space, $\mathcal{P}(n)$ has a basis $\{\mathbb{1}_n\} \cup \{(\delta_i, I) \mid i \in [d], I \subseteq [n], |I| = s\}$.

Assuming first that $\mathcal{P}$ is an operad, we would like to derive some defining equations. By Corollary 4.4, if $I' \subseteq [n]$ such that $|I'| = n - w$, then

$$\Gamma^{I'}((\delta, I)) = \begin{cases} \delta, & I = I', \\ 0, & I \neq I', \end{cases}$$

or, for $J \subseteq [n]$ with $|J| = w$,

$$\pi^J((\delta, I)) = \begin{cases} \delta, & I \cup J = [n], \\ 0, & I \cup J \neq [n]. \end{cases}$$

Following Lemma 4.9, we set

$$(\delta, I) * \sigma = (\delta * \Gamma^{\sigma^{-1}(I)}, \sigma^{-1}(I))$$

for all $\delta \in V$ and I. Together with the trivial $\mathbb{S}_n$ on $\Bbbk \mathbb{1}_n$, this defines $\mathbb{S}_n$-module structure on $\mathcal{P}(n)$.

Next we consider partial compositions. Similar to Example 2.4 (1), we set

$$(\delta, I) \underset{s}{\circ} (\delta', I') = 0$$

because, for every $|J| = w$,

$$\pi^J((\delta, I) \underset{s}{\circ} (\delta', I')) = 0.$$

Write $I = \{i_1, \cdots, i_{n-w}\} \subseteq [n]$. Define

$$\mathbb{1}_m \underset{s}{\circ} \mathbb{1}_n = \mathbb{1}_{m+n-1},$$
$$\mathbb{1}_m \underset{s}{\circ} (\delta, I) = (\delta, I'),$$

where $I' = \{1, \cdots, s-1, I+(s-1), n+s, \cdots, n+m-1\}$, and

$$(\delta, I) \underset{s}{\circ} \mathbb{1}_m = \begin{cases} (\delta, \bar{I}), & s \in I, \\ \sum_{u=1}^{m} (\delta, I_u), & s \notin I, \end{cases}$$

where

$$\bar{I} = \{i_1, \cdots, i_{f-1}, s, s+1, \cdots, s+m-1, i_{f+1}+m-1, \cdots, i_{n-w}+m-1\}$$

when $s = i_f$ for some f, and where

$$I_u = \{i_1, \cdots, i_{f-1}, s, \cdots, \widehat{s+u-1}, \cdots, s+m-1, i_f+m-1, i_{f+1}+m-1, \cdots, i_{n-w}+m-1\}$$

when $i_{f-1} < s < i_f$. Now it is routine to check that $\mathcal{P}$ is a 2-unitary operad with given signature. Conversely, any operad with signature $\{0, \cdots, 0, d, 0, \cdots, \}$ is given in this way.

Theorem 6.10. *Let $w \geq 2$. Every $\mathcal{C}om$-augmented operad of signature $\{0, \cdots, 0, d_w, 0, \cdots\}$ is of the form given in (E6.9.1).*

Proof. The proof is similar to the proof of Theorem 6.3. We omit the proof due to its length. □

Now we can prove the main result in this subsection.

Theorem 6.11. *Let $\mathcal{C}om\downarrow\mathsf{Op}$ be the category of $\mathcal{C}om$-augmented operads.*

(1) *For every sequence of non-negative integers $\mathbf{d} := \{d_1, d_2, \cdots\}$, there is an operad $\mathcal{P}$ in $\mathcal{C}om\downarrow\mathsf{Op}$ such that $\mathcal{S}(\mathcal{P}) = \mathbf{d}$.*
(2) *Exponent* exp *of (E0.0.4) is a surjective map from $\mathcal{C}om\downarrow\mathsf{Op}$ or from the category of 2-unitary operads to $\{1\} \cup [2, \infty]$.*

Proof. (1) For each d_w for $w \geq 1$, pick a trivial $\mathbb{S}_w$-module V_w of dimension d_w. By Example 6.9, there is a $\mathcal{C}om$-augmented (thus 2-unitary) operad $\mathcal{P}_w$ with signature $\{0, \cdots, 0, d_w, 0, \cdots\}$. By Lemma 6.8(3), $\bigoplus_w \mathcal{P}_w$ has the required signature.

(2) Take a sequence $\mathcal{S}(\mathcal{P})$ with $\exp(\mathcal{S}(\mathcal{P})) = \infty$, then $\exp(\mathcal{P}) = \infty$. One such example is $\mathcal{P} = \mathcal{A}ss$.

We know that $\exp(\mathcal{C}om) = 1$. Let $\mathcal{P}$ be an 2-unitary operad with $\mathcal{S}(\mathcal{P}) = \{b_1, \cdots, b_w, \cdots\}$. If $b_n = 0$ for all $n \gg 0$, then $\exp(\mathcal{S}(\mathcal{P})) = 0$ and by Lemma 5.1(1),

$\exp(\mathcal{P}) = \exp(\{\dim \mathcal{P}(n)\}_{n\geq 0}) = 1$ since $\{\dim \mathcal{P}(n)\}_{n\geq 0}$ is the inverse binomial transform of $\mathcal{S}(\mathcal{P})$, see (E5.3.1). Otherwise, $\exp(\mathcal{S}(\mathcal{P})) = 1$ and by Lemma 5.1(1), $\exp(\mathcal{P}) \geq 2$.

It remains to show that for each $r \geq 2$, there is a 2-unitary operad (in fact, a $\mathcal{C}om$-augmented operad) $\mathcal{P}$ such that $\exp(\mathcal{P}) = r$. Let $d_w = \lfloor (r-1)^w \rfloor$ for each $w \geq 1$. By part (1), there is a $\mathcal{C}om$-augmented operad $\mathcal{P}$ such that $\mathcal{S}(\mathcal{P}) = \{d_1, d_2, \cdots, d_w, \cdots\}$. Thus $\exp(\mathcal{S}(\mathcal{P})) = r-1$. By Lemma 5.1(1), $\exp(\mathcal{P}) = r$ as required. □

We conclude this section with an easy corollary.

Corollary 6.12. *Let* $\mathbf{d} := \{d_i\}_{i\geq 1}$ *be any sequence of non-negative integers. Then there is a unitary operad* $\mathcal{P}$ *such that* $G_{\mathcal{P}}(t) = 1 + (d_1+1)t + \sum_{i=2}^{\infty} d_i t^i$.

Proof. By Theorem 6.11(1), there is a 2-unitary operad $\mathcal{Q}$ such that $\mathcal{S}(\mathcal{Q}) = \mathbf{d}$. Let $\mathcal{P} = \Bbbk 1_1 \oplus \bigoplus_{i=0}^{\infty} {}^{i}\Upsilon_{\mathcal{Q}}(i)$. By Proposition 3.13(2), $\mathcal{P}$ is a unitary operad. By the definition of signature, we see that

$$G_{\mathcal{P}}(t) = 1 + (d_1+1)t + \sum_{i=2}^{\infty} d_i t^i. \quad \square$$

7. Truncatified operads

The truncation of a unitary operad $\mathcal{P}$ defines a descending filtration on $\mathcal{P}$ which induces an associated operad, called a truncatified operad, as we will define next.

Definition 7.1. A unitary operad $\mathcal{P}$ is called *truncatified* if the following hold.

(1) For each n, $\mathcal{P}(n)$ has a decomposition of $\mathbb{S}_n$-submodules,

$$\mathcal{P}(n) = \bigoplus_{i=0}^{n} \mathcal{P}(n)_i.$$

(2) For all k and all $n \geq k$,

$${}^{k}\Upsilon(n) = \bigoplus_{i=k}^{n} \mathcal{P}(n)_i.$$

(3) Let $\mu \in \mathcal{P}(n)_{n_0}$ and $\nu \in \mathcal{P}(m)_{m_0}$. Suppose $1 \leq i \leq n$.

(3a) If $n_0, m_0 \geq 1$, then

$$\mu \underset{i}{\circ} \nu \in \mathcal{P}(n+m-1)_{n_0+m_0-1}.$$

(3b) If $m_0 = 0$ or $n_0 = 0$, then

$$\mu \underset{i}{\circ} \nu \in \mathcal{P}(n+m-1)_{m_0+n_0}.$$

Remark 7.2. A truncatified operad in the above definition may be called a *truncated* operad since it is induced by the truncation (see also Lemma 7.3). However, the notion of a *truncated* operad has been defined in [16, Definition 4.2.1] and been used in some other papers [35]. To avoid possible confusions, we create a new word, "truncatified", in Definition 7.1. Note that every truncatified operad is either $\mathcal{C}om$-augmented or $\mathcal{U}ni$-augmented.

It is easy to check that the operads in Examples 2.4 and 6.9 are truncatified. Truncatified operads can be constructed from a non-truncatified operad.

Lemma 7.3. *Let $\mathcal{Q}$ be a unitary operad and $\{{}^i\Upsilon_{\mathcal{Q}}\}_{i\geq 0}$ be the truncation of $\mathcal{Q}$. For each $n \geq 0$, let $\mathcal{P}(n)$ denote the $\Bbbk$-linear space $\bigoplus_{i=0}^{\infty} {}^i\Upsilon_{\mathcal{Q}}(n)/{}^{i+1}\Upsilon_{\mathcal{Q}}(n)$. Then $\mathcal{P} := \{\mathcal{P}(n)\}_{n\geq 0}$ is a truncatified operad.*

Proof. Let $\mathcal{P}(n)_i := {}^i\Upsilon_{\mathcal{Q}}(n)/{}^{i+1}\Upsilon_{\mathcal{Q}}(n)$ for all n, i. For the rest of the proof, i, j, k, m, n and s are non-negative integers. Assume that $1 \leq s \leq m$.

Let $\mu \in \mathcal{P}(m)_i$ and $\nu \in \mathcal{P}(n)_j$. Then μ is the image of some $a \in {}^i\Upsilon_{\mathcal{Q}}(m)$ and ν is the image of some $b \in {}^j\Upsilon_{\mathcal{Q}}(n)$. Define $\mu \underset{s}{\circ} \nu$ to be the image of $a \underset{s}{\circ} b$ in $\mathcal{P}(m+n-1)_{i+j-1} := {}^{i+j-1}\Upsilon_{\mathcal{Q}}(m+n-1)/{}^{i+j}\Upsilon_{\mathcal{Q}}(m+n-1)$ (or in $\mathcal{P}(m+n-1)_{i+j}$ if either i or j is zero). It is routine to check that $\mathcal{P}$ is a unitary operad using the partial definition Definition 1.2.

Next we show (1), (2) and (3) in Definition 7.1.

(1) Since ${}^i\Upsilon_{\mathcal{Q}}(n) = 0$ for all $i > n$, we have

$$\mathcal{P}(n) = \bigoplus_{i=0}^{\infty} {}^i\Upsilon_{\mathcal{Q}}(n)/{}^{i+1}\Upsilon_{\mathcal{Q}}(n) = \bigoplus_{i=0}^{n} {}^i\Upsilon_{\mathcal{Q}}(n)/{}^{i+1}\Upsilon_{\mathcal{Q}}(n) = \bigoplus_{i=0}^{n} \mathcal{P}(n)_i.$$

Since each $\mathcal{P}(n)_i$ is clearly an $\mathbb{S}_n$-module, (1) holds.

(2) Denote $T^n_{n-k} = \{K \subset [n] \mid |K| = k\}$. (Note that T^n_k is defined before Lemma 4.10.)

Let θ be an element in ${}^i\Upsilon_{\mathcal{Q}}(n)$ such that $\theta \notin {}^{i+1}\Upsilon_{\mathcal{Q}}(n)$. If $k < i$, by definition, we have $\pi^K_{\mathcal{Q}}(\theta) = 0$ for all $K \in T^n_{n-k}$. If $k \geq i$, we have $\pi^K_{\mathcal{Q}}(\theta) \in {}^i\Upsilon_{\mathcal{Q}}(k)$ for all $K \in T^n_{n-k}$, and there exists some $K_0 \in T^n_{n-k}$ such that $\pi^{K_0}_{\mathcal{Q}}(\theta) \notin {}^{i+1}\Upsilon_{\mathcal{Q}}(k)$. In fact, since $\theta \notin {}^{i+1}\Upsilon_{\mathcal{Q}}(n)$, there exists some $I \in T^n_{n-i}$ such that $\pi^I_{\mathcal{Q}}(\theta) \neq 0$. Then for every K_0 with $I \subseteq K_0 \in T^n_{n-k}$, we have $\pi^{K_0}_{\mathcal{Q}}(\theta) \notin {}^{i+1}\Upsilon_{\mathcal{Q}}(k)$.

Return to consider the restricted operator $\pi^I_{\mathcal{P}} \colon \mathcal{P}(n) \to \mathcal{P}(|I|)$. Pick any nonzero element μ in ${}^i\Upsilon_{\mathcal{Q}}(n)/{}^{i+1}\Upsilon_{\mathcal{Q}}(n)$ and write it as $\mu = \theta + {}^{i+1}\Upsilon_{\mathcal{Q}}(n) \neq \bar{0}$ for some $\theta \in {}^i\Upsilon_{\mathcal{Q}}(n)$. If $i > k-1$, then, for every $I \in T^n_{n-(k-1)}$, we have

$$\pi^I_{\mathcal{P}}(\theta + {}^{i+1}\Upsilon_{\mathcal{Q}}(n)) = \bar{0} \in \mathcal{P}(k-1).$$

This implies that, for any $i \geq k$, $\mathcal{P}(n)_i = {}^i\Upsilon_{\mathcal{Q}}(n)/{}^{i+1}\Upsilon_{\mathcal{Q}}(n) \subset {}^k\Upsilon_{\mathcal{P}}(n)$.

On the other hand, if $i < k$, then, for every nonzero element $\mu := \theta + {}^{i+1}\Upsilon_{\mathcal{Q}}(n) \in \mathcal{P}(n)_i$, there exists $I_0 \in T^n_{n-(k-1)}$ such that

$$\pi_{\mathcal{P}}^{I_0}(\mu) = \pi_{\mathcal{P}}^{I_0}(\theta + {}^{i+1}\Upsilon_{\mathcal{Q}}(n)) = \pi_{\mathcal{Q}}^{I_0}(\theta) + {}^{i+1}\Upsilon_{\mathcal{Q}}(k-1) \neq \bar{0}$$

in $\mathcal{P}(k-1)$. It follows that ${}^k\Upsilon_{\mathcal{P}}(n) \subset \bigoplus_{i=k}^{n} \mathcal{P}(n)_i$.

(3) Note that (3a) and (3b) follow from the proof of Proposition 3.1(2). □

In the setting of Lemma 7.3, we say that $\mathcal{P}$ is the *associated truncatified operad* of $\mathcal{Q}$, and denoted it by $\mathrm{trc}\mathcal{Q}$. The process from $\mathcal{Q}$ to $\mathrm{trc}\mathcal{Q}$ is called *truncatifying*.

It follows from Lemma 7.3 that a unitary operad $\mathcal{P}$ is truncatified if and only if $\mathcal{P} \cong \mathrm{trc}(\mathcal{P})$. As a consequence, $\mathrm{trc}(\mathrm{trc}(\mathcal{P})) \cong \mathrm{trc}(\mathcal{P})$ for all unitary operads $\mathcal{P}$.

Next we show that $\mathcal{P}ois$ is the associated truncatified operad of $\mathcal{A}ss$. For any unitary operad $\mathcal{P}$, let $\mathcal{P}_{\geq 1}$ be the non-unitary version of $\mathcal{P}$, namely,

$$\mathcal{P}_{\geq 1}(n) = \begin{cases} 0 & n = 0, \\ \mathcal{P}(n) & n \geq 1. \end{cases}$$

Note that $\mathcal{P}ois_{\geq 1}$ agrees with the non-unitary version of the Poisson operad, and $\mathcal{P}ois_{\geq 1}$ is denoted by $\mathcal{P}ois$ in [9, Section 1.2.12] and [26, Section 13.3.3]. On the other hand, the unitary version of the Poisson operad (namely, our $\mathcal{P}ois$) is denoted by $\mathcal{P}ois_{+}$ in the book [9].

Lemma 7.4. *Let $\mathcal{A}ss$ be the operad for the unital associative algebras and $\mathcal{P}ois$ be the operad for unital commutative Poisson algebras. Then* $\mathrm{trc}\mathcal{A}ss \cong \mathcal{P}ois$.

Proof. Denote by ${}^k\Upsilon$ the k-th truncation ideal of $\mathcal{A}ss$. By Lemma 3.4, we have ${}^1\Upsilon = {}^2\Upsilon$. By definition, we have $\mathrm{trc}\mathcal{A}ss(2) = \Bbbk\bar{1}_2 \oplus \Bbbk\bar{\Phi}_2$, where $\bar{1}_2 = 1_2 + {}^1\Upsilon(2)$ and $\bar{\Phi}_2 \in \mathrm{trc}\mathcal{A}ss(2)$ is the corresponding element of $(1_2 - (21)) \in {}^2\Upsilon(2)$. Clearly, $\bar{1}_2 * (21) = \bar{1}_2$ and $\bar{\Phi}_2 * (21) = -\bar{\Phi}_2$, and they satisfy the following relations

$$\bar{1}_2 \underset{1}{\circ} \bar{1}_2 = \bar{1}_2 \underset{2}{\circ} \bar{1}_2, \tag{E7.4.1}$$

$$\bar{\Phi}_2 \underset{1}{\circ} \bar{1}_2 = \bar{1}_2 \underset{2}{\circ} \bar{\Phi}_2 + (\bar{1}_2 \underset{2}{\circ} \bar{\Phi}_2) * (213), \tag{E7.4.2}$$

$$\bar{\Phi}_2 \underset{2}{\circ} \bar{\Phi}_2 = \bar{\Phi}_2 \underset{1}{\circ} \bar{\Phi}_2 + (\bar{\Phi}_2 \underset{2}{\circ} \bar{\Phi}_2) * (213) \tag{E7.4.3}$$

which are exactly the defining relations of $\mathcal{P}ois_{\geq 1}$, see [9, Section 1.2.12]. Observe that $\mathrm{trc}\mathcal{A}ss$ is generated by $\Bbbk\bar{1}_2 \oplus \Bbbk\bar{\Phi}_2$. In fact, from Theorem 4.6(1), we know ${}^k\Upsilon(n)/{}^{k+1}\Upsilon(n)$ admits a $\Bbbk$-linear basis

$$B_k(n) = \{1_2 \circ (\theta_i^k, 1_{n-k}) * c_I \mid 1 \leq i \leq z_k, I \in T_k^n\},$$

where $\{\theta_1^k, \cdots, \theta_{z_k}^k\}$ is a $\Bbbk$-basis of ${}^k\Upsilon(k)$. Furthermore, for every $k \geq 3$, we have ${}^k\Upsilon(k) \subset {}^2\Upsilon(k)$. By Lemma 3.8, ${}^2\Upsilon(2)$ is generated by Φ_2. Note that 1_2 generates 1_n for all $n \geq 2$. By the proof of Lemma 3.8, for every $k \geq 3$, ${}^k\Upsilon(k)$ is generated by $\{1_n\}_{n\geq 2}$ and Φ_2.

Therefore ${}^{k}\Upsilon(n)/{}^{k+1}\Upsilon(n)$ can be generated by $\bar{1}_2$ and $\bar{\Phi}_2$ for any $n \geq k \geq 2$. It follows that $(\mathrm{trc}\mathcal{A}ss)_{\geq 1}$ can be generated by $\bar{1}_2$ and $\bar{\Phi}_2$. The above argument shows that there is a canonical epimorphism $\mathcal{P} := \mathscr{T}(E)/(R) \to (\mathrm{trc}\mathcal{A}ss)_{\geq 1}$, where $\mathscr{T}(E)/(R)$ be the quotient operad of the free operad $\mathscr{T}(E)$ on the $\Bbbk\mathbb{S}$-module $E = (0, 0, \Bbbk\bar{1}_2 \oplus \Bbbk\bar{\Phi}_2, 0, \cdots)$ modulo relations (E7.4.1)-(E7.4.3). By [9, Section 1.2.12], $\mathcal{P} \cong \mathcal{P}ois_{\geq 1}$. By the fact that

$$\dim \mathcal{P}(n) = \dim \mathcal{P}ois(n) = n! = \dim \mathcal{A}ss(n) = \dim \mathrm{trc}\mathcal{A}ss(n)$$

for all $n \geq 1$ [26, Section 13.3.3], we have $(\mathrm{trc}\mathcal{A}ss)_{\geq 1} = \mathcal{P}$, which is isomorphic to the Poisson operad $\mathcal{P}ois_{\geq 1}$. Therefore we obtain that $(\mathrm{trc}\mathcal{A}ss)_{\geq 1} = \mathcal{P}ois_{\geq 1}$. It remains to verify that 0-ary operations of $\mathrm{trc}\mathcal{A}ss$ and $\mathcal{P}ois$ agree. We can easily see that, in $\mathrm{trc}\mathcal{A}ss$,

$$\bar{1}_2 \underset{i}{\circ} \bar{1}_0 = \bar{1}_1$$

and

$$\bar{\Phi}_2 \underset{i}{\circ} \bar{1}_0 = 0$$

for $i = 1, 2$. This is also how we define the unitary Poisson operad $\mathcal{P}ois$. This finishes the proof. $\square$

Remark 7.5. We make some comments about the above lemma.

(1) The result in Lemma 7.4 may be well-known, possibly in a different language. Similar ideas appeared in [23,5,27].
(2) By Livernet-Loday [23], $\mathcal{A}ss$ is a deformation of $\mathcal{P}ois$, in the sense that there is a family of operads, denoted by $\mathcal{LL}_q$, such that $\mathcal{P}ois \cong \mathcal{LL}_0$ and that $\mathcal{A}ss \cong \mathcal{LL}_q$ for any $q \neq 0$. Further study in this direction can be found in [5,27] and [26, Section 13.3.4]. Lemma 7.4 gives an explanation why $\mathcal{A}ss$ is a deformation of $\mathcal{P}ois$. We refer to [27, Example 4 and Theorem 5] for some interesting connections with deformation quantization.
(3) Related to combinatorics, the dimension of ${}^{k}\Upsilon(k)$ of either $\mathcal{A}ss$ and $\mathcal{P}ois$ is the number of derangements of a set of size k.
(4) It would be interesting to determine associated truncatified operads of other unitary operads.

8. Appendix

In this part, we mainly rewrite some conventions and facts on operads, see [26] or [9,10].

8.1. Symmetric groups, permutations and block permutations

We use $\mathbb{S}_n$ to denote the symmetric group, namely, the set of bijections, on the set $[n]$. Note that both $\mathbb{S}_0$ and $\mathbb{S}_1$ are isomorphic to the trivial group with one element.

Following convention in the book [26], we identify $\mathbb{S}_n$ with the set of permutations of $[n]$ by assigning each $\sigma \in \mathbb{S}_n$ the sequence $(\sigma^{-1}(1), \sigma^{-1}(2), \cdots, \sigma^{-1}(n))$. This assignment is convenient when we use other convention such as (E2.1.3). Equivalently, each permutation $(i_1, i_2, \cdots, i_n)$ of $[n]$ corresponds to the $\sigma \in \mathbb{S}_n$ given by $\sigma(i_k) = k$ for all $1 \leq k \leq n$.

Let $n > 0$, $k_1, k_2 \cdots, k_n \geq 0$ be integers. For simplicity we write $m = k_1 + k_2 + \cdots + k_n$, $m_1 = 0$, and $m_i = k_1 + \cdots + k_{i-1}$ for $2 \leq i \leq n$. We may divide $(1, 2, \cdots, m)$ into n-blocks $(B_1, B_2, \cdots, B_n)$, where $B_i = (m_i + 1, \cdots, m_i + k_i)$ for $1 \leq i \leq n$. Now each $\mathbb{S}_{k_i}$ acts on the block B_i, and each element in $\mathbb{S}_n$ acts on $[m]$ naturally by permuting the blocks. More precisely, we have the following natural map

$$\begin{aligned} \vartheta_{n;k_1,\cdots,k_n} \colon \mathbb{S}_n \times \mathbb{S}_{k_1} \times \cdots \times \mathbb{S}_{k_n} &\to \mathbb{S}_m, \\ (\sigma, \sigma_1, \cdots, \sigma_n) &\mapsto (\tilde{B}_{\sigma^{-1}(1)}, \cdots, \tilde{B}_{\sigma^{-1}(n)}) \end{aligned}$$

for all $\sigma \in \mathbb{S}_n$ and $\sigma_i \in \mathbb{S}_{k_i}$ for $1 \leq i \leq n$, where each

$$\tilde{B}_i = m_i + (\sigma_i^{-1}(1), \cdots, \sigma_i^{-1}(k_i)) = (m_i + \sigma_i^{-1}(1), \cdots, m_i + \sigma_i^{-1}(k_i)) \tag{E8.0.1}$$

is the sequence corresponding to σ_i.

The following lemma is easy.

Lemma 8.1. *Retain the above notation.*

(1) *We have*

$$\begin{aligned} &\vartheta_{n;k_1,\cdots,k_n}(\tau\sigma, \tau_1\sigma_1, \cdots, \tau_n\sigma_n) \\ &\quad = \vartheta_{n;k_{\sigma^{-1}(1)},\cdots,k_{\sigma^{-1}(n)}}(\tau, \tau_{\sigma^{-1}(1)}, \cdots, \tau_{\sigma^{-1}(n)}) \vartheta_{n;k_1,\cdots,k_n}(\sigma, \sigma_1, \cdots, \sigma_n) \end{aligned} \tag{E8.1.1}$$

for all $\sigma, \tau \in \mathbb{S}_n$, and $\sigma_i, \tau_i \in \mathbb{S}_{k_i}$, $1 \leq i \leq n$.

(2) *In particular,*

$$\begin{aligned} &\vartheta_{n;k_1,\cdots,k_n}(\sigma, \sigma_1, \cdots, \sigma_n) \\ &\quad = \vartheta_{n;k_1,\cdots,k_n}(\sigma, 1, \cdots, 1) \vartheta_{n;k_1,\cdots,k_n}(1, \sigma_1, \cdots, \sigma_n) \\ &\quad = \vartheta_{n;k_{\sigma^{-1}(1)},\cdots,k_{\sigma^{-1}(n)}}(1, \sigma_{\sigma^{-1}(1)}, \cdots, \sigma_{\sigma^{-1}(n)}) \vartheta_{n;k_1,\cdots,k_n}(\sigma, 1, \cdots, 1) \end{aligned} \tag{E8.1.2}$$

where 1 in different positions represents the identity map of $[k_i]$ or $[n]$.

Proof. We first prove part (2). For any $\sigma \in \mathbb{S}_n$, $\sigma_i \in \mathbb{S}_{k_i}$, $1 \leq i \leq n$, using notation in (E8.0.1), we have

$$\begin{aligned}
\vartheta_{n;k_1,\cdots,k_n}(\sigma, 1, \cdots, 1)\vartheta_{n;k_1,\cdots,k_n}(1, \sigma_1, \cdots, \sigma_n)&\\
&= \vartheta_{n;k_1,\cdots,k_n}(\sigma, 1, \cdots, 1)(\tilde{B}_1, \cdots, \tilde{B}_n)\\
&= (\tilde{B}_{\sigma^{-1}(1)}, \cdots, \tilde{B}_{\sigma^{-1}(n)})\\
&= \vartheta_{n;k_1,\cdots,k_n}(\sigma, \sigma_1, \cdots, \sigma_n),
\end{aligned}$$

and

$$\begin{aligned}
\vartheta_{n;k_{\sigma^{-1}(1)},\cdots,k_{\sigma^{-1}(n)}}(1, \sigma_{\sigma^{-1}(1)}, \cdots, \sigma_{\sigma^{-1}(n)})\vartheta_{n;k_1,\cdots,k_n}(\sigma, 1, \cdots, 1)&\\
&= \vartheta_{n;k_{\sigma^{-1}(1)},\cdots,k_{\sigma^{-1}(n)}}(1, \sigma_{\sigma^{-1}(1)}, \cdots, \sigma_{\sigma^{-1}(n)})(B_{\sigma^{-1}(1)}, \cdots, B_{\sigma^{-1}(n)})\\
&= (\tilde{B}_{\sigma^{-1}(1)}, \cdots, \tilde{B}_{\sigma^{-1}(n)})\\
&= \vartheta_{n;k_1,\cdots,k_n}(\sigma, \sigma_1, \cdots, \sigma_n).
\end{aligned}$$

For part (1), by part (2), we have

$$\begin{aligned}
&\vartheta_{n;k_1,\cdots,k_n}(\tau\sigma, \tau_1\sigma_1, \cdots, \tau_n\sigma_n)\\
&\quad= \vartheta_{n;k_1,\cdots,k_n}(\tau\sigma, 1, \cdots, 1)\vartheta_{n;k_1,\cdots,k_n}(1, \tau_1\sigma_1, \cdots, \tau_n\sigma_n)\\
&\quad= \vartheta_{n;k_{\sigma^{-1}(1)},\cdots,k_{\sigma^{-1}(n)}}(\tau, 1, \cdots, 1)\vartheta_{n;k_1,\cdots,k_n}(\sigma, 1, \cdots, 1)\\
&\qquad\quad \vartheta_{n;k_1,\cdots,k_n}(1, \tau_1, \cdots, \tau_n)\vartheta_{n;k_1,\cdots,k_n}(1, \sigma_1, \cdots, \sigma_n)\\
&\quad= \vartheta_{n;k_{\sigma^{-1}(1)},\cdots,k_{\sigma^{-1}(n)}}(\tau, 1, \cdots, 1)\vartheta_{n;k_{\sigma^{-1}(1)},\cdots,k_{\sigma^{-1}(n)}}(1, \tau_{\sigma^{-1}(1)}, \cdots, \tau_{\sigma^{-1}(n)})\\
&\qquad\quad \vartheta_{n;k_1,\cdots,k_n}(\sigma, 1, \cdots, 1)\vartheta_{n;k_1,\cdots,k_n}(1, \sigma_1, \cdots, \sigma_n)\\
&\quad= \vartheta_{n;k_{\sigma^{-1}(1)},\cdots,k_{\sigma^{-1}(n)}}(\tau, \tau_{\sigma^{-1}(1)}, \cdots, \tau_{\sigma^{-1}(n)})\vartheta_{n;k_1,\cdots,k_n}(\sigma, \sigma_1, \cdots, \sigma_n)
\end{aligned}$$

for all $\tau, \sigma \in \mathbb{S}_n$, $\tau_i, \sigma_i \in \mathbb{S}_{k_i}$, $1 \leq i \leq n$. $\square$

For convenience, we introduce the following maps obtained from $\vartheta_{n;k_1,\cdots,k_n}$:

$$\begin{aligned}
&\vartheta_{k_1,\cdots,k_n} : \mathbb{S}_n \to \mathbb{S}_m, \sigma \mapsto \vartheta_{n;k_1,\cdots,k_n}(\sigma, 1, \cdots, 1),\\
&\vartheta^i_{k_1,\cdots,k_n} : \mathbb{S}_{k_i} \to \mathbb{S}_m, \sigma_i \mapsto \vartheta_{n;k_1,\cdots,k_n}(1, 1, \cdots, 1, \sigma_i, 1, \cdots, 1).
\end{aligned} \tag{E8.1.3}$$

Note that $\mathbb{S}_{k_1} \times \cdots \times \mathbb{S}_{k_n}$ can be viewed as a subgroup of $\mathbb{S}_m$ via the embedding maps $\vartheta^i_{k_1,\cdots,k_n}$. While in general, $\vartheta_{k_1,\cdots,k_n}$ is not an embedding of groups. It is the case if and only if all blocks have the same size, that is, $k_1 = k_2 = \cdots = k_n$.

8.2. Multilinear maps, compositions and symmetric group action

Let V be a vector space and $n > 0$ an integer. Denote by $V^{\otimes n}$ the tensor space $V \otimes V \otimes \cdots \otimes V$ with n factors. For any $v_1, \cdots, v_n \in V$, we simply denote $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ by $[v_1, v_2, \cdots, v_n]$. Let $\mathbf{B} \subset V$ be a $\Bbbk$-linear basis of V, then $V^{\otimes n}$ has a $\Bbbk$-linear basis

$$\{[v_1, v_2, \cdots, v_n] \mid v_i \in \mathbf{B},\ 1 \le i \le n\}.$$

For consistency, we set $V^{\otimes 0} = \Bbbk$, and denote by $[\,]$ a fixed basis element of $V^{\otimes 0}$. Under the map $[v_1, \cdots, v_i] \otimes [\,] \otimes [v_{i+1}, \cdots, v_{i+j}] \mapsto [v_1, \cdots, v_{i+j}]$, we may identify $V^{\otimes i} \otimes V^{\otimes 0} \otimes V^{\otimes j}$ with $V^{\otimes (i+j)}$.

For each $n \ge 0$, let $\mathcal{E}nd_V(n)$ denote the $\Bbbk$-vector space $\mathrm{Hom}_{\Bbbk}(V^{\otimes n}, V)$ of multilinear operators on V. Clearly, $\mathcal{E}nd_V(0) \cong V$ under the mapping $f \mapsto f([\,])$.

It is standard that $\mathbb{S}_n$ acts on $V^{\otimes n}$ on the left by permuting the factors, more precisely

$$\sigma \cdot [x_1, x_2, \cdots, x_n] = [x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \cdots, x_{\sigma^{-1}(n)}] \tag{E8.1.4}$$

for all $\sigma \in \mathbb{S}_n$, and $x_1, x_2 \cdots, x_n \in V$ [26, p. xxiv and p. 164]. This convention could be different from the one used by some researchers. This action induces a right action of $\mathbb{S}_n$ on $\mathcal{E}nd_V(n)$ by

$$(f * \sigma)(X) = f(\sigma X)$$

for all $\sigma \in \mathbb{S}_n$, $f \in \mathcal{E}nd_V(n)$ and $X \in V^{\otimes n}$. Here $*$ denotes the (right) $\mathbb{S}_n$-action.

Consider the composition map

$$\begin{aligned} \circ\colon \mathcal{E}nd_V(n) \otimes \mathcal{E}nd_V(k_1) \otimes \cdots \otimes \mathcal{E}nd_V(k_n) &\longrightarrow \mathcal{E}nd_V(k_1 + \cdots + k_n), \\ (f, f_1, \cdots, f_n) \mapsto f \circ (f_1, \cdots, f_n) &:= f \bullet (f_1 \otimes \cdots \otimes f_n), \end{aligned} \tag{E8.1.5}$$

where

$$\begin{aligned} &f \bullet (f_1 \otimes \cdots \otimes f_n)([x_{1,1}, \cdots, x_{1,k_1}, \cdots, x_{n,1}, \cdots, x_{n,k_n}]) \\ &\qquad = f(f_1([x_{1,1}, \cdots, x_{1,k_1}]) \otimes \cdots \otimes f_n([x_{n,1}, \cdots, x_{n,k_n}])), \end{aligned} \tag{E8.1.6}$$

for all $f \in \mathcal{E}nd_V(n)$, $f_i \in \mathcal{E}nd_V(k_i)$ and $x_{ij} \in V$. Here $\bullet$ denotes an ordinary composition of two functions and $\circ$ denotes the composition map of an operad. The composition map $\circ$ is compatible with the symmetric group actions. The following is clear.

Lemma 8.2. *Keep the above notation. Then*

$$\begin{aligned} &(f * \sigma) \circ (f_1 * \tau_1, \cdots, f_n * \tau_n) \\ =&(f \circ (f_{\sigma^{-1}(1)}, \cdots, f_{\sigma^{-1}(n)})) * \vartheta_{n;k_1,\cdots,k_n}(\sigma, \tau_1, \cdots, \tau_n) \end{aligned} \tag{E8.2.1}$$

for all $\sigma \in \mathbb{S}_n$, and $\tau_i \in \mathbb{S}_{k_i}$, $1 \le i \le n$.

Proof. We write $m_1 = 0$, $m_i = k_1 + \cdots + k_{i-1}$ for $2 \le i \le n$, and $m = k_1 + \cdots + k_n$. Then

$$\begin{aligned}
&(f * \sigma)\circ(f_1 * \tau_1, \cdots, f_n * \tau_n)[x_1, \cdots, x_n] \\
&\quad = (f * \sigma)([(f_1 * \tau_1)([x_1, \cdots, x_{k_1}]), \cdots, (f_n * \tau_n)([x_{m_n+1}, \cdots, x_{m_n+k_n}])]) \\
&\quad = f([(f_{\sigma^{-1}(1)} * \tau_{\sigma^{-1}(1)})([x_{m_{\sigma^{-1}(1)}+1}, \cdots, x_{m_{\sigma^{-1}(1)}+k_{\sigma^{-1}(1)}}]), \cdots, \\
&\qquad (f_{\sigma^{-1}(n)} * \tau_{\sigma^{-1}(n)})([x_{m_{\sigma^{-1}(n)}+1}, \cdots, x_{m_{\sigma^{-1}(n)}+k_{\sigma^{-1}(n)}}])]) \\
&\quad = f([f_{\sigma^{-1}(1)}([x_{m_{\sigma^{-1}(1)}+\tau^{-1}_{\sigma^{-1}(1)}(1)}, \cdots, x_{m_{\sigma^{-1}(1)}+\tau^{-1}_{\sigma^{-1}(1)}(k_{\sigma^{-1}(1)})}]), \cdots, \\
&\qquad f_{\sigma^{-1}(n)}([x_{m_{\sigma^{-1}(n)}+\tau^{-1}_{\sigma^{-1}(n)}(1)}, \cdots, x_{m_{\sigma^{-1}(n)}+\tau^{-1}_{\sigma^{-1}(n)}(k_{\sigma^{-1}(n)})}])]) \\
&\quad = ((f \circ (f_{\sigma^{-1}(1)}, \cdots, f_{\sigma^{-1}(n)})) * \vartheta_{n;k_1,\cdots,k_n}(\sigma, \tau_1, \cdots, \tau_n))([x_1, \cdots, x_m])
\end{aligned}$$

This completes the proof. □

Moreover, denote by $\mathbb{1} \in \mathcal{E}nd_V(1)$ the identity map on V. Clearly, the composition $\circ$ satisfies the following coherence axioms:

(1) (Identity)

$$f \circ (\mathbb{1}, \mathbb{1}, \cdots, \mathbb{1}) = f = \mathbb{1} \circ f;$$

(2) (Associativity)

$$\begin{aligned}
&f \circ (f_1 \circ (f_{1,1}, \cdots, f_{1,k_1}), \cdots, f_n \circ (f_{n,1}, \cdots, f_{n,k_n})) \\
=&(f \circ (f_1, \cdots, f_n)) \circ (f_{1,1}, \cdots, f_{1,k_1}, \cdots, f_{n,1}, \cdots, f_{n,k_n}).
\end{aligned}$$

8.3. Associative algebras and the operad $\mathcal{A}ss$

Recall that an *associative algebra* (over $\mathbb{k}$) is a $\mathbb{k}$-vector space A equipped with a binary operation,

$$\mu\colon A \otimes A \to A, \qquad \mu(a, b) = ab$$

satisfying the associative law $\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu)$. If moreover, there exists a linear map $u\colon \mathbb{k} \to A$ such that $\mu \circ (u \otimes \mathrm{id}_A) = \mathrm{id}_A = \mu \circ (\mathrm{id}_A \otimes u)$, then A is said to be *unital*.

The famous operad $\mathcal{A}ss$ encodes the category of unital associative algebras, namely, unital associative algebras are exactly $\mathcal{A}ss$-algebras. Recall that, for each $n \ge 0$, $\mathcal{A}ss(n) = \mathbb{k}\mathbb{S}_n$ as a right $\mathbb{S}_n$-module, and the composition $\circ$ is given by

$$\sigma \circ (\sigma_1, \cdots, \sigma_n) = \vartheta_{n;k_1,\cdots,k_n}(\sigma, \sigma_1, \cdots, \sigma_n)$$

for all $n > 0$, $k_1, \cdots, k_n \geq 0$, and $\sigma \in \mathbb{S}_n$ and $\sigma_i \in \mathbb{S}_{k_i}$ for $1 \leq i \leq n$. It is direct to verify that $\mathcal{A}ss$ is an operad with the identity $\mathbb{1}_1 := 1_{\mathbb{S}_1} \in \mathcal{A}ss(1)$ [26, Section 9.1.3]. From now on, we denote $1_{\mathbb{S}_n}$ by 1_n (or $\mathbb{1}_n$) for short for all $n \geq 0$.

Let (A, γ) be an $\mathcal{A}ss$-algebra. Clearly $\mu := \gamma(1_2)$ gives a binary operator on A, which is associative since

$$1_2 \circ (1_2, 1_1) = 1_3 = 1_2 \circ (1_1, 1_2). \tag{E8.2.2}$$

Moreover, 1_0 gives a linear map $u := \gamma(1_0)\colon \mathbb{k} \to A$, and the fact that

$$1_2 \circ (1_0, 1_1) = 1_1 = 1_2 \circ (1_1, 1_0) \tag{E8.2.3}$$

means that u is the unit map of A. Thus (A, μ, u) is a unital algebra. Conversely, for every unital associative algebra (A, μ, u), we may define $\gamma\colon \mathcal{A}ss \to \mathcal{E}nd_A$ as follows. By definition, $\gamma(1_0) = u$, and for each $n > 0$ and each $\sigma \in \mathbb{S}_n$, $\gamma(\sigma)$ is given by

$$\gamma(\sigma)\colon A^{\otimes n} \to A, \quad a_1 \otimes \cdots \otimes a_n \mapsto a_{\sigma^{-1}(1)} a_{\sigma^{-1}(2)} \cdots a_{\sigma^{-1}(n)}$$

for all $a_1, \cdots, a_n \in A$, where the right hand side in the above formula means the multiplication in A. It is direct to check that γ is a morphism of operads (these are standard facts in the book [26]).

8.4. Magmatic algebras and the operad $\mathcal{M}ag$

Recall that a *magmatic algebra* is a vector space equipped with a binary operad $\nu\colon A \otimes A \to A$ with no relation. If moreover, there exists a linear map $u\colon \mathbb{k} \to A$, such that

$$\nu \circ (u \otimes \mathrm{id}_A) = \mathrm{id}_A = \nu \circ (\mathrm{id}_A \otimes u),$$

then A is said to be *unital*.

The operad $\mathcal{M}ag$ encodes the category of unital magmatic algebras. In fact, the operad $\mathcal{M}ag$ is the operad generated by the $\mathbb{S}$-module $(\mathbb{k}u, \mathbb{k}\mathbb{1}, \mathbb{k}\mathbb{S}_2\nu, 0, 0, \cdots)$ with relations

$$\nu \underset{i}{\circ} u = \mathbb{1}, \quad (i = 1, 2),$$

where $\mathbb{k}\mathbb{S}_2\nu$ is the regular $\mathbb{k}\mathbb{S}_2$-module with the basis ν. To be precise, $\mathcal{M}ag(0) = \mathbb{k}u$, $\mathcal{M}ag(1) = \mathbb{k}\mathbb{1}$, and for each $n \geq 2$,

$$\mathcal{M}ag(n) = \mathbb{k}[PBT_n] \otimes_{\mathbb{k}} \mathbb{k}\mathbb{S}_n,$$

where PBT_n is the set of planar binary rooted trees with n leaves, and the partial composition is given by the grafting of the trees, see [26, Appendix C.1.1] for details. In this paper we use $\mathbb{1}_0$ for u and $\mathbb{1}_2$ for v as in the proof of Lemma 2.1.

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