

# A PROBABILISTIC VARIANT OF SPERNER'S THEOREM AND OF MAXIMAL $r$ -COVER FREE FAMILIES

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ABSTRACT. A family of sets is called  $r$ -cover free if no set in the family is contained in the union of  $r$  (or less) other sets in the family. A 1-cover free family is simply an antichain with respect to set inclusion. Thus, Sperner's classical result determines the maximal cardinality of a 1-cover free family of subsets of an  $n$ -element set. Estimating the maximal cardinality of an  $r$ -cover free family of subsets of an  $n$ -element set for  $r > 1$  was also studied. In this note we are interested in the following probabilistic variant of this problem. Let  $S_0, S_1, \dots, S_r$  be independent and identically distributed random subsets of an  $n$ -element set. Which distribution minimizes the probability that  $S_0 \subseteq \bigcup_{i=1}^r S_i$ ? A natural candidate is the uniform distribution on an  $r$ -cover-free family of maximal cardinality. We show that for  $r = 1$  such distribution is indeed best possible. In a complete contrast, we also show that this is far from being true for every  $r > 1$  and  $n$  large enough.

## 1. INTRODUCTION

For every positive integer  $n$ , let  $\Omega_n$  be the set of all subsets of some fixed  $n$ -element set. For a positive integer  $r$ , a family  $\mathcal{F} \subseteq \Omega_n$  is called  $r$ -cover free if no set in  $\mathcal{F}$  is contained in the union of  $r$  (or less) other sets in  $\mathcal{F}$ . Let us denote by  $g_r(n)$  the maximal cardinality of an  $r$ -cover free family in  $\Omega_n$ . A 1-cover free family in  $\Omega_n$  is just an antichain in  $\Omega_n$ , with respect to set inclusion. Hence  $g_1(n) = \binom{n}{\lfloor n/2 \rfloor}$ , by the classical result of Sperner ([7]). For  $r = 2$  it was shown in [2] that  $1.134^n < g_2(n) < O(\sqrt{n}) \left(\frac{5}{4}\right)^n$  and in the subsequent paper [3], the same authors showed that for every  $r$ ,

$$(1) \quad \left(1 + \frac{1}{4r^2}\right)^n < g_r(n) \leq \sum_{k=1}^n \frac{\binom{n}{\lceil k/r \rceil}}{\binom{k-1}{\lceil k/r \rceil-1}}.$$

A different upper bound, which is better for large  $r$ , was obtained in [1]. In [6], this bound was given a simpler proof and the following, more explicit, form: for every  $r \geq 2$  and  $n$  large enough,

$$(2) \quad g_r(n) \leq r^{8n/r^2}.$$

We will now describe a probabilistic variant of  $r$ -cover free families of maximal cardinality. Let  $\mathcal{P}_n := \{p : \Omega_n \rightarrow [0, \infty) : \sum_{A \in \Omega_n} p(A) = 1\}$  be the family of probability distributions on  $\Omega_n$ . For a positive integer  $r$  and  $p \in \mathcal{P}_n$ , let  $\tau_r(p)$  be the probability that  $S_0 \subseteq \bigcup_{i=1}^r S_i$ , where  $S_0, S_1, \dots, S_r$  are random sets, drawn independently from  $\Omega_n$  according to the distribution  $p$ . Natural candidates to minimize  $\tau_r$  are distributions in the set  $\mathcal{CF}_{n,r} := \{p \in \mathcal{P}_n : p \text{ is supported on an } r\text{-cover free family}\}$  (in which case, one only has to worry about choosing the same set twice).

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Clearly,  $\min_{p \in \mathcal{CF}_{n,1}} \tau_1(p) = \frac{1}{\binom{n}{\lfloor n/2 \rfloor}}$  where the minimum is attained for any distribution which is uniformly supported on a maximal antichain in  $\Omega_n$ . Our first result is that for  $n \geq 2$  this is indeed the minimum of  $\tau_1$  over all  $\mathcal{P}_n$ .

**Theorem 1.** *Suppose that  $n \geq 2$ . Then  $\tau_1(p) \geq \frac{1}{\binom{n}{\lfloor n/2 \rfloor}}$  for every  $p \in \mathcal{P}_n$  and consequently,  $\min_{p \in \mathcal{P}_n} \tau_1(p) = \min_{p \in \mathcal{CF}_{n,1}} \tau_1(p)$ .*

We note that the weaker statement that  $\Pr(S_0 \subseteq S_1 \text{ or } S_0 \supseteq S_1) \geq \frac{1}{\binom{n}{\lfloor n/2 \rfloor}}$  for every independent identically distributed random sets  $S_0, S_1$  in  $\Omega_n$ , readily follows from the fact that  $\Omega_n$  may be covered by  $\binom{n}{\lfloor n/2 \rfloor}$  chains (with respect to set inclusion). This symmetric version of Theorem 1 may be generalized as follows. For a property  $P$  of families of sets, let  $ex(n, P)$  denote the maximum possible cardinality of a family of sets in  $\Omega_n$  satisfying  $P$  and let  $ex(n, k, P)$ , for  $0 \leq k \leq n$ , denote the maximum possible cardinality of a family of  $k$ -element sets in  $\Omega_n$  satisfying  $P$ . Thus, for example, if  $P_1$  is the property of being an antichain then  $ex(n, P_1) = \binom{n}{\lfloor n/2 \rfloor}$  by Sperner's Theorem, if  $P_2$  is the property of being an intersecting family and  $n \geq 2k$  then  $ex(n, k, P_2) = \binom{n-1}{k-1}$  by the Erdős-Ko-Rado Theorem [4], and if  $P_3$  is the property of not containing two sets whose symmetric difference has cardinality smaller than  $d$ , then  $ex(n, P_3)$  is the maximum possible cardinality of an error correcting code with length  $n$  and minimum distance  $d$ . Similarly,  $ex(n, k, P)$  is the maximum cardinality of the corresponding constant weight code.

**Theorem 2.** *Let  $\mathcal{H}$  be a family of unordered pairs of distinct sets in  $\Omega_n$  and let  $P_{\mathcal{H}}$  be the property of containing no pair from  $\mathcal{H}$ . For  $p \in \mathcal{P}_n$ , let  $\tau_{\mathcal{H}}(p) := \Pr(\{S_0, S_1\} \in \mathcal{H} \text{ or } S_0 = S_1)$ , where  $S_0, S_1$  are random sets, drawn independently from  $\Omega_n$  according to the distribution  $p$ . Then  $\min_{p \in \mathcal{P}_n} \tau_{\mathcal{H}}(p) = \frac{1}{ex(n, P_{\mathcal{H}})}$ . Similarly, for every  $0 \leq k \leq n$ , the minimum of  $\tau_{\mathcal{H}}(p)$  over distributions  $\mathcal{P}_n$  whose support is a subset of  $\{A \in \Omega_n : |A| = k\}$  is  $\frac{1}{ex(n, k, P_{\mathcal{H}})}$ .*

The examples mentioned above provide several specific applications of the theorem, and it is not difficult to describe others.

In a complete contrast to Theorem 1, we show that for every  $r > 1$  (and  $n$  large enough), the minimum of  $\tau_r$  on  $\mathcal{P}_n$  is much smaller than the minimum of  $\tau_r$  over  $\mathcal{CF}_{n,r}$ . For every  $0 \leq \ell \leq n$ , let  $p_{\ell}$  be the probability distribution in  $\mathcal{P}_n$  uniformly supported on the family of all  $\ell$ -element sets in  $\Omega_n$ .

**Theorem 3.** *Suppose that  $r \geq 2$ . There is  $0 < \mu_r < 1$  such that for every  $n$  large enough,  $\min_{0 < \ell < \frac{n}{r}} \tau_r(p_{\ell}) < \mu_r^n \min_{p \in \mathcal{CF}_{n,r}} \tau_r(p)$  and consequently,  $\min_{p \in \mathcal{P}_n} \tau_r(p) < \mu_r^n \min_{p \in \mathcal{CF}_{n,r}} \tau_r(p)$ .*

For every  $r \geq 2$ , Theorem 3 shows that  $\min_{p \in \mathcal{P}_n} \tau_r(p)$  is (much) smaller than  $\min_{p \in \mathcal{CF}_{n,r}} \tau_r(p)$ , which is at most  $1 - \left(1 - \frac{1}{g_r(n)}\right)^r < \frac{r}{g_r(n)}$ , as shown by considering any probability distribution uniformly supported on an  $r$ -cover free family of maximal cardinality. A lower bound for  $\min_{p \in \mathcal{P}_n} \tau_r(p)$  is given in the following theorem.

**Theorem 4.** *Suppose that  $r \geq 2$ . There is  $C_r > 0$  such that  $\min_{p \in \mathcal{P}_n} \tau_r(p) \geq \frac{C_r}{(g_r(n))^r}$  and hence, for  $n$  large enough, by (2),  $\min_{p \in \mathcal{P}_n} \tau_r(p) \geq \frac{C_r}{r^{8n/r}}$ .*

We prove Theorems 1 and 2 in Section 2 and Theorems 3 and 4 in Section 3.

2. THE CASE  $r = 1$ 

*Proof of Theorem 1.* Let  $p \in \mathcal{P}_n$ . Let  $\mathcal{C}$  be the set of all maximal chains in  $\Omega_n$ , with respect to set inclusion. Every  $A \in \Omega_n$  belongs to exactly  $\frac{|\mathcal{C}|}{\binom{n}{|A|}}$  maximal chains. Therefore,  $\frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \sum_{A \in C} \binom{n}{|A|} p(A) = \sum_{A \in \Omega_n} p(A) = 1$  and since  $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$  for every  $0 \leq k \leq n$ ,

$$(3) \quad \binom{n}{\lfloor \frac{n}{2} \rfloor} \sum_{A \in \Omega_n} p(A)^2 \geq \sum_{A \in \Omega_n} \binom{n}{|A|} p(A)^2 = \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \sum_{A \in C} \binom{n}{|A|}^2 p(A)^2.$$

Similarly, every pair  $A_0 \subsetneq A_1$  of sets in  $\Omega_n$  belong to exactly  $\frac{|\mathcal{C}|}{\binom{n}{|A_1|} \binom{|A_1|}{|A_0|}}$  maximal chains.

Therefore, since  $\binom{k}{\ell} \leq \frac{1}{2} \binom{n}{\lfloor \frac{n}{2} \rfloor}$  for every  $0 \leq k < \ell \leq n$ ,

$$(4) \quad \begin{aligned} \binom{n}{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{(A_0, A_1) \in \Omega_n^2 \\ A_0 \subsetneq A_1}} p(A_0)p(A_1) &\geq 2 \sum_{\substack{(A_0, A_1) \in \Omega_n^2 \\ A_0 \subsetneq A_1}} \frac{\binom{n}{|A_0|}}{\binom{|A_1|}{|A_0|}} p(A_0)p(A_1) \\ &= \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \sum_{\substack{(A_0, A_1) \in C^2 \\ A_0 \neq A_1}} \binom{n}{|A_0|} \binom{n}{|A_1|} p(A_0)p(A_1). \end{aligned}$$

Summing up (3) and (4) yields

$$\begin{aligned} \binom{n}{\lfloor \frac{n}{2} \rfloor} \tau_1(p) &= \binom{n}{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{(A_0, A_1) \in \Omega_n^2 \\ A_0 \subseteq A_1}} p(A_0)p(A_1) \geq \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \sum_{(A_0, A_1) \in C^2} \binom{n}{|A_0|} \binom{n}{|A_1|} p(A_0)p(A_1) \\ &= \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \left( \sum_{A \in C} \binom{n}{|A|} p(A) \right)^2 \geq \left( \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \sum_{A \in C} \binom{n}{|A|} p(A) \right)^2 = 1. \end{aligned}$$

as claimed.  $\square$

*Proof of Theorem 2.* Let  $G$  be the complement of the graph  $(\Omega_n, \mathcal{H})$ . The size of the maximum clique in  $G$  is clearly  $ex(n, P_{\mathcal{H}})$ . Therefore, by a theorem of Motzkin and Straus [5, Theorem 1],

$$\min_{p \in \mathcal{P}_n} \tau_{\mathcal{H}}(p_{\mathcal{H}}) = 1 - 2 \max_{p \in \mathcal{P}_n} \sum_{\{A_0, A_1\} \text{ is an edge of } G} p(A_0)p(A_1) = \frac{1}{ex(n, P_{\mathcal{H}})}.$$

The second statement follows similarly, by considering the graph induced by  $G$  on the vertex set  $\{A \in \Omega_n : |A| = k\}$ .  $\square$

3. THE CASE  $r > 1$ 

Note that if  $p \in \mathcal{P}_n$  is supported on an  $r$ -cover free family  $\mathcal{F}$ , then

$$1 - \tau_r(p) = \sum_{F \in \mathcal{F}} p(F) (1 - p(F))^r \leq \sum_{F \in \mathcal{F}} p(F) (1 - p(F)) \leq 1 - \frac{1}{|\mathcal{F}|},$$

and hence  $\min_{p \in \mathcal{CF}_{n,r}} \tau_r(p) \geq \frac{1}{g_r(n)}$ . Therefore, to prove Theorem 3 for some  $r \geq 2$ , it is enough to show that there is  $0 < \mu_r < 1$  such that for  $n$  large enough,

$$(5) \quad \min_{0 < \ell < \frac{n}{r}} \tau_r(p_\ell) < \mu_r^n \frac{1}{g_r(n)}.$$

For large  $r$  this may be easily deduced as follows. For  $\ell := \lfloor \frac{n}{er} \rfloor$ , clearly

$$\tau_r(p_\ell) \leq \frac{\binom{r\ell}{\ell}}{\binom{n}{\ell}} \leq \left( \frac{r\ell}{n} \right)^\ell \leq \frac{1}{e^\ell} < e \frac{1}{e^{\frac{n}{er}}} = e \left( e^{-\frac{1}{e}} r^{\frac{8}{r}} \right)^{\frac{n}{r}} \frac{1}{r^{\frac{8n}{r^2}}}.$$

Therefore, by (2), for  $n$  large enough

$$(6) \quad \min_{0 < \ell < \frac{n}{r}} \tau_r(p_\ell) < e \left( e^{-\frac{1}{e}} r^{\frac{8}{r}} \right)^{\frac{n}{r}} \frac{1}{r^{\frac{8n}{r^2}}} < e \left( e^{-\frac{1}{e}} r^{\frac{8}{r}} \right)^{\frac{n}{r}} \frac{1}{g_r(n)}.$$

It can be verified that  $e^{-\frac{1}{e}} r^{\frac{8}{r}} < 1$  for every  $r \geq 101$ . Thus, (6) confirms (5), and hence Theorem 3, for  $r \geq 101$ . We proceed to describe the proof Theorem 3 for general  $r \geq 2$ .

*Proof of Theorem 3.* Let  $\ell$  be an integer in the interval  $[0, \frac{n}{r})$  for which  $\binom{n}{\ell+1}/\binom{r\ell}{\ell}$  is maximal. It is simple to verify that if  $n$  is large enough, then the sequence  $\left( \binom{n}{j+1}/\binom{rj}{j} \right)_{j=0}^{\lfloor n/4r \rfloor + 1}$  is increasing and hence  $\ell > \frac{n}{4r}$ .

Let  $S_0, S_1, \dots, S_r$  be random sets chosen, independently and uniformly, from all the  $\ell$ -element sets in  $\Omega_n$ .

Let  $t := \lfloor \ell^2/n \rfloor$  and let  $\mathcal{E}$  be the event:  $|\bigcup_{i=1}^r S_i| > r\ell - t$ . It is easy to verify that the sequence  $(\Pr(S_1 \cup S_2 = k))_{k=2\ell-t}^{2\ell}$  is decreasing, and hence

$$\Pr(\mathcal{E}) \leq \Pr(|S_1 \cup S_2| > 2\ell - t) \leq t \Pr(|S_1 \cup S_2| = 2\ell - t) = t \frac{\binom{n-\ell}{\ell-t} \binom{\ell}{t}}{\binom{n}{\ell}}.$$

Therefore, by (1),

$$\begin{aligned} \tau_r(p_\ell) &= \Pr \left( S_0 \subseteq \bigcup_{i=1}^r S_i \right) \\ &= \Pr(\mathcal{E}) \Pr \left( S_0 \subseteq \bigcup_{i=1}^r S_i \mid \mathcal{E} \right) + \Pr(\Omega_n \setminus \mathcal{E}) \Pr \left( S_0 \subseteq \bigcup_{i=1}^r S_i \mid \Omega_n \setminus \mathcal{E} \right) \\ &\leq t \frac{\binom{n-\ell}{\ell-t} \binom{\ell}{t}}{\binom{n}{\ell}} \cdot \frac{\binom{r\ell}{\ell}}{\binom{n}{\ell}} + 1 \cdot \frac{\binom{r\ell-t}{\ell}}{\binom{n}{\ell}} = \left( t \frac{\binom{n-\ell}{\ell-t} \binom{\ell}{t}}{\binom{n}{\ell}} + \frac{\binom{r\ell-t}{\ell}}{\binom{n}{\ell}} \right) \frac{n-\ell}{\ell+1} \cdot \frac{\binom{r\ell}{\ell}}{\binom{n}{\ell+1}} \\ &\leq \left( t \frac{\binom{n-\ell}{\ell-t} \binom{\ell}{t}}{\binom{n}{\ell}} + \frac{\binom{r\ell-t}{\ell}}{\binom{n}{\ell}} \right) \frac{(n-\ell)n}{\ell+1} \cdot \frac{1}{g_r(n)}, \end{aligned}$$

and (5) follows by using standard estimates on binomial coefficients. This completes the proof of the theorem.  $\square$

Finally, we prove Theorem 4.

*Proof of Theorem 4.* Let  $p \in \mathcal{P}_n$ , let  $N := 2g_r(n)$ , let  $S_1, \dots, S_N$  be random sets, drawn independently from  $\Omega_n$  according to the distribution  $p$ , and consider the random variable

$$I := \{i \in [N] : \text{there is } J \subset [N] \setminus \{i\} \text{ of cardinality } r \text{ such that } S_i \subseteq \bigcup_{j \in J} S_j\}.$$

The family  $\{S_i\}_{i \in [N] \setminus I}$  is clearly  $r$ -cover free, therefore  $N - |I| = |[N] \setminus I| \leq g_r(n)$  and hence  $\mathbb{E}|I| \geq N - g_r(n) = g_r(n)$ . On the other hand, clearly  $\mathbb{E}|I| \leq N \binom{N-1}{r} \tau_r(p)$ . Hence

$$\tau_r(p) \geq \frac{g_r(n)}{N \binom{N-1}{r}} \geq \frac{r! g_r(n)}{N^{r+1}} = \frac{r!}{2^{r+1} g_r(n)^r}$$

and the result follows.  $\square$

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