

A PROBABILISTIC VARIANT OF SPERNER'S THEOREM AND OF MAXIMAL r -COVER FREE FAMILIES

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ABSTRACT. A family of sets is called r -cover free if no set in the family is contained in the union of r (or less) other sets in the family. A 1-cover free family is simply an antichain with respect to set inclusion. Thus, Sperner's classical result determines the maximal cardinality of a 1-cover free family of subsets of an n -element set. Estimating the maximal cardinality of an r -cover free family of subsets of an n -element set for $r > 1$ was also studied. In this note we are interested in the following probabilistic variant of this problem. Let S_0, S_1, \dots, S_r be independent and identically distributed random subsets of an n -element set. Which distribution minimizes the probability that $S_0 \subseteq \bigcup_{i=1}^r S_i$? A natural candidate is the uniform distribution on an r -cover-free family of maximal cardinality. We show that for $r = 1$ such distribution is indeed best possible. In a complete contrast, we also show that this is far from being true for every $r > 1$ and n large enough.

1. INTRODUCTION

For every positive integer n , let Ω_n be the set of all subsets of some fixed n -element set. For a positive integer r , a family $\mathcal{F} \subseteq \Omega_n$ is called r -cover free if no set in \mathcal{F} is contained in the union of r (or less) other sets in \mathcal{F} . Let us denote by $g_r(n)$ the maximal cardinality of an r -cover free family in Ω_n . A 1-cover free family in Ω_n is just an antichain in Ω_n , with respect to set inclusion. Hence $g_1(n) = \binom{n}{\lfloor n/2 \rfloor}$, by the classical result of Sperner ([7]). For $r = 2$ it was shown in [2] that $1.134^n < g_2(n) < O(\sqrt{n}) \left(\frac{5}{4}\right)^n$ and in the subsequent paper [3], the same authors showed that for every r ,

$$(1) \quad \left(1 + \frac{1}{4r^2}\right)^n < g_r(n) \leq \sum_{k=1}^n \frac{\binom{n}{\lceil k/r \rceil}}{\binom{k-1}{\lceil k/r \rceil - 1}}.$$

A different upper bound, which is better for large r , was obtained in [1]. In [6], this bound was given a simpler proof and the following, more explicit, form: for every $r \geq 2$ and n large enough,

$$(2) \quad g_r(n) \leq r^{8n/r^2}.$$

We will now describe a probabilistic variant of r -cover free families of maximal cardinality. Let $\mathcal{P}_n := \{p : \Omega_n \rightarrow [0, \infty) : \sum_{A \in \Omega_n} p(A) = 1\}$ be the family of probability distributions on Ω_n . For a positive integer r and $p \in \mathcal{P}_n$, let $\tau_r(p)$ be the probability that $S_0 \subseteq \bigcup_{i=1}^r S_i$, where S_0, S_1, \dots, S_r are random sets, drawn independently from Ω_n according to the distribution p . Natural candidates to minimize τ_r are distributions in the set $\mathcal{CF}_{n,r} := \{p \in \mathcal{P}_n : p \text{ is supported on an } r\text{-cover free family}\}$ (in which case, one only has to worry about choosing the same set twice).

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Clearly, $\min_{p \in \mathcal{CF}_{n,1}} \tau_1(p) = \frac{1}{\binom{n}{\lfloor n/2 \rfloor}}$ where the minimum is attained for any distribution which is uniformly supported on a maximal antichain in Ω_n . Our first result is that for $n \geq 2$ this is indeed the minimum of τ_1 over all \mathcal{P}_n .

Theorem 1. *Suppose that $n \geq 2$. Then $\tau_1(p) \geq \frac{1}{\binom{n}{\lfloor n/2 \rfloor}}$ for every $p \in \mathcal{P}_n$ and consequently, $\min_{p \in \mathcal{P}_n} \tau_1(p) = \min_{p \in \mathcal{CF}_{n,1}} \tau_1(p)$.*

We note that the weaker statement that $\Pr(S_0 \subseteq S_1 \text{ or } S_0 \supseteq S_1) \geq \frac{1}{\binom{n}{\lfloor n/2 \rfloor}}$ for every independent identically distributed random sets S_0, S_1 in Ω_n , readily follows from the fact that Ω_n may be covered by $\binom{n}{\lfloor n/2 \rfloor}$ chains (with respect to set inclusion). This symmetric version of Theorem 1 may be generalized as follows. For a property P of families of sets, let $ex(n, P)$ denote the maximum possible cardinality of a family of sets in Ω_n satisfying P and let $ex(n, k, P)$, for $0 \leq k \leq n$, denote the maximum possible cardinality of a family of k -element sets in Ω_n satisfying P . Thus, for example, if P_1 is the property of being an antichain then $ex(n, P_1) = \binom{n}{\lfloor n/2 \rfloor}$ by Sperner's Theorem, if P_2 is the property of being an intersecting family and $n \geq 2k$ then $ex(n, k, P_2) = \binom{n-1}{k-1}$ by the Erdős-Ko-Rado Theorem [4], and if P_3 is the property of not containing two sets whose symmetric difference has cardinality smaller than d , then $ex(n, P_3)$ is the maximum possible cardinality of an error correcting code with length n and minimum distance d . Similarly, $ex(n, k, P)$ is the maximum cardinality of the corresponding constant weight code.

Theorem 2. *Let \mathcal{H} be a family of unordered pairs of distinct sets in Ω_n and let $P_{\mathcal{H}}$ be the property of containing no pair from \mathcal{H} . For $p \in \mathcal{P}_n$, let $\tau_{\mathcal{H}}(p) := \Pr(\{S_0, S_1\} \in \mathcal{H} \text{ or } S_0 = S_1)$, where S_0, S_1 are random sets, drawn independently from Ω_n according to the distribution p . Then $\min_{p \in \mathcal{P}_n} \tau_{\mathcal{H}}(p) = \frac{1}{ex(n, P_{\mathcal{H}})}$. Similarly, for every $0 \leq k \leq n$, the minimum of $\tau_{\mathcal{H}}(p)$ over distributions \mathcal{P}_n whose support is a subset of $\{A \in \Omega_n : |A| = k\}$ is $\frac{1}{ex(n, k, P_{\mathcal{H}})}$.*

The examples mentioned above provide several specific applications of the theorem, and it is not difficult to describe others.

In a complete contrast to Theorem 1, we show that for every $r > 1$ (and n large enough), the minimum of τ_r on \mathcal{P}_n is much smaller than the minimum of τ_r over $\mathcal{CF}_{n,r}$. For every $0 \leq \ell \leq n$, let p_{ℓ} be the probability distribution in \mathcal{P}_n uniformly supported on the family of all ℓ -element sets in Ω_n .

Theorem 3. *Suppose that $r \geq 2$. There is $0 < \mu_r < 1$ such that for every n large enough, $\min_{0 < \ell < \frac{n}{r}} \tau_r(p_{\ell}) < \mu_r^n \min_{p \in \mathcal{CF}_{n,r}} \tau_r(p)$ and consequently, $\min_{p \in \mathcal{P}_n} \tau_r(p) < \mu_r^n \min_{p \in \mathcal{CF}_{n,r}} \tau_r(p)$.*

For every $r \geq 2$, Theorem 3 shows that $\min_{p \in \mathcal{P}_n} \tau_r(p)$ is (much) smaller than $\min_{p \in \mathcal{CF}_{n,r}} \tau_r(p)$, which is at most $1 - \left(1 - \frac{1}{g_r(n)}\right)^r < \frac{r}{g_r(n)}$, as shown by considering any probability distribution uniformly supported on an r -cover free family of maximal cardinality. A lower bound for $\min_{p \in \mathcal{P}_n} \tau_r(p)$ is given in the following theorem.

Theorem 4. *Suppose that $r \geq 2$. There is $C_r > 0$ such that $\min_{p \in \mathcal{P}_n} \tau_r(p) \geq \frac{C_r}{(g_r(n))^r}$ and hence, for n large enough, by (2), $\min_{p \in \mathcal{P}_n} \tau_r(p) \geq \frac{C_r}{r^{8n/r}}$.*

We prove Theorems 1 and 2 in Section 2 and Theorems 3 and 4 in Section 3.

2. THE CASE $r = 1$

Proof of Theorem 1. Let $p \in \mathcal{P}_n$. Let \mathcal{C} be the set of all maximal chains in Ω_n , with respect to set inclusion. Every $A \in \Omega_n$ belongs to exactly $\frac{|\mathcal{C}|}{\binom{n}{|A|}}$ maximal chains. Therefore, $\frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \sum_{A \in C} \binom{n}{|A|} p(A) = \sum_{A \in \Omega_n} p(A) = 1$ and since $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$ for every $0 \leq k \leq n$,

$$(3) \quad \binom{n}{\lfloor \frac{n}{2} \rfloor} \sum_{A \in \Omega_n} p(A)^2 \geq \sum_{A \in \Omega_n} \binom{n}{|A|} p(A)^2 = \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \sum_{A \in C} \binom{n}{|A|}^2 p(A)^2.$$

Similarly, every pair $A_0 \subsetneq A_1$ of sets in Ω_n belong to exactly $\frac{|\mathcal{C}|}{\binom{n}{|A_1|} \binom{|A_1|}{|A_0|}}$ maximal chains.

Therefore, since $\frac{\binom{n}{k}}{\binom{n}{\ell}} \leq \frac{1}{2} \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for every $0 \leq k < \ell \leq n$,

$$(4) \quad \begin{aligned} \binom{n}{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{(A_0, A_1) \in \Omega_n^2 \\ A_0 \subsetneq A_1}} p(A_0) p(A_1) &\geq 2 \sum_{\substack{(A_0, A_1) \in \Omega_n^2 \\ A_0 \subsetneq A_1}} \frac{\binom{n}{|A_0|}}{\binom{|A_1|}{|A_0|}} p(A_0) p(A_1) \\ &= \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \sum_{\substack{(A_0, A_1) \in C^2 \\ A_0 \neq A_1}} \binom{n}{|A_0|} \binom{n}{|A_1|} p(A_0) p(A_1). \end{aligned}$$

Summing up (3) and (4) yields

$$\begin{aligned} \binom{n}{\lfloor \frac{n}{2} \rfloor} \tau_1(p) &= \binom{n}{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{(A_0, A_1) \in \Omega_n^2 \\ A_0 \subsetneq A_1}} p(A_0) p(A_1) \geq \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \sum_{(A_0, A_1) \in C^2} \binom{n}{|A_0|} \binom{n}{|A_1|} p(A_0) p(A_1) \\ &= \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \left(\sum_{A \in C} \binom{n}{|A|} p(A) \right)^2 \geq \left(\frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \sum_{A \in C} \binom{n}{|A|} p(A) \right)^2 = 1. \end{aligned}$$

as claimed. \square

Proof of Theorem 2. Let G be the complement of the graph (Ω_n, \mathcal{H}) . The size of the maximum clique in G is clearly $ex(n, P_{\mathcal{H}})$. Therefore, by a theorem of Motzkin and Straus [5, Theorem 1],

$$\min_{p \in \mathcal{P}_n} \tau_{\mathcal{H}}(p) = 1 - 2 \max_{\substack{p \in \mathcal{P}_n \\ \{A_0, A_1\} \text{ is an edge of } G}} \sum p(A_0) p(A_1) = \frac{1}{ex(n, P_{\mathcal{H}})}.$$

The second statement follows similarly, by considering the graph induced by G on the vertex set $\{A \in \Omega_n : |A| = k\}$. \square

3. THE CASE $r > 1$

Note that if $p \in \mathcal{P}_n$ is supported on an r -cover free family \mathcal{F} , then

$$1 - \tau_r(p) = \sum_{F \in \mathcal{F}} p(F) (1 - p(F))^r \leq \sum_{F \in \mathcal{F}} p(F) (1 - p(F)) \leq 1 - \frac{1}{|\mathcal{F}|},$$

and hence $\min_{p \in \mathcal{CF}_{n,r}} \tau_r(p) \geq \frac{1}{g_r(n)}$. Therefore, to prove Theorem 3 for some $r \geq 2$, it is enough to show that there is $0 < \mu_r < 1$ such that for n large enough,

$$(5) \quad \min_{0 < \ell < \frac{n}{r}} \tau_r(p_\ell) < \mu_r^n \frac{1}{g_r(n)}.$$

For large r this may be easily deduced as follows. For $\ell := \lfloor \frac{n}{er} \rfloor$, clearly

$$\tau_r(p_\ell) \leq \frac{\binom{r\ell}{\ell}}{\binom{n}{\ell}} \leq \left(\frac{r\ell}{n} \right)^\ell \leq \frac{1}{e^\ell} < e^{\frac{1}{e^\ell}} = e \left(e^{-\frac{1}{e}} r^{\frac{8}{r}} \right)^{\frac{n}{r}} \frac{1}{r^{\frac{8n}{r^2}}}.$$

Therefore, by (2), for n large enough

$$(6) \quad \min_{0 < \ell < \frac{n}{r}} \tau_r(p_\ell) < e \left(e^{-\frac{1}{e}} r^{\frac{8}{r}} \right)^{\frac{n}{r}} \frac{1}{r^{\frac{8n}{r^2}}} < e \left(e^{-\frac{1}{e}} r^{\frac{8}{r}} \right)^{\frac{n}{r}} \frac{1}{g_r(n)}.$$

It can be verified that $e^{-\frac{1}{e}} r^{\frac{8}{r}} < 1$ for every $r \geq 101$. Thus, (6) confirms (5), and hence Theorem 3, for $r \geq 101$. We proceed to describe the proof Theorem 3 for general $r \geq 2$.

Proof of Theorem 3. Let ℓ be an integer in the interval $[0, \frac{n}{r})$ for which $\binom{n}{\ell+1} / \binom{r\ell}{\ell}$ is maximal. It is simple to verify that if n is large enough, then the sequence $\left(\binom{n}{j+1} / \binom{rj}{j} \right)_{j=0}^{\lfloor n/4r \rfloor + 1}$ is increasing and hence $\ell > \frac{n}{4r}$.

Let S_0, S_1, \dots, S_r be random sets chosen, independently and uniformly, from all the ℓ -element sets in Ω_n .

Let $t := \lfloor \ell^2/n \rfloor$ and let \mathcal{E} be the event: $|\bigcup_{i=1}^r S_i| > r\ell - t$. It is easy to verify that the sequence $(\Pr(S_1 \cup S_2 = k))_{k=2\ell-t}^{2\ell}$ is decreasing, and hence

$$\Pr(\mathcal{E}) \leq \Pr(|S_1 \cup S_2| > 2\ell - t) \leq t \Pr(|S_1 \cup S_2| = 2\ell - t) = t \frac{\binom{n-\ell}{\ell-t} \binom{\ell}{t}}{\binom{n}{\ell}}.$$

Therefore, by (1),

$$\begin{aligned} \tau_r(p_\ell) &= \Pr \left(S_0 \subseteq \bigcup_{i=1}^r S_i \right) \\ &= \Pr(\mathcal{E}) \Pr \left(S_0 \subseteq \bigcup_{i=1}^r S_i \mid \mathcal{E} \right) + \Pr(\Omega_n \setminus \mathcal{E}) \Pr \left(S_0 \subseteq \bigcup_{i=1}^r S_i \mid \Omega_n \setminus \mathcal{E} \right) \\ &\leq t \frac{\binom{n-\ell}{\ell-t} \binom{\ell}{t}}{\binom{n}{\ell}} \cdot \frac{\binom{r\ell}{\ell}}{\binom{n}{\ell}} + 1 \cdot \frac{\binom{r\ell-t}{\ell}}{\binom{n}{\ell}} = \left(t \frac{\binom{n-\ell}{\ell-t} \binom{\ell}{t}}{\binom{n}{\ell}} + \frac{\binom{r\ell-t}{\ell}}{\binom{r\ell}{\ell}} \right) \frac{n-\ell}{\ell+1} \cdot \frac{\binom{r\ell}{\ell}}{\binom{n}{\ell+1}} \\ &\leq \left(t \frac{\binom{n-\ell}{\ell-t} \binom{\ell}{t}}{\binom{n}{\ell}} + \frac{\binom{r\ell-t}{\ell}}{\binom{r\ell}{\ell}} \right) \frac{(n-\ell)n}{\ell+1} \cdot \frac{1}{g_r(n)}, \end{aligned}$$

and (5) follows by using standard estimates on binomial coefficients. This completes the proof of the theorem. \square

Finally, we prove Theorem 4.

Proof of Theorem 4. Let $p \in \mathcal{P}_n$, let $N := 2g_r(n)$, let S_1, \dots, S_N be random sets, drawn independently from Ω_n according to the distribution p , and consider the random variable

$$I := \{i \in [N] : \text{there is } J \subset [N] \setminus \{i\} \text{ of cardinality } r \text{ such that } S_i \subseteq \bigcup_{j \in J} S_j\}.$$

The family $\{S_i\}_{i \in [N] \setminus I}$ is clearly r -cover free, therefore $N - |I| = |[N] \setminus I| \leq g_r(n)$ and hence $\mathbb{E}|I| \geq N - g_r(n) = g_r(n)$. On the other hand, clearly $\mathbb{E}|I| \leq N \binom{N-1}{r} \tau_r(p)$. Hence

$$\tau_r(p) \geq \frac{g_r(n)}{N \binom{N-1}{r}} \geq \frac{r! g_r(n)}{N^{r+1}} = \frac{r!}{2^{r+1} g_r(n)^r}$$

and the result follows. \square

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