List Ramsey numbers

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Abstract

We introduce a list colouring extension of classical Ramsey numbers. We investigate when the two Ramsey numbers are equal, and in general, how far apart they can be from each other. We find graph sequences where the two are equal and where they are far apart. For ℓ-uniform cliques we prove that the list Ramsey number is bounded by an exponential function, while it is well-known that the Ramsey number is super-exponential for uniformity at least 3. This is in great contrast to the graph case where we cannot even decide the question of equality for cliques.

1 Introduction

The notion of proper colouring and the corresponding parameter of the chromatic number is one of the most applicable and widely-studied topics in (hyper)graph theory. In some of these applications the list-colouring extension of the notion is necessary to describe the situation appropriately. A *colouring* of a hypergraph $H = (V, E)$ is a function $c : V \to \mathbb{N}$. A colouring is called *proper* if no hyperedge $e \in E$ is monochromatic. For an assignment $L : V \to 2^{\mathbb{N}}$ of a subset $L_v \subseteq \mathbb{N}$ of colours to each vertex $v \in V$, we call a colouring $c : V \to \mathbb{N}$ an *L-colouring* if $c(v) \in L_v$ for every $v \in V$. When $L_v = [k]$ for every $v \in V$, an L-colouring is called a *k-colouring*.

The chromatic number $\chi(H)$ is the smallest integer k such that there exists a proper k-colouring of H and the list-chromatic number (or choice number) $\chi_\ell(H)$ is the smallest integer k such that for every assignment L of lists of size k to the vertices of H there is a proper L-colouring. By definition $\chi(H) \leq \chi_\ell(H)$ for every graph H. Under what circumstances are the two parameters equal and how far they can be from each other? These fundamental questions are the subject of vigorous research, see, e.g., [8], Chapter 14 and the references therein. A notorious open question in this direction is the List Colouring Conjecture suggested independently by various researchers including Vizing, Albertson, Collins, Tucker and Gupta, which appeared first in print in the paper of Bollobás and Harris [7] and states that the list-chromatic number is equal to the chromatic number for line-graphs. This conjecture was proved by Galvin [18] for bipartite graphs, by Häggkvist and Janssen [21] for cliques of odd order, by Alon and Tarsi [1] for cubic bridgeless planar graphs, by Ellingham and Goddyn [14] for regular class-1 planar multigraphs and by Kahn [22] asymptotically, but is very much open in general. Even

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for cliques K_n of even order it is not known whether the list-chromatic number of its line graph is n or $n-1$.

A particularly interesting instance of hypergraph colouring arises from Ramsey theory, which is concerned with the proper colouring of very specific hypergraphs. Ramsey's Theorem states that for any r-uniform hypergraph (or r-graph) G and number k of colours any k-colouring of the r-subsets of $[n]$ contains a monochromatic copy of the hypergraph G, provided n is large enough depending on G and k. The smallest such integer n is usually called the k-colour Ramsey number of the hypergraph G.

Definition. The k-colour (ordinary) *Ramsey number* of an r-graph G is defined as

$$R(G,k) := \min\{n \mid \forall k\text{-colouring of } E(K_n^{(r)}),\ \exists \text{ a monochromatic copy of } G\}.$$

The study of Ramsey numbers has attracted a lot of attention over the years and many natural generalisations and extensions of Ramsey numbers were considered, for excellent surveys see [19], [12] and the references therein. In this paper we study a new variant, a list colouring version of the Ramsey problem. In particular, when is it possible to assign lists of size k to the edges of $K_n^{(r)}$ in such a way that if we colour each edge with a colour from its list we can always find a monochromatic copy of a given graph. If we require all lists to be the same we recover the ordinary Ramsey number. This gives rise to the following list-colouring variant of the Ramsey number.

Definition. The k-colour *list Ramsey number* of an r-uniform hypergraph G is defined by

$$R_\ell(G,k) = \min\{n \mid \exists L : E(K_n^{(r)}) \to \binom{\mathbb{N}}{k} \text{ s.t. } \forall\ L\text{-colouring of } E(K_n^{(r)})\ \exists \text{ a monochromatic copy of } G\}.$$

A first observation, immediate from the definition, is that for every G and k, we have

$$R_\ell(G,k) \le R(G,k). \qquad (1)$$

In our paper we will be investigating when this inequality is an equality and, more generally, when the two quantities are close to each other and when they are far apart, how far apart can they be. This question for specific families of graphs turns out to be related to several long standing open problems such as the aforementioned list colouring conjecture, we give the details in the following subsections.

Remark. Notion of the list Ramsey number was suggested at https://mathoverflow.net/questions/298778/list-ramsey-numbers, where some basic observations were made, as well as a conjecture, which we disprove, that inequality (1) is actually always an equality.

1.1 Results

1.1.1 Stars

Any edge-colouring of a graph contains no monochromatic copy of $K_{1,2}$ if and only if it is proper. Therefore the k-colour Ramsey number (list Ramsey number) of $K_{1,2}$ is equal to the smallest number n such that $\chi'(K_n) > k$ ($\chi'_\ell(K_n) > k$, respectively), where here $\chi'(G)$ denotes the edge chromatic number of G which can be defined as the chromatic number of its line graph and similarly for χ'_ℓ. Hence the question whether the two Ramsey numbers of $K_{1,2}$ are equal for an arbitrary number k of colours is essentially equivalent to the aforementioned List Colouring Conjecture for cliques. It was proved by Häggkvist and Janssen that $\chi'_\ell(K_n) \le n$ for every n, which implies that the list chromatic index $\chi'_\ell(K_n)$ is equal to the chromatic index $\chi'(K_n)$ for odd n. The question whether $\chi'_\ell(K_n)$ is equal to $\chi'(K_n)$ for even n is still open. Consequently we know that $R_\ell(K_{1,2}, k) = k + 1 = R(K_{1,2}, k)$ when k is even, but we do not know whether $R_\ell(K_{1,2}, k)$ is $k + 1$ or $k + 2$ when k is odd.

The multicolour Ramsey number for stars of arbitrary size was determined by Burr and Roberts [9]. They showed that

$$(r-1)k + 1 \le R(K_{1,r}, k) \le (r-1)k + 2, \tag{2}$$

and that the lower bound is tight if and only if both r and k are even.

In our first theorem we extend the validity of the lower bound to the list Ramsey number, thus establishing that the lower bound is tight when both r and k are even. Furthermore, we show for any fixed number k of colours, that for large enough r the upper bound is tight.

Theorem 1. *For any k and $r \in \mathbb{N}$, except possibly finitely many integers r for each odd k, we have $R_\ell(K_{1,r}, k) = R(K_{1,r}, k)$. More precisely,*

(a) For every $r, k \in \mathbb{N}$, we have

$$(r-1)k + 1 \le R_\ell(K_{1,r}, k). \tag{3}$$

In particular, $R_\ell(K_{1,r}, k) = (r-1)k + 1 = R(K_{1,r}, k)$ whenever both r and k are even.

(b) For every $k \in \mathbb{N}$ there exists $w(k) \in \mathbb{N}$ such that the following holds. For every k and $r \ge w(k)$ that are not both even, we have

$$R_\ell(K_{1,r}, k) = (r-1)k + 2 = R(K_{1,r}, k).$$

Our theorem fails to give a full characterisation of the tightness of the lower bound in (3). For two colours we can give such a characterisation and find that the two Ramsey numbers are always equal.

Theorem 2. *For every $r \in \mathbb{N}$ we have*

$$R_\ell(K_{1,r}, 2) = R(K_{1,r}, 2) = \begin{cases} 2r - 1 & \text{if } r \text{ is even} \\ 2r & \text{if } r \text{ is odd.} \end{cases}$$

1.1.2 Matchings

We saw above that for stars the two Ramsey numbers are equal, possibly up to an additive constant one. Next we consider matchings and find that, unlike for stars, the ordinary Ramsey number is significantly larger than the list Ramsey number for most values of the parameters.

Ramsey numbers of matchings were determined in 1975 by Cockayne and Lorimer [10]. They showed that for every $r, k \in \mathbb{N}$,

$$R(rK_2, k) = rk + r - k + 1. \tag{4}$$

A trivial lower bound on the list Ramsey number $R_\ell(rK_2, k)$ is $2r$: if we were to find a matching of size r in K_n, monochromatic or not, then n better be at least the number of vertices in rK_2. It turns out that if the number k of colours is not too large compared to r, then this trivial lower bound is asymptotically tight! That is, even if n is just slightly larger than $2r$, there exists an assignment of lists of size k to the edges of K_n, such that any list-colouring of the edges contains a monochromatic rK_2 (i.e., an almost perfect matching which is monochromatic). Note that by (4), using the same k

colours on each edge one *can* colour a much larger clique without a monochromatic rK_2. In particular we show that for any fixed number k of colours the two Ramsey numbers are a constant factor $\frac{k+1}{2}$ away from each other asymptotically, as r tends to infinity.

The number k of colours becomes more visible in the value of the list Ramsey number once k is larger than a logarithmic function of the size r of the matching. In particular for any fixed r, we determine the growth rate of the k-colour list Ramsey number up to an absolute constant factor and find that the ratio of the two Ramsey numbers grows as $\Theta(\log k)$.

Theorem 3. *For any fixed $k \geq 2$ and r tending to infinity, we have $R_\ell(rK_2, k) = 2r + o(r)$. In particular*

$$\frac{R(rK_2, k)}{R_\ell(rK_2, k)} = \frac{k+1}{2} + o(1).$$

For any fixed $r \geq 1$ and k tending to infinity, we have $R_\ell(rK_2, k) = \Theta(k/\log k)$. In particular

$$\frac{R(rK_2, k)}{R_\ell(rK_2, k)} = \Theta(\log k).$$

In fact we determine the list Ramsey number of matchings for all values of r and k up to a constant factor and when r is sufficiently bigger than k even up to an additive lower order term. For more details see Subsection 2.2.

1.1.3 Cliques and Hypergraphs

Some of the most famous open problems in Ramsey theory involve cliques. The proofs of the classic probabilistic lower bounds on $R(K_r, 2)$ all go through in the list chromatic setting, hence

$$2^{r/2} < R_\ell(K_r, 2) \leq R(K_r, 2) < 2^{2r}.$$

Not unexpectedly, we cannot improve on the lower bound. It is not difficult to see that $R_\ell(K_3, 2) = 6 = R(K_3, 2)$, but for $r > 3$ we cannot even decide the equality of the two Ramsey numbers of K_r when $k = 2$.

For hypergraphs of uniformity $\ell \geq 3$ however, we are able to show an exponential (or even larger, depending on the uniformity) separation between the ordinary and the list Ramsey numbers. On the one hand it is known via the stepping-up lemma of Erdős and Hajnal (see Chapter 4.7 of [19]) that the Ramsey numbers of cliques are super-polynomial in the exponent whenever $\ell \geq 4$ or $\ell = 3, k \geq 3$ (Conlon, Fox, Sudakov [11] for $k = \ell = 3$) and in fact grow at least as fast as a tower of height $\ell - 2$. For the list Ramsey number on the other hand we can show that for fixed uniformity and number of colours it is upper bounded by an exponential in a polynomial in r.

Theorem 4. *For arbitrary positive integers $r \geq \ell$ and $k \in \mathbb{N}$ we have*

$$R_\ell(K_r^{(\ell)}, k) \leq 2^{4r^{3\ell-1}+4kr^{\ell-1}\log_2 r}.$$

This theorem obviously provides an upper bound on the list Ramsey number of any fixed ℓ-graph H, which is an exponential function of k. For a growing number of colours the base of the exponent can be strengthened. In order to state our result, we need to introduce a few standard parameters. Let $\mathrm{ex}(H, n)$ denote the maximum number of edges in an H-free ℓ-graph on n vertices and let $\pi(H) =$

$\lim_{n\to\infty} \mathrm{ex}(H,n)/\binom{n}{\ell}$. Assuming H has at least 2 edges let

$$m(H) = \max_{H'\subset H, e(H')>1} \frac{e(H')-1}{v(H')-\ell}.$$

Theorem 5. *Let H be an ℓ-uniform hypergraph. Then, as k tends to infinity, we have*

$$R_\ell(H,k) \le (1-\pi(H)+o(1))^{-km(H)}.$$

For the particular case of k-colour list Ramsey number of the triangle the theorem gives the exponential upper bound $R_\ell(K_3,k) \le (4+o(1))^k$.

The behaviour of the ordinary k-colour Ramsey number $R(K_3,k)$ is related to other open problems, most notably the question if the maximum possible Shannon capacity of a graph with independence number 2 is finite, see [16], [2]. It is one of the notorious open problems of combinatorics to decide whether its growth rate is exponential or superexponential. Erdős offers \$100 for its resolution and \$250 for the determination of the limit $\lim\limits_{k\to\infty} \sqrt[k]{R(K_3,k)}$ provided it exists. The current best lower bound is $R(K_3,k) \ge 3.199^k$ (see [24]), so not large enough for us to conclude that the ordinary and the list Ramsey numbers are different.

For the list Ramsey number we can only give a much weaker lower bound, where the exponent is the square root of the number of colours.

Theorem 6. *If H is an ℓ-uniform hypergraph with $\chi(H) > r$, then we have*

$$R_\ell(H,k) \ge e^{\sqrt{k\log r/(4\ell)}}.$$

In particular $R_\ell(K_3,k) > e^{\sqrt{k}/4}$.

Note that this theorem gives a lower bound exponential in the square root of k for every non-2-colourable ℓ-graph H. Our argument extends to every non-ℓ-partite ℓ-graph, even if they are 2-colourable, with a somewhat worse constant factor in the exponent.

Theorem 7. *Let H be an ℓ-uniform hypergraph which is not ℓ-partite. We have*

$$R_\ell(H,k) \ge e^{c_\ell\sqrt{k}},$$

where $1/c_\ell = 2\ell e^{\ell/2}$.

Our proof method works most efficiently when the ordinary Ramsey number of H is small. It is known that the multicolour Ramsey number of an ℓ-graph H is polynomial in k if and only if H is ℓ-partite. For them we determine the list Ramsey number up to a poly-logarithmic factor.

Theorem 8. *Let H be an ℓ-partite ℓ-uniform hypergraph with parts of size at most r. There is a constant $c = c(r,\ell)$ such that*

$$R(H, \lfloor ck/\log k\rfloor) \le R_\ell(H,k) \le R(H,k).$$

In particular, if $\mathrm{ex}(H,n) = \tilde{\Theta}(n^{\ell-\varepsilon(H)})$,[1] *for some $\varepsilon(H) > 0$, then*

$$R_\ell(H,k) = \tilde{\Theta}(R(H,k)) = \tilde{\Theta}(k^{1/\varepsilon(H)}).$$

[1] Here $f = \tilde{\Theta}(g)$ means, as usual, that f and g are equal up to polylogarithmic factors.

This theorem can be considered an extension of the second part of Theorem 3, where we determine that the ordinary and the list Ramsey numbers of matchings are exactly a $\log k$ factor away from each other. For several bipartite graphs (for example for complete bipartite graphs $K_{r,s}$ for $s > (r-1)!$, even cycles C_6 and C_{10} or general trees) the asymptotic behaviour of the ordinary Ramsey number is known up to a polylogarithmic factor and hence by Theorem 8 so is the list Ramsey number.

The rest of this paper is organised as follows. In Subsection 2.1 we prove our results for stars. In Subsection 2.2 we prove the results for matchings, demonstrating on a relatively simple example the methods we are going to use in Subsection 2.3 to prove the bounds for list Ramsey numbers of general graphs. In Section 3 we give concluding remarks and present some open problems. All our logarithms are natural unless explicitly indicated otherwise.

2 Bounds for list Ramsey numbers

2.1 Stars

Let us start with a few preliminaries and tools which we will use throughout this subsection.

Theorem 9 (Galvin [18])**.** *If G is a bipartite graph of maximal degree Δ then $\chi'_\ell(G) = \Delta$.*

To show $R_\ell(G,k) > n$ we need to show that for any assignment of lists of size k to the edges of K_n we can choose the colours from the lists in such a way that we create no monochromatic copy of G. We distinguish two cases depending on parity. The following simple observation will enable us to give lower bounds on $R_\ell(K_{1,r},k)$.

Lemma 10. *Let us assume that graphs $G_1,\dots,G_t$ partition the edge set of K_n. If $\chi'_\ell(G_i) \le k$ for all i and each vertex belongs to at most $r-1$ of the G_i's then $R_\ell(K_{1,r},k) > n$.*

Proof. Let L be an assignment of lists of size k to the edges of K_n. By the assumption that $\chi'_\ell(G_i) \le k$ there is a proper L-colouring c_i of each G_i. Let us define an L-colouring c of $E(K_n)$ by $c(e) = c_i(e)$, where i is the index of the unique G_i containing e. Note that since any vertex v belongs to at most $r-1$ G_i's we know that edges incident to v are using colours from at most $r-1$ c_i's. Since each c_i is proper this means that for any fixed colour v is incident to at most $r-1$ edges of this colour, showing there can be no monochromatic $K_{1,r}$ under c as desired. □

We begin with the case of 2 colours, by proving Theorem 2.

Theorem 2. *For every $r \in \mathbb{N}$ we have*

$$R_\ell(K_{1,r},2) = R(K_{1,r},2) = \begin{cases} 2r-1 & \text{if } r \text{ is even} \\ 2r & \text{if } r \text{ is odd.} \end{cases}$$

Proof. It is well known that the standard Ramsey number satisfies the same equalities, [9]. So by (1) we only have to show the corresponding lower bounds.

Case 1: r even.

We will make use of the following fact proved by Alspach and Gavlas [3].

Proposition. *Let n be an even integer. K_n can be partitioned into a single perfect matching and Hamilton cycles.*

Let $n = 2r - 2$. By the above proposition we can partition K_n into a perfect matching G_1 and $r-2$ Hamilton cycles $G_2, \dots, G_{r-1}$. By Galvin's theorem [18] we know that $\chi'_\ell(G_i) \leq 2$ and each vertex belongs to exactly $r-1$ of the G_i's so by Lemma 10 we are done.

Case 2: r odd.

In this case we make use of a different partitioning result of Alspach and Gavlas [3].

Proposition. *Let n be an odd integer and m an integer satisfying $4 \leq m \leq n$. K_n can be partitioned into cycles of length m if and only if m divides the number of edges of K_n.*

Let $n = 2r - 1$. Let us first assume that $r \geq 5$. Since $|E(K_n)| = n(n-1)/2 = (2r-1)(r-1)$ by the above result we can partition K_n into cycles of length $r-1$. Since r is odd these cycles are bipartite and have $\chi'_\ell(C_{r-1}) = 2$. As they are 2-regular and partition $E(K_n)$ we know that each vertex belongs to exactly $r-1$ of these cycles. Therefore, we are done by Lemma 10.

The case $r = 1$ is immediate, so we are left with the case $r = 3$. Let L be an assignment of lists of size 2 to the edges of K_5. Partition K_5 into two 5-cycles C_1, C_2. If we can properly colour both C_1 and C_2 using colours from the lists we are done. It is well-known and easy to see that the only way in which a 5-cycle does not admit a 2-colouring from its lists is if the lists are all the same. Therefore, we may assume that edges of one cycle, say C_1 have the same lists. We now colour all edges of C_1 using a single colour c from their list and colour all edges of C_2 by arbitrary colours in their lists which differ from c. In this colouring there is no monochromatic $K_{1,3}$ as desired. $\square$

Let us now consider the case of more colours. As in the case of 2-colours all our upper bounds come from the ordinary Ramsey numbers, which were determined by Burr and Roberts in [9] and the trivial inequality (1). The following two lemmas establish the two lower bounds claimed in Theorem 1, completing its proof.

Lemma 11.

$$(r-1)k + 1 \leq R_\ell(K_{1,r}, k).$$

Proof. Let $n = (r-1)k$, partition the vertices of K_n into sets $V_1, \dots, V_{r-1}$, each of size k. We let G_i be the subgraph induced by V_i and for $i \neq j$ we let $G_{i,j}$ be the complete bipartite subgraph with parts V_i, V_j. By Theorem 9 we know that $\chi'_\ell(G_{i,j}) \leq k$ and since by a result of Häggkvist and Janssen [21] we know that $\chi'_\ell(K_k) \leq k$ we also have $\chi'_\ell(G_i) \leq k$. Every vertex belongs to exactly $r-1$ of these subgraphs which partition $E(K_n)$, and we are done by Lemma 10. $\square$

This completes the proof of Theorem 1 part (a). Before turning to part (b) we state a packing result of Gustavsson [20] which we will use for its proof.

Theorem (Gustavsson [20])**.** *For any graph F there exists an $\varepsilon = \varepsilon(F) > 0$ and $n_0 = n_0(F)$ such that for any graph G on $n \geq n_0$ vertices with minimum degree at least $(1-\varepsilon)n$ one can partition the edge set of G into copies of F, provided:*

- $e(F) \mid e(G)$ *and*
- $\gcd(F) \mid \gcd(G)$,

where $e(H)$ denotes the number of edges of a graph H and $\gcd(H)$ denotes the greatest common divisor of the degrees of vertices in H.

Lemma 12. *For every $k \in \mathbb{N}$ there exists $w(k) \in \mathbb{N}$ such that the following holds. For every k and $r \geq w(k)$ that are not both even, we have*

$$R_\ell(K_{1,r}, k) = (r-1)k + 2.$$

Proof. Let $n = (r-1)k+1$. Therefore, $e(K_n) = \binom{n}{2} = (r-1)k((r-1)k+1)/2$. Since, if k is even r must be odd and in particular, $2|r-1$ we know that $k \mid e(K_n)$. Let $t \equiv e(K_n)/k \bmod k$. Let $G_1, \dots, G_t$ be vertex disjoint subgraphs of K_n each isomorphic to $K_{k+1,k+1}$ with a perfect matching removed, where we require $w(k) \geq 2k+1$ in order to have enough room ($n = (r-1)k+1 \geq (w(k)-1)k+1 \geq 2k^2+1 \geq 2(k+1)t$, since $t \leq k-1$). Let G be the subgraph of K_n obtained by removing the edges of all G_i's and let $F = K_{k,k}$. Note that $e(G) \equiv tk - tk(k+1) \equiv 0 \bmod k^2$ so $e(F) = k^2 \mid e(G)$. Furthermore, every vertex of K_n not in any G_i still has degree $(r-1)k$ in G while any vertex of G_i has degree $(r-1)k - k$ so $\gcd(G) = k = \gcd(F)$. Therefore, if we let $\varepsilon = \varepsilon(K_{k,k})$ and $n_0 = n_0(K_{k,k})$ given by the above theorem, then for $w(k) \geq \max(n_0/k, 2/\varepsilon)$ the above theorem applies, implying that $E(G)$ can be partitioned into $G_{t+1}, \dots, G_q$ all isomorphic to $K_{k,k}$.

Since each G_i is a k-regular bipartite graph Galvin's Theorem implies $\chi'_\ell(G_i) \leq k$ and since G_i's partition $E(G)$ we know that each vertex belongs to at most $(n-1)/k = r-1$ of the G_i's so our Lemma 10 applies and implies the result. □

2.2 Matchings

In this section we will show the following bounds on the list Ramsey number of matchings.

Theorem 13. *Let $r, k \in \mathbb{N}$. If $2(k+1) \leq \log r$ then*

$$2r \leq R_\ell(rK_2, k) \leq 2r + 42r^{k/(k+1)}.$$

If $2(k+1) > \log r > 0$ then

$$\frac{rk}{4\log(rk)} \leq R_\ell(rK_2, k) \leq \frac{34rk}{\log(rk)}.$$

Theorem 3 is now an immediate consequence of Theorem 13 and (4).

The proof of Theorem 13 appears in the following two lemmas. Our arguments below aim to illustrate as well the ideas we apply for the general setting in the next subsection, hence they are slightly more complicated than necessary.

We start with the lower bound.

Lemma 14. *Assuming $r, k \in \mathbb{N}$ such that $rk > 1$ we have*

$$R_\ell(rK_2, k) \geq \max\left(2r,\ \frac{(r-1)k}{2\log(rk)}\right).$$

Proof. Let $n = \max\left(2r-1,\ (r-1)\cdot\left\lfloor\frac{k}{2\log(rk)}\right\rfloor + r\right)$. Our task is to show that for any assignment L of lists of size k to $E(K_n)$ we can choose an L-colouring without a monochromatic rK_2. This is clear if the first term of the maximum is greater or equal than the second, because then rK_2 has more vertices than K_n. So we may assume $\frac{k}{2\log(rk)} \geq 2$. Let $t = \left\lfloor\frac{k}{2\log(rk)}\right\rfloor \geq 2$.

Let $c : E(K_n) \to [t]$ be a t-colouring of $E(K_n)$ without a monochromatic rK_2, which exists since $R(rK_2, t) = (r-1)t + r + 1 > n$, using (4).

We split all colours in $\cup_{e\in E(K_n)} L_e$ into t types indexed by $[t]$, with each colour being assigned a type independently and uniformly at random. Let B_e denote the event that no colour in L_e got assigned the type $c(e)$. Then

$$\mathbb{P}(B_e) = \left(1 - \frac{1}{t}\right)^k \leq \left(1 - \frac{2\log(rk)}{k}\right)^k \leq \frac{1}{r^2k^2}.$$

So by the union bound we obtain:

$$\bigcup_{e\in E(K_n)} \mathbb{P}(B_e) \leq \frac{n^2}{2r^2k^2} < 1,$$

where we used $k \geq 2$, which follows from $\frac{k}{2\log(rk)} \geq 2$.

Thus there is an assignment of types to colours appearing in the lists such that for every $e \in E(K_n)$ there is a colour $c'(e)$ of type $c(e)$ in L_e. Note that c' is an L-colouring of K_n with no monochromatic rK_2, since otherwise there would be a monochromatic rK_2 using only one type of colours, contradicting our choice of c. □

We now turn to the upper bounds. Once again we need to distinguish between the two regimes.

Lemma 15. *Let $r, k \in \mathbb{N}$. If $2(k+1) \leq \log r$ then we have*

$$R_\ell(rK_2, k) \leq 2r + 42r^{k/(k+1)},$$

and else we have

$$R_\ell(rK_2, k) \leq 34rk/\log(rk).$$

Proof. First notice that when $r = 1$ or $k = 1$ the result is immediate, so we assume $r, k \geq 2$ throughout the proof. In order to show an upper bound $R_\ell(G, k) \leq n$, we need to find a list assignment L of lists of size k to each edge of K_n in such a way that there is no way of L-colouring K_n without having a monochromatic copy of G.

Before proceeding with the proof, let us give some intuition for the next step. We are going to choose the lists by assigning each edge a uniformly random subset of colours from a slightly larger universe. Our goal then is to show that with probability less than one our random assignment of lists L has an L-colouring having no monochromatic rK_2. We now take a union bound over all possible colourings of K_n having no monochromatic rK_2 and check what is the probability that a fixed one is an L-colouring. The fact every edge misses a random set of colours makes this probability rather low.

For each edge of K_n we choose independently and uniformly at random a list of size k from the universe U of $k+t$ colours. For now we do not specify the values of n and t since they will differ depending on which of the two regimes we are considering, we will however assume that n is even.

Let B denote the event that there is a colouring c from our lists having no monochromatic rK_2. Our goal is to show $\mathbb{P}(B) < 1$. Let us restrict attention to the complete bipartite graph $H = K_{n/2,n/2}$ within our K_n. If B happens this means that there is an edge colouring c of H for which every colour class contains no matching of size r. Since H is bipartite König's theorem implies that every colour class has a cover of size at most $r-1$.

For any subset S of vertices of H of size $|S| = r-1$ consider the subgraph of H on the same vertex set containing all the edges of H incident to a vertex in S. Denote these subgraphs by $C_1, \cdots C_m$, where $m = \binom{n}{r-1}$.

The above observation implies that if B happens, every colour class of c on H is completely contained within some C_i. For all $i \in U$ we denote by c_i the subgraph of H made by the i-th colour class of c. Then

$$\begin{aligned}
\mathbb{P}(B) &\leq \mathbb{P}(\exists \text{ an } L\text{-colouring } c : E(H) \to U \text{ s.t. } \forall i \in U,\, \exists j \in [m] : c_i \subseteq C_j) \\
&\leq \sum_{j_1,\ldots,j_{k+t} \in [m]} \mathbb{P}(\exists \text{ an } L\text{-colouring } c : E(H) \to U \text{ s.t. } \forall i \in U : c_i \subseteq C_{j_i}) \\
&\leq m^{k+t} \max_{j_1,\ldots,j_{k+t} \in [m]} \mathbb{P}(\exists \text{ an } L\text{-colouring } c : E(H) \to U \text{ s.t. } \forall i \in U : c_i \subseteq C_{j_i}) \\
&= m^{k+t} \max_{j_1,\ldots,j_{k+t} \in [m]} \mathbb{P}(\forall e \in E(H), \exists i \in L_e : e \in C_{j_i}).
\end{aligned} \tag{5}$$

Let us now bound the last term. For fixed values $j_1, \ldots, j_{k+t}$, let d_e denote the number of C_{j_i} to which edge e belongs. As each C_j has at most $(r-1)n/2$ edges, we have that $\sum_{e \in E(H)} d_e \leq (k+t)(r-1)n/2$.

$$\begin{aligned}
\mathbb{P}(\forall e \in E(H), \exists i \in L_e : e \in C_{j_i}) &= \prod_{e \in E(H)} \mathbb{P}(\exists i \in L_e : e \in C_{j_i}) \\
&= \prod_{e \in E(H)} \left(1 - \binom{k+t-d_e}{k} \Big/ \binom{k+t}{k}\right) \\
&= \prod_{e \in E(H)} \left(1 - \left(1 - \frac{d_e}{k+t}\right) \cdots \left(1 - \frac{d_e}{t+1}\right)\right) \\
&\leq \prod_{e \in E(H)} \left(1 - \left(1 - \frac{d_e}{t+1}\right)^k\right) \\
&\leq \left(1 - \left(1 - \frac{\widetilde{d_e}}{t+1}\right)^k\right)^{n^2/4},
\end{aligned} \tag{6}$$

where in the first inequality we used the independence of the assignment of lists between edges, $\widetilde{d_e} :=$

$\frac{\sum_{e\in E(H)} d_e}{|E(H)|} \leq \frac{(k+t)(r-1)}{n/2}$ and we used Jensen's inequality. Combining (5) and (6) we obtain:

$$\begin{aligned}
\mathbb{P}(B) &\leq \binom{n}{r-1}^{k+t}\left(1-\left(1-\frac{\widetilde{d_e}}{t+1}\right)^k\right)^{n^2/4} \\
&\leq \left(\frac{en}{r-1}\right)^{(r-1)(k+t)}\left(1-\left(1-\frac{2(r-1)(k+t)}{n(t+1)}\right)^k\right)^{n^2/4}. \qquad (7)
\end{aligned}$$

At this point we proceed differently depending on the relation between k and r. In the first case we will assume k to be significantly smaller than r, specifically we assume $2(k+1) \leq \log r$. We choose $t = (k-1)\cdot\left\lceil \frac{n}{20r^{k/(k+1)}}\right\rceil - 1$ and our goal is to show that for $n = 2r + 2\cdot\left\lceil 20r^{k/(k+1)}\right\rceil$ we have $\mathbb{P}(B) < 1$.

$$\begin{aligned}
\log\mathbb{P}(B) &\leq \log\left(\left(\frac{en}{r-1}\right)^{(r-1)(k+t)}\left(1-\left(1-\frac{2(r-1)(k+t)}{n(t+1)}\right)^k\right)^{n^2/4}\right) \\
&< \left(1+\log\frac{n}{r-1}\right)(r-1)(k+t) - \frac{n^2}{4}\cdot\left(\frac{n-2r}{n}-\frac{k-1}{t+1}\right)^k \\
&< 6r(k-1)\left(1+\left\lceil\frac{n}{20r^{k/(k+1)}}\right\rceil\right) - \frac{n^2}{4}\cdot\left(\frac{20r^{k/(k+1)}}{n}\right)^k \\
&< \frac{12r(k-1)n}{20r^{k/(k+1)}} - r^2\cdot\left(\frac{20r^{k/(k+1)}}{n}\right)^k \\
&= \frac{rn}{20r^{k/(k+1)}}\cdot\left(12(k-1)-\left(\frac{20r}{n}\right)^{k+1}\right) \\
&< \frac{rn}{20r^{k/(k+1)}}\cdot\left(12(k-1)-2.5^{k+1}\right) < 0,
\end{aligned}$$

where in the second inequality for the second term we used $\log(1-x) \leq -x$, for $x < 1$, and $1 - \frac{2(r-1)(k+t)}{n(t+1)} \geq 1 - \frac{2r(k+t)}{n(t+1)} = \frac{n-2r}{n} - \frac{2r(k-1)}{n(t+1)} > \frac{n-2r}{n} - \frac{k-1}{t+1}$ where the last inequality follows since $n > 2r$. In the third inequality for the first term we used $\left(1+\log\frac{n}{r-1}\right) \leq 1+\log 88 \leq 6$. In the fourth inequality we used $1+\lceil x\rceil < 2x$ when $x > 2$ for the first term and $n > 2r$ for the second, while in the last inequality we used $\log r \geq 2(k+2)$, to get: $\frac{20r}{n} \geq \frac{10}{1+20r^{-1/(k+1)}+(2r)^{-1}} \geq \frac{10}{1+21/e^2} > 2.5$.

For the second case, when $\log r < 2k+2$, we let $n = 2\left\lceil 16rk/\log(rk)\right\rceil$ and $t = k$ and use (7) to get:

$$\begin{aligned}
\log\mathbb{P}(B) &\leq \log\left(\left(\frac{en}{r-1}\right)^{(r-1)(k+t)}\left(1-\left(1-\frac{2(r-1)(k+t)}{nt}\right)^k\right)^{n^2/4}\right) \\
&\leq 2rk\log\left(\frac{en}{r-1}\right) + \frac{n^2}{4}\log\left(1-\left(1-\frac{\log(rk)}{8k}\right)^k\right) \\
&\leq 2rk(8+\log k) + \frac{n^2}{4}\log\left(1-e^{-\log(rk)/4}\right) \\
&\leq 2rk(8+\log k) - (rk)^{-1/4}n^2/4 \\
&\leq 16rk\left(k^{1/4} - 16(rk)^{3/4}/\log^2(rk)\right) \\
&\leq 16rk\left(k^{1/4} - (rk)^{1/4}\right) < 0,
\end{aligned}$$

where in the first term of the third inequality we used $\log\left(\frac{en}{r-1}\right) \leq 1 + \log(128k) \leq 8 + \log k$, while in the second term we used $(1-x) \geq e^{-2x}$, given $x \leq 1/2$, with $x = \frac{\log rk}{8k} \leq \frac{2k+2+\log k}{8k} \leq 1/2$. In the fifth inequality we used $8 + \log k \leq 8k^{1/4}$ and in the sixth $\log x \leq 4x^{1/4}$. $\square$

2.3 General bounds.

In this subsection we give our bounds for general graphs and hypergraphs.

2.3.1 Upper bounds.

We start with upper bounds. The idea closely follows the one presented in the previous section with the main distinction that now it is not so easy to find the appropriate sets C_j. Note that the only property we required from C_i's is that the edge set of every graph not containing a copy of rK_2 is contained in some C_i. In the general setting we will find such sets by using the container method introduced by Saxton and Thomason [23] and Balogh, Morris and Samotij [5]. Specifically, we make use of the following theorem (Theorem 2.3 in [23]).

Theorem 16. *Let H be an ℓ-graph with $|E(H)| \geq 2$ and let $\varepsilon > 0$. There is a constant $c > 0$ such that for any $n \geq c$ there is a collection of ℓ-graphs $C_1, \dots, C_m$ on the vertex set $[n]$, such that*

(a) every H-free ℓ-graph on the vertex set $[n]$ is contained within some C_i,

(b) $|E(C_i)| \leq (\pi(H) + \varepsilon)\binom{n}{\ell}$ and

(c) $\log m \leq cn^{\ell - 1/m(H)} \log n$.

We now give an upper bound on the list Ramsey number for a fixed graph as the number of colours becomes large.

Theorem 5. *Let H be an ℓ-uniform hypergraph. Then, as k tends to infinity, we have*

$$R_\ell(H,k) \leq (1 - \pi(H) + o(1))^{-km(H)}.$$

Proof. We once again choose the lists for each edge uniformly at random out of the universe of $k+t$ colours. As before, B will denote the event that there is a colouring from our lists having no monochromatic H. Once again our goal is to show $\mathbb{P}(B) < 1$.

Let $\varepsilon > 0$, Theorem 16 provides us with a constant $c = c(\varepsilon, H)$ and a collection of ℓ-graphs $C_1, \dots, C_m$ satisfying the conditions (a),(b) and (c), where we will choose the value of $n \geq c$ later. We once again obtain as in (5)

$$\mathbb{P}(B) \leq m^{k+t} \max_{j_1,\dots,j_{k+t} \in [m]} \mathbb{P}(\forall e \in E(H), \exists i \in L_e : e \in C_{j_i}).$$

Once again for fixed values of j_i we define d_e to be the number of C_{j_i} that contain the edge e, and denote $\widetilde{d_e} = \sum_{e \in E\left(K_n^{(\ell)}\right)} d_e / \binom{n}{\ell} \leq (k+t)(\pi(H) + \varepsilon)$, where the last inequality follows from (b). Once again as in (6) we obtain:

$$\mathbb{P}(\forall e \in E(H), \exists i \in L_e : e \in C_{j_i}) \leq \left(1 - \left(1 - \frac{\widetilde{d_e}}{t+1}\right)^k\right)^{\binom{n}{\ell}}.$$

We choose $t = \lceil k/\varepsilon \rceil$, and require $2\varepsilon < 1 - \pi(H)$ to get

$$
\begin{aligned}
\log \mathbb{P}(B) &\leq (k+t)\log m + \binom{n}{\ell} \log\left(1 - \left(1 - \frac{\widetilde{d}_e}{t+1}\right)^k\right) \\
&\leq (k+t)cn^{\ell - 1/m(H)} \log n - \left(1 - \frac{(k+t)(\pi(H)+\varepsilon)}{t}\right)^k \binom{n}{\ell} \\
&\leq ck(1+2/\varepsilon)n^{\ell-1/m(H)} \log n - (1-\pi(H)-2\varepsilon)^k \frac{n^\ell}{\ell^\ell}.
\end{aligned}
$$

where we used $\frac{(k+t)(\pi(H)+\varepsilon)}{t} \leq (1+\varepsilon)(\pi(H)+\varepsilon) \leq \pi(H) + 2\varepsilon$ where in the last inequality we used $\pi(H) + \varepsilon < 1 - \varepsilon$. The last expression will be less than 0 provided

$$
\frac{ck(1+2/\varepsilon)\ell^\ell}{(1-\pi(H)-2\varepsilon)^k} < n^{1/m(H)}/\log n. \tag{8}
$$

Given $3\varepsilon < 1 - \pi(H)$, for large enough value of k this holds for $n = (1-\pi(H)-3\varepsilon)^{-km(H)}$. □

In the above argument it was important that H was fixed, since the constant c coming from the container theorem depends on H. The dependence of c on H is somewhat complicated, but by analysing the proof of Theorem 16 it should be possible to obtain good bounds for various families of graphs. We illustrate this by obtaining an explicit bound on $R_\ell(K_r^{(\ell)}, k)$. We start with a slightly weaker version of Theorem 16, which is a special case of Theorem 9.2 of [23].[2]

Theorem 17. *Let $H = K_r^{(\ell)}$ with $r > \ell$ and let $\delta > 0$. For any positive integer n there exists a collection of ℓ-graphs $C_1, \ldots, C_m$ on the vertex set $[n]$, such that*

(a) every H-free ℓ-graph on the vertex set $[n]$ is contained within some C_i,

(b') Each C_i contains at most $\delta\binom{n}{r}$ copies of H and

(c) $\log m \leq \frac{1}{\delta} \log \frac{1}{\delta} 2^{10\binom{r}{\ell}^2} n^{\ell-1/m(H)} \log n$.

Apart from an explicit constant in part (c) the main difference compared to Theorem 16 is that in the condition (b') rather than bounding the number of edges in each container we bound the number of copies of the forbidden graph H it contains. It is not hard to obtain condition (b) from (b') by making use of the Erdős-Simonovits supersaturation lemma, but requiring an explicit constant makes it slightly messy. We start with the standard bound of De Caen on $\mathrm{ex}(K_r^{(\ell)}, n)$.

Theorem 18 (De Caen [13])**.**

$$
\mathrm{ex}(K_r^{(\ell)}, n) \leq \left(1 - \frac{n-r+1}{n-\ell+1} / \binom{r-1}{\ell-1}\right) \binom{n}{\ell}.
$$

We now state the Erdős-Simonovits supersaturation lemma, keeping track of the constants.

Theorem 19 (Erdős-Simonovits [17])**.** *Let H be an ℓ-graph with r vertices, $x, \varepsilon > 0$ and $m \in \mathbb{N}$. Given $\mathrm{ex}(H, m) < x\binom{m}{\ell}$ we have that if an ℓ-uniform hypergraph on n vertices contains at least $(x+\varepsilon)\binom{n}{\ell}$ edges then it contains more than $\varepsilon\binom{m}{r}^{-1}\binom{n}{r}$ copies of H.*

[2] Where we plugged in the explicit values given in their Corollary 3.6 and Theorem 9.3 to obtain our explicit constant.

Combining the last three theorems gives us the following explicit version of Theorem 16 for the complete ℓ-graph.

Theorem 20. *Let $H = K_r^{(\ell)}$ with $r > \ell$ and let $\delta > 0$. For any positive integer n there exists a collection of ℓ-graphs $C_1, \ldots, C_m$ on the vertex set $[n]$, such that*

(a) every H-free ℓ-graph on the vertex set $[n]$ is contained within some C_i,

(b) $|E(C_i)| \leq \left(1 - \frac{2}{3}\binom{r-1}{\ell-1}^{-1}\right)\binom{n}{\ell}$ and

(c) $\log m \leq 2^{13\binom{r}{\ell}^2} n^{\ell-1/m(H)} \log n$.

Proof. Let $x = 1 - \frac{5}{6}\binom{r-1}{\ell-1}^{-1}, \varepsilon = \frac{1}{6}\binom{r-1}{\ell-1}^{-1}$ and $m = 6r$. By Theorem 18 we know that $\mathrm{ex}(H,m) < x\binom{m}{\ell}$ so Theorem 19 applies showing that any ℓ graph on n vertices with more than $(x+\varepsilon)\binom{n}{\ell} = \left(1 - \frac{2}{3}\binom{r-1}{\ell-1}^{-1}\right)\binom{n}{\ell}$ edges contains at least $\delta\binom{n}{r}$ copies of H, where $1/\delta := \varepsilon^{-1}\binom{m}{r} \leq 6\binom{r-1}{\ell-1}\binom{6r}{r} \leq 2^{2\binom{r}{\ell}^2}$. Using this value of δ in Theorem 17 we obtain the result. $\square$

We are now ready to obtain the bound on $R_\ell(K_r^{(\ell)}, k)$ promised in the introduction.

Theorem 4. *For arbitrary positive integers $r \geq \ell$ and $k \in \mathbb{N}$ we have*

$$R_\ell(K_r^{(\ell)}, k) \leq 2^{4r^{3\ell-1} + 4kr^{\ell-1}\log_2 r}.$$

Proof. We may assume $r > \ell \geq 2$ and $k \geq 2$, as otherwise the inequality is clearly true.

Repeating the argument that lead to (8) with $1 - \binom{r-1}{\ell-1}^{-1}$ in place of $\pi(H)$, $\varepsilon = \frac{1}{3}\binom{r-1}{\ell-1}^{-1}$ and using Theorem 20 instead of Theorem 16 we obtain that $R_\ell(K_r^{(\ell)}, k) \leq n$ given:

$$2^{13\binom{r}{\ell}^2} \cdot k \cdot \left(1 + 6\binom{r-1}{\ell-1}\right) \cdot \ell^\ell \cdot \left(3\binom{r-1}{\ell-1}\right)^k < n^{1/m(H)}/\log n.$$

Which, using $m(H) = \frac{\binom{r}{\ell}-1}{r-\ell} \leq r^{\ell-1}/(l-1)$ holds for $n = 2^{4r^{3\ell-1}+4kr^{\ell-1}\log_2 r}$, to see this notice that $\log n \leq 10r^{3\ell-1}k$; $k^2\left(3\binom{r-1}{\ell-1}\right)^k \leq k^2 r^{\ell k} \leq 2^{4k(\ell-1)\log_2 r}$; $10r^{3\ell-1}\left(1+6\binom{r-1}{\ell-1}\right)\ell^\ell \leq r^{5r} \leq 2^{3r^2} \leq 2^{3\binom{r}{\ell}^2}$; and $2^{16\binom{r}{\ell}^2} \leq 2^{4r^{2\ell}}$. $\square$

After this paper was submitted Balogh and Samotij obtained a more efficient container lemma in [6]. This can be used to obtain a slight improvement in the bound of the above theorem.

2.3.2 Lower bounds.

Let us now turn towards lower bounds. The main tool is the following lemma, giving us a lower bound for $R_\ell(H,k)$ in terms of the ordinary Ramsey number, but with fewer colours.

Theorem 21. *If $R(H, \lfloor k/(\ell\log n)\rfloor) > n$ then:*

$$R_\ell(H,k) > n.$$

Proof. The proof will proceed along similar lines as that of Lemma 14. Let $m = \lfloor k/(\ell \log n) \rfloor$. Consider a colouring $c : E\left(K_n^{(\ell)}\right) \to [m]$, without a monochromatic H, which exists because $n < R(H, m)$.

Let each edge e of $K_n^{(\ell)}$ be assigned a list L_e of size k, our goal is to show that we can pick colours from the lists avoiding a monochromatic copy of H.

We assign to each colour a type from $[m]$, independently and uniformly at random. Let B_e be the event that no colour in L_e got assigned type $c(e)$. Then

$$\mathbb{P}(B_e) \le (1 - 1/m)^k \le (1 - \ell \log n/k)^k \le 1/n^\ell.$$

So by the union bound we obtain:

$$\bigcup_{e \in E\left(K_n^{(\ell)}\right)} \mathbb{P}(B_e) \le \binom{n}{\ell} \cdot 1/n^\ell < 1.$$

Thus there is an assignment of types for which every $e \in E\left(K_n^{(\ell)}\right)$ has at least one colour of type $c(e)$ in its list and we colour e in one such colour. In this colouring there can be no monochromatic copy of H since otherwise there would be a monochromatic copy of H under c, contradicting our choice of c. $\square$

We can now deduce all our lower bounds from the introduction.

Proof of Theorem 6. Let us first show that $R(H, k) > r^k$. In order to do this we exhibit a colouring of $G = K_{r^k}^{(\ell)}$ without a monochromatic copy of H. We split G into r equal parts and colour all edges not completely within one of the parts using colour 1, then we repeat within each of the parts. Notice that since $\chi(H) > r$ there can be no monochromatic copy of H in this colouring, implying the claim.

Choosing $n = r^{\left\lfloor \sqrt{k/(\ell \log r)} \right\rfloor}$ we have that

$$R(H, \lfloor k/(\ell \log n) \rfloor) > r^{\lfloor k/(\ell \log n) \rfloor} \ge r^{\left\lfloor \sqrt{k/(\ell \log r)} \right\rfloor} = n.$$

Hence Theorem 21 applies, giving us the desired inequality. $\square$

Proof of Theorem 7. Axenovich, Gyárfás, Liu and Mubayi [4] showed that if an ℓ-graph H is not ℓ-partite then

$$R(H, k) \ge e^{k/((\ell+1)e^\ell)}. \tag{9}$$

Then for $n = \left\lfloor e^{c_\ell \sqrt{k}} \right\rfloor$ we have that

$$R(G, \lfloor k/(\ell \log n) \rfloor) \ge e^{\lfloor k/(\ell \log n) \rfloor/((\ell+1)e^\ell)} \ge e^{c_\ell \sqrt{k}} \ge n,$$

so Theorem 21 applies and gives us the desired inequality. $\square$

Proof of Theorem 8. The upper bound is the trivial inequality (1). For the lower bound we set $n = R(H, \lfloor ck/\log k \rfloor) - 1$, which implies $\left\lfloor \frac{ck}{\log k} \right\rfloor \mathrm{ex}(H, n) \ge \binom{n}{\ell}$, since each colour class is H-free. Using

Erdős' upper bound [15, Theorem 1] on the Turán number of ℓ-partite ℓ-graphs one obtains

$$R(H,k) \leq (k\ell^\ell)^{r^{\ell-1}}, \tag{10}$$

for any ℓ-partite ℓ-graph H with each part of size at most r. Substituting $1/c := 2r^{\ell-1}\ell^2 \log \ell$ we get that

$$\lfloor k/(\ell \log n) \rfloor \geq \lfloor k/(\ell \log (k\ell^\ell)^{r^{\ell-1}}) \rfloor \geq \lfloor ck/\log k \rfloor .$$

So we obtain that

$$R(H, \lfloor k/(\ell \log n)\rfloor) \geq R(H, \lfloor ck/\log k \rfloor) > n.$$

Hence, Theorem 21 implies the result.

To deduce the second part, note that from $\mathrm{ex}(H,n) = \tilde{\Theta}(n^{\ell-\varepsilon(H)})$ it is not hard to deduce that $R(H,k) = \tilde{\Theta}(k^{1/\varepsilon(H)})$, for example it follows from Lemma 15 of [4]. Combining this and the first part of the theorem the result follows. □

3 Concluding remarks and open problems

In this paper we initiate the systematic study of list Ramsey numbers of graphs and hypergraphs. We obtain several general bounds and reach a good understanding of how the list Ramsey number relates to the ordinary Ramsey number for some families of graphs. There are plenty of very natural further questions that arise.

For stars we have shown that the list Ramsey number is at most one smaller than the Ramsey number. We showed that they are equal in the case of two colours or when the size of the star is sufficiently large compared to the number of colours. Actually, we could not show them to differ for any values of the parameters, and we tend to conjecture that they are always equal.

Conjecture 1. *For any $r,k \in \mathbb{N}$*

$$R_\ell(K_{1,r},k) = R(K_{1,r},k).$$

Proving this conjecture for small r, in particular for $r = 2$, seems to be difficult, since that is equivalent to the well-studied and still open List Colouring Conjecture for cliques. That said, it would also be really interesting to show the conjecture for any $r \geq 3$, because this already seems to require new ideas.

For matchings we determine the list Ramsey number up to a constant factor. While our approach is very similar to the one we use in the general setting, we obtain very good bounds by exploiting the very simple structure of matchings. It would be interesting, but again probably hard, to determine the list Ramsey number of matchings exactly. We actually obtain the list Ramsey number of matchings up to a smaller order additive term when the size of the matching is sufficiently larger than the number of colours. When the number of colours is large enough compared to the size then we could obtain tight bounds only up to a multiplicative constant factor. It would be highly desirable to prove bounds which are correct up to a lower order term.

Question 2. *Does the limit*

$$\lim_{k\to\infty} R_\ell(rK_2,k)/(k/\log k)$$

exist and if it does what is its value?

If this limit exists we have shown that it is between $r/4$ and $34r$. While we did not make a serious attempt to optimise these constant factors and it is not hard to improve them by being more careful with our arguments, finding the precise constant factor seems to require new ideas.

There are many other families of graphs for which pretty good bounds are known for the Ramsey number, such as paths or cycles, and which might exhibit interesting behaviour in the list Ramsey setting.

In the case of general graphs and hypergraphs we have shown that the list Ramsey number is bounded above by a single exponential function in terms of the number of colours, which for higher uniformity hypergraphs is in stark contrast to the ordinary Ramsey number, which is known to exhibit an iterated exponential behaviour. In the case of ℓ-partite ℓ-graphs we showed that the list Ramsey number is in fact a polynomial function of the number of colours and that it is close to the ordinary Ramsey number. For non ℓ-partite ℓ-graphs we have shown a lower bound which is exponential in the square root of the number of colours. It would be interesting to ascertain whether this lower bound or the exponential upper bound is closer to the truth, even only for some specific families (of non-ℓ-partite ℓ-graphs) such as cliques. In fact for the case of $\ell = 2$, that is, for graphs, it is still open whether the k-colour list Ramsey number of cliques is always equal to its ordinary Ramsey counterpart.

Question 3. *Is it true that for any $r, k \in \mathbb{N}$*

$$R_\ell(K_r, k) = R(K_r, k)?$$

We have shown how list Ramsey numbers connect to various interesting problems and sometimes exhibit very different behaviour when compared to their ordinary Ramsey counterparts. Such information may give some indication for the original Ramsey problem as well. For example, since $R_\ell(K_3, k) \leq (4 + o(1))^k$ if one wishes to construct an example showing $R(K_3, k)$ is super-exponential in k (and in the process win a \$100 prize from Erdős) one needs to ensure this example does not also work in the case of list Ramsey numbers.

Ramsey theory is very rich in attractive problems and there are many such problems which may prove to be interesting in the list Ramsey setting as well. Some classical examples that come to mind are Schur's or Van der Waerden's Theorems.

Acknowledgements The research on this project was initiated during a joint research workshop of Tel Aviv University and the Free University of Berlin on Graph and Hypergraph Colouring Problems, held in Berlin in August 2018, and supported by a GIF grant number G-1347-304.6/2016. We would like to thank the German-Israeli Foundation (GIF) and both institutions for their support.

We are extremely grateful to the anonymous referees for their many useful suggestions and comments.

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