

Monochromatic subgraphs in iterated triangulations

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Abstract

For integers $n \geq 0$, an iterated triangulation $\text{Tr}(n)$ is defined recursively as follows: $\text{Tr}(0)$ is the plane triangulation on three vertices and, for $n \geq 1$, $\text{Tr}(n)$ is the plane triangulation obtained from the plane triangulation $\text{Tr}(n-1)$ by, for each inner face F of $\text{Tr}(n-1)$, adding inside F a new vertex and three edges joining this new vertex to the three vertices incident with F .

In this paper, we show that there exists a 2-edge-coloring of $\text{Tr}(n)$ such that $\text{Tr}(n)$ contains no monochromatic copy of the cycle C_k for any $k \geq 5$. As a consequence, the answer to one of two questions asked by Axenovich et al. is negative. We also determine the radius 2 graphs H for which there exists n such that every 2-edge-coloring of $\text{Tr}(n)$ contains a monochromatic copy of H , extending a result of Axenovich et al. for radius 2 trees.

Mathematics Subject Classifications: 05C55, 05C10, 05D10

1 Introduction

For graphs G and H , we write $G \rightarrow H$ if, for any 2-edge-coloring of G , there is a monochromatic copy of H . Otherwise, we write $G \not\rightarrow H$. We say that H is *planar unavoidable* if there exists a planar graph G such that $G \rightarrow H$. Otherwise, we say H is *planar avoidable*. This notion is introduced and studied in [4].

Deciding if $G \not\rightarrow H$ is clearly equivalent to asking whether a graph G admits a decomposition (i.e., an edge-decomposition) such that none of the two graphs in the decomposition contains the given graph H . The well-known Four Color Theorem [2, 3] (also see [10])

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implies that every planar graph admits a decomposition into two bipartite graphs; so planar unavoidable graphs must be bipartite. A result of Gonçalves [5] says that every planar graph admits a decomposition into two outer planar graphs; so planar unavoidable graphs must be also outer planar. There are a number of interesting results about decomposing planar graphs, see [1, 6–9].

For any positive integer n , let P_n denote the path on n vertices, K_n denote the complete graph on n vertices, and $K_{n,m}$ denote the complete bipartite graph with two parts of sizes n and m . For integer $n \geq 3$, we use C_n to denote the cycle on n vertices. It is shown in [4] that P_n , C_4 , and all trees with radius at most 2 are planar unavoidable. This is done by analyzing several sequences of graphs.

In this paper, we investigate one such sequence – the iterated triangulations, which is of particular interest as suggested in [4]. Let $n \geq 0$ be an integer. An *iterated triangulation* $\text{Tr}(n)$ is a plane graph defined as follows: $\text{Tr}(0) \cong K_3$ is the plane triangulation with exactly two 3-faces. For each $i \geq 0$, let $\text{Tr}(i+1)$ be obtained from the plane triangulation $\text{Tr}(i)$ by adding a new vertex in each of the inner faces of $\text{Tr}(i)$ and connecting this vertex with edges to the three vertices in the boundary of their respective face. The authors of [4] asked whether for any planar unavoidable graph H there is an integer n such that $\text{Tr}(n) \rightarrow H$. They also asked whether there exists an integer $k \geq 3$ such that the even cycle C_{2k} is planar-unavoidable.

Our first result indicates that a positive answer to one of the above questions implies a negative answer to the other. Let H^+ be the graph with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_5, v_2v_6\}$.

Theorem 1.1. *For all positive integers n , $\text{Tr}(n) \not\rightarrow C_k$ for $k \geq 5$, $\text{Tr}(n) \not\rightarrow H^+$, and $\text{Tr}(n) \not\rightarrow K_{2,3}$*

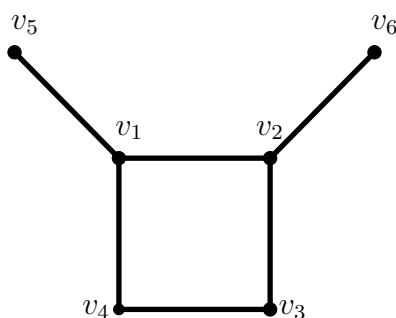


Figure 1: H^+

As another direct consequence, we see that if B is a bipartite graph and $\text{Tr}(n) \rightarrow B$ for some n , then every block of B must be a C_4 or K_2 . This can be used to characterize all radius 2 graphs B for which there exists n such that $\text{Tr}(n) \rightarrow B$, generalizing a result in [4] for radius 2 trees. Before we state this characterization, it is worth mentioning that the authors in [4] show that there is a planar avoidable tree of radius 3. We need some additional notation. A *flower* F_k is a collection of k copies of C_4 s sharing a common

vertex, which is called the *center*. A k -ary tree of radius 2 means a rooted tree such that every non-leaf vertex has degree k and every leaf has depth 2. A *jellyfish* J_k is obtained from F_k and a k -ary tree of radius 2 by identifying the center of F_k with the root of the k -ary tree. A *bistar* B_k is obtained from one C_4 and two disjoint $K_{1,k}$ s by identifying the roots of the $K_{1,k}$ s with two non-adjacent vertices of C_4 , respectively.

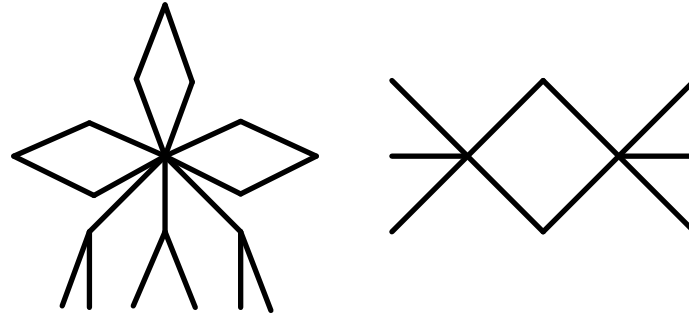


Figure 2: J_3 and B_3

Theorem 1.2. *Let L be a graph with radius 2. Then there exists n such that $\text{Tr}(n) \rightarrow L$ if, and only if, L is a subgraph of a jellyfish or bistar.*

We organize this paper as follows. In Section 2, we prove $\text{Tr}(n) \not\rightarrow C_k$ for $k \geq 5$ and $\text{Tr}(n) \not\rightarrow H^+$ by finding a special edge-coloring scheme for $\text{Tr}(n)$. In Section 3, we complete the proof of Theorem 1.1 by using another edge-coloring scheme on $\text{Tr}(n)$. From Theorem 1.1, we can derive the following: if L has radius 2 and $\text{Tr}(n) \rightarrow L$ for some n , then L is a subgraph of a jellyfish or bistar. Hence to prove Theorem 1.2, it suffices to show that for any $k \geq 1$ there exists some n such that $\text{Tr}(n) \rightarrow J_k$ and $\text{Tr}(n) \rightarrow B_k$. We prove the former statement in Section 4 and the latter one in Section 5 by showing that we can choose n to be linear in k .

2 H^+ and C_k for $k \geq 5$

In this section, we prove Theorem 1.1 for H^+ and C_k , with $k \geq 5$. First, we describe the 2-edge-coloring of $\text{Tr}(n)$ that we will use. Let $\sigma : E(\text{Tr}(n)) \rightarrow \{0, 1\}$ be defined inductively for all $n \geq 1$ as follows:

- (i) Fix an arbitrary triangle T bounding an inner face of $\text{Tr}(1)$, and let $\sigma(e) = 0$ if $e \in E(T)$ and $\sigma(e) = 1$ if $e \in E(\text{Tr}(1)) \setminus E(T)$.
- (ii) Suppose for some $1 \leq i < n$, we have defined $\sigma(e)$ for all $e \in E(\text{Tr}(i))$. We extend σ to $E(\text{Tr}(i+1))$ as following. Let $x \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$ be arbitrary, let $v_0v_1v_2v_0$ denote the triangle bounding the inner face of $\text{Tr}(i-1)$ containing x , and fix a labeling so that $\sigma(xv_1) = \sigma(xv_2)$.

- (iii) Let $x_j \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$ be such that x_j is inside the face of $\text{Tr}(i)$ bounded by the triangle $xv_jv_{j+1}x$, where $j = 0, 1, 2$ and the subscripts are taken modulo 3. Define $\sigma(xv_0) = \sigma(x_0v_0) = \sigma(x_2v_0) = \sigma(x_jx)$ for all $j = 0, 1, 2$, and $\sigma(xv_1) = \sigma(x_0v_1) = \sigma(x_1v_1) = \sigma(x_2v_1)$.

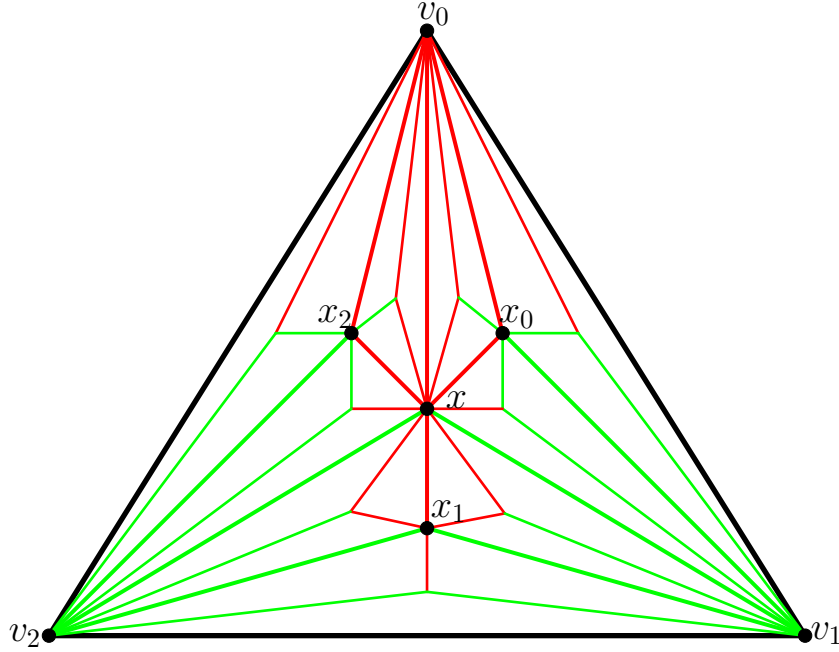


Figure 3: 2 edge-coloring scheme

We now proceed by a sequence of claims to show that σ has no monochromatic C_k for $k \geq 5$ nor monochromatic H^+ , thereby proving $\text{Tr}(n) \not\rightarrow C_k$ for $k \geq 5$ and $\text{Tr}(n) \not\rightarrow H^+$. The first claim is immediate from (iii) so we omit its proof.

Claim 1. For $1 \leq i \leq n$ and $x \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$, $|\{\sigma(xv) : v \in V(\text{Tr}(i-1))\}| = 2$.

Claim 2. Let $v_0v_1v_2v_0$ be a triangle bounding an inner face of $\text{Tr}(i)$, where $0 \leq i < n$, let $v \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$ with v inside $v_0v_1v_2v_0$. Then, for any $v_0w \in E(\text{Tr}(n))$ with w inside $v_0v_1v_2v_0$, $\sigma(v_0w) = \sigma(v_0v)$.

Proof. Let $v_0w \in E(\text{Tr}(n))$ with w inside $v_0v_1v_2v_0$. Then there exists $k \geq 0$ with $i+k+1 \leq n$, such that $w \in V(\text{Tr}(i+k+1)) \setminus V(\text{Tr}(i+k))$. We prove Claim 2 by applying induction on k . The basis case is trivial because $k = 0$ implies $w = v$.

So assume $k \geq 1$. Let $v_0v_3v_4v_0$ be the triangle bounding an inner face of $\text{Tr}(i+k-1)$ with w inside $v_0v_3v_4v_0$, and let $v_5 \in V(\text{Tr}(i+k)) \setminus V(\text{Tr}(i+k-1))$ that is inside $v_0v_3v_4v_0$. By symmetry, assume w is inside $v_0v_5v_4v_0$. By induction hypothesis, $\sigma(v_0v_5) = \sigma(v_0v)$.

Suppose $\sigma(v_4v_5) = \sigma(v_0v_5)$. Hence by (ii) and (iii), $\sigma(v_0w) = \sigma(wv_4) = \sigma(v_0v_5)$. Thus $\sigma(v_0w) = \sigma(v_0v)$. Now assume $\sigma(v_4v_5) \neq \sigma(v_0v_5)$. Then $\sigma(v_3v_5) = \sigma(v_0v_5)$ or $\sigma(v_3v_5) = \sigma(v_4v_5)$. It follows from (iii) that $\sigma(v_0w) = \sigma(v_0v_5)$. Hence, $\sigma(v_0w) = \sigma(v_0v)$. ■

Claim 3. Let $v_0v_1v_2v_0$ be a triangle bounding an inner face of $\text{Tr}(i)$ with $0 \leq i \leq n-2$, and let $v \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$ such that v is inside $v_0v_1v_2v_0$ and $\sigma(vv_0) \neq \sigma(vv_1) = \sigma(vv_2)$. Then for any $vw \in E(\text{Tr}(n))$ with w inside $v_0v_1v_2v_0$, $\sigma(vw) = \sigma(vv_0)$.

Proof. To prove Claim 3, let $\{w_0, w_1, w_2\} \subseteq V(\text{Tr}(i+2)) \setminus V(\text{Tr}(i+1))$ such that w_j is inside $vv_jv_{j+1}v$ for $j = 0, 1, 2$, with subscripts modulo 3. By (ii) and (iii), $\sigma(vw_0) = \sigma(vw_2) = \sigma(vw_1) = \sigma(vv_0)$. By Claim 2, there exists some $j \in \{0, 1, 2\}$ with $\sigma(vw) = \sigma(vw_j)$. Hence, $\sigma(vw) = \sigma(vv_0)$. ■

Claim 4. Let $v_0v_1v_2v_0$ be a triangle bounding an inner face of $\text{Tr}(i)$, where $0 \leq i \leq n-2$, and let $v \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$ such that v is inside $v_0v_1v_2v_0$ and $\sigma(vv_0) \in \{\sigma(vv_1), \sigma(vv_2)\}$. Then for any $w \in (N(v) \cap N(v_0)) \setminus \{v_1, v_2\}$, $\sigma(vv_0) \neq \sigma(wv)$.

Proof. To prove Claim 4, we may assume by symmetry and Claim 1 that $\sigma(vv_2) \neq \sigma(vv_0) = \sigma(vv_1)$. Then $\sigma(wv_0) = \sigma(vv_0)$ by Claim 2, and $\sigma(wv) = \sigma(vv_2)$ by Claim 3. Hence, $\sigma(wv_0) \neq \sigma(wv)$. ■

Claim 5. Suppose upv is a monochromatic path of length two in $\text{Tr}(n)$ with $uv \in E(\text{Tr}(i+1))$ and $p \in V(\text{Tr}(n)) \setminus V(\text{Tr}(i+1))$. Then any monochromatic path in $\text{Tr}(n)$ between u and v and of the color $\sigma(up)$ has length at most two.

Proof. Consider any monochromatic path $P = a_0a_1 \dots a_r$ of the color $\sigma(up)$ with $a_0 = v$ and $a_r = u$. First, suppose $uv \in E(\text{Tr}(0))$. Let $\text{Tr}(0) = uvwu$ and $x \in V(\text{Tr}(1)) \setminus V(\text{Tr}(0))$. By Claim 2, $\sigma(ux) = \sigma(up)$ and $\sigma(vx) = \sigma(vp)$; so $\sigma(xu) = \sigma(xv)$. Thus, by (i), $\sigma(wx) = \sigma(wu) = \sigma(wv) \neq \sigma(xu)$. Let $v_0v_1 \dots v_n$ be a path in $\text{Tr}(n)$ with $v_0 = w$, $v_1 = x$ and for $1 \leq i \leq n$, $v_i \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$ is inside $v_{i-1}uvv_{i-1}$. By (ii) and (iii), $\sigma(v_iu) = \sigma(v_iv) = \sigma(vx)$ for $1 \leq i \leq n$, and $\sigma(v_iv_{i+1}) = \sigma(xw)$ for $0 \leq i \leq n-1$. By planarity, P is contained in the closed region bounded by $uvwu$. So either $P = uv$ or there exists some $1 \leq k \leq r-1$ such that $a_k \in \{v_0, \dots, v_n\}$. We may assume the latter case occurs. If $\{a_{k-1}, a_{k+1}\} = \{u, v\}$, then $r = 2$. Hence without loss of generality, let $a_{k-1} \notin \{u, v\}$. Then by Claim 2 and Claim 3, $\sigma(a_{k-1}a_k) = \sigma(v_iv_{i+1}) \neq \sigma(pu)$ for $i \in \{0, 1, \dots, n-1\}$, a contradiction. Hence $r \leq 2$. We remark that this paragraph also shows that such uv in $E(\text{Tr}(0))$ cannot be in a monochromatic C_4 .

Thus, we may assume $uv \notin E(\text{Tr}(0))$. By symmetry, we may assume that $v \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$ for some $0 \leq i < n$ and v is inside the triangle $u_1u_2u_3u_1$ bounding an inner face of $\text{Tr}(i)$ and $u_1 = u$. By Claim 4, $\sigma(u_1v) \neq \sigma(u_2v) = \sigma(u_3v)$.

If a_1 is inside vu_2u_3v then there exists $1 \leq k < r$ such that a_k is inside vu_3u_2v and $a_{k+1} \in \{u_2, u_3\}$; so by Claim 2, $\sigma(a_ka_{k+1}) = \sigma(vu_2) = \sigma(vu_3) \neq \sigma(u_1v) = \sigma(pu)$, a contradiction.

Therefore, suppose that $P \neq uv$, by symmetry, we may assume that a_1 is inside $u_1vu_2u_1$. Let $v_0 = u_2$ and let $v_1v_2 \dots v_{n-i-1}$ be the path in $\text{Tr}(n)$ such that, for $1 \leq \ell \leq n-i-1$, $v_\ell \in V(\text{Tr}(i+\ell+1)) \setminus V(\text{Tr}(i+\ell))$ is inside $u_1v_{\ell-1}vu_1$.

By (ii) and (iii), $\sigma(v_\ell u_1) = \sigma(v_\ell v) = \sigma(u_1v)$ for $1 \leq \ell \leq n-i-1$, and $\sigma(v_\ell v_{\ell+1}) = \sigma(vu_2) \neq \sigma(vu_1)$ for $0 \leq \ell \leq n-i-2$. If a_1 is inside $v_\ell v_{\ell+1}vv_\ell$ for some ℓ with $0 \leq \ell \leq n-i-2$, then exists $1 \leq k \leq r$ such that a_k is inside $v_\ell v_{\ell+1}vv_\ell$ and $a_{k+1} \in \{v_\ell, v_{\ell+1}\}$;

so by Claim 3 $\sigma(a_k a_{k+1}) = \sigma(v_\ell v_{\ell+1})$, a contradiction. So $a_1 = v_\ell$ for some ℓ with $1 \leq \ell \leq n - i - 1$. Then as $\sigma(a_1 a_2) = \sigma(u_1 v)$ and by Claim 3, we have $a_2 = u_1$. Therefore, $r = 2$, proving Claim 5. ■

Claim 6. If C_k is monochromatic in $\text{Tr}(n)$ then $k \leq 4$.

Proof. Let $C_k = a_1 a_2 \dots a_k a_1$ be a monochromatic cycle in $\text{Tr}(n)$. By (i), $E(C_k) \not\subseteq E(\text{Tr}(0))$. So we may assume that there exists some $1 \leq i \leq k$ such that $a_{i+1} \in V(\text{Tr}(\ell + 1)) \setminus V(\text{Tr}(\ell))$ is inside the triangle $a_i u v a_i$ which bounds an inner face of some $\text{Tr}(\ell)$. We may further assume that $\ell \leq n - 2$, as otherwise, we could consider $\text{Tr}(n + 1)$ instead of $\text{Tr}(n)$.¹

Suppose $\sigma(a_i a_{i+1}) \in \{\sigma(a_{i+1} u), \sigma(a_{i+1} v)\}$. By symmetry, we may assume $\sigma(a_i a_{i+1}) = \sigma(a_{i+1} u)$. Then $a_{i+2} = u$ by Claim 3. Hence, by Claim 5, any monochromatic path in C_k between a_i and $a_{i+2} = u$ has length at most 2. So $k \leq 4$.

Thus, we may assume $\sigma(a_i a_{i+1}) \notin \{\sigma(a_{i+1} u), \sigma(a_{i+1} v)\}$; hence, $\sigma(a_{i+1} u) = \sigma(a_{i+1} v)$. Let $w \in V(\text{Tr}(\ell + 2)) \setminus V(\text{Tr}(\ell + 1))$ be inside the triangle $a_i u a_{i+1} a_i$. By (ii) and (iii), $\sigma(w a_i) = \sigma(w a_{i+1}) = \sigma(a_i a_{i+1})$. Hence, by Claim 5, the monochromatic path $C_k - a_i a_{i+1}$ in $\text{Tr}(n)$ of the color $\sigma(a_i a_{i+1}) = \sigma(w a_i)$ has length at most 2; so $k = 3$. ■

Claim 7. There is no monochromatic H^+ in $\text{Tr}(n)$.

Proof. Suppose that there is a monochromatic copy of H^+ on $\{v_i : 1 \leq i \leq 6\}$ in which $v_1 v_2 v_3 v_4 v_1$ is a 4-cycle and $v_1 v_5, v_2 v_6$ are edges. If $v_1 v_2 \in E(\text{Tr}(0))$, then $v_1 v_2$ satisfies the conditions of Claim 5 and by the footnote from the proof of Claim 5, there is no monochromatic C_4 containing $v_1 v_2$, a contradiction. So $v_1 v_2 \notin E(\text{Tr}(0))$. By symmetry, we may assume that $v_2 \in V(\text{Tr}(i + 1)) \setminus V(\text{Tr}(i))$ for some i and that $v_1 u w v_1$ is the triangle bounding the inner face of $\text{Tr}(i)$ containing v_2 . Again as before we may assume that $0 \leq i \leq n - 2$.

If $\sigma(v_2 u) = \sigma(v_2 w)$, then there exists some $p \in V(\text{Tr}(n)) \setminus V(\text{Tr}(i + 1))$ such that $v_1 p v_2$ has the same color as $\sigma(v_1 v_2)$. But $v_1 v_4 v_3 v_2$ is a monochromatic path of length 3 in $\text{Tr}(n)$ between v_1 and v_2 and of the color $\sigma(v_1 v_2)$, a contradiction to Claim 5.

Hence, $\sigma(v_1 v_2) \in \{\sigma(v_2 u), \sigma(v_2 w)\}$ and by symmetry, we may assume $\sigma(v_1 v_2) = \sigma(v_2 u)$. Then by Claim 1, $\sigma(v_1 v_2) \neq \sigma(v_2 w)$ and thus $\sigma(v_2 v_3) = \sigma(v_2 v_6) \neq \sigma(v_2 w)$. This shows $w \notin \{v_3, v_6\}$. So there exists $y \in \{v_3, v_6\} \setminus \{u, w\}$. By Claim 3, $\sigma(v_2 y) = \sigma(v_2 w)$, a contradiction. ■

This completes the proof of Theorem 1.1 for H^+ and C_k , with $k \geq 5$.

3 Monochromatic $K_{2,3}$

In this section, we prove Theorem 1.1 for $K_{2,3}$ using a different coloring scheme on $\text{Tr}(n)$ described below. Let $\sigma : E(\text{Tr}(n)) \rightarrow \{0, 1\}$ be defined inductively as follows:

¹This is fair because $\text{Tr}(n + 1) \not\rightarrow C_k$ implies $\text{Tr}(n) \not\rightarrow C_k$.

- (i) Fix a triangle T bounding an inner face of $\text{Tr}(1)$, and let $\sigma(e) = 0$ if $e \in E(T)$ and $\sigma(e) = 1$ if $e \in E(\text{Tr}(1)) \setminus E(T)$.
- (ii) Suppose for some $1 \leq i < n$, we have defined $\sigma(e)$ for all $e \in E(\text{Tr}(i))$. We now extend σ to $E(\text{Tr}(i+1))$. Let $x \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$ be arbitrary, let $v_0v_1v_2v_0$ denote the triangle bounding the inner face of $\text{Tr}(i-1)$ containing x , with v_0, v_1, v_2 on the triangle in clockwise order, and let $\sigma(xv_1) = \sigma(xv_2)$.
- (iii) Let $x_j \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$ such that x_j is inside the face of $\text{Tr}(i)$ bounded by the triangle $xv_jv_{j+1}x$, where $j = 0, 1, 2$ and the subscripts are taken modulo 3. Define $\sigma(v_0x) = \sigma(v_0x_0) = \sigma(v_0x_2) = \sigma(xx_2) = \sigma(x_1v_1)$, and $\sigma(v_2x) = \sigma(v_2x_1) = \sigma(v_2x_2) = \sigma(xx_1) = \sigma(x_0v_1)$.

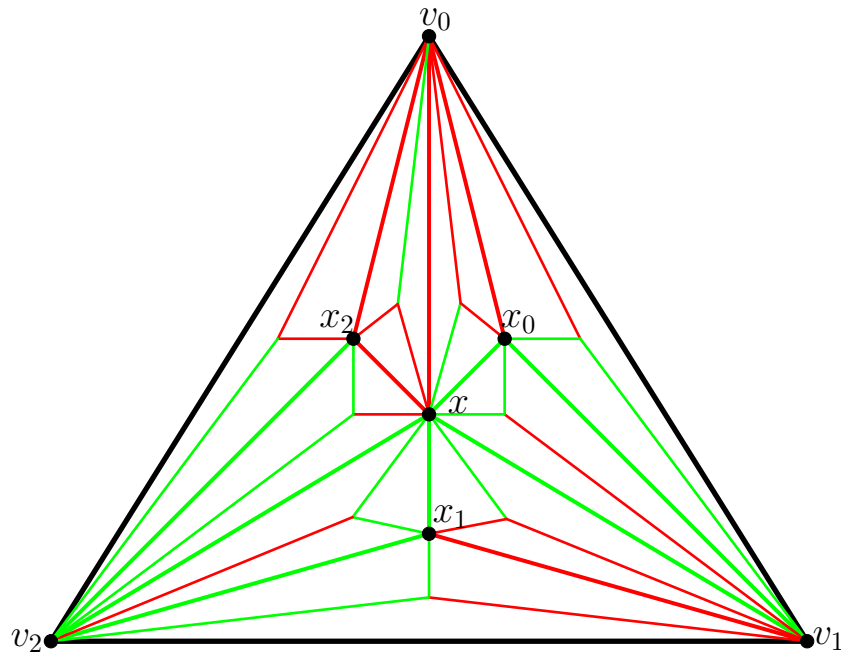


Figure 4: 2 edge-coloring scheme

Note that in (ii) we have $|\{\sigma(xv_j) : j = 0, 1, 2\}| = 2$ and that in (iii) we have $\sigma(x_jv_j) \neq \sigma(x_jv_{j+1})$ for $j = 0, 1, 2$. Hence, inductively, we have

- (1) For $1 \leq i \leq n$ and $x \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$, $|\{\sigma(xv) : v \in V(\text{Tr}(i-1))\}| = 2$.
- (2) If $x_1x_2x_3x_1$ is a triangle which bounds an inner face of $\text{Tr}(i)$ for some $1 \leq i \leq n-2$, and if $x \in V(\text{Tr}(n)) \setminus V(\text{Tr}(i+1))$ is inside $x_1x_2x_3x_1$ with $xx_1, xx_2 \in E(\text{Tr}(n))$, then $\sigma(xx_1) \neq \sigma(xx_2)$.

These two claims are straightforward so we omit their proofs.

- (3) For any $x_1x_2 \in E(\text{Tr}(n))$, $|\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 0\}| \leq 2$ and $|\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 1\}| \leq 2$.

First, suppose $x_1x_2 \in E(\text{Tr}(0))$. Then by (i) and (2), $|\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 0\}| \leq 1$ and $|\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 1\}| \leq 1$.

So we may assume that x_1vwx_1 bounds an inner face of $\text{Tr}(i)$ and $x_2 \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$ inside x_1vwx_1 . Let $v_1 \in \text{Tr}(i+2)$ be inside $x_1vx_2x_1$ and $w_1 \in \text{Tr}(i+2)$ be inside $x_1wx_2x_1$. By (iii), $\sigma(w_1x_1) \neq \sigma(w_1x_2)$ or $\sigma(v_1x_1) \neq \sigma(v_1x_2)$. By (2), for any $x \in V(\text{Tr}(n)) \setminus V(\text{Tr}(i+2))$ inside x_1vwx_1 with $xx_1, xx_2 \in E(\text{Tr}(n))$, we have $\sigma(xx_1) \neq \sigma(xx_2)$. Hence, if (3) fails, then we may assume by symmetry between w_1 and v_1 that $\sigma(vx_1) = \sigma(vx_2) = \sigma(wx_1) = \sigma(wx_2) = \sigma(v_1x_1) = \sigma(v_1x_2)$, and $\sigma(w_1x_1) \neq \sigma(w_1x_2)$. Then, by (1), $\sigma(x_1x_2) \neq \sigma(x_2v) = \sigma(x_2w)$. Now by (iii), at least one of the two edges v_1x_1 and v_1x_2 has the same color as x_1x_2 , a contradiction. This proves (3).

- (4) If $x_1x_2x_3x_4x_1$ is a 4-cycle in $\text{Tr}(n)$, then $x_1x_3 \in E(\text{Tr}(n))$ or $x_2x_4 \in E(\text{Tr}(n))$.

We may assume that $\{x_1, x_2, x_3, x_4\} \subseteq V(\text{Tr}(i+1))$ and $x_j \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$ for some $0 \leq i < n$ and $j \in [4]$. Let $uvwu$ be the triangle bounding an inner face of $\text{Tr}(i)$ such that x_j is inside it. Then $\{x_{j-1}, x_{j+1}\} \subseteq \{u, v, w\}$, implying that $x_{j-1}x_{j+1} \in E(\text{Tr}(n))$. $\square$

- (5) There is no monochromatic $K_{2,3}$ in $\text{Tr}(n)$.

For, suppose $\text{Tr}(n)$ has a monochromatic copy of $K_{2,3}$ on $\{v_1, v_2, v_3, v_4, v_5\}$ with $v_4v_i, v_5v_i \in E(\text{Tr}(n))$ for all $i = 1, 2, 3$. Then $v_4v_5 \notin E(\text{Tr}(n))$ by (3) and, hence, it follows from (4) that $v_1v_2, v_2v_3, v_3v_1 \in E(\text{Tr}(n))$. By planarity, $v_1v_2v_3v_1$ bounds an inner face of $\text{Tr}(i)$ for some i with $1 \leq i < n$ and, by the symmetry between v_4 and v_5 , we may assume that v_4 is inside $v_1v_2v_3v_1$. Then $v_4 \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$. However, this contradicts (1), as $\sigma(v_4v_1) = \sigma(v_4v_2) = \sigma(v_4v_3)$. We have completed the proof of Theorem 1.1. $\blacksquare$

4 Monochromatic J_k

In this section we prove that $\text{Tr}(100k) \rightarrow J_k$ holds for any positive integer k .

We need the following result, which is Lemma 9 in [4]. The original statement in [4] states $\text{Tr}(16) \rightarrow C_4$, but the same proof in [4] actually gives the following stronger version.

Lemma 4.1. *If $xyzx$ bounds the outer face of $\text{Tr}(16)$, then any 2-edge-coloring of $\text{Tr}(16)$ gives a monochromatic C_4 that intersects $\{x, y\}$.*

Note that if the triangle $xyzx$ bounds the outer face of $\text{Tr}(n)$ and $v \in V(\text{Tr}(1)) \setminus V(\text{Tr}(0))$ then the subgraph of $\text{Tr}(n)$ contained in the closed disc bounded by $vxyv$ is isomorphic to $\text{Tr}(n-1)$. Hence, the following is an easy consequence of Lemma 4.1.

Corollary 4.2. *If $xyzx$ bounds the outer face of $\text{Tr}(17)$ then any 2-edge-coloring of $\text{Tr}(17)$ gives a monochromatic C_4 that intersects $\{x, y\}$ and avoids z .*

Lemma 4.3. *For any positive integer k , $\text{Tr}(38k) \rightarrow F_k$*

Proof. Let $\sigma : E(\text{Tr}(38k)) \rightarrow \{0, 1\}$ be an arbitrary 2-edge coloring. Let $uvwu$ be the triangle bounding the outer face of $\text{Tr}(38k)$. Let $x_0 := w$ and, for $1 \leq \ell \leq 2k$, let $x_\ell \in V(\text{Tr}(\ell)) \setminus V(\text{Tr}(\ell - 1))$ such that x_ℓ is inside $x_{\ell-1}uvx_{\ell-1}$. Let $y_{i,0} := x_i$ for $i \in \{0, 1, \dots, 2k - 1\}$ and, for $\ell \in \{1, \dots, 36k\}$, let $y_{i,\ell} \in V(\text{Tr}(i + 1 + \ell)) \setminus V(\text{Tr}(i + \ell))$ such that $y_{i,\ell}$ is inside $y_{i,\ell-1}ux_{i+1}y_{i,\ell-1}$.

Suppose for each $0 \leq i \leq 2k - 1$ there exists a monochromatic C_4 inside $x_iux_{i+1}x_i$ that contains u and avoids x_i . By pigeonhole principle, at least k of these C_4 s are of the same color, which form a monochromatic F_k centered at u .

Hence, we may assume that there exists some $i \in \{0, 1, \dots, 2k - 1\}$ such that no monochromatic C_4 inside $x_iux_{i+1}x_i$ contains u and avoids x_i . Since $i \leq 2k - 1$, $x_iux_{i+1}x_i$ bounds the outer face of a $\text{Tr}(36k)$ that is contained in $\text{Tr}(38k)$.

Now for each $h \in \{0, 1, \dots, 2k - 1\}$, we view the region enclosed by u , x_{i+1} and $y_{i,18h}$ without the closed region enclosed by u , x_{i+1} and $y_{i,18(h+1)}$ as a $\text{Tr}(17)$. Note that these copies of $\text{Tr}(17)$ share u, x_{i+1} as the only common vertices. Taking $y_{i,18h}$ to be the vertex z in Corollary 4.2, we conclude from Corollary 4.2 that there is a monochromatic C_4 in the $\text{Tr}(17)$. We denote this C_4 by G_h . Then $x_{i+1} \in V(G_h)$ and $\{u, y_{i,18h}\} \cap V(G_h) = \emptyset$. By pigeonhole principle, at least k of these C_4 s are of the same color, which clearly form a monochromatic F_k centered at x_{i+1} . ■

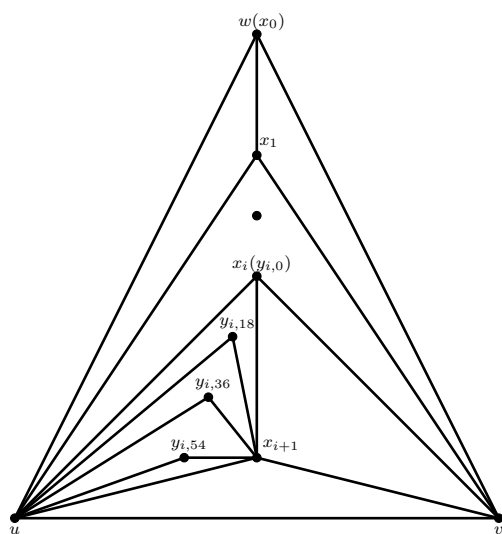


Figure 5: Lemma 4.3

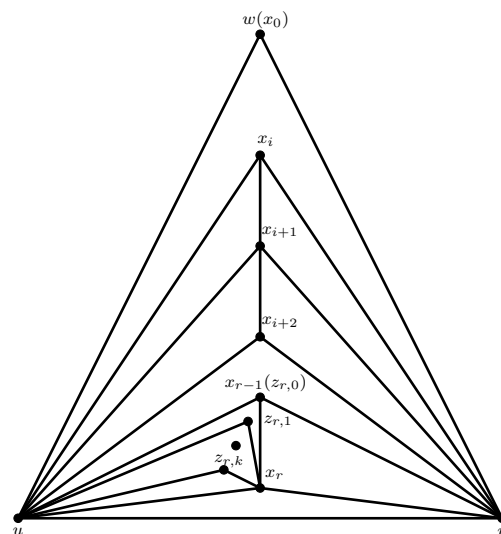


Figure 6: Lemma 4.4

Lemma 4.4. *Let k be a positive integer and let $uvwu$ bound the outer face of $\text{Tr}(9k + 2)$. Suppose $\sigma : E(\text{Tr}(9k + 2)) \rightarrow \{0, 1\}$ is a 2-edge-coloring such that $|\{\sigma(ux) : x \in V(\text{Tr}(9k + 2))\}| = 1$ and there is no monochromatic C_4 containing u . Then $\text{Tr}(9k + 2)$ contains monochromatic J_k centered at v .*

Proof. Without loss of generality, assume $\sigma(uv) = 0$. Then $\sigma(uy) = 0$ for all $y \in N(u)$. Let $x_0 := w$ and, for $1 \leq i \leq 8k + 1$, let $x_i \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$ such that x_i is inside $x_{i-1}uvx_{i-1}$. Since no monochromatic C_4 in $\text{Tr}(9k+2)$ contains u , there is at most one $i \in \{0, 1, 2, \dots, 8k+1\}$ such that $\sigma(x_iv) = 0$. Hence, there exists $i \in \{0, 1, \dots, 4k+2\}$ such that $\sigma(vx_j) = 1$ for $j \in \{i, i+1, \dots, i+4k-1\}$. We now make the following claim.

Claim. The subgraph of $\text{Tr}(9k+2)$ contained in the closed disc bounded by $vx_i \dots x_{i+3k-1}v$ has a monochromatic F_k of color 1 and centered at v , which we denote by F_v .

To show this, it suffices to show that for each r with $0 \leq r \leq k-1$, the subgraph of $\text{Tr}(9k+2)$ inside $vx_{i+3r}x_{i+3r+1}x_{i+3r+2}v$ (inclusive) contains a monochromatic C_4 of color 1 and containing v , as the union of such C_4 is an F_k centered at v . So fix an arbitrary r , with $0 \leq r \leq k-1$. Note that $\sigma(x_{i+3r}x_{i+3r+1}) = 1$ or $\sigma(x_{i+3r+1}x_{i+3r+2}) = 1$, for $0 \leq r \leq k-1$; for, otherwise, $x_{i+3r}x_{i+3r+1}x_{i+3r+2}ux_{i+3r}$ is a monochromatic C_4 of color 0 and containing u , a contradiction. Without loss of generality, assume $\sigma(x_{i+3r}x_{i+3r+1}) = 1$.

Let $y \in V(\text{Tr}(i+3r+2)) \setminus V(\text{Tr}(i+3r+1))$ such that y is inside $x_{i+3r}x_{i+3r+1}vx_{i+3r}$. If there are two edges in $\{yx_{i+3r}, yx_{i+3r+1}, yv\}$ of color 0, then one can easily find a monochromatic C_4 of color 0 and containing u , a contradiction. Hence, at least two of $\{\sigma(yx_{i+3r}), \sigma(yx_{i+3r+1}), \sigma(yv)\}$ are 1. So $\{y, x_{i+3r}, x_{i+3r+1}, v\}$ induces a subgraph which contains a monochromatic C_4 of color 1. This proves the claim. $\square$

Note that for $i+3k \leq r \leq i+4k-1$, $ux_r x_{r-1}u$ bounds the outer face of a $\text{Tr}(k+1)$. Let $z_{r,0} := x_{r-1}$ and, for $r \in \{i+3k, i+3k+1, \dots, i+4k-1\}$ and $\ell \in \{1, 2, \dots, k\}$, let $z_{r,\ell} \in V(\text{Tr}(r+\ell)) \setminus V(\text{Tr}(r+\ell-1))$ such that $z_{r,\ell}$ is inside $z_{r,\ell-1}x_r u z_{r,\ell-1}$. Because $\sigma(uz_{r,j}) = 0$ (by assumption) and $\text{Tr}(9k+2)$ has no monochromatic C_4 containing u , there is at most one $y \in \{z_{r,1}, z_{r,2}, \dots, z_{r,k}\}$ such that $\sigma(yx_r) = 0$. So there exists $k-1$ vertices in $\{z_{r,1}, \dots, z_{r,k}\}$ which together with $x_r v$ form a monochromatic $K_{1,k}$ of color 1 centered at x_r , which we denote by H_r . Now $H_{i+3k}, H_{i+3k+1}, \dots, H_{i+4k-1}$ form a monochromatic k -ary radius 2 tree rooted at v of color 1. This radius 2 tree and F_v form a monochromatic J_k of color 1, completing the proof of Lemma 4.4. $\blacksquare$

Now we are ready to prove the main result of this section, that is $\text{Tr}(100k) \rightarrow J_k$. Let $\sigma : E(\text{Tr}(100k)) \rightarrow \{0, 1\}$ be arbitrary. We show that σ always contains a monochromatic J_k . By Lemma 4.3, $\text{Tr}(76k)$ contains monochromatic copy of F_{2k} , say F , and, without loss of generality, assume it is of color 1. Let the C_4 s in F be $xa_{i,1}a_{i,2}a_{i,3}x$ for $i \in [2k]$. For $i \in [2k]$, let $b_i \in V(\text{Tr}(76k+1)) \setminus V(\text{Tr}(76k))$ such that b_i is inside $xa_{i,1}a_{i,2}a_{i,3}x$ and $a_{i,1}a_{i,2}b_i a_{i,1}$ bounds an inner face of $\text{Tr}(76k+1)$. Let A_i be the family of all vertices $a \in N(a_{i,1})$ inside $a_{i,1}a_{i,2}b_i a_{i,1}$ and satisfying $\sigma(aa_{i,1}) = 1$.

- (1) There exists some $i \in \{k+1, k+2, \dots, 2k\}$ such that $|A_i| < k$.

Otherwise, suppose that $|A_i| \geq k$ for all $i \in \{k+1, k+2, \dots, 2k\}$. Then let $Z_i := \{z_{i,1}, z_{i,2}, \dots, z_{i,k-1}\} \subseteq A_i$. Now, for each $i \in \{k+1, \dots, 2k\}$, $\{x, a_{i,1}\} \cup Z_i$ induces a graph containing a monochromatic $K_{1,k}$. Those $K_{1,k}$ s form a monochromatic radius-two

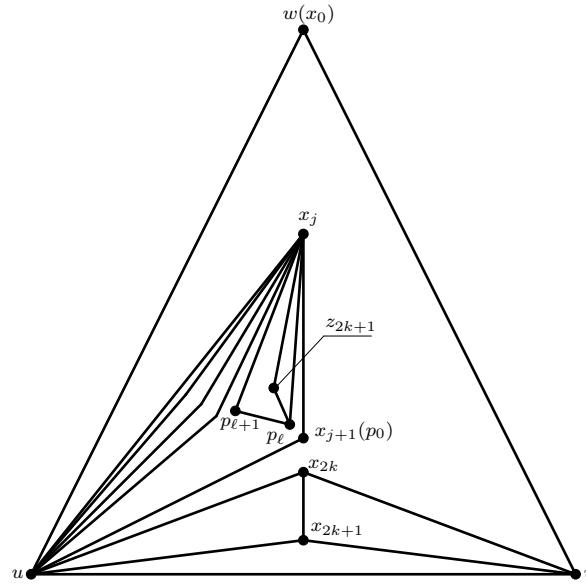


Figure 7

k -ary tree of color 1 and rooted at x , which we denote by T_x . The k four-cycles $xa_{i,1}a_{i,2}a_{i,3}x$ for $i \in [k]$ form a monochromatic F_k . Now $F_k \cup T_x$ is a monochromatic J_k . $\square$

Let $u := a_{i,1}$. By (1), there exists an edge $vw \in \text{Tr}(78k)$ such that $uvwu$ bounds an inner face of $\text{Tr}(78k)$ and $\sigma(uy) = 0$ for any $y \in N(u)$ in the closed disc bounded by $uvwu$.

Let G be the subgraph of $\text{Tr}(78k)$ contained in the closed disc bounded by $uvwu$ (see Figure 5). Clearly G is isomorphic to a copy of $\text{Tr}(22k)$. In the rest of the proof, we should only discuss the graph G and all $\text{Tr}(i)$ will be referred to this copy of $\text{Tr}(22k)$. Let $x_0 := w$ and for $i \in [4k]$, let $x_i \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$ such that x_i is inside $ux_{i-1}vu$.

- (2) G contains a monochromatic copy of F_k , say F' , which has color 0 and center u and is disjoint from the union of closed regions bounded by $ux_ix_{i+1}u$ over all $0 \leq i \leq 2k-1$.

If for each $i \in \{k, k+1, \dots, 2k-1\}$ there exists a monochromatic C_4 inside $ux_{2i}x_{2i+1}u$ and containing u , then these k monochromatic C_4 s of color 0 form a desired monochromatic F_k centered at u and thus (2) holds. Otherwise, since $ux_{2i}x_{2i+1}u$ bounds the outer face of a $\text{Tr}(9k+2)$, it follows from Lemma 4.4 that there exists a monochromatic J_k in G . $\square$

For $j \in \{0, 1, \dots, 2k-1\}$, let B_j be the family of all vertices $x \in N(x_j)$ inside $ux_jx_{j+1}u$ and satisfying $\sigma(xx_j) = 0$.

- (3) There exists some $j \in \{0, 1, \dots, k-1\}$ such that $|B_j| < k$.

Suppose to the contrary that there exist subsets $Z_j \subseteq B_j$ of size k for all $j \in \{0, 1, \dots, k-1\}$. Then each $Z_j \cup \{u, x_j\}$ induces a graph containing a monochromatic $K_{1,k}$ which is

centered at x_j and has color 0. These $K_{1,k}$ s together with F' form a monochromatic J_k of color 0. This proves (3). $\square$

Let $p_0 := x_{j+1}$ and for $1 \leq \ell \leq 4k$, let $p_\ell \in V(\text{Tr}(j + \ell + 1)) \setminus V(\text{Tr}(j + \ell))$ such that p_ℓ is inside $ux_j p_{\ell-1} u$. By (3), there exists some $0 \leq \ell \leq 4k - 1$ such that $\sigma(zx_j) = 1$ for any $z \in N(x_j)$ in the closed disc bounded by $x_j p_\ell p_{\ell+1} x_j$.

(4) There is a monochromatic F_k inside $x_j p_\ell p_{\ell+1} x_j$, say F'' , with color 1 and center x_j .

Let $z_0 := p_{\ell+1}$ and for $s \in [2k + 1]$, let $z_s \in V(\text{Tr}(j + \ell + s + 2)) \setminus V(\text{Tr}(j + \ell + s + 1))$ such that z_s is inside $x_j z_{s-1} p_\ell x_j$. Note that each $x_j z_{2s} z_{2s+1} x_j$ bounds a $\text{Tr}(9k + 3)$. If for each $s \in [k]$ there exists a monochromatic C_4 of color 1 inside $x_j z_{2s} z_{2s+1} x_j$ and containing x_j , then these monochromatic copies of C_4 form the desired monochromatic F_k centered at x_j . Otherwise, it follows from Lemma 4.4 that there exists a monochromatic J_k . $\square$

As $|B_j| < k$, there exists a subset $A \subseteq \{p_1, p_2, \dots, p_{4k}\}$ of size $2k$ such that $\sigma(\alpha x_j) = 1$ for each $\alpha \in A$ and moreover, there is no neighbors of A belonging to $V(F'')$. Let $A := \{\alpha_1, \dots, \alpha_{2k}\}$. Note that for each $h \in [2k]$, we have $\sigma(\alpha_h u) = 0$ and $\sigma(\alpha_h x_j) = 1$.

It is easy to see that there exist pairwise disjoint sets $N_h \subseteq N(\alpha_h)$ of size $2k$ for $h \in [2k]$. Then there exists $M_h \subseteq N_h$ such that $|M_h| = k - 1$ and $\sigma(x\alpha_h)$ is the same for all $x \in M_h$. This gives $2k$ monochromatic copies of $K_{1,k-1}$ with centers α_h for $h \in [2k]$. At least k of them (say with centers α_h for $h \in [k]$) have the same color. If this color is 0, these copies together with $\{u\alpha_h : h \in [k]\}$ and F' give a monochromatic J_k with color 0 and center u . Otherwise, this color is 1. Then these copies together with $\{x_j\alpha_h : h \in [k]\}$ and F'' give a monochromatic J_k with color 1 and center u . This proves $\text{Tr}(100k) \rightarrow J_k$. ■

5 Monochromatic bistar

In this section we prove $\text{Tr}(6k + 30) \rightarrow B_k$. We first establish the following lemma.

Lemma 5.1. *Let $uvwu$ be the triangle bounding the outer face of $\text{Tr}(k + 10)$. Let $\sigma : E(\text{Tr}(k + 10)) \rightarrow \{0, 1\}$ such that $|\{\sigma(ux) : x \in V(\text{Tr}(k + 10))\}| = 1$ and there is no monochromatic C_4 containing u . Then $\text{Tr}(k + 10)$ contains a monochromatic B_k .*

Proof. Without loss of generality, let $\sigma(uv) = 0$. Let $x_0 := w$ and, for $i \in [6]$, let $x_i \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i - 1))$ such that x_i is inside $uvx_{i-1}u$.

Since $\text{Tr}(k + 10)$ has no monochromatic C_4 containing u , we see that $|\{0 \leq i \leq 6 : \sigma(vx_i) = 0\}| \leq 1$. So there exists some $i \in \{0, 1, 2, 3, 4\}$ such that $\sigma(vx_i) = \sigma(vx_{i+1}) = \sigma(vx_{i+2}) = 1$. We have either $\sigma(x_i x_{i+1}) = 1$ or $\sigma(x_{i+1} x_{i+2}) = 1$; as otherwise $ux_i x_{i+1} x_{i+2} u$ is a monochromatic C_4 of color 0 and containing u , a contradiction. We consider two cases.

Case 1. $\sigma(x_i x_{i+1}) = \sigma(x_{i+1} x_{i+2})$.

In this case, we have $\sigma(x_i x_{i+1}) = \sigma(x_{i+1} x_{i+2}) = 1$. So $x_i x_{i+1} x_{i+2} v x_i$ is a monochromatic C_4 of color 1. Let $y_0 := x_{i+1}$ and for $\ell \in [k + 1]$, let $y_\ell \in V(\text{Tr}(i + 1 + \ell)) \setminus V(\text{Tr}(i + \ell))$

such that y_ℓ is inside $uy_{\ell-1}x_iu$. Similarly let $z_0 := x_{i+1}$ and for $\ell \in [k+1]$, let $z_\ell \in V(\text{Tr}(i+2+\ell)) \setminus V(\text{Tr}(i+1+\ell))$ such that z_ℓ is inside $uz_{\ell-1}x_{i+2}u$.

Since $\text{Tr}(k+10)$ has no monochromatic C_4 containing u , this shows that $|\{\ell \in [k+1] : \sigma(x_i y_\ell) = 0\}| \leq 1$ and $|\{\ell \in [k+1] : \sigma(x_{i+2} z_\ell) = 0\}| \leq 1$. Therefore, there exist $Y \subseteq \{y_\ell : \ell \in [k+1]\}$ and $Z \subseteq \{z_\ell : \ell \in [k+1]\}$ such that $|Y| = |Z| = k$, $\sigma(yx_i) = 1$ for each $y \in Y$ and $\sigma(zx_{i+2}) = 1$ for each $z \in Z$. Hence, $\text{Tr}(k+10)$ has two monochromatic $K_{1,k}$ s of color 1 with centers x_i, x_{i+1} and leave sets Y, Z , respectively. These two $K_{1,k}$ s together with $vx_i x_{i+1} x_{i+2} v$ form a monochromatic B_k of color 1.

Case 2. $\sigma(x_i x_{i+1}) \neq \sigma(x_{i+1} x_{i+2})$.

Without loss of generality, let $\sigma(x_i x_{i+1}) = 0$ and $\sigma(x_{i+1} x_{i+2}) = 1$. Let $y \in V(\text{Tr}(i+2)) \setminus V(\text{Tr}(i+1))$ be inside $ux_i x_{i+1} u$. Because $\sigma(uy) = 0$ and $\text{Tr}(k+10)$ has no monochromatic C_4 containing u , $\sigma(yx_i) = \sigma(yx_{i+1}) = 1$. Therefore, $yx_{i+1} vx_i y$ is a monochromatic C_4 of color 1. Let $y_0 := y$ and, for $\ell \in [k+1]$, let $y_\ell \in V(\text{Tr}(i+2+\ell)) \setminus V(\text{Tr}(i+1+\ell))$ such that y_ℓ is inside $uy_{\ell-1}x_iu$. Let $z_0 := y$ and, for $\ell \in [k+1]$, let $z_\ell \in V(\text{Tr}(i+2+\ell)) \setminus V(\text{Tr}(i+1+\ell))$ such that z_ℓ is inside $uz_{\ell-1}x_{i+1}u$.

The remaining proof is similar as in Case 1. We observe that $|\{\ell \in [k+1] : \sigma(x_i y_\ell) = 0\}| \leq 1$ and $|\{\ell \in [k+1] : \sigma(x_{i+1} z_\ell) = 0\}| \leq 1$. Therefore, there exist $Y \subseteq \{y_\ell : \ell \in [k+1]\}$ and $Z \subseteq \{z_\ell : \ell \in [k+1]\}$ such that $|Y| = |Z| = k$, $\sigma(yx_i) = 1$ for $y \in Y$, and $\sigma(zx_{i+1}) = 1$ for $z \in Z$. Hence, $\text{Tr}(k+10)$ has two monochromatic $K_{1,k}$ s of color 1 with centers x_i, x_{i+1} and leave sets Y, Z , respectively. These two $K_{1,k}$ s together with $yx_{i+1} vx_i y$ form a monochromatic B_k of color 1. This proves Lemma 5.1. ■

We are ready to prove $\text{Tr}(6k+30) \rightarrow B_k$. Let $\sigma : E(\text{Tr}(6k+30)) \rightarrow \{0, 1\}$. By Lemma 4.1, each copy of $\text{Tr}(16)$ with the same outer face as of $\text{Tr}(6k+30)$ contains a monochromatic C_4 , say $u_1 u_2 u_3 u_4 u_1$ of color 1 (see Figure 6). For each $i \in \{1, 3\}$, let $v_i w_i$ be an edge in $\text{Tr}(18)$ such that $u_i v_i w_i u_i$ is a triangle inside $u_1 u_2 u_3 u_4 u_1$. Note that $u_i v_i w_i u_i$ bounds the outer face of a $\text{Tr}(6k+12)$. Let A_i be the family of all vertices $x \in N(u_i)$ inside $u_i v_i w_i u_i$ and satisfying $\sigma(xu_i) = 1$. If $|A_1| \geq k$ and $|A_3| \geq k$, then together with the monochromatic 4-cycle $u_1 u_2 u_3 u_4 u_1$, it is easy to form a monochromatic B_k of color 1.

Hence by symmetry, we may assume that $|A_1| < k$. Then there exists an edge vw in $\text{Tr}(18+k)$ such that $u_1 v w u_1$ bounds an inner face of $\text{Tr}(18+k)$ and $\sigma(u_1 x) = 0$ for all $x \in N(u_1)$ in the closed disc bounded by $u_1 v w u_1$. We may assume that the induced subgraph contained in the closed disc bounded by $u_1 v w u_1$ has a monochromatic C_4 say $u_1 x y z u_1$ (as otherwise, it contains a B_k by Lemma 5.1). Furthermore, we have $\{x, y, z\} \subseteq V(\text{Tr}(2k+28))$.

Let $\{p_0, q_0\} \subseteq V(\text{Tr}(2k+29)) \setminus V(\text{Tr}(2k+28))$ such that both xyp_0x and yzq_0y bound two inner faces of $\text{Tr}(2k+29)$. For $\ell \in [3k]$, let $p_\ell \in V(\text{Tr}(2k+29+\ell)) \setminus V(\text{Tr}(2k+28+\ell))$ such that p_ℓ is inside $xp_{\ell-1}yx$. Similarly, for $\ell \in [3k]$, let $q_\ell \in V(\text{Tr}(2k+29+\ell)) \setminus V(\text{Tr}(2k+28+\ell))$ such that q_ℓ is inside $yq_{\ell-1}zy$. Moreover, let

$$B_1 := \{p \in N(x) : p \text{ is inside } xp_0yx \text{ and } \sigma(xp) = 0\},$$

$$B_2 := \{q \in N(z) : q \text{ is inside } yq_0zy \text{ and } \sigma(zq) = 0\}.$$

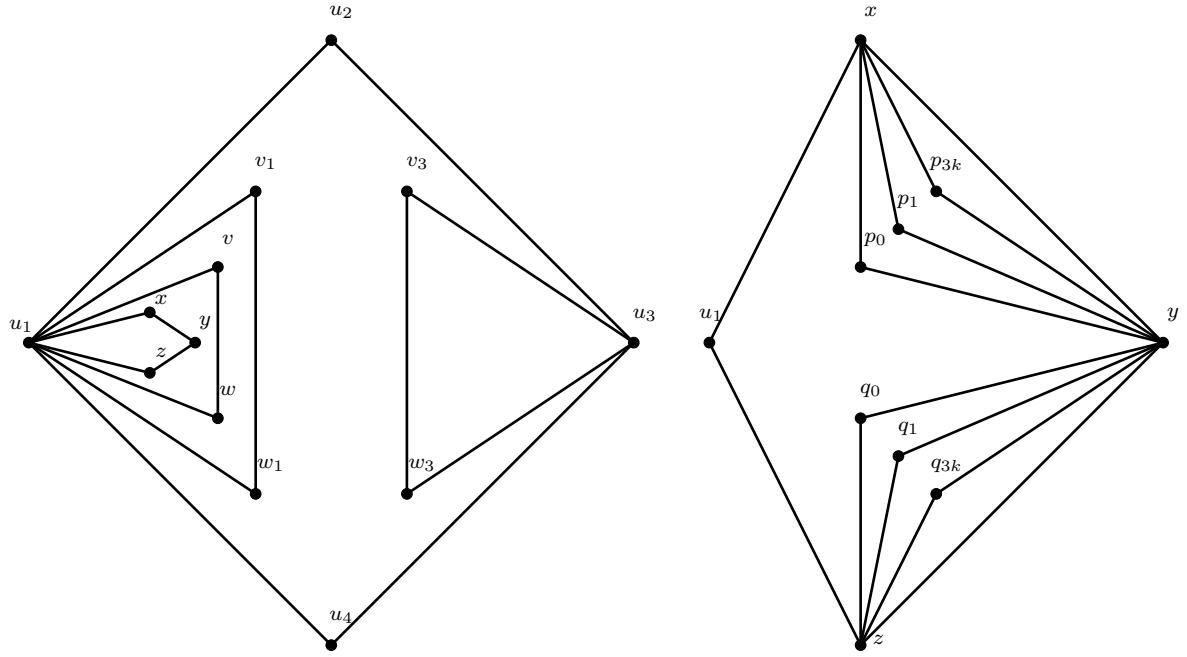


Figure 8

If $|B_1| \geq k$ and $|B_2| \geq k$, we can find two monochromatic $K_{1,k}$ s of color 0, one inside xp_0yx rooted at x and one inside yq_0zy rooted at z ; these two $K_{1,k}$ s and $u_1xyz u_1$ form a monochromatic B_k of color 0. So we may assume, without loss of generality, that $|B_1| < k$.

Let $C := \{\ell \in [3k] : \sigma(y p_\ell) = 0\}$. We claim $|C| < k$. Suppose to the contrary that $|C| \geq k$. Then there is a monochromatic $K_{1,k}$ with root y and leaves in C of color 0. Since $\sigma(u_1 p) = 0$ for all $p \in N(u_1)$ inside u_1vw , there is also a monochromatic $K_{1,k}$ with root u_1 and leaves inside $u_1xyz u_1$ of color 0. Now these two $K_{1,k}$ s and $u_1xyz u_1$ form a monochromatic B_k of color 0.

So $|B_1| < k$ and $|C| < k$. Then there exist p_h, p_s with $h, s \in [3k]$ such that $\sigma(p_h x) = \sigma(p_h y) = \sigma(p_s x) = \sigma(p_s y) = 1$. Because xp_0p_1x bounds an inner face of $\text{Tr}(2k + 30)$, it also bounds the outer face of a $\text{Tr}(4k)$. As $|B_1| < k$, there exists a monochromatic $K_{1,k}$ of color 1 with the root x and k leaves inside xp_0p_1x . Similarly, as $|C| < k$, there exists a monochromatic $K_{1,k}$ of color 1 with root y and k leaves inside yp_0p_1y . Now these two $K_{1,k}$ s and the 4-cycle xp_hyp_sx form a monochromatic B_k of color 1. This proves that $\text{Tr}(6k + 30) \rightarrow B_k$ and thus completes the proof of Theorem 1.2. ■

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