The Spectrum Problem for Some Digraphs of Order 4 and Size 6

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Abstract

Consider the multigraph obtained by adding a double edge to $K_4 - e$. Now, let D be a directed graph obtained by orientating the edges of that multigraph. We establish necessary and sufficient conditions on n for the existence of a (K_n^*, D)-design for four such orientations.

1 Introduction

Let $\mathbb{Z}_m$ denote the set of integers modulo m. For a graph H, let $V(H)$ and $E(H)$ denote the vertex set of H and the edge set of H, respectively. Similarly, for a digraph D, let $V(D)$ and $A(D)$ denote the vertex set of D and the arc set of D, respectively. The *order* and the *size* of a graph H (or digraph D) are $|V(H)|$ and $|E(H)|$ (or $|V(D)|$ and $|A(D)|$), respectively.

We denote the complete multipartite graph with parts of sizes a_i for $1 \leq i \leq m$ by $K_{a_1,a_2,\ldots,a_m}$. If $a_i = a$ for all $i \in \{1,\ldots,m\}$,

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then we use the notation $K_{m\times a}$. Furthermore, let $V(K_{m\times a}) = \mathbb{Z}_{ma}$ with vertex partition $\{V_0, V_1, \ldots, V_{m-1}\}$ where $V_i = \{j \in \mathbb{Z}_{ma} : j \equiv i \pmod{m}\}$. Then $E(K_{m\times a})$ consists of all edges $\{i, j\}$ such that $i \not\equiv j \pmod{m}$.

The *complete graph of order n with a hole of size t*, denoted $K_n \setminus K_t$, is the graph with vertex set V and edge set $\{\,\{a, b\} : a \in V,\, b \in V \setminus U,\, a \neq b\,\}$ where $|V| = n$, $U \subseteq V$, and $|U| = t$. The vertices in U are said to be the *vertices in the hole.*

Let tG denote the graph consisting of t vertex-disjoint copies of G. The *join* of two vertex-disjoint graphs G and H, denoted $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{a, b\} : a \in V(G), b \in V(H)\}$. For example, K_{5x+1} could be described as $(xK_5 \vee K_1) \cup K_{x\times 5}$. Note that, by convention, the union of two graphs implies the graphs are edge-disjoint, but not necessarily vertex-disjoint.

Let H be a graph and let $\mathcal{G}$ be a set of subgraphs of H. We will refer to a graph $G \in \mathcal{G}$ as a *G-block.* A *$\mathcal{G}$-decomposition of H* is a set $\Delta = \{G_1, G_2, \ldots, G_r\}$ of pairwise edge-disjoint subgraphs of H such that for every $i \in [1, r]$, $G_i \cong G$ for some $G \in \mathcal{G}$ and such that $E(H) = \bigcup_{i=1}^{r} E(G_i)$. Of particular importance is when $\mathcal{G} = \{G\}$, in which case we write "G-decomposition of H" instead of "$\{G\}$-decomposition of H." A G-decomposition of K_n is also known as a *(K_n, G)-design.* The set of all n for which K_n admits a G-decomposition is called the *spectrum of G.* The spectrum has been determined for many classes of graphs, including all graphs on at most 4 vertices [4] and all graphs on 5 vertices (see [3] and [12]). We direct the reader to [2] and [5] for recent surveys on graph decompositions.

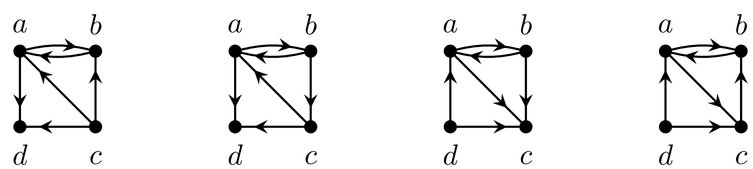
1.1 Definitions for Digraphs

Similar concepts to those defined above for undirected graphs can be defined for digraphs. First, we introduce additional notation. For an undirected graph G, let G^* denote the digraph obtained from G by replacing each edge $\{u, v\} \in E(G)$ with the arcs (u, v) and (v, u). Thus K_n^*, the *complete digraph of order n*, is the digraph on n vertices with the arcs (u, v) and (v, u) between every pair of distinct vertices u and v.

Let H and D be digraphs such that D is a subgraph of H. A *D-decomposition* of H is a set $\Delta = \{D_1, D_2, \ldots, D_r\}$ of pairwise arc-disjoint subgraphs of H each of which is isomorphic to D and such that $A(H) = \bigcup_{i=1}^{r} A(D_i)$. As with the undirected case, a D-decomposition of K_n^* is also known as a *(K_n^*, D)-design*, and the set of all n for which K_n^* admits a D-decomposition is called the *spectrum of D*.

The spectra for several digraphs of small order at most 4 have been determined. This includes the spectra for all digraphs on at most 3 vertices [14], all bipartite digraphs on 4 vertices (see [9]), the orientations of a triangle with a pendent edge (see [6] and [8]), and several of the orientations of $K_4 - e$ (see [7]).

In this paper, we extend the known results on the spectra of digraphs of order 4 by determining the spectra for the four digraphs found in Figure 1. We use the naming convention found in *An Atlas of Graphs* [15] by Read and Wilson. For example, D113$[a,b,c,d]$ refers to the digraph with vertex set $\{a,b,c,d\}$ and arc set $\big\{(a,b), (a,d), (b,a), (c,a), (c,b), (c,d)\big\}$.



D113$[a,b,c,d]$ D119$[a,b,c,d]$ D121$[a,b,c,d]$ D147$[a,b,c,d]$

Figure 1: The 4 digraphs for which we settle the spectrum. Note that these are 4 possible orientations of a multigraph obtained from adding a double edge to $K_4 - e$.

1.2 Some Basic Results

The obvious necessary conditions for a digraph D to decompose K_n^* are

(A) $|V(D)| \le n$,
(B) $|A(D)|$ divides $|A(K_n^*)| = n(n-1)$, and
(C) both $\gcd\{\text{outdegree}(v) : v \in V(D)\}$ and $\gcd\{\text{indegree}(v) : v \in V(D)\}$ divide $n-1$.

Applying these necessary conditions to the 4 digraphs under consideration, we obtain the following necessary conditions:

Lemma 1. *For $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$, a (K_n^*, D)-design exists only if $n \geq 7$ and $n \equiv 1$ or $3 \pmod 6$.*

Given a digraph D, the *reverse orientation of* D, denoted $\mathrm{Rev}(D)$, is the digraph with vertex set $V(D)$ and arc set $\{(v,u) : (u,v) \in A(D)\}$. We make use of the following fact that was first noted in [9]:

Observation 2 ([9])**.** Let D and H be digraphs. A D-decomposition of H exists if and only if a $(\mathrm{Rev}(D))$-decomposition of $\mathrm{Rev}(H)$ exists.

The fact that $K_n^* \cong \mathrm{Rev}(K_n^*)$ leads to the following corollary:

Corollary 3. *Let D be a digraph. A (K_n^*, D)-design exists if and only if a $(K_n^*, \mathrm{Rev}(D))$-design exists.*

Note that the 4 digraphs of interest in this paper occur in pairs with respect to their reverse orientations. Namely, D113 $\cong$ Rev(D121) and D119 $\cong$ Rev(D147).

The following theorems on decompositions of complete graphs and complete multipartite graphs are crucial for proving our main results. Note that these background results concern graphs, as opposed to digraphs. All of these results can be found in the *Handbook of Combinatorial Designs* [10] (see for example [1] and [11]).

Theorem 4 ([10])**.** *If n is an odd positive integer, then there exists a $\{K_3, K_5\}$-decomposition of K_n.*

Theorem 5 ([10])**.** *Let $t \geq 3$. There exists a K_3-decomposition of $K_{t\times 2}$ if $t \equiv 0$ or $1 \pmod 3$ and of $K_{4,\,(t-2)\times 2}$ if $t \equiv 2 \pmod 3$.*

Theorem 6 ([10])**.** *Let $t \geq 4$. There exists a K_4-decomposition of $K_{t\times 3}$ if $t \equiv 0$ or $1 \pmod 4$ and of $K_{6,\,(t-2)\times 3}$ if $t \equiv 2$ or $3 \pmod 4$ and $t \neq 6$.*

The following is a well-known result that is a special case of Wilson's Fundamental Construction (see [13]).

Theorem 7 ([13])**.** *Let m, n, r, s, and t be positive integers. If there exists a $(K_{t\times m}, K_n)$-design, then there exists a $(K_{t\times ms}, K_{n\times s})$-design. Similarly, if there exists a $(K_{r,\,t\times m}, K_n)$-design, then there exists a $(K_{rs,\,t\times ms}, K_{n\times s})$-design.*

2 Examples of Small Designs

We now turn our attention to the designs of small order which will be used for the general constructions.

Given a digraph represented by the notation $D[a, b, c, d]$ and some $i \in \mathbb{Z}_n$, we define $D[a, b, c, d] + i = D[a + i, b + i, c + i, d + i]$ where all addition is performed in $\mathbb{Z}_n$. By convention, define $\infty + 1 = \infty$.

Example 1. *There exists a (K_7^*, D)-design for $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.*

Let $V(K_7^*) = \mathbb{Z}_7$.
A $(K_7^*, \mathrm{D113})$-design is given by $\{\mathrm{D113}[0, 1, 4, 2] + i : i \in \mathbb{Z}_7\}$.
A $(K_7^*, \mathrm{D119})$-design is given by $\{\mathrm{D119}[0, 1, 5, 3] + i : i \in \mathbb{Z}_7\}$.
Applying Corollary 3, we obtain the remaining designs.

Example 2. *There exists a (K_9^*, D)-design for $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.*

Let $V(K_9^*) = \mathbb{Z}_2 \times \mathbb{Z}_4 \cup \{\infty\}$.
A $(K_9^*, \mathrm{D113})$-design is given by

$$\begin{aligned} &\{\mathrm{D113}[(1, 1 + i), (1, 0 + i), (0, 0 + i), (0, 1 + i)] : i \in \mathbb{Z}_4\} \\ &\quad \cup \{\mathrm{D113}[(0, 3 + i), (1, 2 + i), (1, 0 + i), (0, 2 + i)] : i \in \mathbb{Z}_4\} \\ &\qquad \cup \{\mathrm{D113}[\infty, (0, 3 + i), (0, 1 + i), (1, 3 + i)] : i \in \mathbb{Z}_4\}. \end{aligned}$$

A $(K_9^*, \mathrm{D119})$-design is given by

$$\begin{aligned} &\{\mathrm{D119}[(0, 3 + i), (0, 0 + i), (1, 1 + i), (1, 2 + i)] : i \in \mathbb{Z}_4\} \\ &\quad \cup \{\mathrm{D119}[(0, 3 + i), (1, 3 + i), (1, 2 + i), (0, 1 + i)] : i \in \mathbb{Z}_4\} \\ &\qquad \cup \{\mathrm{D119}[\infty, (0, 3 + i), (1, 1 + i), (1, 3 + i)] : i \in \mathbb{Z}_4\}. \end{aligned}$$

Applying Corollary 3, we obtain the remaining designs.

Example 3. *There exists a (K_{13}^*, D)-design for $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.*

Let $V(K_{13}^*) = \mathbb{Z}_{13}$.
A $(K_{13}^*, \mathrm{D113})$-design is given by

$$\{\mathrm{D113}[0, 4, 6, 8] + i : i \in \mathbb{Z}_{13}\} \cup \{\mathrm{D113}[1, 0, 8, 11] + i : i \in \mathbb{Z}_{13}\}.$$

A $(K_{13}^*, \mathrm{D119})$-design is given by

$$\{\mathrm{D119}[0, 4, 5, 7] + i : i \in \mathbb{Z}_{13}\} \cup \{\mathrm{D119}[9, 12, 10, 2] + i : i \in \mathbb{Z}_{13}\}.$$

Applying Corollary 3, we obtain the remaining designs.

Example 4. *There exists a (K_{15}^*, D)-design for $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.*

Let $V(K_{15}^*) = \mathbb{Z}_2 \times \mathbb{Z}_7 \cup \{\infty\}$.
A $(K_{15}^*, \mathrm{D113})$-design is given by

$$\begin{aligned}
&\{\mathrm{D113}[(0,4+i),(0,1+i),(1,0+i),(1,1+i)] : i \in \mathbb{Z}_7\}\\
&\quad \cup \{\mathrm{D113}[(0,6+i),(0,0+i),(1,4+i),(1,1+i)] : i \in \mathbb{Z}_7\}\\
&\quad\quad \cup \{\mathrm{D113}[(1,2+i),(1,1+i),(0,2+i),(0,0+i)] : i \in \mathbb{Z}_7\}\\
&\quad\quad\quad \cup \{\mathrm{D113}[(1,4+i),(0,3+i),(0,1+i),(1,0+i)] : i \in \mathbb{Z}_7\}\\
&\quad\quad\quad\quad \cup \{\mathrm{D113}[\infty,(1,6+i),(1,0+i),(0,0+i)] : i \in \mathbb{Z}_7\}.
\end{aligned}$$

A $(K_{15}^*, \mathrm{D119})$-design is given by

$$\begin{aligned}
&\{\mathrm{D119}[(0,6+i),(0,0+i),(1,2+i),(1,3+i)] : i \in \mathbb{Z}_7\}\\
&\quad \cup \{\mathrm{D119}[(0,3+i),(0,0+i),(1,0+i),(1,6+i)] : i \in \mathbb{Z}_7\}\\
&\quad\quad \cup \{\mathrm{D119}[(0,5+i),(1,3+i),(1,0+i),(0,0+i)] : i \in \mathbb{Z}_7\}\\
&\quad\quad\quad \cup \{\mathrm{D119}[(1,2+i),(0,4+i),(1,5+i),(1,0+i)] : i \in \mathbb{Z}_7\}\\
&\quad\quad\quad\quad \cup \{\mathrm{D119}[\infty,(1,1+i),(0,0+i),(0,5+i)] : i \in \mathbb{Z}_7\}.
\end{aligned}$$

Applying Corollary 3, we obtain the remaining designs.

Example 5. *There exists a (K_{21}^*, D)-design for $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.*

Let $V(K_{21}^*) = \mathbb{Z}_{21}$.
A $(K_{21}^*, \mathrm{D113})$-design is given by

$$\begin{aligned}
&\{\mathrm{D113}[0,3,9,1], \mathrm{D113}[0,4,15,2], \mathrm{D113}[0,7,12,5],\\
&\ \mathrm{D113}[0,11,1,19], \mathrm{D113}[0,17,13,20], \mathrm{D113}[1,7,5,12],\\
&\ \mathrm{D113}[1,2,8,6], \mathrm{D113}[1,14,11,3], \mathrm{D113}[1,17,19,10],\\
&\ \mathrm{D113}[2,14,17,18]\}.
\end{aligned}$$

A $(K_{21}^*, \mathrm{D119})$-design is given by

$$\begin{aligned}
&\{\mathrm{D119}[7,15,18,5], \mathrm{D119}[2,16,0,4], \mathrm{D119}[4,3,18,13],\\
&\ \mathrm{D119}[0,9,7,18], \mathrm{D119}[0,5,4,14], \mathrm{D119}[0,11,19,20],\\
&\ \mathrm{D119}[0,17,8,6], \mathrm{D119}[1,4,17,13], \mathrm{D119}[1,7,11,17],\\
&\ \mathrm{D119}[2,20,8,9]\}.
\end{aligned}$$

Applying Corollary 3, we obtain the remaining designs.

Example 6. *There exists a (K_{25}^*, D)-design for $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.*

Let $V(K_{25}^*) = \mathbb{Z}_{25}$.
A $(K_{25}^*, \mathrm{D113})$-design is given by

$$\{\mathrm{D113}[0,1,13,19] + i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D113}[0,3,17,21] + i : i \in \mathbb{Z}_{25}\}$$
$$\cup \{\mathrm{D113}[0,5,15,17] + i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D113}[0,7,16,14] + i : i \in \mathbb{Z}_{25}\}.$$

A $(K_{25}^*, \mathrm{D119})$-design is given by

$$\{\mathrm{D119}[0,12,2,8] + i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D119}[0,24,6,10] + i : i \in \mathbb{Z}_{25}\}$$
$$\cup \{\mathrm{D119}[0,22,14,16] + i : i \in \mathbb{Z}_{25}\} \cup \{\mathrm{D119}[0,20,4,18] + i : i \in \mathbb{Z}_{25}\}.$$

Applying Corollary 3, we obtain the remaining designs.

Example 7. *There exists a (K_{39}^*, D)-design for $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.*

Let $V(K_{39}^*) = \mathbb{Z}_{39}$.
A $(K_{39}^*, \mathrm{D113})$-design is given by

$$\{\mathrm{D113}[25,24,19,11], \mathrm{D113}[13,25,23,36], \mathrm{D113}[26,20,3,5],$$
$$\mathrm{D113}[17,9,34,27], \mathrm{D113}[34,13,21,35], \mathrm{D113}[18,6,29,7], \mathrm{D113}[0,10,17,5],$$
$$\mathrm{D113}[4,6,5,0], \mathrm{D113}[0,3,9,15], \mathrm{D113}[0,18,31,9], \mathrm{D113}[0,19,15,22],$$
$$\mathrm{D113}[0,20,35,11], \mathrm{D113}[0,32,2,38], \mathrm{D113}[1,25,17,38], \mathrm{D113}[1,31,6,35],$$
$$\mathrm{D113}[1,5,35,8], \mathrm{D113}[1,14,37,34], \mathrm{D113}[1,20,24,11], \mathrm{D113}[1,29,26,12]\}.$$

A $(K_{39}^*, \mathrm{D119})$-design is given by

$$\{\mathrm{D119}[25,29,12,37], \mathrm{D119}[35,34,26,15], \mathrm{D119}[5,25,24,36],$$
$$\mathrm{D119}[23,17,28,13], \mathrm{D119}[5,18,36,8], \mathrm{D119}[4,2,7,33], \mathrm{D119}[0,3,9,24],$$
$$\mathrm{D119}[0,4,6,27], \mathrm{D119}[0,7,30,1], \mathrm{D119}[0,19,22,16], \mathrm{D119}[0,28,34,2],$$
$$\mathrm{D119}[0,31,2,29], \mathrm{D119}[0,23,14,35], \mathrm{D119}[0,32,5,22], \mathrm{D119}[0,38,35,14],$$
$$\mathrm{D119}[1,19,32,16], \mathrm{D119}[1,31,6,23], \mathrm{D119}[1,26,13,29], \mathrm{D119}[2,26,36,34]\}.$$

Applying Corollary 3, we obtain the remaining designs.

Example 8. *There exists a* $(K^*_{4\times 2}, D)$*-design* $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.

Let $V(K^*_{4\times 2}) = \mathbb{Z}_8$ with partition $\big\{\{0,4\},\{1,5\},\{2,6\},\{3,7\}\big\}$.
A $(K^*_{4\times 2}, \mathrm{D113})$-design is given by

$$\{\mathrm{D113}[0,1,2,5], \mathrm{D113}[0,6,3,2], \mathrm{D113}[0,7,5,3], \mathrm{D113}[4,1,3,5], \mathrm{D113}[4,2,5,6], \mathrm{D113}[4,7,6,3], \mathrm{D113}[2,7,1,3], \mathrm{D113}[6,1,7,5]\}.$$

A $(K^*_{4\times 2}, \mathrm{D119})$-design is given by

$$\{\mathrm{D119}[0,3,1,2] + i : i \in \mathbb{Z}_8\}.$$

Applying Corollary 3, we obtain the remaining designs.

Example 9. *There exists a* $(K^*_{3\times 6}, D)$*-design for* $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.

Let $V(K^*_{3\times 6}) = \mathbb{Z}_{18}$ with partition $\big\{\{j \in \mathbb{Z}_{18} : j \equiv i \pmod 3\} : i \in \mathbb{Z}_3\big\}$.
A $(K^*_{3\times 6}, \mathrm{D113})$-design is given by

$$\{\mathrm{D113}[2,12,13,15] + i : i \in \mathbb{Z}_{18}\} \cup \{\mathrm{D113}[1,15,14,12] + i : i \in \mathbb{Z}_{18}\}.$$

A $(K^*_{3\times 6}, \mathrm{D119})$-design is given by

$$\{\mathrm{D119}[2,12,13,15] + i : i \in \mathbb{Z}_{18}\} \cup \{\mathrm{D119}[1,15,14,12] + i : i \in \mathbb{Z}_{18}\}.$$

Applying Corollary 3, we obtain the remaining designs.

Example 10. *There exists a* $(K^*_{5\times 6}, D)$*-design for* $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.

Let $V(K^*_{5\times 6}) = \mathbb{Z}_{30}$ with partition $\big\{\{j \in \mathbb{Z}_{30} : j \equiv i \pmod 5\} : i \in \mathbb{Z}_5\big\}$.
A $(K^*_{5\times 6}, \mathrm{D113})$-design is given by

$$\{\mathrm{D113}[0,2,21,7] + i : i \in \mathbb{Z}_{30}\} \cup \{\mathrm{D113}[1,7,8,22] + i : i \in \mathbb{Z}_{30}\} \cup \{\mathrm{D113}[3,16,15,11] + i : i \in \mathbb{Z}_{30}\} \cup \{\mathrm{D113}[2,29,10,14] + i : i \in \mathbb{Z}_{30}\}.$$

A $(K^*_{5\times 6}, \mathrm{D119})$-design is given by

$$\{\mathrm{D119}[0,2,21,7] + i : i \in \mathbb{Z}_{30}\} \cup \{\mathrm{D119}[1,7,8,22] + i : i \in \mathbb{Z}_{30}\} \cup \{\mathrm{D119}[3,16,15,11] + i : i \in \mathbb{Z}_{30}\} \cup \{\mathrm{D119}[2,29,10,14] + i : i \in \mathbb{Z}_{30}\}.$$

Applying Corollary 3, we obtain the remaining designs.

Example 11. *There exists a* $(K_9^* \setminus K_3^*, D)$*-design for* $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.

Let $V((K_9^* \setminus K_3^*) = \mathbb{Z}_6 \cup \{\infty_1, \infty_2, \infty_3\}$ where ∞_1, ∞_2, and ∞_3 are the vertices in the hole.
A $(K_9^* \setminus K_3^*, \mathrm{D113})$-design is given by

$$\begin{aligned}&\{\mathrm{D113}[0,2,5,\infty_3], \mathrm{D113}[5,\infty_1,0,\infty_2], \mathrm{D113}[\infty_1,1,2,3],\\ &\ \mathrm{D113}[\infty_2,1,3,2], \mathrm{D113}[\infty_3,2,1,3], \mathrm{D113}[1,0,\infty_3,5],\\ &\ \mathrm{D113}[3,0,\infty_2,5], \mathrm{D113}[4,\infty_2,2,5], \mathrm{D113}[4,\infty_3,3,\infty_1],\\ &\ \mathrm{D113}[4,0,\infty_1,2], \mathrm{D113}[4,1,5,3]\}.\end{aligned}$$

A $(K_9^* \setminus K_3^*, \mathrm{D119})$-design is given by

$$\begin{aligned}&\{\mathrm{D119}[0,4,\infty_1,1], \mathrm{D119}[\infty_1,5,1,2], \mathrm{D119}[0,3,2,\infty_1],\\ &\ \mathrm{D119}[3,\infty_1,4,1], \mathrm{D119}[\infty_2,0,2,3], \mathrm{D119}[\infty_2,5,3,4],\\ &\ \mathrm{D119}[\infty_2,1,4,2], \mathrm{D119}[5,2,4,\infty_3], \mathrm{D119}[5,0,\infty_3,4],\\ &\ \mathrm{D119}[\infty_3,2,1,0], \mathrm{D119}[3,\infty_3,1,5]\}.\end{aligned}$$

Applying Corollary 3, we obtain the remaining designs.

3 Main Results

We finally address the general constructions needed to piece together the small designs mentioned previously to prove sufficiency of the necessary conditions.

Theorem 8. *If* $n \equiv 1 \pmod 6$ *and* $n \geq 7$*, then a* (K_n^*, D)*-design exists for* $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.

Proof. Let $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$ and let $n = 6x + 1$ for some positive integer x. When x is 1, 2, or 4, the result follows from Examples 1, 3, and 6, respectively. The remainder of the proof breaks into two cases.

Case 1: x is odd with $x \geq 3$.
By Theorem 4 there exists a $\{K_3, K_5\}$-decomposition of K_x. Thus, by Theorem 7, there exists a $\{K_{3\times 6}, K_{5\times 6}\}$-decomposition of $K_{x\times 6}$. Note that $K_{6x+1} = (xK_6 \vee K_1) \cup K_{x\times 6} = K_{x\times 6} \cup \bigcup_{i=1}^{x} K_7$. Thus, $K_n^* = K_{x\times 6}^* \cup \bigcup_{i=1}^{x} K_7^*$. Since there exists a $(K_{3\times 6}^*, D)$-design (by Example 9) and there exists a $(K_{5\times 6}^*, D)$-design (by Example 10),

there exists a $(K^*_{x\times 6}, D)$-design. Since there also exists a (K^*_7, D)-design (by Example 1), there exists a (K^*_n, D)-design.

Case 2: x is even with $x \geq 6$.
Let $x = 2y$ for some integer $y \geq 3$. Hence, $n = 6(2y) + 1 = 12y + 1$.

Subcase 2.1: $y \equiv 0$ or $1 \pmod 3$.
By Theorem 5 there exists a K_3-decomposition of $K_{y\times 2}$. Thus, by Theorem 7, there exists a $K_{3\times 6}$-decomposition of $K_{y\times 12}$. Note that $K_{12y+1} = (yK_{12} \vee K_1) \cup K_{y\times 12} = K_{y\times 12} \cup \bigcup_{i=1}^{y} K_{13}$. Thus, $K^*_n = K^*_{y\times 12} \cup \bigcup_{i=1}^{y} K^*_{13}$. Since there exists a $(K^*_{3\times 6}, D)$-design (by Example 9), there exists a $(K^*_{y\times 12}, D)$-design. Since there also exists a (K^*_{13}, D)-design (by Example 3), there exists a (K^*_n, D)-design.

Subcase 2.2: $y \equiv 2 \pmod 3$.
By Theorem 5 there exists a K_3-decomposition of $K_{4,\,(y-2)\times 2}$. Thus, by Theorem 7, there exists a $K_{3\times 6}$-decomposition of $K_{24,\,(y-2)\times 12}$. Note that $K_{12y+1} = \big(\big(K_{24} \cup (y-2)K_{12}\big) \vee K_1\big) \cup K_{24,\,(y-2)\times 12} = K_{24,\,(y-2)\times 12} \cup K_{25} \cup \bigcup_{i=1}^{y-2} K_{13}$. Thus, $K^*_n = K^*_{24,\,(y-2)\times 12} \cup K^*_{25} \cup \bigcup_{i=1}^{y-2} K^*_{13}$. Since there exists a $(K^*_{3\times 6}, D)$-design (by Example 9), there exists a $(K^*_{24,\,(y-2)\times 12}, D)$-design. Since there also exist (K^*_{25}, D)- and (K^*_{13}, D)-designs (by Examples 6 and 3), there exists a (K^*_n, D)-design. ∎

Theorem 9. *If $n \equiv 3 \pmod 6$ and $n \geq 9$, then a (K^*_n, D)-design exists for $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$.*

Proof. Let $D \in \{\mathrm{D113}, \mathrm{D119}, \mathrm{D121}, \mathrm{D147}\}$ and let $n = 6x + 3$ for some positive integer x. When x is 1, 2, 3, or 6, the result follows from Examples 2, 4, 5, and 7, respectively. The remainder of the proof breaks into two cases.

Case 1: $x \equiv 0$ or $1 \pmod 4$ with $x \geq 4$.
By Theorem 6 there exists a K_4-decomposition of $K_{x\times 3}$. Thus, by Theorem 7, there exists a $K_{4\times 2}$-decomposition of $K_{x\times 6}$. Note that $K_{6x+3} = (xK_6 \vee K_3) \cup K_{x\times 6} = K_{x\times 6} \cup K_9 \cup \bigcup_{i=1}^{x-1}(K_9 \setminus K_3)$. Thus, $K^*_n = K^*_{x\times 6} \cup K^*_9 \cup \bigcup_{i=1}^{x-1}(K^*_9 \setminus K^*_3)$. Since there exists a $(K^*_{4\times 2}, D)$-design (by Example 8), there exists a $(K^*_{x\times 6}, D)$-design. Since there also exists a (K^*_9, D)-design (by Example 2) and a $(K^*_9 \setminus K^*_3, D)$-design (by Example 11), there exists a (K^*_n, D)-design.

Case 2: $x \equiv 2$ or $3 \pmod 4$ with $x \geq 7$.
By Theorem 6 there exists a K_4-decomposition of $K_{6,(x-2)\times 3}$. Thus, by Theorem 7, there exists a $K_{4\times 2}$-decomposition of $K_{12,(x-2)\times 6}$. Note that $K_{6x+3} = \big((K_{12} \cup (x-2)K_6) \vee K_3\big) \cup K_{12,(x-2)\times 6} = K_{12,(x-2)\times 6} \cup K_{15} \cup \bigcup_{i=1}^{x-2}(K_9 \setminus K_3)$. Thus, $K_n^* = K_{12,(x-2)\times 6}^* \cup K_{15}^* \cup \bigcup_{i=1}^{x-2}(K_9^* \setminus K_3^*)$. Since there exists a $(K_{4\times 2}^*, D)$-design (by Example 8), there exists a $(K_{12,(x-2)\times 6}^*, D)$-design. Since there also exists a (K_{15}^*, D)-design (by Example 4) and a $(K_9^* \setminus K_3^*, D)$-design (by Example 11), there exists a (K_n^*, D)-design. ■

Hence, our main result can be summarized as Theorem 10.

Theorem 10. *For $D \in \{\text{D113, D119, D121, D147}\}$, there exists a (K_n^*, D)-design if and only if $n \equiv 1$ or $3 \pmod 6$ and $n \geq 7$.*

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