Discrete Mathematics 344 (2021) 112375

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A formula for enumerating permutations with a fixed pinnacle set

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ARTICLE INFO

Article history:
Received 24 April 2020
Received in revised form 4 December 2020
Accepted 2 March 2021
Available online xxxx

Keywords:
Pinnacles of permutations
Peaks of permutations
Foata–Strehl group action on $\mathfrak{S}_n$

ABSTRACT

In 2017 Davis, Nelson, Petersen, and Tenner pioneered the study of pinnacle sets of permutations and asked whether there exists a class of operations, which applied to a permutation in $\mathfrak{S}_n$, can produce any other permutation with the same pinnacle set and no others. In this paper, we adapt a group action defined by Foata and Strehl to provide a way to generate all permutations with a given pinnacle set. From this we give an answer to a second question asked by Davis, Nelsen, Peterson, and Tenner, which asks for a closed non-recursive formula enumerating permutations with a given pinnacle set.

1. Introduction

Let $\mathbb{N}$ denote the set of the nonnegative integers. For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n$ denote the set of permutations $\pi = \pi_1\pi_2 \cdots \pi_n$ of $[n]$. Given a permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$, throughout we let $\pi_0 = \pi_{n+1} = \infty$. When using two digit numbers for any of the π_i (as in Example 4.5), we use the notation $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$ to avoid confusion. A permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ has a **descent** at a nonzero index i if $\pi_i > \pi_{i+1}$, an **ascent** at a nonzero index i if $\pi_i < \pi_{i+1}$, a **peak** at a nonzero index i if $\pi_{i-1} < \pi_i > \pi_{i+1}$, and a **valley** at a nonzero index i if $\pi_{i-1} > \pi_i < \pi_{i+1}$. Whereas, the value π_i of a permutation is a **pinnacle** if $\pi_{i-1} < \pi_i > \pi_{i+1}$, and a **vale** if $\pi_{i-1} > \pi_i < \pi_{i+1}$. Then the **peak set** of a permutation $\pi \in \mathfrak{S}_n$ is $\mathrm{Pk}(\pi) = \{i \in [n] : i \text{ is a peak of } \pi\}$, the **pinnacle set** of π is the set

$$\mathrm{Pin}(\pi) = \{i \in [n] : i \text{ is a pinnacle of } \pi\},$$

the **valley set** of π is $\mathrm{Vy}(\pi) = \{i \in [n] : i \text{ is a valley of } \pi\}$, and the **vale set** of π is the set

$$\mathrm{Vale}(\pi) = \{i \in [n] : i \text{ is a vale of } \pi\}.$$

Note that we can also think of the pinnacle set as the image of the peak set under the function π, and we can think of the vale set as the image of the valley set under the function π. For example, the permutation $\pi = 15264387$ has pinnacle set $\mathrm{Pin}(\pi) = \{5, 6, 8\}$, peak set $\mathrm{Pk}(\pi) = \{2, 4, 7\}$, vale set $\mathrm{Vale}(\pi) = \{1, 2, 3, 7\}$ and valley set $\mathrm{Vy}(\pi) = \{1, 3, 6, 8\}$.

Although the notions of pinnacles and peaks (resp. vales and valleys) capture a sense of a rise and fall (resp. fall and rise) in a permutation, they behave rather differently. To capture this difference, we consider $T \subset [n]$ and let

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https://doi.org/10.1016/j.disc.2021.112375
0012-365X/

$\mathrm{Pk}(T; n) = \{\pi \in \mathfrak{S}_n : \mathrm{Pk}(\pi) = T\}$, $\mathrm{Pin}(T; n) = \{\pi \in \mathfrak{S}_n : \mathrm{Pin}(\pi) = T\}$, and present some previous results in the study of peaks and pinnacles of permutations. In 2013, Billey, Burdzy, and Sagan presented a result regarding the enumeration of permutations in $\mathfrak{S}_n$ with a specified n-admissible peak set T, that is, $T \subset [n]$ such that $\mathrm{Pk}(T; n) \neq \emptyset$. Their main result is as follows.

Theorem 1.1 (*Billey, Burdzy, and Sagan 2013 [1]*)**.** *If $T = \{i_1 < \cdots < i_s\}$ is an n-admissible peak set, then*

$$|\mathrm{Pk}(T; n)| = p(n)2^{n-|T|-1} \tag{1}$$

where $p(n)$ is a polynomial depending on T such that $p(m)$ is an integer for all integral m and $\deg p(n) = i_s - 1$.

In 2017, Davis, Nelson, Petersen, and Tenner determined bounds for the number of permutations with a specified n-admissible pinnacle set $P \subset [n]$. That is, $P \subset [n]$ such that $\mathrm{Pin}(P; n) \neq \emptyset$. Their main result is as follows.

Theorem 1.2 (*Davis, Nelson, Petersen, and Tenner 2017 [3]*)**.** *If P is an admissible pinnacle set, then*

$$2^{n-|P|-1} \;\leq\; |\mathrm{Pin}(P; n)| \;\leq\; |P|! \cdot (|P| + 1)! \cdot 2^{n-2|P|-1} \cdot \mathcal{S}(n - |P|, |P| + 1)$$

where $\mathcal{S}(r, s)$ denotes the Stirling number of the second kind. Moreover, these bounds are sharp.

Davis et al. posed the question of whether there exists a class of operations which, applied to a permutation in $\mathfrak{S}_n$, can produce any other permutation with the same pinnacle set and no others [3, Question 4.2]. In this paper, we provide a way to generate all permutations with a given pinnacle set by using a group action on permutations called the dual Foata–Strehl action, which we define in Section 2. Specifically, this action partitions the set $\mathrm{Pin}(P; n)$ into disjoint orbits, and we generate one permutation in each orbit. From this we then prove Theorem 1.3 to provide an answer to [3, Question 4.4] which asks for a closed non-recursive formula for the total number of permutations with a given pinnacle set. To state this result, for a given pinnacle set P, we define $\mathcal{V}(P)$ to be the set of all vale sets, $V \subseteq ([n] \setminus P)$, so that P and V are an n-admissible pinnacle and vale set combination, i.e. there are permutations in $\mathfrak{S}_n$ with P as their pinnacle set and V as their vale set.

Theorem 1.3 (*Corollary 4.6 in this paper*)**.** *If P is an n-admissible pinnacle set, then*

$$\mathrm{Pin}(P; n) = 2^{n-|P|-1} \sum_{V \in \mathcal{V}(P)} \left(\prod_{p \in P} \binom{N_{PV}(p)}{2} \prod_{x \in [n] \setminus (P \,\cup\, V)} N_{PV}(x) \right),$$

where $V_k = \{v \in V : v < k\}$, $P_k = \{p \in P : p < k\}$, and $N_{PV}(k) = |V_k| - |P_k|$, counting the number of vales less than k, minus the number of pinnacles less than k.

This work is organized as follows. In Section 2 we define the dual Foata–Strehl group action on permutations, recall some known characteristics of this action, and establish that the dual Foata–Strehl group action on permutations preserves pinnacle sets (Theorem 2.3). In Section 3 we describe a unique representative from each orbit under the dual Foata–Strehl action (Theorem 3.6). In Section 4 we construct and count permutations with a fixed pinnacle set (Corollary 4.6) and provide a way to construct and count all vale sets in $\mathcal{V}(P)$ (Proposition 4.7). In Section 5 we present computational evidence that the algorithm based on our constructions in Section 4 is drastically faster than the naive algorithm for generating $\mathrm{Pin}(P; n)$. In Section 6 we present a few open problems for further study.

2. The dual Foata–Strehl group action on $\mathfrak{S}_n$

Let $\pi \in \mathfrak{S}_n$ and $x \in [n]$. We can write $\pi = w_1 w_2 x w_4 w_5$ where w_2 is the longest contiguous subword immediately to the left of x such that all values are less than x and w_4 is the longest contiguous subword immediately to the right of x such that all letters of w_4 are less than x. Call this the *x-factorization of* π, then let $\varphi_x(\pi) = w_1 w_4 x w_2 w_5$, which defines an involution on $\mathfrak{S}_n$. Note that if x is a vale, then $w_2 = \emptyset = w_4$, where $\emptyset$ denotes the empty word, and $\varphi_x(\pi) = \pi$.

The map φ_x is a modified version of the map Foata and Strehl defined in [4]. In their paper, the x-factorization of w was defined by letting w_2 be the longest contiguous subword immediately to the left of x such that all values are greater than x and w_4 is the longest contiguous subword immediately to the right of x such that all letters of w_4 are greater than x. Then they use their x-factorization to define the map $\phi_x(\pi) = \phi_x(w_1 w_2 x w_4 w_5) = w_1 w_4 x w_2 w_5$.

In that sense, φ_x and ϕ_x only differ in that one switches the values near x that are less than x and the other switches the values near x that are greater than x. If we let w_0 be the longest word of $\mathfrak{S}_n$, namely $w_0 = n(n-1)\ldots 1$, and if $\pi = \pi_1 \pi_2 \cdots \pi_n$ then $w_0(\pi_i) = n - \pi_i + 1$ for all $1 \leq i \leq n$. Hence, for any $x \in [n]$, we have that

$$\varphi_x(\pi) = w_0(\phi_{w_0(x)}(w_0 \pi)). \tag{2}$$

Geometrically, this equation states that to obtain $\varphi_x(\pi)$ we can first flip the graph of π vertically along the $y = (n+1)/2$ line, which is achieved by multiplying π by w_0 on the left. Then, we apply the map $\phi_{w_0(x)}$, and finally flip the permutation vertically again along the same line.

In the next example, we provide some computations of the maps φ_x and ϕ_x. We then notice a commutativity property that is later proved in Lemma 2.2.

Example 2.1. If $\pi = 6534127$, then

$$\varphi_4(\pi) = \varphi_4(\underbrace{65}_{w_1}\underbrace{3}_{w_2}\,4\,\underbrace{12}_{w_4}\underbrace{7}_{w_5}) = 6512437$$
$$\varphi_5(\pi) = \varphi_5(\underbrace{6}_{w_1}\underbrace{\emptyset}_{w_2}\,5\,\underbrace{3412}_{w_4}\underbrace{7}_{w_5}) = 6341257,$$

and

$$\phi_4(\pi) = \phi_4(\underbrace{653}_{w_1}\underbrace{\emptyset}_{w_2}\,4\,\underbrace{\emptyset}_{w_4}\underbrace{127}_{w_5}) = 6534127$$
$$\phi_5(\pi) = \phi_5(\underbrace{\emptyset}_{w_1}\underbrace{6}_{w_2}\,5\,\underbrace{\emptyset}_{w_4}\underbrace{34127}_{w_5}) = 5634127.$$

Repeating this process shows that $\varphi_5(\varphi_4(\pi)) = 6124357 = \varphi_4(\varphi_5(\pi))$ and $\phi_5(\phi_4(\pi)) = 5634127 = \phi_4(\phi_5(\pi))$.

In Section 2 of [4] Foata and Strehl prove that for any $x, y \in [n]$ and any permutation π we have that $\phi_x(\phi_y(\pi)) = \phi_y(\phi_x(\pi))$. We use this result together with Eq. (2) to show the equivalent result for φ_x and φ_y.

Lemma 2.2. *If $x, y \in [n]$, then $\varphi_x(\varphi_y(\pi)) = \varphi_y(\varphi_x(\pi))$ for any $\pi \in \mathfrak{S}_n$.*

Proof. Let $\pi \in \mathfrak{S}_n$, then

$$\begin{aligned}
\varphi_x(\varphi_y(\pi)) &= \varphi_x\big(w_0(\phi_{w_0(y)}(w_0\pi))\big) && \text{by (2) applied to } \varphi_y\\
&= w_0\Big(\phi_{w_0(x)}\Big(w_0\big(w_0\phi_{w_0(y)}(w_0\pi)\big)\Big)\Big) && \text{by (2) applied to } \varphi_x\\
&= w_0\Big(\phi_{w_0(x)}\Big(\phi_{w_0(y)}(w_0\pi)\Big)\Big) && \text{as } w_0 \text{ is an idempotent}\\
&= w_0\Big(\phi_{w_0(y)}\Big(\phi_{w_0(x)}(w_0\pi)\Big)\Big) && \text{since } \phi_{w_0(x)} \text{ and } \phi_{w_0(y)} \text{ commute}\\
&= w_0\Big(\phi_{w_0(y)}\Big(w_0\big(w_0\phi_{w_0(x)}(w_0\pi)\big)\Big)\Big) && \text{as } w_0 \text{ is an idempotent}\\
&= \varphi_y\big(w_0(\phi_{w_0(x)}(w_0\pi))\big) && \text{by (2) applied to } \varphi_y\\
&= \varphi_y(\varphi_x(\pi)) && \text{by (2) applied to } \varphi_x. \quad \square
\end{aligned}$$

Given $S \subseteq [n]$, Foata and Strehl [4] define

$$\phi_S(\pi) = \prod_{x \in S} \phi_x(\pi)$$

where the product notation denotes the composition of the functions ϕ_x for all $x \in S$, and if $S = \emptyset$, then ϕ_S is the identity map on $\mathfrak{S}_n$. Since ϕ_x and ϕ_y commute for all $x, y \in [n]$, then $\phi_S(\pi)$ is well defined. This can be interpreted as a group action $\phi : \mathbb{Z}_2^n \times \mathfrak{S}_n \to \mathfrak{S}_n$ defined by $\phi(\mathbf{a}, \pi) = \phi_{X_{\mathbf{a}}}(\pi)$ where $X_{\mathbf{a}} := \{i : a_i = 1\}$. We call ϕ_S the **Foata–Strehl action**.

Given $S \subseteq [n]$, we can similarly define

$$\varphi_S(\pi) = \prod_{x \in S} \varphi_x(\pi)$$

where the product notation denotes the composition of the functions φ_x for all $x \in S$. When $S = \emptyset$, define φ_S to be the identity map on $\mathfrak{S}_n$. Since φ_x and φ_y commute for all $x, y \in [n]$, then φ_S is well defined. Similarly, the group $\mathbb{Z}_2^n$ acts on the symmetric group $\mathfrak{S}_n$ via the function φ_S. To be precise, $\varphi : \mathbb{Z}_2^n \times \mathfrak{S}_n \to \mathfrak{S}_n$ defined by $\varphi(\mathbf{a}, \pi) = \varphi_{X_{\mathbf{a}}}(\pi)$ where $X_{\mathbf{a}} := \{i : a_i = 1\}$ is a group action. We henceforth refer to φ_S as the **dual Foata–Strehl action**.

Our first result establishes that the dual Foata–Strehl action preserves the pinnacle set of a permutation.

Theorem 2.3. *For any $S \subset [n]$ and any $\pi \in \mathfrak{S}_n$, $\mathrm{Pin}(\pi) = \mathrm{Pin}(\varphi_S(\pi))$.*

Proof. First, note that it is enough to show that for any $x \in [n]$, $\mathrm{Pin}(\pi) = \mathrm{Pin}(\varphi_x(\pi))$. We write

$$\pi = w_1 w_2 x w_4 w_5 \tag{3}$$
$$\varphi_x(\pi) = w_1 w_4 x w_2 w_5. \tag{4}$$

Since φ_x is an involution, it is enough to prove that $\mathrm{Pin}(\pi) \subset \mathrm{Pin}(\varphi_x(\pi))$. Let $y \in \mathrm{Pin}(\pi)$. We show $y \in \mathrm{Pin}(\varphi_x(\pi))$. First, consider the case that $y = x$. It is clear that y will still be a pinnacle of $\varphi_x(\pi)$, as the subwords w_2 and w_4 are defined to be strictly smaller than y, and swapping the two words around y will preserve the fact that y is still a pinnacle.

Table 1
Partitioning of $\mathfrak{S}_4$ by the orbits of the dual Foata–Strehl action.

Pinnacle set	Equivalence class							
$P = \emptyset$	1234	2134	3124	4123	3214	4213	4312	4321
$P = \{3\}$	1324	2314	4132	4231				
$P = \{4\}$	1243	2143	3412	3421				
$P = \{4\}$	1342	3142	2413	2431				
$P = \{4\}$	1423	1432	2341	3241				

Now, we consider the cases such that the pinnacle y is contained in subwords w_1, w_2, w_4, or w_5. Note that it is enough to consider the cases when y is on one of the ends of the words that comprise the factorization. Indeed, since the action preserves the structure of the subwords themselves, changes to the pinnacle set will only arise at the junctions between the subwords.

We first consider the case that $y \in w_1$. Since the left-most letter of w_1 cannot be a pinnacle by definition, we consider the case when y is the right-most letter of w_1. If y is a pinnacle of π, it must be greater than its neighbor to the left in w_1, which remains the same in $\varphi_x(\pi)$. Since $y \in w_1$, it must be the case that $y > x$, by definition of the x-factorization. Moreover, all letters of w_4 will be less than x, which is also less than y. Furthermore, if w_4 is empty, then the neighbor to the right of y in $\varphi_x(\pi)$ is x itself. So, y is always greater than its neighbor to the right in $\varphi_x(\pi)$ and is thus a pinnacle. This argument similarly applies to the case that y is the left-most letter of w_5.

We claim that it is impossible to have a pinnacle on the ends of w_2 and w_4. We consider the case of w_2 and note that an analogous argument applies to w_4. Suppose y is a pinnacle sitting at the right-most end of w_2. Because it is a pinnacle, y must be greater than its neighbor to the right, namely x. However, by definition of the x-factorization, the letter y would not be in w_2, as w_2 is the longest contiguous word to the left of x whose letters are all *less* than x. Now suppose y is a pinnacle sitting at the left-most end of w_2. By definition of pinnacle, y must be greater than its left neighbor, which is in w_1. On the other hand, by definition of the x-factorization, all letters of w_2 – and thus y – are less than x, and the neighbor to the left of y in w_1 must be greater than x and thus greater than y. We have arrived at a contradiction and conclude that the left-most end of w_2 cannot be a pinnacle.

Thus, we have shown that $\text{Pin}(\pi) \subset \text{Pin}(\varphi_x(\pi))$, which implies that $\text{Pin}(\pi) = \text{Pin}(\varphi_x(\pi))$ for arbitrary $x \in [n]$. Thus, we can conclude that the dual Foata–Strehl action preserves pinnacle sets. □

Let $\sim$ be the equivalence relation on $\mathfrak{S}_n$ defined by the action of $\mathbb{Z}_2^n$. Namely, $\pi \sim \tau$ if and only if there exists $\mathbf{a} \in \mathbb{Z}_2^n$ such that $\varphi_{X_{\mathbf{a}}}(\pi) = \tau$. The equivalence classes under this relation are precisely the orbits of the dual Foata–Strehl action. In light of Theorem 2.3, we know that these orbits partition $\mathfrak{S}_n$ into subsets of permutations sharing a pinnacle set.

The following example illustrates that there may be multiple equivalence classes with the same pinnacle set.

Example 2.4. In Table 1, each row represents an equivalence class of $\mathfrak{S}_4$ arising from the dual Foata–Strehl action, and we have labeled the pinnacle set of each class at the left of the row. Note that there are three equivalence classes with the same pinnacle set $P = \{4\}$.

Next we measure the size of each equivalence class and do so by examining the relationship between pinnacles and vales of permutations.

In what follows we let $v(\pi)$ denote the number of vales in π.

Lemma 2.5. *If P is an n-admissible pinnacle set, then $v(\pi) = |P| + 1$ for all $\pi \in \text{Pin}(P; n)$.*

Proof. Since $\pi_0 = \pi_{n+1} = \infty$, and since vales and pinnacles alternate we know there will be one more vale than pinnacles. □

For any $\pi \in \mathfrak{S}_n$, let $\text{Orb}_\varphi(\pi) := \{\varphi_S(\pi) : S \subseteq [n]\}$ denote the orbit of π under the dual Foata–Strehl action φ. Similarly, let $\text{Orb}_\phi(\pi) := \{\phi_S(\pi) : S \subseteq [n]\}$ denote the orbit of π under the Foata–Strehl action ϕ. In [4, Section 3], Foata and Strehl proved that

$$|\text{Orb}_\phi(\pi)| = 2^{n-v(\pi)}.$$

We now prove the analogous result for $\text{Orb}_\varphi(\pi)$.

Proposition 2.6. *If $\pi \in \mathfrak{S}_n$, then $|\text{Orb}_\varphi(\pi)| = 2^{n-v(\pi)}$.*

Proof. For a set $S \subseteq [n]$, let $w_0(S) = \{w_0(s) : s \in S\}$, where $w_0(s) = n - s + 1$. We now create a bijection between $\text{Orb}_\varphi(\pi)$ and $\text{Orb}_\phi(w_0\pi)$. Let

$$F : \text{Orb}_\varphi(\pi) \to \text{Orb}_\phi(w_0\pi) \text{ such that } F(\varphi_S(\pi)) = \phi_{w_0(S)}(w_0\pi)$$

and

$$G : \mathrm{Orb}_{\phi}(w_0\pi) \to \mathrm{Orb}_{\varphi}(\pi) \text{ such that } G(\phi_S(w_0\pi)) = \varphi_{w_0(S)}(\pi).$$

Then $F \circ G$ and $G \circ F$ are the identity maps on $\mathrm{Orb}_{\phi}(w_0\pi)$ and $\mathrm{Orb}_{\varphi}(\pi)$, respectively. Thus,

$$|\mathrm{Orb}_{\varphi}(\pi)| = |\mathrm{Orb}_{\phi}(w_0\pi)| = 2^{n-v(\pi)}. \quad \square$$

Remark 2.7. We remark that Foata and Strehl determined that the number of orbits under ϕ is given by the nth tangent or secant number, depending on whether n is odd or even [4]. The nth tangent number (resp. nth secant number) is defined as the coefficient of $u^n/n!$ in the exponential power series of the tangent function (resp. secant function), i.e., the coefficient of $u^n/n!$ in the series

$$\tan(u) = 1 \cdot \frac{u}{1!} + 2 \cdot \frac{u^3}{3!} + 16 \cdot \frac{u^5}{5!} + 272 \cdot \frac{u^7}{7!} + \cdots \quad \text{and} \quad \sec(u) = 1 + 1 \cdot \frac{u^2}{2!} + 5 \cdot \frac{u^4}{4!} + 61 \cdot \frac{u^6}{6!} + \cdots.$$

By Proposition 2.6, the same is true for the number of orbits under φ. In the subsequent sections of this paper, we count the number of orbits of φ that have a prescribed pinnacle set P.

Remark 2.8. In [2], Petter Brändén defined a modified function, which we call φ'_x, such that $\varphi'_x(\pi) = \varphi_x(\pi)$ if x is neither a pinnacle nor a vale, and $\varphi'_x(\pi) = \pi$ if x is a pinnacle or a vale. Similar to ϕ and φ, the author defines φ' as an action of $\mathbb{Z}_2^n$ on $\mathfrak{S}_n$ and uses it to prove that for any $T \subseteq \mathfrak{S}_n$, the polynomial defined by

$$A(T; x) = \sum_{\pi \in T} x^{\mathrm{des}(\pi)}$$

is γ-nonnegative, where $\mathrm{des}(\pi) = |\{i \in [n] \mid \pi_i > \pi_{i+1}\}|$. In [6], Postnikov, Reiner and Williams defined a modified function, which we call φ'', such that $\varphi''_x(\pi) = \phi_x(\pi)$ if x is neither a pinnacle nor a vale and $\varphi''(\pi) = \pi$ if x is a pinnacle or a vale. Similar to ϕ, φ, and φ', they define an action φ'' of $\mathbb{Z}_2^n$ on $\mathfrak{S}_n$ and use it to prove Gal's conjecture for the chordal nestohedra, [6, Theorem 11.6].

3. Representatives of dual Foata–Strehl orbits

In this section, we describe a collection of permutations, called ***FS*-minimal permutations**, that characterize the orbits of the dual Foata–Strehl action φ. Then in Section 4, we provide a construction of all *FS*-minimal permutations with a given pinnacle set. These results will allow us to count all permutations with a given pinnacle set.

Definition 3.1 (*Admissibility*)**.** A pair of sets (P, V) is considered **admissible** if there is a permutation with pinnacle set P and vale set V. Given a pinnacle set P, define $\mathcal{V}(P)$ to be the set of all vale sets V for which the pair (P, V) is admissible.

Throughout the section, let π be a permutation with pinnacle set $P = \{p_1, \ldots, p_\ell\}$ and vale set $V = \{v_1, \ldots, v_{\ell+1}\}$, respectively. We will often list the pinnacles and vales in the order in which they appear in π, from left to right. We will also restrict π to permutations of the sets P, V, and $P \cup V \subseteq [n]$. For instance, we write $\pi|_P = p_1p_2\cdots p_\ell$ to denote the restriction of the permutation π to just the values at which π has pinnacles, which we list in the order they appear in π. Similarly, $\pi|_V = v_1v_2\cdots v_{\ell+1}$ denotes the restriction of the permutation π to just the values at which π has vales, which we list in the order they appear in π. Similarly, we let

$$\pi|_{P\cup V} = v_1p_1v_2p_2\cdots p_\ell v_{\ell+1}$$

denote the restriction of π to just the values at which π has vales and pinnacles, listed in the order they appear in π. For example, if $\pi = 32814756$, then $P = \{7, 8\}$, $V = \{1, 2, 5\}$, $\pi|_P = 87$, $\pi|_V = 215$, and $\pi|_{P\cup V} = 28175$.

In what follows, we present three technical lemmas used to prove the main theorem of the section, Theorem 3.6.

Lemma 3.2. *If π is a permutation with pinnacle set $P = \{p_1, \ldots, p_\ell\}$, then for all $i \in [\ell]$, π and $\varphi_{p_i}(\pi)$ have the same number of descents.*

Proof. For any $i \in [\ell]$, consider the p_i-factorization of π,

$$\pi = w_1w_2p_iw_4w_5 = \underbrace{\pi_1\cdots\pi_{k_1}}_{w_1}\underbrace{\pi_{k_1+1}\cdots\pi_{k_2}}_{w_2}\,p_i\,\underbrace{\pi_{k_4}\cdots\pi_{k_5-1}}_{w_4}\underbrace{\pi_{k_5}\cdots\pi_n}_{w_5}.$$

By the definition of this factorization $p_i > \max(w_2)$, $p_i > \max(w_4)$ and $\pi_{k_1} > p_i < \pi_{k_5}$. Applying φ_{p_i} we get

$$\varphi_{p_i}(\pi) = w_1w_4p_iw_2w_5 = \underbrace{\pi_1\cdots\pi_{k_1}}_{w_1}\underbrace{\pi_{k_4}\cdots\pi_{k_5-1}}_{w_4}\,p_i\,\underbrace{\pi_{k_1+1}\cdots\pi_{k_2}}_{w_2}\underbrace{\pi_{k_5}\cdots\pi_n}_{w_5}.$$

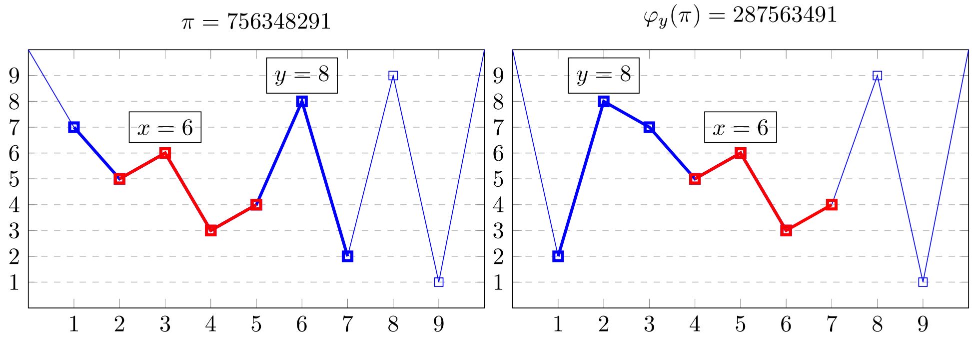

Fig. 1. Let $x = 6$, $y = 8$. In the left, we show the permutation $\pi = 756348291$, bolding the subpermutation w_2yw_4 in the y-factorization of π and highlighting in red the subpermutation v_2xv_4 in the x-factorization of π. In the right, we show the permutation $\varphi_y(\pi)$, bolding the subpermutation $\beta_2y\beta_4$ in the y-factorization of $\varphi_y(\pi)$ and highlighting in red the subpermutation $\alpha_2x\alpha_4$ in the x-factorization of $\varphi_y(\pi)$.

Since the content in w_1, w_2, w_4, w_5 did not change, it is enough to study the places where these subwords meet in $\varphi_{p_i}(\pi)$, namely $\pi_{k_1}\pi_{k_4}, \pi_{k_5-1}p_i, p_i\pi_{k_1+1}$, and $\pi_{k_2}\pi_{k_5}$. Since $\pi_{k_4} < p_i < \pi_{k_1}$ and $p_i > \pi_{k_1+1}$, the descents $\pi_{k_1}\pi_{k_1+1}$ and $p_i\pi_{k_4}$ in π got replaced by the descents $\pi_{k_1}\pi_{k_4}$ and $p_i\pi_{k_1+1}$ in $\varphi_{p_i}(\pi)$, respectively. Similarly, the ascents $\pi_{k_2}p_i$ and $\pi_{k_5-1}\pi_{k_5}$ in π got replaced by the ascents $\pi_{k_5-1}p_i$ and $\pi_{k_2}\pi_{k_5}$ in $\varphi_{p_i}(\pi)$, respectively. Thus, the number of descents remained constant. □

Lemma 3.3. *Let $\pi \in \mathfrak{S}_n$ and let x, y be two distinct elements in $[n]$. If*

$v_1\, v_2\, x\, v_4\, v_5$ *is the x-factorization of π and*

$\alpha_1\, \alpha_2\, x\, \alpha_4\, \alpha_5$ *is the x-factorization of $\varphi_y(\pi)$,*

then $\max(v_2) = \max(\alpha_2)$ *and* $\max(v_4) = \max(\alpha_4)$.

Proof. Let x, y be two distinct elements in $[n]$. Let

$v_1\, v_2\, x\, v_4\, v_5$ denote the x-factorization of π,

$w_1\, w_2\, y\, w_4\, w_5$ denote the y-factorization of π,

$\alpha_1\, \alpha_2\, x\, \alpha_4\, \alpha_5$ denote the x-factorization of $\varphi_y(\pi)$, and

$\beta_1\, \beta_2\, y\, \beta_4\, \beta_5$ denote the y-factorization of $\varphi_y(\pi)$.

There are six possible cases to consider. In the first four cases, detailed below, the subword v_2xv_4 remains unchanged in $\varphi_y(\pi)$, hence $v_2xv_4 = \alpha_2x\alpha_4$.

(1) If v_2xv_4 lies in w_2 then the subword v_2xv_4 remains together, but is moved to within β_4 in $\varphi_y(\pi)$. In this case $v_2xv_4 = \alpha_2x\alpha_4$. An example of this is shown in Fig. 1.
(2) If v_2xv_4 lies in w_4 then v_2xv_4 remains together, but is moved to β_2 in $\varphi_y(\pi)$. In this case $v_2xv_4 = \alpha_2x\alpha_4$.
(3) If v_2xv_4 lies in w_1 then v_2xv_4 remains in β_1 in $\varphi_y(\pi)$. In this case $v_2xv_4 = \alpha_2x\alpha_4$.
(4) If v_2xv_4 lies in w_5 then v_2xv_4 remains together in β_5 in $\varphi_y(\pi)$.

In the last two cases, described below, either v_2 or v_4 is rearranged slightly in α_2 or α_4, but this does not affect the maximum element of α_2 or α_4 in $\varphi_y(\pi)$.

(5) If v_2x lies in w_1 but v_4 does not lie entirely in w_1 then y is contained in v_4. In this case the subword v_2x remains unchanged in $\varphi_y(\pi)$ in the sense that $v_2x = \alpha_2x$, and v_4 has some of its elements rearranged by φ_y but the set of elements appearing in α_4 remains the same (i.e. $v_2 = \alpha_2$ and the underlying set of v_4 is equal to the underlying set of α_4). Hence $\max(v_2) = \max(\alpha_2)$ and $\max(v_4) = \max(\alpha_4)$ in this case. An example of this is shown in Fig. 2.
(6) If xv_4 lies in w_5 but v_2 does not lie entirely in w_5, then y is contained in v_2. In this case the word xv_4 remains unchanged in $\varphi_y(\pi)$ in the sense that $xv_4 = x\alpha_4$, and v_2 has some of its elements rearranged by φ_y but the set of elements appearing in v_2 remains the same. Hence $\max(v_2) = \max(\alpha_2)$ and $\max(v_4) = \max(\alpha_4)$ in this case. □

We can also define an x-factorization of any subword of a permutation. That is, given a subword $\sigma = s_1s_2\cdots s_\ell$ of a permutation $\pi \in \mathfrak{S}_n$, and $x = s_i$ for some $1 \le i \le \ell$, the x-factorization of σ is $w_1w_2xw_4w_5$ where w_2 is the longest contiguous subword immediately to the left of x such that all values are less than x and w_4 is the longest contiguous subword immediately to the right of x such that all letters of w_4 are less than x. We then define $\varphi_x(\sigma)$ to be

$$\varphi_x(\sigma) = w_1w_4xw_2w_5.$$

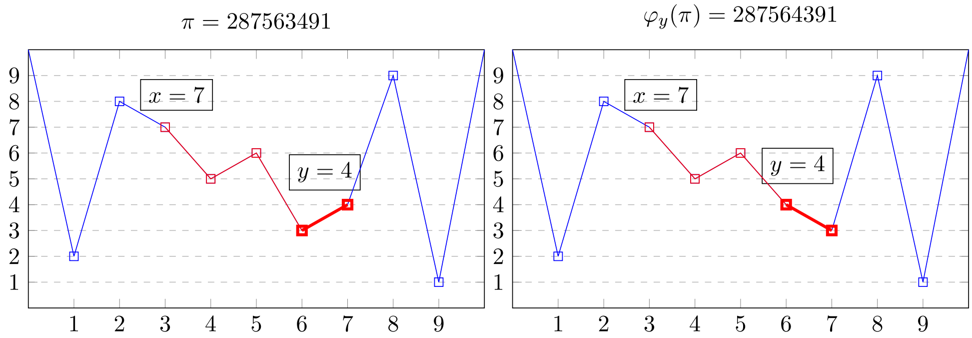

Fig. 2. Let $x = 7$, $y = 4$. In the left, we show the permutation $\pi = 287563491$, bolding the subpermutation w_2yw_4 in the y-factorization of π and highlighting in red the subpermutation v_2xv_4 in the x-factorization of π. In the right, we show the permutation $\varphi_y(\pi)$, bolding the subpermutation $\beta_2y\beta_4$ in the y-factorization of $\varphi_y(\pi)$ and highlighting in red the subpermutation $\alpha_2x\alpha_4$ in the x-factorization of $\varphi_y(\pi)$.

Lemma 3.4. *Let π be a permutation with pinnacle set P and vale set V. If $\pi|_P = p_1p_2\cdots p_\ell$ and $\pi|_V = v_1v_2\cdots v_{\ell+1}$, then $\varphi_{p_i}(\pi)|_{P\cup V} = \varphi_{p_i}(\pi|_{P\cup V})$ for any $i \in [\ell]$.*

Proof. Fix a pinnacle p_i in π and consider the p_i-factorization $\pi = w_1w_2p_iw_4w_5$. Now define $\alpha_i = w_i|_{P\cup V}$ for each $i = 1, 2, 4, 5$. Then,

$$\begin{aligned}\varphi_{p_i}(\pi)|_{P\cup V} &= (w_1\, w_4\, p_i\, w_2\, w_5)|_{P\cup V}\\ &= w_1|_{P\cup V}\ w_4|_{P\cup V}\ p_i\ w_2|_{P\cup V}\ w_5|_{P\cup V}\\ &= \alpha_1\ \alpha_4\ p_i\ \alpha_2\ \alpha_5.\end{aligned}$$

In the case where neither w_1 nor w_5 are empty, suppose p' is the right most pinnacle in w_1, and p'' is the left most pinnacle in w_5. Hence, $p' > p_i$ and $p'' > p_i$. Now consider

$$\pi|_{P\cup V} = v_1\, p_1\, v_2\, p_2\ \ldots\ v_i\, p_i\, v_{i+1}\ \ldots\ p_\ell\, v_{\ell+1}.$$

Let $\pi|_{P\cup V} = w_1'w_2'p_iw_4'w_5'$ be the p_i-factorization of $\pi|_{P\cup V}$. Since $p' > p_i$ and $p'' > p_i$, then $p' \in w_1'$ and $p'' \in w_5'$. It now follows that $w_1' = w_1|_{P\cup V} = \alpha_1$ and $w_5' = w_5|_{P\cup V} = \alpha_5$. Thus, the p_i-factorization of $\pi|_{P\cup V}$ is

$$\pi|_{P\cup V} = \alpha_1\alpha_2p_i\alpha_4\alpha_5.$$

Therefore

$$\varphi_{p_i}(\pi|_{P\cup V}) = \alpha_1\alpha_4p_i\alpha_2\alpha_5 = \varphi_{p_i}(\pi)|_{P\cup V}.$$

Note that w_2, w_4 cannot be empty as p_i is a pinnacle, and so the proof is complete by noting that whenever w_1 or w_5 are empty, it implies $w_i = w_i' = \alpha_i = \emptyset$ for $i = 1, 5$, respectively. □

We now define the notion of *FS*-minimal permutations and proceed to show our main theorem of the section, that there is a unique *FS*-minimal permutation in each dual Foata–Strehl orbit of $\mathfrak{S}_n$.

Definition 3.5. A permutation π is ***FS*-minimal** if π contains no double descents and for each $p \in \mathrm{Pin}(\pi)$ the p-factorization $w_1\, w_2\, p\, w_4\, w_5$ of $\pi|_{P\cup V}$ satisfies $\max(w_2) < \max(w_4)$.

Theorem 3.6. *If π is a permutation with pinnacle set P and vale set V, then there is a unique FS-minimal permutation in the dual Foata–Strehl orbit of π.*

Proof. We first show there is an *FS*-minimal permutation in each orbit and then show this permutation is unique. Let π be a permutation with

$$\pi|_P = p_1p_2\cdots p_\ell, \quad \pi|_V = v_1v_2\ldots v_{\ell+1}, \quad \text{and} \quad \pi|_{P\cup V} = v_1p_1v_2p_2\cdots v_\ell p_\ell v_{\ell+1}.$$

Let

$$R = \{r \in [n] \setminus (P \cup V) : r \text{ appears left of } v_1 \text{ or between } p_k \text{ and } v_{k+1} \text{ for some } 1 \le k \le \ell\},$$

that is, r is either in the beginning descending segment of π or in a descending segment strictly between a pinnacle and a vale. Note that this implies that π has $|P| + |R|$ descents. The r-factorization of π is then $w_1\emptyset rw_4w_5$ and

$$\varphi_r(\pi) = w_1w_4r\emptyset w_5.$$

In $\varphi_r(\pi)$ we solely moved r from a descending segment to an ascending segment and left the rest of π unchanged. Hence, $\varphi_r(\pi)$ has one fewer descent than π, since the relative order of the entries in w_4 remains unchanged. Then let $\rho(\pi) := \prod_{r\in R}\varphi_r(\pi)$. By this construction, $\rho(\pi)$ has only $|P|$ descents occurring only at the indices of pinnacles (at the peak set of π), and none of these descents occur consecutively, i.e. there are no double descents.

Let

$$T = \{p \in P : \text{ the } p\text{-factorization } w_1w_2pw_4w_5 \text{ of } \pi|_{P\cup V} \text{ satisfies } \max(w_2) > \max(w_4)\},$$

and define $\tau(\rho(\pi)) := \prod_{t\in T}\varphi_t(\rho(\pi))$. We now claim that $\tau(\rho(\pi))$ is FS-minimal. Since $\rho(\pi)$ has no double descents, then by Lemma 3.2, $\tau(\rho(\pi))$ has no double descents.

Let $t \in T$. By definition of the dual Foata–Strehl action, $\varphi_t(\rho(\pi))$ satisfies that $\max(w_2) < \max(w_4)$ in the t-factorization of $\varphi_t(\rho(\pi))|_{P\cup V}$. By Lemma 3.3, for all other pinnacles $p \in P$, applying φ_t to $\rho(\pi)$ does not change $\max(w_2)$ nor $\max(w_4)$ in the p-factorization of $\rho(\pi)$. Repeating this argument for all other elements of T and using the fact that by Lemma 3.4 we can apply the dual Foata–Strehl action and then restrict to $P\cup V$ or restrict to $P\cup V$ and then apply the dual Foata–Strehl action and the result is the same, shows that $\tau(\rho(\pi))$ is FS-minimal.

To show this permutation is unique, suppose π and σ are both FS-minimal and lie in the same dual Foata–Strehl orbit. Then $\pi = \varphi_S(\sigma)$, for some $S \subseteq [n]$. We will show that $S \subseteq V$, and since $\varphi_v(\sigma) = \sigma$ for all $v \in V$, then $\pi = \varphi_S(\sigma) = \sigma$.

Suppose $p \in P$. We will first show $P \cap S = \emptyset$. If $w_1'w_2'pw_4'w_5'$ is the p-factorization of $\varphi_p(\sigma)|_{P\cup V}$, then $\max(w_2') > \max(w_4')$, since σ is FS-minimal. Lemma 3.3 shows that for any $k \in S$, applying φ_k to $\varphi_p(\sigma)|_{P\cup V}$ would not change this inequality, thus $p \notin S$ as otherwise $\max(w_2') > \max(w_4')$ in the p-factorization of π, contradicting that it is FS-minimal. Hence, $P \cap S = \emptyset$.

Since applying the dual Foata–Strehl action at a vale leaves a permutation unchanged, it suffices to show $S \cap ([n]\setminus(P\cup V)) = \emptyset$ to conclude $S \subset V$ and $\pi = \sigma_S(\sigma) = \sigma$. Suppose by contradiction that there is an element r in $[n]\setminus(P\cup V)$ that lies in S. Since σ has no double descents, r must belong to an ascending segment, i.e., the r-factorization of σ is then $w_1w_2r\emptyset w_5$ and

$$\varphi_r(\sigma) = w_1\emptyset rw_2w_5.$$

In $\varphi_r(\sigma)$ we solely moved r from an ascending segment to a descending segment and left the rest of σ unchanged. Applying the dual Foata–Strehl action at any other element of $[n]\setminus(P\cup V)$ will simply move an element from an ascending segment to a descending segment, hence it will not remove the double descent created in $\varphi_r(\sigma)$. Thus, $\pi = \varphi_S(\sigma)$ will contain a double descent, which contradicts the fact it is FS-minimal. We conclude that $S \subset V$ and $\pi = \sigma_S(\sigma) = \sigma$. □

4. Constructing and counting permutations with a fixed pinnacle set

In this section we count the number of dual Foata–Strehl orbits with permutations having pinnacle set P by counting the number of FS-minimal permutations with pinnacle set P in $\mathfrak{S}_n$. Recall that a pair of sets (P, V) is considered admissible if there is a permutation with pinnacle set P and vale set V. Given an admissible tuple (P, V) and a fixed integer $k \in [n]$, we set the following notation:

- Given a nonempty word w of some letters in $[n]$, let $\max(w)$ be the largest number that appears in the word w.
- Let $V_k = \{v \in V : v < k\}$.
- Let $P_k = \{p \in P : p < k\}$.
- Let $N_{PV}(k) = |V_k| - |P_k|$.

Lemma 4.1. *If π is a permutation with pinnacle set $P = \{p_1 < p_2 < \cdots < p_\ell\}$ and vale set $V = \{v_1 < v_2 < \cdots < v_{\ell+1}\}$, then for all $1 \le i \le \ell - 1$,*

(a) $1 \in V$,
(b) $2 \le N_{PV}(p_i)$,
(c) $v_{i+1} < p_i$ *for all* $i \in \{1, 2, \ldots, \ell\}$,
(d) $N_{PV}(p_i) \le N_{PV}(p_{i+1}) + 1$,
(e) $N_{PV}(p_\ell) = 2$.

Furthermore, properties (b) and (c) are equivalent.

Proof. For part (a), since 1 appears in π, we must have $\pi_i = 1$ for some $i \in [n]$. Since $\pi_{i-1} > 1 < \pi_{i+1}$ then $1 \in V$.

For part (b), let p_i be any pinnacle in P. Consider the set $P' = \{p_1, \ldots, p_{i-1}, p_i\} \subseteq P$. Because each pinnacle has a vale smaller than it to its left and one to its right, and there is a vale between any two pinnacles, then there are at least $i+1$ vales (those around the i pinnacles in P') smaller than p_i in π. Thus,

$$N_{PV}(p_i) = |V_{p_i}| - |P_{p_i}| \ge (i+1) - (i-1) = 2.$$

For part (c), since $N_{PV}(p_i) = |V_{p_i}| - |P_{p_i}| \ge 2$ and $P_{p_i} = i - 1$ for each i in $\{1, 2, \ldots, \ell\}$, then $V_{p_i} \ge i + 1$. That is, there are at least $i+1$ vales less than p_i. Hence, $v_{i+1} < p_i$.

For part (d), note that since $p_{i+1} > p_i$, we must have $|V_{p_{i+1}}| \geq |V_{p_i}|$. Since there are $j-1$ pinnacles smaller than p_j for any $p_j \in P$, we get $|P_{p_i}| = i-1$ and $|P_{p_{i+1}}| = i$. Thus,

$$N_{PV}(p_{i+1}) = |V_{p_{i+1}}| - |P_{p_{i+1}}| \geq |V_{p_i}| - (|P_{p_i}| + 1) = |V_{p_i}| - |P_{p_i}| - 1 = N_{PV}(p_i) - 1.$$

For part (e), since $v_{\ell+1} < p_\ell$ by property (c), then $V_{p_\ell} = V$ and

$$N_{PV}(p_\ell) = |V_{p_\ell}| - |P_{p_\ell}| = (\ell + 1) - (\ell - 1) = 2.$$

To prove the last statement note that we already showed (b) $\Rightarrow$ (c). For the reverse, if $v_{i+1} < p_i$ then $N_{PV}(p_i) = |V_{p_i}| - |P_{p_i}| \geq i + 1 - (i-1) = 2$. □

We now describe which pairs (P, V) are admissible.

Proposition 4.2. *Let (P, V) be a pair of disjoint subsets of $[n]$ with $|V| = |P| + 1$. Then (P, V) is admissible if and only if properties (a) and (b) or, equivalently, properties (a) and (c) from Lemma 4.1 hold.*

Proof. The forward direction is proven in Lemma 4.1. For the backward direction, suppose P and V satisfy properties (a) and (b) from Lemma 4.1. Thus, P and V can be written as $P = \{p_1 < p_2 < \cdots < p_\ell\}$ and $V = \{v_1 < v_2 < \cdots < v_{\ell+1}\}$ with $v_1 = 1$. We need to create a permutation with pinnacle set P and vale set V.

Let $n = p_\ell$. Let α be defined as follows:

$$\alpha = v_1\, a_1\, p_1\, v_2\, a_2\, p_2\, \ldots\, v_\ell\, a_\ell\, p_\ell\, v_{\ell+1}.$$

where each a_i is the ascending sequence containing the elements in $[n] \setminus (P \cup V)$ between v_i and v_{i+1}. Since properties (b) and (c) are equivalent, then $v_{i+1} < p_i$ for all $i \in \{1, 2, \ldots, \ell\}$. Thus, α has pinnacle set P and vale set V. □

4.1. Creating and counting the number of permutations with a fixed pinnacle and vale set

Given an admissible pair (P, V) we define a ***PV*-arrangement** α to be a permutation of the elements of $P \cup V$ such that every element $p \in P$ is a pinnacle in α and every element $v \in V$ is a vale in α. We say that a PV-arrangement α is **canonical** if for each $p \in P$ the p-factorization $w_1 w_2 p w_4 w_5$ of α satisfies $\max(w_2) < \max(w_4)$.

Lemma 4.3 (*Counting Canonical PV-Arrangements*)**.** *For an admissible pair (P, V) the number of canonical PV-arrangements is*

$$\prod_{p \in P} \binom{N_{PV}(p)}{2}. \tag{5}$$

Proof. We prove the result by induction on $|P|$. If $P = \emptyset$ then V is a one element set $V = \{v\}$. In this case the only PV-arrangement is $\alpha = v$, which is canonical and is counted by the empty product in (5). For a nontrivial illustration, we show the case $|P| = 1$. If $P = \{p\}$ then V is a set with two elements by Lemma 2.5, so $V = \{v_1, v_2\}$ for two elements $v_1, v_2 \in [n-1]$. Without loss of generality, let $v_1 < v_2$. Then, the only PV-arrangements one could make are $\alpha_1 = v_1 p v_2$ and $\alpha_2 = v_2 p v_1$, of which only α_1 is a canonical PV-arrangement. By Lemma 4.1(e), the product in (5) is $\binom{N_{PV}(p)}{2} = \binom{2}{2} = 1$, as desired.

Suppose the result is true for all pinnacle sets P with cardinality $\ell - 1$. Then if $|P| = \ell$, write $P = \{p_1, \ldots, p_\ell\}$ with $p_1 < p_2 < \cdots < p_\ell$. Choose any two elements $v_1, v_2 \in V$ such that $v_1 < p_1$ and $v_2 < p_1$ and let

$$P' = P \setminus \{p_1\} \quad \text{and} \quad V' = (V \setminus \{v_1, v_2\}) \cup \{p_1\}.$$

Note that there are

$$\binom{|V_{p_1}|}{2} = \binom{N_{PV}(p_1)}{2}$$

choices of v_1 and v_2. By Lemma 4.1(b), the number of choices is always at least 1.

For each such choice and for each canonical $P'V'$-arrangement, we will create a unique canonical PV-arrangement and show that every canonical PV-arrangement is created in such manner. By induction, this would imply that the number of canonical PV-arrangements is

$$\binom{N_{PV}(p_1)}{2} \prod_{p \in P'} \binom{N_{P'V'}(p)}{2}. \tag{6}$$

Since for $p \in P'$,

$$N_{P'V'}(p) = |V'_p| - |P'_p| = (|V_p| - 1) - (|P_p| - 1) = |V_p| - |P_p| = N_{PV}(p),$$

then the expression in (6) equals our desired result

$$\binom{N_{PV}(p_1)}{2}\prod_{p\in P'}\binom{N_{P'V'}(p)}{2}=\binom{N_{PV}(p_1)}{2}\prod_{p\in P'}\binom{N_{PV}(p)}{2}=\prod_{p\in P}\binom{N_{PV}(p)}{2}.$$

To prove our claim, suppose without loss of generality that $v_1 < v_2$. Let α' be a canonical $P'V'$-arrangement (hence, $\mathrm{Pin}(\alpha') = P'$ and $\mathrm{Vale}(\alpha') = V'$). Thus, the element $p_1 \in V'$ is a vale in α', so

$$\alpha' = \cdots\ v_k\, p_i\, p_1\, p_j\, v_{k+1}\ \ldots,$$

for some pinnacles $p_i, p_j \in P'$ and vales $v_k, v_{k+1} \in V'$. Insert v_1 to the left of p_1 and v_2 to the right of p_1 to create the permutation

$$\alpha = \cdots v_k\, p_i\, v_1\, p_1\, v_2\, p_j\, v_{k+1}\ \ldots.$$

Note that α is a PV-arrangement, that is, a permutation of $P \cup V$ with pinnacle set P and vale set V. We claim this permutation is a canonical PV-arrangement. Since α' was a canonical $P'V'$-arrangement and we only added two numbers v_1, v_2 that are less than p_1 and adjacent to p_1, then $\max w_2$ is still less than $\max w_4$ in the p'-factorization of α for any $p' \in P'$. For the p-factorization of α, note that $w_2 = v_1$ and $w_4 = v_2$ because all other peaks, particularly p_i and p_j are greater than p_1. Hence, $\max(w_2) < \max(w_4)$ in this factorization as well. Thus, α is a canonical PV-arrangement. Since this construction does not change α' other than by introducing v_1 and v_2 to each side of p_1, it follows that distinct $P'V'$-arrangements create distinct PV-arrangements.

To finish the proof, we need to show that all canonical PV-arrangements are created in this manner. For any canonical PV-arrangement α, it must contain a subsequence $v_ip_1v_j$ with $v_i, v_j \in V_{p_1}$ and $v_i < v_j$. If we remove v_i and v_j from α, we get a canonical $P'V'$-arrangement α'. Hence, α was obtained via our construction by choosing v_i, v_j from V_{p_1} and inserting v_i and v_j before and after p_1, respectively. Thus, our construction gives all canonical PV-arrangements. □

We are ready to prove the main theorem of the section. In it, we count the number of FS-minimal permutations with a given fixed pinnacle set P. By Theorem 3.6, this also counts the number of dual Foata–Strehl orbits containing permutations with pinnacle set P.

Theorem 4.4 (*FS-minimal Permutations for Each PV-arrangement*)**.** *For an admissible pair (P, V), given a canonical PV-arrangement α, the number of FS-minimal permutations π with $\pi|_{P\cup V}= \alpha$, denoted O_{PV}, is*

$$O_{PV} = \prod_{r\in[n]\setminus(P\cup V)} N_{PV}(r).$$

Furthermore, the number of all FS-minimal permutations with pinnacle set P, denoted O_P, is

$$O_P = \sum_{V\in\mathcal{V}(P)}\prod_{p\in P}\binom{N_{PV}(p)}{2}\prod_{r\in[n]\setminus(P\cup V)} N_{PV}(r),$$

where $\mathcal{V}(P)$ is the set of all vale sets V for which the pair (P, V) is admissible.

Proof. Let $\alpha = v_1p_1v_2p_2\cdots v_\ell p_\ell v_{\ell+1}$ be a canonical PV-arrangement. Since in an FS-minimal permutation each pinnacle is immediately followed by a vale, to count the number of FS-minimal permutations π with $\pi|_{P\cup V}= \alpha$, note that each element $r \in ([n] \setminus (P \cup V))$ must appear to the right of a vale v_i less than r and to the left of a pinnacle p_i greater than r. The number of such indices i so that r satisfies $v_i < r < p_i$ is precisely $|V_r| - |P_r| = N_{PV}(r)$. The total number of choices over all $r \in ([n] \setminus (P \cup V))$ is then

$$\prod_{r\in[n]\setminus(P\cup V)} N_{PV}(r).$$

Once these choices are made, all of the elements (if any) between each vale v_i and pinnacle p_i must appear in ascending order by the definition of an FS-minimal permutation. This proves the first statement. The last statement follows by summing through all the canonical PV-arrangements and using Lemma 4.3. □

Example 4.5. A canonical PV-arrangement α for $P = \{7, 10, 12\}$ and $V = \{1, 3, 5, 8\}$ is depicted in Fig. 3. To construct FS-minimal permutations from α we insert each element $r \in [n] \setminus (P \cup V) = \{2, 4, 6, 9, 11\}$ in ascending order on the slopes between v_i and p_i satisfying $v_i < r < p_i$.

Table 2 describes the possible locations where the elements $r \in [n] \setminus (P \cup V) = \{2, 4, 6, 9, 11\}$ can be located in an FS-minimal permutation. For example, if we choose the first possible location listed in the third column in Table 2, the resulting FS-minimal permutation is $[8, 9, 11, 12, 5, 6, 10, 1, 2, 4, 7, 3]$.

Hence, in this case, there are $2 \cdot 3 \cdot 3 \cdot 2 = 36$ FS-minimal permutations π with $\pi|_{P\cup V}= \alpha$.

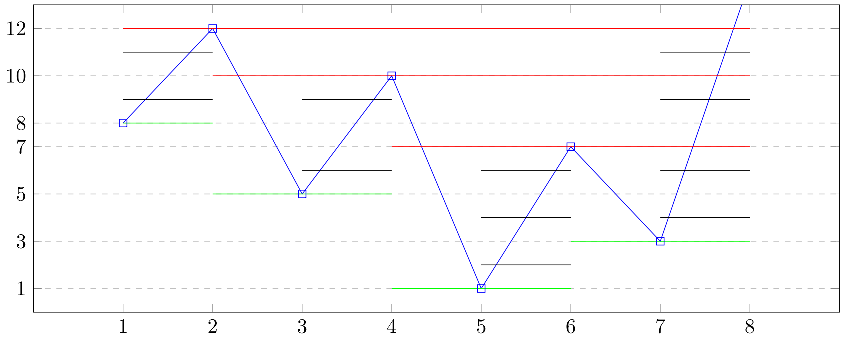

Fig. 3. A canonical PV-arrangement $\alpha = [8, 12, 5, 10, 1, 7, 3]$ for $P = \{7, 10, 12\}$ and $V = \{1, 3, 5, 8\}$.

Table 2
Example of possible locations of where to insert certain values to create an FS-minimal permutation.

r	$N_{PV}(r)$	Possible locations of r in FS-minimal permutations
2	1	between $(1, 7)$
4	2	between $(1, 7)$ or right of 3
6	3	between $(5, 10)$, between $(1, 7)$, or right of 3
9	3	between $(8, 12)$, between $(5, 10)$, or right of 3
11	2	between $(8, 12)$ or right of 3

Corollary 4.6. *If P is an admissible pinnacle set, then*

$$|\mathrm{Pin}(P; n)| = 2^{n-|P|-1} \cdot O_P = 2^{n-|P|-1} \left(\sum_{V \in \mathcal{V}(P)} \prod_{p \in P} \binom{N_{PV}(p)}{2} \prod_{r \in [n] \setminus (P \cup V)} N_{PV}(r) \right).$$

Proof. By Proposition 2.6, each orbit of the dual Foata–Strehl action has $2^{n-|P|-1}$ elements. By Theorem 3.6 there is a unique FS-minimal permutation in each orbit. By Theorem 4.4, the number of FS-minimal permutations with pinnacle set P, denoted O_P, is

$$O_P = \sum_{V \in \mathcal{V}(P)} \prod_{p \in P} \binom{N_{PV}(p)}{2} \prod_{r \in [n] \setminus (P \cup V)} N_{PV}(r).$$

Hence, we obtain our result. □

Since computing the number of permutations with a given pinnacle set P depends on the number of admissible vale sets, we now construct and establish a count for the number of vale sets V such that (P, V) is admissible.

4.2. Creating and counting the number of admissible vale sets

Given a nonnegative integer ℓ, we recall that a weak composition of ℓ is a sequence of nonnegative integers whose sum is ℓ. For $\ell \geq 1$, define

$$C(\ell) = \left\{ \mathbf{t} = (t_1, t_2, \ldots, t_\ell) \;\middle|\; \sum_{i=1}^{\ell} t_i = \ell \text{ and } \sum_{i=1}^{k} t_i \geq k \text{ for each } k \text{ in } \{1, 2, \ldots, \ell\} \right\}$$

and $C(0) = \{()\}$ just contains the empty tuple. Given a pinnacle set $P = \{p_1 < p_2 < \cdots < p_\ell\}$, for $1 \leq i \leq \ell$, let $G_i = \{j \in [n] \setminus P : p_{i-1} < j < p_i\}$, where $p_0 = 1$. We call G_i the ***i*th gap set**. Let $g_i = |G_i|$. For example, if $P = \{4, 8, 11\}$ then $G_1 = \{2, 3\}$, $G_2 = \{5, 6, 7\}$ and $G_3 = \{9, 10\}$. We now proceed to create and count all possible vale sets given a fixed pinnacle set.

Proposition 4.7. *If $P = \{p_1 < p_2 < \cdots < p_\ell\}$ is an admissible pinnacle set, then the number of sets V such that (P, V) is admissible is*

$$|\mathcal{V}(P)| = \sum_{\mathbf{t} \in C(\ell)} \prod_{i=1}^{\ell} \binom{g_i}{t_i}.$$

Proof. For each weak composition $\mathbf{t} = (t_1, t_2, \ldots, t_\ell) \in C(\ell)$, we will construct $\prod_{i=1}^{\ell} \binom{g_i}{t_i}$ admissible vale sets. By definition note $G_i \cap G_j = \emptyset$ whenever $i \neq j$.

For each G_i, we choose a subset $T_i \subseteq G_i$ of cardinality t_i, recalling that for each $1 \leq i \leq \ell$, we have that $\sum_{i=1}^{k} t_i \geq k$, for all $k \in \{1, \ldots, \ell\}$. There are $\prod_{i=1}^{\ell} \binom{g_i}{t_i}$ different choices for the collection of subsets $T_i \subseteq G_i$.

Once all the T_i are chosen, let $V = \{1\} \cup \bigcup_{i=1}^{\ell} T_i$. Since $T_i \cap T_j = \emptyset$ then $|V| = 1 + \sum_{i=1}^{\ell} t_i = 1 + \ell$. We now prove that (P, V) is admissible by creating a permutation with pinnacle set P and vale set V. Sort V such that $V = \{v_0, v_1, v_2, \ldots, v_\ell\}$ with $v_0 = 1$ and $v_{i-1} < v_i$ for $i \in \{1, \ldots, \ell\}$. Let

$$\alpha = v_0 p_1 v_1 p_2 v_2 \cdots p_\ell v_\ell.$$

Note that α is a permutation of $P \cup V$ with pinnacle set P and vale set V because for any pinnacle p_k, by the defining condition of $C(\ell)$, we have chosen $\sum_{i=1}^{k} t_i$ elements in V, all of which are less than p_k. Thus, the vales $1, v_1, \ldots, v_k$ are all less than p_k, which implies that the pinnacle set (resp. vale set) of α is P (resp. V). To complete the claim, extend α to a permutation of $[n]$ by inserting each element $r \in [n] \setminus (P \cup V)$ in ascending order between v_i and p_i if $v_i < r < p_{i+1}$. The resulting permutation π has pinnacle set P and vale set V. Hence, (P, V) is admissible.

To show these are the only admissible pairs, let V' be any set such that (P, V') is admissible. Then there is a permutation π such that $\operatorname{Pin}(\pi) = P$ and $\operatorname{Vale}(\pi) = V'$. Partition V' as $V' = \{1\} \cup \bigcup_{i=1}^{\ell} T_i'$ where for $1 \leq i \leq \ell$, T_i' is defined as

$$T_i' = \{v \in V' : p_{i-1} < v < p_i\} \text{ with } p_0 = 1.$$

Note that, by definition of G_i, we have $T_i' \subseteq G_i$ and set $t_i' = |T_i'|$. Since $T_i' \cap T_j' = \emptyset$ whenever $i \neq j$ and $\cup_{i=1}^{\ell} T_i' = V' \setminus \{1\}$, we have that $\sum_{i=1}^{\ell} t_i' = |V'| - 1 = \ell$. Thus, $\mathbf{t} = (t_1', \ldots, t_\ell')$ is a weak composition of ℓ. Further, $\mathbf{t}$ is an element of $C(\ell)$ since for each k in $\{1, 2, \ldots, \ell\}$ the elements in $\cup_{i=1}^{k} T_i'$ correspond to choices of vales not equal to 1 that are less than p_k, thus

$$\sum_{i=1}^{k} t_i' = |V_{p_k}| - 1 = |V_{p_k}| - |P_{p_k}| + (k-1) - 1 = k - 2 + N_{PV}(p_k) \geq k,$$

where the last inequality follows from Lemma 4.1(b). Hence, the set V' is created via the construction described in this result by starting with $\mathbf{t} \in C(\ell)$ and making the choices so that $T_i = T_i' \subseteq G_i$. Therefore, we have constructed all vale sets V such that (P, V) is admissible. □

We now give two examples to compute the number of permutations with pinnacle sets $P = \{5\}$ and $P = \{4, 8, 11\}$.

Example 4.8. Let $n = 8$ and consider the admissible pinnacle set $P = \{5\}$. As 1 is always a vale, and $G_1 = \{2, 3, 4\}$ the possible vale sets are $\mathcal{V}(P) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. Since $N_{PV}(5) = 2$, by Corollary 4.6 we have that

$$\begin{aligned}
|\operatorname{Pin}(P; 8)| &= 2^6 \left(\sum_{V \in \mathcal{V}(P)} \binom{N_{PV}(5)}{2} \prod_{r \in [n] \setminus (P \cup V)} N_{PV}(r) \right) \\
&= 2^6 \left[\underbrace{\binom{2}{2} \prod_{r \in \{3,4,6,7,8\}} N_{PV}(r)}_{V=\{1,2\}} + \underbrace{\binom{2}{2} \prod_{r \in \{2,4,6,7,8\}} N_{PV}(r)}_{V=\{1,3\}} + \underbrace{\binom{2}{2} \prod_{r \in \{2,3,6,7,8\}} N_{PV}(r)}_{V=\{1,4\}} \right] \\
&= 2^6 [1 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1] \\
&= 2^6 (2^2 + 2 + 1) \\
&= 448.
\end{aligned}$$

Example 4.9. Let $n = 12$ and consider $P = \{4, 8, 11\}$ so $\ell = 3$. Then

$$C(\ell) = \{(1, 1, 1), (2, 0, 1), (2, 1, 0), (1, 2, 0), (3, 0, 0)\}.$$

The gaps are $G_1 = \{2, 3\}$, $G_2 = \{5, 6, 7\}$, and $G_3 = \{9, 10\}$, so $(g_1, g_2, g_3) = (2, 3, 2)$. The number of admissible vale sets is

$$|\mathcal{V}(P)| = \binom{2}{1}\binom{3}{1}\binom{2}{1} + \binom{2}{2}\binom{3}{0}\binom{2}{1} + \binom{2}{2}\binom{3}{1}\binom{2}{0} + \binom{2}{1}\binom{3}{2}\binom{2}{0} + \binom{2}{3}\binom{3}{0}\binom{2}{0} = 23.$$

Note that the term $\binom{2}{1}\binom{3}{1}\binom{2}{1}$ counts the vale sets with one element coming from each set of gaps, while the term $\binom{2}{2}\binom{3}{0}\binom{2}{1}$ counts the vale sets where 2 elements come from the first set of gaps, 0 come from the second set of gaps, and 1 comes from the third set of gaps, etc. Thus

$$\mathcal{V}(P) = \left\{ \begin{matrix} \{1,2,5,9\}, \{1,2,5,10\}, \{1,2,6,9\}, \{1,2,6,10\}, \{1,2,7,9\}, \{1,2,7,10\}, \\ \{1,3,5,9\}, \{1,3,5,10\}, \{1,3,6,9\}, \{1,3,6,10\}, \{1,3,7,9\}, \{1,3,7,10\}, \\ \{1,2,3,9\}, \{1,2,3,10\}, \{1,2,3,5\}, \{1,2,3,6\}, \{1,2,3,7\}, \\ \{1,2,5,6\}, \{1,2,5,7\}, \{1,2,6,7\}, \{1,3,5,6\}, \{1,3,5,7\}, \{1,3,6,7\} \end{matrix} \right\}. \tag{7}$$

From this, a straight forward computation using Corollary 4.6 yields $|\text{Pin}(\{4,8,11\};12)| = 132,480$.

Remark 4.10. The partitions $t \in C(\ell)$ indexing the sum in Proposition 4.7 are counted by Catalan numbers. Particularly, $|C(\ell)|$ is the ℓth Catalan number. We refer the interested reader to Exercise 86 of Richard Stanley's book on Catalan numbers [9].

5. Algorithms to generate all permutations with a given pinnacle set

As we saw in Section 2, the dual Foata–Strehl action preserved the pinnacles of a permutation, but the orbits did not encompass all elements having the same set of pinnacles. In this section, we describe two algorithms that generate the set $\text{Pin}(P;n)$, and we compare their computational run times.

Given a pinnacle set P, let Algorithm 1 be the naive algorithm that runs through all permutations of S_n, computes their pinnacle sets and returns those permutations with pinnacle set P. Let Algorithm 2 be the algorithm that replicates the constructions detailed in Section 4. More specifically, given a set P, it first runs through all admissible vale sets V using the criteria in Proposition 4.2. Then, for a given pair (P,V), it constructs all canonical PV-arrangements using the recursive construction described in the proof of Lemma 4.3. Then, it creates all FS-minimal permutations from the canonical PV-arrangements as described in the proof of Theorem 4.4. Finally, it applies the dual Foata–Strehl action on the FS-minimal permutations to create all permutations with pinnacle set P and vale set V, as guaranteed by Theorem 3.6.

In Table 3, we provide the run times of Algorithms 1 and 2 applied to all pinnacle sets of permutations in $\mathfrak{S}_8$. The code and sample computations for these algorithms are provided at github.com/8080509/Pinnacles_of_Permutations.

6. Future directions

We end with a few open problems for further study.

Problem 6.1. Algorithm 2 provides an efficient algorithm to generate $\text{Pin}(P;n)$. Are there any other algorithms for generating $\text{Pin}(P;n)$ that are more efficient than Algorithm 2?

A couple of days after we made a preprint of this work available, Irena Rusu presented a preprint showing a concatenation of three algorithms that allow you to transform any permutation with pinnacle set P to any other permutation with the same pinnacle set. We refer the interested reader to [7]. Shortly thereafter Rusu and Tenner presented a preprint describing, given a pinnacle set P, which ordering of the pinnacles can actually appear in permutations with pinnacle set P. For additional open problems related to this, see [8].

In [3], Davis et al. give explicit formulas for the number of permutations with pinnacle sets of size 0, 1, and 2 as well as two extremal cases.

Problem 6.2. Theorem 4.4 provides an expression for the number O_P of orbits containing permutations with pinnacle set P under the dual Foata–Strehl action. Find explicit expressions (only depending on n) for $\text{Pin}(P;n)$ with $|P| \geq 3$ using Corollary 4.6.

If P is a pinnacle set and S is a peak set, by Theorem 1.1 and Corollary 4.6, we know

$$|\text{Pin}(P;n)| = 2^{n-|P|-1}O_P \qquad \text{and} \qquad |\text{Pk}(S,n)| = 2^{n-|S|-1}p_S(n),$$

where $p_S(n)$ is the peak polynomial of S and O_P is given by

$$O_P = \sum_{V \in \mathcal{V}(P)} \prod_{p \in P} \binom{N_{PV}(p)}{2} \prod_{r \in [n] \setminus (P \cup V)} N_{PV}(r).$$

In light of the similarity between these equations and the fact that in the pinnacle setting the power of two describes the size of each dual Foata–Strehl orbit in $|\text{Pin}(P;n)|$, and O_P counts number of orbits, we pose the following question.

Problem 6.3. Is there a group action on permutations which preserves peaks sets, such that there are exactly $p_S(n)$ many orbits each of size $2^{n-|S|-1}$?

We have presented the following conjecture at several talks concerning peaks, descents, and pinnacles of permutations over the past year. An elegant proof of this conjecture was recently given in the preprint [5, Corollary 10]. We present the conjecture here to have it recorded in the literature.

Table 3
Run times of four algorithms constructing all permutations in $\mathfrak{S}_8$ with pinnacle set P.

n	P	$\lvert\mathrm{Pin}(P; n)\rvert$	Run time Algorithm 1	Run time Algorithm 2
8	∅	128	327.32 ms	0.30 ms
	{3}	64	286.39 ms	0.21 ms
	{4}	192	300.00 ms	0.63 ms
	{5}	448	346.05 ms	1.12 ms
	{6}	960	360.80 ms	2.65 ms
	{7}	1984	293.21 ms	6.78 ms
	{8}	4032	271.34 ms	9.45 ms
	{3, 5}	32	411.87 ms	0.11 ms
	{3, 6}	96	480.09 ms	0.54 ms
	{3, 7}	224	436.53 ms	0.59 ms
	{3, 8}	480	275.81 ms	1.13 ms
	{4, 5}	96	309.15 ms	0.43 ms
	{4, 6}	288	306.61 ms	1.31 ms
	{4, 7}	672	280.14 ms	1.64 ms
	{4, 8}	1440	291.30 ms	3.67 ms
	{5, 6}	576	324.70 ms	1.66 ms
	{5, 7}	1376	307.15 ms	3.75 ms
	{5, 8}	2976	285.69 ms	10.77 ms
	{6, 7}	2400	297.79 ms	6.08 ms
	{6, 8}	5280	338.10 ms	14.92 ms
	{7, 8}	8640	341.82 ms	20.45 ms
	{3, 5, 7}	16	298.76 ms	0.13 ms
	{3, 5, 8}	48	282.71 ms	0.19 ms
	{3, 6, 7}	48	269.83 ms	0.20 ms
	{3, 6, 8}	144	328.53 ms	0.53 ms
	{3, 7, 8}	288	305.97 ms	0.84 ms
	{4, 5, 7}	48	296.36 ms	0.20 ms
	{4, 5, 8}	144	341.79 ms	0.71 ms
	{4, 6, 7}	144	294.63 ms	0.47 ms
	{4, 6, 8}	432	296.74 ms	1.31 ms
	{4, 7, 8}	864	358.77 ms	2.68 ms
	{5, 6, 7}	288	294.47 ms	0.99 ms
	{5, 6, 8}	864	305.26 ms	2.44 ms
	{5, 7, 8}	1728	334.79 ms	4.98 ms
	{6, 7, 8}	2880	276.56 ms	8.08 ms

Conjecture 6.4. *If S is an admissible peak set, then the set $\mathrm{Pk}(S, n)$ of permutations with peak set S in $\mathfrak{S}_n$ can be partitioned into subsets of permutations of the same length, and the size of these subsets is palindromic about the value $\binom{n}{2}$.*

We remark that as the sets $\mathrm{Pin}(P; n)$ are preserved by multiplying by w_0 on the right i.e., reversing the order of the permutations, the sets $\mathrm{Pin}(P; n)$ have the same palindromicity property as $\mathrm{Pk}(S, n)$. However, unlike $\mathrm{Pk}(S; n)$, they are not unimodal or log-concave in general.

Problem 6.5. For what pinnacle sets P are the cardinalities of the sets $\mathrm{Pin}(P; n)$ a unimodal sequence as you vary n?

Acknowledgements

We thank the anonymous referees for their careful revisions that undoubtedly helped us improve the quality of this article. Erik Insko was partially supported by the FGCU Office of Undergraduate Scholarship and the FGCU Seidler Collaboration Fellowship.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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