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Brief paper

Stability of stochastic functional differential equations with random switching and applications ☆

Nguyen H. Du [a], Dang H. Nguyen [b], Nhu N. Nguyen [c], George Yin [c,*]

[a] Department of Mathematics–Mechanics–Informatics, Hanoi University of Science, HaNoi, Viet Nam
[b] Department of Mathematics, University of Alabama, Tuscaloosa, AL, USA
[c] Department of Mathematics, University of Connecticut, Storrs, CT, USA

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ABSTRACT

This paper develops a new stability theory for stochastic functional differential systems with random switching. It examines general systems that are time inhomogeneous, past-dependent, and perturbed by a Brownian motion and modulated by a switching process. In contrast to the advances in the literature, this paper provides weaker and more verifiable conditions. Examples and discussion are also given to demonstrate the applicability of our results.

1. Introduction

This paper focuses on stability of stochastic functional differential equations (SFDEs) with random switching. The distinct features of the paper include general nonlinear stochastic functional differential equations and interactions with discrete events represented by random switching processes. In many real world scenarios, for example, in queueing systems, biological and ecological systems, finance and economics, control engineering, networked systems, wired and wireless communications, as well as other related fields, delays are often unavoidable. Such dynamic systems have memory and include the past dependence (Kolmanovskii & Myshkis, 1992). Because dynamic systems are often corrupted by noise, stochastic functional differential equations (SFDEs) have been studied extensively in the past decades; see Federico and Øksendal (2011), Mao (1999), Mohammed (1986) and Scheutzow (2005) and references therein. From another angle, in addition to noise appearing in an analog fashion, random switching frequently takes place in a finite set resulting in the systems being hybrid, in which continuous dynamics and discrete events coexist and interact. With the pressing need and taking the above points into consideration, this paper concentrates on the SFDEs with random switching. Many systems are in operation for a long period of time, thus it is necessary to examine stability of the systems. Various notions of stability for stochastic functional differential equations have been considered using Razumikhin methods and Lyapunov functionals in Bao, Yin, and Yuan (2016), Guo, Mao, and Yue (2016), Mao and Yuan (2006), Shaikhet (2013) and Zhao and Deng (2014) for systems without switching and in Li and Mao (2012), Nguyen and Yin (2020) and Yuan and Mao (2004) for systems with switching. In the literature, treating stochastic differential equations evolving delays and switching, the past-dependence often appears to be in certain specific forms; see Li and Mao (2012) and Yuan and Mao (2004) resulting in the problems being still finite dimensional. Similar to the counterpart of functional differential equations, by examining the segment processes, stochastic functional differential equations are generally infinite dimensional. Recently, inspired by the work of Dupire, Nguyen and Yin in Nguyen and Yin (2020) obtained sufficient conditions for stability of SFDEs with regime switching by using appropriate Lyapunov functionals. Nevertheless, the systems considered in Nguyen and Yin (2020) are autonomous, i.e., the coefficients do not depend on the time variable. Moreover, the main theorems in Nguyen and Yin (2020, Theorem 3.2), as well as in Li and Mao (2012, Theorem 1), Yuan and Mao (2004, Theorem 2.1), and references therein, use conditions that require the existence of certain Lyapunov functionals satisfying suitable conditions at each discrete state i together with some uniformity. Such conditions can be restrictive and difficult to verify in applications.

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* Corresponding author.
E-mail addresses: nhdu@viasm.edu.vn (N.H. Du), hnguyen4@ua.edu (D.H. Nguyen), nguyen.nhu@uconn.edu (N.N. Nguyen), gyin@uconn.edu (G. Yin).

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The novelty and contribution of this paper can be summarized as follows. We provide weaker and more verifiable conditions for stability of more general stochastic delay systems allowing the coexistence of continuous dynamics and discrete events with time-varying delays, which generalize the results of Li and Mao (2012), Nguyen and Yin (2020) and Yuan and Mao (2004). Our results can be applied to linear stochastic differential-difference equations, nonlinear stochastic functional systems under linearization, multi-agent systems (Zong, Li, & Zhang, 2019), controls of networked systems (Donkers, Heemels, Bernardini, Bemporad, & Shneer, 2012), and time-varying delay systems (Kao & Lincoln, 2004; Yao, Zhang, & Xie, 2020; Zhou, 2019). In addition, our approach enables one to generalize the results in the aforementioned references for stability of more complex and more general systems.

The rest of the paper is organized as follows. Section 2 is devoted to new conditions for exponential stability in probability of SFDEs with regime switching. Section 3 discusses our results and provides some examples. Further remarks are made in Section 4 to conclude the paper.

2. Stability of stochastic functional differential equations with Markov switching

We work with a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$. Let $\mathbb{R}^n$ be Euclidean space with Euclidean norm $|\cdot|$, $\mathbb{R}_+ := [0,\infty)$, $W(t)$ be a d-dimensional standard Brownian motion, $\alpha(t)$ be an ergodic Markov chain independent of $W(t)$ taking value in a finite set $\mathcal{M}$ with invariant probability measure $\{\nu_k : k \in \mathcal{M}\}$ and generator $Q = (q_{kl})_{k,l\in\mathcal{M}}$, and $r > 0$. For $X(t) : \mathbb{R} \to \mathbb{R}^n$, we use X_t to denote the segment function, i.e., $X_t := \{X(t+s) : s \in [-r, 0]\} \in \mathcal{C} := \mathcal{C}([-r, 0], \mathbb{R}^n)$, the space of continuous functions endowed with the sup-norm $\|\cdot\|$ and let $b(\cdot,\cdot,\cdot) : \mathcal{C}\times\mathcal{M}\times\mathbb{R}_+ \to \mathbb{R}^n$, $\sigma(\cdot,\cdot,\cdot) : \mathcal{C}\times\mathcal{M}\times\mathbb{R}_+ \to \mathbb{R}^{n\times d}$. Consider the stochastic functional differential equation

$$dX(t) = b(X_t, \alpha(t), t)dt + \sigma(X_t, \alpha(t), t)dW(t), \tag{1}$$

and denote by $(X^\phi(t), \alpha^i(t))$ its solution with initial data $X_0 = \phi$, $\alpha(0) = i$. Use $\mathbb{P}_{\phi,i}$ and $\mathbb{E}_{\phi,i}$ to denote the probability and expectation corresponding to initial data (ϕ, i). To simplify the notation, in what follows, the solution of (1) will be denoted by $(X(t), \alpha(t))$; we will only use the notation $(X^\phi(t), \alpha^i(t))$ when it is necessary.

Suppose $V(x) \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$, the set of twice continuously differentiable functions, $V(x) = 0$ only if $x = 0$, we define the operator on $\mathcal{C}\times\mathbb{R}_+$ by

$$[\mathcal{L}_i V](\phi, t) = V_x(\phi(0))b(\phi, i, t) + \frac{1}{2}\mathrm{tr}(V_{xx}(\phi(0))\Sigma(\phi, i, t)).$$

where $\Sigma(\phi, i, t) := \sigma(\phi, i, t)\sigma^\top(\phi, i, t)$. Note that (ϕ, t) in $[\mathcal{L}_i V](\phi, t)$ is the variables of $[\mathcal{L}_i V]$ instead of V. Throughout this paper, we assume the following assumptions hold.

Assumption 1. For each $i \in \mathcal{M}$, the functions $b(\phi, i, t)$ and $\sigma(\phi, i, t)$ are locally Lipschitz continuous with respect to the first variable.

Assumption 2. Let $V : \mathbb{R}^n \mapsto \mathbb{R}_+ := [0,\infty)$ be a twice continuously differentiable function satisfying

$$c_1|x|^2 \le V(x) \le c_2|x|^2, \forall x \in \mathbb{R}^n \text{ for some } c_1, c_2 > 0. \tag{2}$$

Suppose that there exist $\Delta_0 > 0$, $m(i) \ge 0$, and $a(i) \in \mathbb{R}$ for each $i \in \mathcal{M}$, $q \in [0, 1]$, and a probability measure μ_t (allowed to depend on t) on $[-r, 0]$ such that if $\|\phi\| \le \Delta_0$

$$[\mathcal{L}_i V](\phi, t) \le a(i)V(\phi(0)) + m(i)\int_{-r}^0 [V(\phi(0))]^q[V(\phi(s))]^{1-q}\mu_t(ds), \tag{3}$$

and

$$\sum_{i\in\mathcal{M}}(a(i) + m(i)\bar\mu)\nu_i < 0, \tag{4}$$

where $\gamma := 0 \wedge \min_{i\in\mathcal{M}}\{a(i) + m(i)\}$, and $\bar\mu := \max_{\{t\ge 0\}}\int_{-r}^0 e^{\gamma(1-q)s}\mu_t(ds)$.

Assumption 3. There exist a constant $\widetilde C > 0$, $\widetilde q \in [0, 1]$, and a probability measure $\widetilde\mu_t$ on $[-r, 0]$ such that for all $(\phi, i, t) \in \mathcal{C}\times\mathcal{M}\times\mathbb{R}_+$

$$|b(\phi, i, t)|^2 + \mathrm{tr}(\Sigma(\phi, i, t)) \le \widetilde C\Big(|\phi(0)|^2 + \int_{-r}^0 |\phi(0)|^{2\widetilde q}|\phi(s)|^{2(1-\widetilde q)}\widetilde\mu_t(ds)\Big).$$

Remark 4. In contrast to the existing works, $a(i)$, $m(i)$ in (3) depend on states of switching rather than being constants for all switching states. One can verify conditions for complex systems at each fixed state i. Moreover, the measures μ_t and $\widetilde\mu_t$ can depend on t. Thus our conditions can be used effectively to treat time-varying delays.

Let $\mathcal{F}_t^1$ and $\mathcal{F}_t^2$ be the filtration generated by $W(t)$ and $\alpha(t)$, respectively. For an event A, let $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise. Let $\Delta \in (0, \Delta_0]$, where Δ_0 is as in Assumption 2. Define $\tau_\Delta = \inf\{t \ge 0 : V(X(t)) \ge \Delta\}$.

Lemma 5. *There is a constant $0 < H_1 < \infty$ independent of Δ such that*

$$\mathbb{E}\ (\mathbf{1}_{\{\tau_\Delta > r+t\}}\|X_{t+r}\|^2|\mathcal{F}_\infty^2) \le H_1 \max_{s\in[t-r,t+r]}\mathbb{E}(\mathbf{1}_{\{\tau_\Delta>s\}}|X(s)|^2|\mathcal{F}_\infty^2),\ t \ge 0.$$

Proof. We obtain from (1) that

$$\sup_{s\in[t,t+r]}|X(s)| \le |X(t)| + \int_t^{t+r}|b(X_s, \alpha(s), s)|ds + \sup_{s\in[t,t+r]}\Big|\int_t^s \sigma(X_u, \alpha(u), u)dW(u)\Big|.$$

Then, an elementary inequality implies that

$$\begin{aligned}(\sup_{s\in[t,t+r]}|X(s)|)^2 &\le 3|X(t)|^2 + 3r\int_t^{t+r}|b(X_s, \alpha(s), s)|^2ds\\ &\quad+3\sup_{s\in[t,t+r]}\Big|\int_t^s \sigma(X_u, \alpha(u), t)dW(u)\Big|^2.\end{aligned}$$

Multiplying both sides by $\mathbf{1}_{\{\tau_\Delta>t+r\}}$, using the fact that $\mathbf{1}_{\{\tau_\Delta>t_1\}} \le \mathbf{1}_{\{\tau_\Delta>t_2\}}$ if $t_1 \ge t_2$, taking conditional expectations on both sides, and using the Burkholder–Davis–Gundy inequality (which is valid owing to the independence of $\alpha(t)$ and $W(t)$), we have that

$$\begin{aligned}&\mathbb{E}(\mathbf{1}_{\{\tau_\Delta>t+r\}}\sup_{s\in[t,t+r]}|X(s)||\mathcal{F}_\infty^2)^2\\
&\le 3(r+1)\mathbb{E}\Big[\ \mathbf{1}_{\{\tau_\Delta>t\}}|X(t)|^2\\
&\quad+\int_t^{t+r}\mathbf{1}_{\{\tau_\Delta>s\}}|b(X_s, \alpha(s), s)|^2ds\\
&\quad+\sup_{u\in[t,t+r]}\Big|\int_t^u \mathbf{1}_{\{\tau_\Delta>s\}}\sigma(X_s, \alpha(s), s)dW(s)\Big|^2|\mathcal{F}_\infty^2\ \Big]\\
&\le 3(r+1)\mathbb{E}\Big[\ \mathbf{1}_{\{\tau_\Delta>t\}}|X(t)|^2\\
&\quad+\int_t^{t+r}\mathbf{1}_{\{\tau_\Delta>s\}}|b(X_s, \alpha(s), s)|^2ds\\
&\quad+4\int_t^{t+r}\mathbf{1}_{\{\tau_\Delta>s\}}\mathrm{tr}(\Sigma(X_s, \alpha(s), s))ds\Big|\mathcal{F}_\infty^2\ \Big]\\
&\le 3(r+1)\mathbb{E}\mathbf{1}_{\{\tau_\Delta>t\}}|X(t)|^2\\
&\quad+12(r+1)\widetilde C\mathbb{E}\Big[\int_t^{t+r}\mathbf{1}_{\{\tau_\Delta>s\}}\Big[\ |X(s)|^2\\
&\qquad+\int_{s-r}^s |X(s)|^{2\widetilde q}|X(u)|^{2(1-\widetilde q)}\widetilde\mu_s(du)\ \Big]\ ds|\mathcal{F}_\infty^2\ \Big]\\
&\le 3(r+1)\mathbb{E}(\mathbf{1}_{\{\tau_\Delta>t\}}|X(t)|^2|\mathcal{F}_\infty^2)\\
&\quad+12(r+1)\widetilde C\int_t^{t+r}\mathbb{E}(\mathbf{1}_{\{\tau_\Delta>s\}}|X(s)|^2|\mathcal{F}_\infty^2)ds\\
&\quad+12(r+1)\widetilde C\int_{s-r}^s([\mathbb{E}(\mathbf{1}_{\{\tau_\Delta>u\}}|X(s)|^2|\mathcal{F}_\infty^2)]^{\widetilde q}\\
&\qquad\times[\mathbb{E}(\mathbf{1}_{\{\tau_\Delta>u\}}|X(s)|^2|\mathcal{F}_\infty^2)\widetilde\mu_s(du)]^{1-\widetilde q})ds\\
&\le H_1\sup_{u\in[t-r,t+r]}\mathbb{E}(\mathbf{1}_{\{\tau_\Delta>u\}}|X(u)|^2|\mathcal{F}_\infty^2),\end{aligned}$$

where H_1 is a finite constant. In the above, we applied Holder's inequality for conditional expectations, $\mathbf{1}_{\{\tau_\Delta>s\}} \le \mathbf{1}_{\{\tau_\Delta>u\}}$ for $s \ge u$,

$$\mathbb{E}\Big(\int_{s-r}^{s} \widetilde{f}(u)\tilde{\mu}_s(du)|\mathcal{F}^2_\infty\Big) = \int_{s-r}^{s} \mathbb{E}(\widetilde{f}(u)|\mathcal{F}^2_\infty)\tilde{\mu}_s(du),$$

and Assumption 3. □

Proposition 6. *There exist $p^* > 0$, $C^* > 0$, and $m^* > 0$ such that $\forall t \ge 0$, $\|\phi\| \le \Delta_0$*

$$\mathbb{E}_{\phi,i}\mathbf{1}_{\{\tau_\Delta>t\}}\|X_t\|^{2p^*} \le C^* \exp\{-m^* t\}\|\phi\|^{2p^*}. \tag{5}$$

Consequently, there is a $T^* > 0$ such that

$$\mathbb{E}_{\phi,i}\mathbf{1}_{\{\tau_\Delta>T^*\}}\|X_{T^*}\|^{2p^*} \le 0.5\|\phi\|^{2p^*},\ \|\phi\| \le \Delta_0.$$

Proof. It is noted that if (4) is satisfied then $\gamma < 0$. Let $c(i) = a(i) + \varepsilon_0 + m(i)\bar{\mu}$, where $\varepsilon_0 > 0$ is sufficiently small such that $\sum_{i\in\mathcal{M}} c(i)\nu_i < 0$. Such an ε_0 exists because of (4). For each (fixed) initial value $X_0 = \phi$, $\alpha(0) = i$, let $\alpha(s) = i$ for $s \in [-r, 0]$ and consider the function

$$G(t) = e^{-\int_0^t c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}[1_{\{\tau_\Delta>t\}}V(X(t))|\mathcal{F}^2_\infty].$$

Since $\mathcal{F}^1_t$ and $\mathcal{F}^2_t$ are independent, $G(t)$ is $\mathcal{F}^1_t$-adapted. We will show that with probability 1, $H(t) := \sup_{s\in[t-r,t]}\{G(s)\}$ is non-increasing.

Let $t_0 > 0$ be fixed. If $G(t_0) < H(t_0)$, it is easy to derive from the continuity of $G(t)$ that $H(t_0+h) < H(t_0)$ for sufficiently small h. Now, consider the case $G(t_0) = H(t_0)$, or equivalently, for all $t \in [t_0 - r, t_0]$,

$$\begin{aligned}&e^{-\int_0^{t_0} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t_0))|\mathcal{F}^2_\infty]\\ &\quad\ge e^{-\int_0^{t} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t\}}V(X(t))|\mathcal{F}^2_\infty],\end{aligned}$$

which together with the fact $c(i) \ge a(i) + m(i) \ge \gamma$ implies that

$$\begin{aligned}&\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t))|\mathcal{F}^2_\infty]\\ &\le \mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t\}}V(X(t))|\mathcal{F}^2_\infty]\\ &\le e^{t_0-t}\max_{i\in\mathcal{M}}\{-c_i\}\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t_0))|\mathcal{F}^2_\infty]\\ &\le \exp(-\gamma(t_0-t))\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t_0))|\mathcal{F}^2_\infty].\end{aligned} \tag{6}$$

Although the derivative of $\int_0^t c(\alpha(s))ds$ is not continuous, it is piecewise constant, thus, we can still obtain (7) by applying Itô's formula successively for $[(t_0+h)\wedge\tau_{t_0}^{(n-1)}, (t_0+h)\wedge\tau_{t_0}^{(n)})$, $n \in \mathbb{Z}_+$, where $\tau_{t_0}^{(n)}$ is the nth jump of $\alpha(t)$ after t_0. By virtue of Itô's formula, we have

$$\begin{aligned}&e^{-\int_0^{t_0+h} c(\alpha(s))ds}V(X(t_0+h))\\ &= e^{-\int_0^{t_0} c(\alpha(s))ds}V(X(t_0))\\ &\quad+\int_{t_0}^{t_0+h} e^{-\int_0^{t} c(\alpha(s))ds}[\mathcal{L}_{\alpha(t)}V](X_t,t)dt\\ &\quad-\int_{t_0}^{t_0+h} c(\alpha(t))e^{-\int_0^{t} c(\alpha(s))ds}V(X(t))dt\\ &\quad+\int_{t_0}^{t_0+h} e^{-\int_0^{t} c(\alpha(s))ds}V_x(X(t))\sigma(X_t,\alpha(t),t)dW(t).\end{aligned} \tag{7}$$

Multiplying both sides by $\mathbf{1}_{\{\tau_\Delta>t_0\}}$ and taking expectation w.r.t. $\mathcal{F}^2_\infty$, we have

$$\begin{aligned}&e^{-\int_0^{t_0+h} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t_0+h))|\mathcal{F}^2_\infty)\\ &\quad-e^{-\int_0^{t_0} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t_0))|\mathcal{F}^2_\infty)\\ &\le \int_{t_0}^{t_0+h} e^{-\int_0^{t} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t_0\}}[\mathcal{L}_{\alpha(t)}V](X_t,t)|\mathcal{F}^2_\infty)dt\\ &\quad-\int_{t_0}^{t_0+h} c(\alpha^i(t))e^{-\int_0^{t} c(\alpha(s))ds}\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t))|\mathcal{F}^2_\infty)dt.\end{aligned} \tag{8}$$

Since $G(t_0) = e^{-\int_0^{t_0} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t_0))|\mathcal{F}^2_\infty)$ and $G(t_0+h) \le e^{-\int_0^{t_0+h} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t_0+h))|\mathcal{F}^2_\infty)$, by applying (8), Assumption 2, and definition of $c(i)$, we have

$$\begin{aligned}&D_+G(t_0)\\ &:= \lim_{h\to 0+}\frac{G(t_0+h)-G(t_0)}{h}\\ &\le e^{-\int_0^{t} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t_0\}}[\mathcal{L}_{\alpha(t_0)}V](X_{t_0},t_0)|\mathcal{F}^2_\infty)\\ &-c(\alpha^i(t_0))e^{-\int_0^{t_0} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t_0))|\mathcal{F}^2_\infty)\\ &\le -[b(\alpha^i(t_0))\textstyle\int_{-r}^0 e^{\gamma(1-q)u}\mu_{t_0}(du)+\varepsilon_0]\\ &\quad\times e^{-\int_0^{t_0} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t_0))|\mathcal{F}^2_\infty]\\ &+b(\alpha^i(t_0))e^{-\int_0^{t_0} c(\alpha^i(s))ds}\textstyle\int_{-r}^0\mu_{t_0}(du)\\ &\quad\times\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t_0\}}[V(X_{t_0}(0))]^q[V(X_{t_0}(u))]^{1-q}|\mathcal{F}^2_\infty].\end{aligned} \tag{9}$$

Moreover, using Holder's inequality and (6), we have

$$\begin{aligned}&\textstyle\int_{-r}^0\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t_0\}}[V(X_{t_0}(0))]^q[V(X_{t_0}(u))]^{1-q}|\mathcal{F}^2_\infty)\mu_{t_0}(du)\\ &\quad\le\ \textstyle\int_{-r}^0(\mathbb{E}_{\phi,i}(1_{\{\tau_\Delta>t_0\}}V(X(t_0))|\mathcal{F}^2_\infty))^q\\ &\qquad\times(\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X_{t_0}(u))|\mathcal{F}^2_\infty))^{1-q}\mu_{t_0}(du)\\ &\quad\le\ \mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t_0))|\mathcal{F}^2_\infty]\textstyle\int_{-r}^0 e^{\gamma(1-q)u}\mu_{t_0}(du).\end{aligned} \tag{10}$$

Combining (9) and (10), we get that $D_+G(t_0) \le 0$. It can be 0 only if $\mathbf{1}_{\{\tau_\Delta>t_0\}}V(X(t_0)) = 0$ and then, $G(t) = 0$, for $t \ge t_0$, it is a trivial case. If $D_+G(t_0) < 0$, $G(t_0+h) \le G(t_0)$ when h is small. Thus, $H(t)$ is not increasing. As a result, for $t \ge 0$

$$\begin{aligned}&e^{-\int_0^{t} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t\}})V(X(u)|\mathcal{F}^2_\infty)\\ &\le \sup_{u\in[-r,0]}\{e^{-\int_0^{u} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>u\}}V(X(u))|\mathcal{F}^2_\infty)\}.\end{aligned} \tag{11}$$

In view of (11), we obtain for $t \ge 0$

$$e^{-\int_0^{t} c(\alpha^i(s))ds}\mathbb{E}_{\phi,i}(\mathbf{1}_{\{\tau_\Delta>t\}}V(X(u))|\mathcal{F}^2_\infty) \le K_2\|\phi\|^2,$$

where $K_2 = c_2e^{r\max_{i\in\mathcal{M}}c(i)}$. As a consequence,

$$\mathbb{E}_{\phi,i}[V(X(t))|\mathcal{F}^2_\infty] \le K_2e^{\int_0^t c(\alpha^i(s))ds}\sup_{u\in[-r,0]}\|V(\phi)\|^2.$$

Therefore, by virtue of Lemma 5 and (2), we obtain that

$$\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>t\}}\|X_t\|^2|\mathcal{F}^2_\infty] \le K_3e^{\int_0^t c(\alpha^i(s))ds}\|\phi\|^2,$$

where $K_3 = \frac{H_1K_2c_2}{c_1}$. Therefore, by conditional Jensen's inequality for $p \in (0,1)$, we have

$$\mathbb{E}_{\phi}[\mathbf{1}_{\{\tau_\Delta>t\}}\|X_t\|^{2p}|\mathcal{F}^2_\infty] \le K_3\|\phi\|^{2p}e^{p\int_0^t c(\alpha^i(s))ds}. \tag{12}$$

Since $\sum c(i)\nu_i < 0$, it follows from Bardet, Guérin, and Malrieu (2010) that there exist $p^* > 0$, $C^* > 0$, and $m^* > 0$ such that

$$\mathbb{E}_ie^{p^*\int_0^t c(\alpha(s))ds} \le \frac{C^*}{K_3}\exp(-m^*t). \tag{13}$$

Combining (12) and (13) implies (5). Finally, let $T^* > 0$ be such that $\frac{C^*}{K_3}\exp(-m^*T^*) < 0.5$, one has

$$\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta>T^*\}}\|X_{T^*}\|^{2p^*}] \le 0.5\|\phi\|^{2p^*}.$$

The proof is complete. □

Lemma 7. *For any p and $T > 0$, there exists a constant H_2 depending on p and T such that*

$$\mathbb{P}_{\phi,i}\{\tau_\Delta \le T\} \le H_2\frac{\|\phi\|^{2p}}{\Delta^p},\ (\phi,i) \in \mathcal{C}\times\mathcal{M}.$$

Proof. Let $\widetilde{c} = \max\{|a_i| + b_i, i \in \mathcal{M}\}$. Define

$$\widetilde{G}(t) = e^{-\widetilde{c}t}\mathbb{E}_{\phi,i}V(X_{t\wedge\tau_\Delta}) \text{ and } \widetilde{H}(t) = \sup_{s\in[t-r,t]} \widetilde{G}(s).$$

Hence, by arguments analogous to those in the proof of Proposition 6 but much simpler (since we do not need to estimate conditional expectation with respect to $\mathcal{F}^2_\infty$), we obtain that $\widetilde{H}(t)$ is a non-increasing function. Therefore, for any $T > 0$, one has $\mathbb{E}_{\phi,i}V(X_{T\wedge\tau_\Delta}) \le H_2\|\phi\|^2$, for some constant $H_2 > 0$. As a consequence, for any $(\phi, i) \in \mathcal{C}\times\mathcal{M}$

$$\mathbb{P}_{\phi,i}\{\tau_\Delta \le T\} \le \frac{\mathbb{E}_{\phi,i}V^p(X_{T\wedge\tau_\Delta})}{\Delta^p} \le H_2\frac{\|\phi\|^{2p}}{\Delta^p}. \quad \square$$

Theorem 8. *For any $\varepsilon > 0$, $0 < \Delta < \Delta_0$ (Δ_0 is as in Assumption 2), there exists a $\delta > 0$ such that*

$$\begin{aligned}&\mathbb{P}_{\phi,i}\{\tau_\Delta = \infty\} \ge 1-\varepsilon \text{ if } \|\phi\| \le \delta, \text{ and}\\ &\mathbb{P}_{\phi,i}\{\tfrac{\ln\|X_t\|}{t} < -\lambda\} \ge 1-\varepsilon \text{ if } \|\phi\| \le \delta,\end{aligned} \tag{14}$$

for some $\lambda > 0$ independent of ε, δ.

Proof. In view of Proposition 6 and the strong Markov property of $(X_t, \alpha(t))$, we have that

$$\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta > kT^*\}}\|X_{kT^*}\|^{2p^*}] \le 2^{-k}\|\phi\|^{2p^*}, k \in \mathbb{Z}_+.$$

We also obtain from the strong Markov property and Lemma 7 that

$$\begin{aligned}&\mathbb{P}_{\phi,i}\{kT^* < \tau_\Delta \le (k+1)T^*\}\\ &=\mathbb{E}_{\phi,i}\mathbf{1}_{\{\tau_\Delta > kT^*\}}\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta \le (k+1)T^*\}}|\mathcal{F}_{kT^*}]\\ &=\mathbb{E}_{\phi,i}\mathbf{1}_{\{\tau_\Delta > kT^*\}}\mathbb{E}_{X_{kT^*},\alpha(kT^*)}\mathbf{1}_{\{\tau_\Delta \le T^*\}}\\ &\le\frac{H_2}{\Delta^{p^*}}\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_\Delta > kT^*\}}\|X_{kT^*}\|^{2p^*}]\\ &=\frac{H_2\|\phi\|^{2p^*}}{\Delta^{p^*}}2^{-k}.\end{aligned} \tag{15}$$

As a result,

$$\begin{aligned}\mathbb{P}_{\phi,i}\{\tau_\Delta < \infty\} &= \textstyle\sum_{k=0}^\infty \mathbb{P}_{\phi,i}\{kT^* < \tau_\Delta \le (k+1)T^*\}\\ &\le \frac{H_2\|\phi\|^{2p^*}}{\Delta^{p^*}}\textstyle\sum_{k=0}^\infty 2^{-k} = \frac{2H_2\|\phi\|^{2p^*}}{\Delta^{p^*}}.\end{aligned}$$

Thus, there exists $\delta > 0$ such that $\mathbb{P}_{\phi,i}\{\tau_\Delta = \infty\} \ge 1 - \varepsilon$ if $\|\phi\| \le \delta$. Now, applying Lemma 7 with Δ replaced by $\Delta_k := (0.75^k H_2\delta^{2p^*})^{1/p^*}$, we have

$$\begin{aligned}\mathbb{P}_{\phi,i}\quad &\{\tau_\Delta = \infty, \tau_{\Delta_k} \le (k+1)T^*\}\\ \le\quad &\mathbb{P}_{\phi,i}\{\tau_\Delta > kT^*, \tau_{\Delta_k} \le (k+1)T^*\}\\ =\quad &\mathbb{E}_{\phi,i}\mathbf{1}_{\{\tau_\Delta \le kT^*\}}\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_{\Delta_k} \le (k+1)T^*\}}|\mathcal{F}_{kT^*}]\\ =\quad &\mathbb{E}_{\phi,i}\mathbf{1}_{\{\tau_\Delta \le kT^*\}}\mathbb{E}_{X_{kT^*},\alpha(kT^*)}\mathbf{1}_{\{\tau_{\Delta_k} \le T^*\}}\\ \le\quad &\frac{H_2}{\Delta^{p^*}}\mathbb{E}_{\phi,i}[\mathbf{1}_{\{\tau_{\Delta_k} > kT^*\}}\|X_{kT^*}\|^{2p^*}]\\ \le\quad &\frac{H_2\|\phi\|^{2p^*}}{\Delta_k^{p^*}}2^{-k} \le (1.5)^{-k}.\end{aligned}$$

Since $\sum_{k=0}^\infty (1.5)^{-k} < \infty$, it follows from the Borel–Cantelli lemma that there exists an integer $m = m(\omega) > 0$ such that with probability 1, we have $\tau_\Delta < \infty$ or $\{\tau_{\Delta_k} \le (k+1)T^*\}$ for any $k > m$. Thus, for almost every $\omega \in \{\tau_\Delta = \infty\}$, we have that $\{\tau_{\Delta_k} \le (k+1)T^*\}$ for any $k > m$ or equivalently, for $t \in [kT^*, (k+1)T^*]$, $k > m$

$$|X(t)|^2 \le \frac{V(X(t))}{c_1} \le \frac{1}{c_1}(0.75^k H_2\delta^{2p^*})^{1/p^*}.$$

As a result, we obtain

$$\mathbb{P}_{\phi,i}\{\lim_{t\to\infty}\frac{\ln\|X(t)\|}{t} < -\lambda\} \ge \mathbb{P}_{\phi,i}\{\tau_\Delta = \infty\} \ge 1-\varepsilon,$$

for some $\lambda > 0$. $\square$

3. Examples

We provide some examples in this section. For demonstration purpose, only relatively simple systems are considered.

Example 9. Consider a linear stochastic delay differential equation with regime-switching of the form:

$$\begin{aligned}dX(t) &= [A(\alpha(t))X(t) + B(\alpha(t))X(t-r)]dt\\ &\quad+ \sum_{j=1}^d [C_j(\alpha(t))X(t) + D_j(\alpha(t))X(t-r)]dW_j(t),\end{aligned} \tag{16}$$

where $X(t) \in \mathbb{R}^n$; $A(\cdot), B(\cdot) : \mathcal{M} \to \mathbb{R}^{n\times n}$, $C_j(\cdot), D_j(\cdot) : \mathcal{M} \to \mathbb{R}^{n\times n}$; $W_j(t)$ are standard and independent Brownian motions for $j = 1,\dots,d$. Moreover, we assume that for each $i \in \mathcal{M}$, $B(i)$ is positive definite. Note that for all $x, y \in \mathbb{R}^n$ and $i \in \mathcal{M}$, $2x^\top B(i)y \le x^\top B(i)x + y^\top B(i)y$. For a symmetric matrix $U \in \mathbb{R}^{n\times n}$, denote $\Lambda^M[U] := \sup\{x^\top Ux : x \in \mathbb{R}^n, |x| = 1\}$ and let

$$\begin{aligned}a(i) &= \Lambda^M[(A^\top(i) + A(i)) + B(i) + \textstyle\sum_{j=1}^d C_j(i)C_j^\top(i)],\\ m(i) &= \Lambda^M[B(i) + \textstyle\sum_{j=1}^d D_j(i)D_j^\top(i)],\end{aligned}$$

Considering a Lyapunov function $V(x) = |x|^2$ and applying Theorem 8, one has that under condition (4) with $a(i), m(i)$ defined above and μ, $\tilde{\mu}$ being measures concentrating on $\{-r\}$, (16) is exponentially stable in probability.

Remark 10. In contrast to the existing results, the coefficients in our conditions can depend on time and switching states. In particular, $m(i)$ in (3) can depend on the switching states whereas its counter-part in Nguyen and Yin (2020) (see also Li & Mao, 2012; Yuan & Mao, 2004) must be a constant. When $m(i)$ is forced to be independent of i, the condition on $a(i)$ will be more restrictive. One can check that our conditions for stability in Example 9 are weaker and easier to verify than that in Nguyen and Yin (2020, Example 3.2) as well as similar examples in Li and Mao (2012) and Yuan and Mao (2004). Moreover, while existing results for stochastic delay systems often cannot handle time-varying delays or are more restrictive (e.g., using uniform conditions (Kao & Lincoln, 2004; Nguyen & Yin, 2020; Yao et al., 2020; Zhou, 2019)), our conditions can handle effectively time-varying delays without required uniformity.

Example 11. Consider again the example above, however, the delay $\tau = \tau(t) \in [0, r]$ depends on time, i.e.,

$$\begin{aligned}dX(t) &= [A(\alpha(t))X(t) + B(\alpha(t))X(t-\tau(t))]dt\\ &\quad+ \sum_{j=1}^d [C_j(\alpha(t))X(t) + D_j(\alpha(t))X(t-\tau(t))]dW_j(t).\end{aligned} \tag{17}$$

In this case, we cannot choose measures μ and $\tilde{\mu}$ uniformly in t as usual. Fortunately, in our conditions, we can choose measures μ_t, $\tilde{\mu}_t$ concentrating on $t - \tau(t)$ to verify the conditions as in Example 9. Note that the measures μ_t and $\tilde{\mu}_t$ in our setup can depend on t.

Compared with the existing results, our results can be readily applied to examine the stability of stochastic delay systems under random switching such as Example 9, stochastic functional systems under linearization, multi-agent systems (Zong et al., 2019), controls of networked systems (Donkers et al., 2012). Next, we demonstrate the utility of our result. For numerous applications, linearization is an important approach. An immediate question is: under what conditions, stability of linearized systems yields the associated nonlinear systems. Consider a stochastic nonlinear functional system of the form

$$dX(t) = b(\zeta(t, X_t), \alpha(t), t)dt + \sigma(\zeta(t, X_t), \alpha(t), t)dW(t), \tag{18}$$

where $\zeta(t, \phi) : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is an M-grid "approximation operator" (for some fixed M), i.e., for each fixed t, $\zeta(t, \phi)$ is defined as follows $\zeta(t, \phi) = \sum_{k=1}^{M} c_k \phi(r_k)\ \forall \phi \in \mathcal{C}$, for some "$M$-grid" points $r_1, \ldots, r_M \in [-r, 0]$ and weights $c_1, \ldots, c_M \in \mathbb{R}$ (depending on t). One cannot store an infinite-dimensional vector X_t in a computer, so a finite dimensional approximation ζ acting on segment function is used as above.

Although system (18) is inhomogeneous and depends on switching, our results allow us verify the condition for stability at a fixed discrete state. In fact, we can linearize the system at fixed t and i as follows

$$b(y, i, t) = \widetilde{b}(i, t)y + o(|y|),$$
$$\sigma(y, i, t) = \left(\widetilde{\sigma}_1(i, t)y, \ldots, \widetilde{\sigma}_d(i, t)y\right) + o(|y|),$$

where $o(\cdot)$ represents high-order terms. The linearized system of (18) leads to consideration of the linear system

$$dX(t) = \widetilde{b}(\alpha(t), t)\zeta(t, X_t)dt + \sum_{j=1}^{d} \widetilde{\sigma}_j(\alpha(t), t)\zeta(t, X_t)dW_j(t). \tag{19}$$

As in Example 9, we can obtain a sufficient condition such that system (19) is exponentially stable in probability; and show that under these conditions, nonlinear system (18) is also exponentially stable in probability.

Applying our results to multi-agent systems, we obtain the conditions for consentability of multi-agent systems, which generalize the results in Zong et al. (2019) because a random switching process is added. The study of the stability of networked control systems considered in Donkers et al. (2012) can also be improved because our setting allows the system to be observed under noises. In contrast to applications in literature, our setting is more general because we allow both discrete and continuous states coexist and we do not require "uniform" conditions and still use "local" conditions. To close this section, we note that our results can be applied effectively to systems in which time-varying delays are unavoidable (see Example 11). Such delays often occur in networked control systems, ethernet, CPU scheduling, etc (see e.g., Lincoln, 2000). While there were many applications on stability of time-varying delay system in the literature (Kao & Lincoln, 2004; Yao et al., 2020; Zhou, 2019), much work was restricted to deterministic cases or systems involving continuous noise only. Our results enable one to capture both discrete and continuous states.

4. Concluding remarks

Devoted to stochastic functional differential equations with random switching, this paper established stability under weaker and more verifiable conditions compared to the existing literature. We established the results by using probabilistic techniques to handle functional hybrid systems. For future study, it would be interesting to treat stochastic functional differential systems with random switching and additional jumps as in the setup on Chen, Chen, Tran, and Yin (2019), in which we face the difficulty of treating non-local behaviors of the underlying systems.

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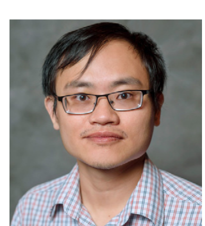
Nguyen H. Du received the B.S. degree in Mathematics from the Mathematics, University of Hanoi in 1979 and Ph.D. in Mathematics from University of Paris VI, France in 1990. He joined Hanoi National University in 1980 and became Professor in Mathematics in 2006. He was Managing Director of Vietnam Institute for Advanced Study in Mathematics from 2013 to 2018. His research interests include differential equations, dynamical system, stochastic systems, stochastic control and their applications, mathematical biology.



Dang H. Nguyen received the B.S. degree in Mathematics from the VNU, Hanoi University of Science in 2008, and Ph.D. in Applied Mathematics from Wayne State University in 2018. He joined University of Alabama in 2018 as an assistant professor. His research interests include stochastic systems and applications, mathematical biology and stochastic control.

Nhu N. Nguyen received the B.S. degree in Mathematics from the Vietnam Nation University of Science in 2015. Now, he is a Ph.D. student in Applied Mathematics at the University of Connecticut. His research interests include probability and applied probability, stochastic approximation, large deviation theory, SDEs, SPDEs and their applications to mathematical biology, mathematical physics, statistics and learning.

George Yin received the B.S. degree in mathematics from the University of Delaware in 1983 and the M.S. degree in electrical engineering and the Ph.D. degree in applied mathematics from Brown University in 1987. He joined Wayne State University, in 1987, and became Professor in 1996 and Distinguished Professor in 2017. He moved to the University of Connecticut in 2020. He is Editor-in-Chief of SIAM Journal on Control and Optimization, and is on the editorial board of many other journals. He is fellow of IFAC, fellow of IEEE, and fellow of SIAM.