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# DYNAMICS OF A VECTOR-HOST MODEL UNDER SWITCHING ENVIRONMENTS

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Abstract. In this paper, the stochastic vector-host model has been proposed and analysed using nice properties of piecewise deterministic Markov processes (PDMPs). A threshold for the stochastic model is derived whose sign determines whether the disease will eventually disappear or persist. We show mathematically the existence of scenarios where switching plays a significant role in surprisingly reversing the long-term properties of deterministic systems.

1. **Introduction.** Vector-borne diseases are infectious diseases, such as yellow fever, malaria, dengue fever, chikungunya which are spread by vectors, e.g., mosquitoes, fleas, ticks, triatomine bugs, etc. It is reported by WHO that about half of the world's population is infected with at least one vector-borne illness [29]. For many vector-borne diseases, there is no vaccine available [22], so it is important to understand the disease transmission dynamics and make efforts to control it. In this context, mathematical modeling has played a very significant role.

Researchers have proposed many epidemic models to understand and control disease dynamics under constant environmental conditions [7, 11, 25, 26, 31, 32]. These kinds of models fall under the category of deterministic modeling (see e.g. [1, 24]). In real life, however, random environmental changes affect the growth of infectious diseases. This gives rise to random switching in the epidemic parameters. For example, in the case of mosquito-borne diseases like dengue fever and malaria, changes in the mosquito population due to fluctuating environments (high temperature, rain fall) lead to significant changes in the disease dynamics. Also, during the course of an epidemic, human behaviour (using control measures) can change the outcome of the disease. It is therefore important to model the disease dynamics under the impact of random fluctuations; see [2, 3, 16].

A common approach for modeling stochasticity in the model is to assume the environment can switch randomly between a number of states depending on changes

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in environmental conditions, such as temperature, humidity, etc. This is called telegraph noise perturbation [10, 16, 21]. In models with telegraph noise, the dynamics of the system together with the driven source of the noise constitute a piecewise deterministic Markov process (PDMP) [9]. The exposition of properties of PDMPs (see [5, 9, 18]) has opened up opportunities for their application in population models. An SIRS epidemic model with random switching is investigated in [16]. Gray et al. [12] have studied the SIS epidemic model under telegraph noise. Cao et al. [8] proposed an SIR model with regime switching by taking into consideration a ratio-dependent incidence rate and degenerate diffusion in the model. Surprising examples of stochastic predator-prey models are described in [19, 28]. Some asymptotic properties of randomly switched Kolmogorov systems are given in [10].

For deterministic epidemic models, the reproduction number $\mathcal{R}_0$ is determined by looking at the spectral radius of the next-generation matrix. However, in a stochastic setting, the next-generation matrix is not constant. Determining the reproduction number by examining the spectrum of the next-generation matrix is therefore not feasible except in some very special cases. Lyapunov functions as well as some martingale inequalities used in [8, 12, 16, 20, 21] do not seem very effective. As an alternative approach (see [4, 6, 14], etc.), we look at the Lyapunov exponent with respect to the invariant measure (which is actually the growth rate of the disease when its density is low) to give a threshold for the dynamics of the epidemic model. After the threshold is introduced, persistence and extinction of the disease can be proved using advanced techniques in stochastic analysis introduced in [4, 6, 9, 14].

The rest of the paper is organized as follows. In section 2, we formulate the stochastic model then transform it into an equivalent model which is easier to deal with. In section 3, we analyze the dynamics of the model on the boundary to obtain a threshold value for extinction and persistence of the disease. Subsequently, the main results are introduced and proved. Section 4 is devoted to a case study of the stochastic model with bilinear incidence rates. We show mathematically that in some scenarios, switching can completely reverse the long-term properties of the deterministic model. Some numerical examples are provided to illustrate these interesting findings by using dengue fever data from the literature. The last section contains some concluding remarks.

## 2. **Formulation.**

The deterministic vector-host model is now presented. Let $N(t)$ be the total host population at time $t$, which is divided into three classes of individuals: susceptible $S_H(t)$, infectious $I_H(t)$ and recovered $R_H(t)$. In an SIR model, the recovered class $R_H$ is comprised of those who have been infected at a previous time and currently have immunity to the disease. Assume that $\omega_1$ is the constant birth rate of susceptible individuals. Let $\mu_1$ be the natural death rate of all host classes and $\gamma$ be the recovery rate of infectious individuals.

As for the vector dynamics, let $S_V(t)$ and $I_V(t)$ be the susceptible and infected vector populations, respectively, and $M(t) = S_V(t) + I_V(t)$. It is assumed that vectors do not recover from the illness once infected. Let $\omega_2$ be the recruitment rate for the vectors, i.e., the natural birth rate, and $\mu_2$ be the natural death rate of the vectors. Further, let $\sigma$ be transmission rate of infection to vectors by infectious hosts.

Assuming all parameters are positive, the following model is formulated:

$$\begin{aligned}
\frac{dS_H}{dt} &= \omega_1 - g(S_H, I_V)I_V - \mu_1 S_H,\\
\frac{dI_H}{dt} &= g(S_H, I_V)I_V - (\gamma + \mu_1)I_H,\\
\frac{dR_H}{dt} &= \gamma I_H - \mu_1 R_H,\\
\frac{dS_V}{dt} &= \omega_2 - \widetilde{g}(S_V, I_H)I_H - \mu_2 S_V,\\
\frac{dI_V}{dt} &= \widetilde{g}(S_V, I_H)I_H - \mu_2 I_V.
\end{aligned} \tag{1}$$

In general, a bilinear incidence rate of the form $g(S_H, I_V) = \beta S_H I_V$ has been used in standard epidemiological models [12]. However, for a large population size, it is not reasonable to consider a bilinear incidence rate. Keeping this fact in mind [15] have considered a saturation incidence (of the form $\frac{\beta S_H I_V}{1+\alpha S_H}$), and [15, 17] use a nonlinear incidence $\beta S_H^p I_V^q$. Accordingly, here a general incidence rate $g(S_H, I_V)$ has been proposed for infected vector to susceptible host interaction. The characteristics of this model, including the reproduction number, are analyzed in [32].

2.1. **Stochastic model.** Suppose the environmental fluctuations make the epidemic dynamics switch between two or more systems of differential equations. We model that effect as follows. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a probability space, and let $(\xi_t)$ be a Markov process on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ taking values in the state space $\mathcal{M} = \{1, 2, \ldots, m_0\}$. Denote by $Q = (q_{kl})_{m_0 \times m_0}$ the generator of the Markov chain$(\xi_t)$. This means that

$$\mathbb{P}\{\xi_{t+\delta} = l | \xi_t = k\} = \begin{cases} q_{kl}\delta + o(\delta) & \text{if } k \neq l,\\ 1 + q_{kk}\delta + o(\delta) & \text{if } k = l, \end{cases}$$

as $\delta \to 0$. Here, $q_{kl}$ is the transition rate from $k$ to $l$ and $q_{kl} \geq 0$ if $k \neq l$, while $q_{kk} = -\sum_{k \neq l} q_{kl}$. We assume that the Markov chain $(\xi_t)$ is irreducible, which means that the system may switch from any regime to any of the other regimes. Under this condition, the Markov chain $(\xi_t)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_{m_0})$. Suppose the incidence rates $g(\xi_t, S_H, I_V)$ and $\widetilde{g}(\xi_t, S_V, I_H)$ and the recovery rate $\gamma(\xi_t)$ depend on the state of $(\xi_t)$.

By the definition of $M$ and $N$,

$$\begin{aligned}
\frac{dN}{dt} &= \omega_1 - \mu_1 N(t),\\
\frac{dM}{dt} &= \omega_2 - \mu_2 M(t).
\end{aligned} \tag{2}$$

We can easily obtain the global positive solutions to (1) in each state. Moreover, because

$$\lim_{t\to\infty} N(t) = \frac{\omega_1}{\mu_1} \text{ and } \lim_{t\to\infty} M(t) = \frac{\omega_2}{\mu_2},$$

we let $M(0) = \frac{\omega_2}{\mu_2}$ and $N(0) = \frac{\omega_1}{\mu_1}$ for simplicity. Note that the behavior of $R_H(t)$ does not affect the dynamics of the other equations when $S_V(t) = \frac{\omega_2}{\mu_2} - I_V(t)$. Let $\gamma_1(\xi_t) = \gamma(\xi_t) + \mu_1$. We restrict our consideration to the following system:

$$\begin{aligned}\frac{dS_H}{dt} &= \omega_1 - g(\xi_t, S_H, I_V)I_V - \mu_1 S_H,\\ \frac{dI_H}{dt} &= g(\xi_t, S_H, I_V)I_V - \gamma_1(\xi_t)I_H,\\ \frac{dI_V}{dt} &= \widetilde{g}\left(\xi_t, \frac{\omega_2}{\mu_2} - I_V, I_H\right)I_H - \mu_2 I_V.\end{aligned} \tag{3}$$

As mentioned in the introduction, in oder to obtain conditions for extinction and persistence of the disease, we wish to consider the dynamics of (3) when $I_H(t)$ is small. We can see from (3) that if $I_H(t)$ converges to 0, so does $I_V(t)$. This suggests we should look at the ratio $\frac{I_V(t)}{I_H(t)}$ when $I_H(t)$ is small. Having this ratio involved will give us an idea to estimate the long-term growth rate of $I_H(t)$ when it is close to 0. This idea is similar to the well-known polar decomposition of linear differential equations. Defining $Y(t) = \frac{I_V(t)}{I_H(t)}$, we convert the system (3) into

$$\begin{aligned}\frac{dS_H}{dt} &= \omega_1 - g(\xi_t, S_H, I_H Y)I_H Y - \mu_1 S_H,\\ \frac{dI_H}{dt} &= g(\xi_t, S_H, I_H Y)I_H Y - \gamma_1(\xi_t)I_H,\\ \frac{dY}{dt} &= -\Big(g(\xi_t, S_H, I_H Y)Y - \gamma_1(\xi_t)\Big)Y + \left(\widetilde{g}\left(\xi_t, \frac{\omega_2}{\mu_2} - I_H Y, I_H\right) - \mu_2 Y\right).\end{aligned} \tag{4}$$

2.2. **Existence and uniqueness.** Throughout this paper, we denote

$$\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$$

and $\mathbb{R}^{n,\circ}_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}$.

**Theorem 2.1.** *Suppose that $g$ and $\widetilde{g}$: $\mathcal{M} \times \mathbb{R}^2_+ \mapsto \mathbb{R}_+$ are non-negative, locally Lipschitz functions and $g(\xi_t, S_H, I_V) = 0$ only if $S_H = 0$, $\widetilde{g}(\xi_t, S_H, I_V) = 0$ only if $S_H = 0$.*

1. *There exists $s_0 > 0$ such that (3) has a unique solution in the invariant set $D \times \mathcal{M}$, where*

$$D := \left\{(S_H, I_H, I_V) \in \mathbb{R}^{3,\circ}_+ : s_0 < S_H, S_H + I_H \leq \frac{\omega_1}{\mu_1}; I_V \leq \frac{\omega_2}{\mu_2}\right\}.$$

2. *There exists $s_0, Y_0 > 0$ such that (4) has a unique solution in the invariant set $\Delta \times \mathcal{M}$, where*

$$\Delta := \left\{(S_H, I_H, Y) \in \mathbb{R}^3_+ : s_0 \leq S_H \leq S_H + I_H \leq \frac{\omega_1}{\mu_1}, I_H Y \leq \frac{\omega_2}{\mu_2}, Y \leq Y_0\right\}.$$

*Proof.* The set $\left\{(S_H, I_H, I_V) \in \mathbb{R}^{3,\circ}_+ : S_H + I_H \leq \frac{\omega_1}{\mu_1}, I_V \leq \frac{\omega_2}{\mu_2}\right\}$ is invariant for (3) because we can obtain from (2) that $\frac{d}{dt}(S_H + I_H) < 0$ if $S_H + I_H \geq \frac{\omega_1}{\mu_1}$ and $I_H > 0$, and $\frac{dI_V}{dt} < 0$ if $I_V \geq \frac{\omega_2}{\mu_2}$. Moreover, since $g(\xi_t, 0, S_H) = 0$, there exists $s_0 > 0$ such that if $S_H \leq s_0$ then

$$\frac{dS_H}{dt} = \omega_1 - g(\xi_t, S_H, I_V)I_V - \mu_1 S_H > 0.$$

As a result, we can assume without loss of generality that $S_H(t) \geq s_0$ and the smaller invariant set we are working on is $D$.

In a similar way, the set $\left\{(S_H, I_H, Y) \in \mathbb{R}^{3,\circ}_+ : s_0 \le S_H \le S_H + I_H \le \frac{\omega_1}{\mu_1}, I_H Y \le \frac{\omega_2}{\mu_2}\right\}$ is invariant for (4). Since

$$\inf\{g(k, S_H, I_H Y) : k \in \mathcal{M}, I_H \ge 0, s_0 \le S_H \le S_H \le S_H + I_H Y \le \frac{\omega_1}{\mu_1}\} > 0,$$

there exists $Y_0 > 0$ such that $\frac{dY}{dt} < 0$ if $Y > Y_0$. Then

$$\Delta^\circ := \left\{(S_H, I_H, Y) \in \mathbb{R}^{3,\circ}_+ : s_0 \le S_H \le S_H + I_H \le \frac{\omega_1}{\mu_1}, I_H Y \le \frac{\omega_2}{\mu_2}, Y \le Y_0\right\}$$

is an invariant set of (4) that attracts all positive solutions of (4). Because the coefficients of (4) are locally Lipschitz, and because

$$\begin{cases} \frac{dI_H(t)}{dt} = 0 & \text{if } I_H(t) = 0, \\ \frac{dS_H}{dt} > 0 & \text{if } S_H(t) \le s_0, \\ \frac{dS_H}{dt} \le 0 & \text{if } S_H(t) = \frac{\omega_1}{\mu_1} \text{ and } I_H(t) = 0, \text{ and} \\ \frac{dY(t)}{dt} \le Y_0 & \text{if } Y(t) \ge Y_0, S_H(t) \ge s_0 \text{ and } S_H(t) + I_H(t) \le \frac{\omega_1}{\mu_1}, \end{cases}$$

we can easily see that for $I_H(0) = 0, S_H(0) \in [s_0, \frac{\omega_1}{\mu_1}], Y(0) \in [0, Y_0]$, (4) has a unique solution satisfying $I_H(t) = 0, S_H(t) \in [s_0, \frac{\omega_1}{\mu_1}], Y(t) \in [0, Y_0]$, for all $t \ge 0$. As a result, the process $(S_H(t), I_H(t), Y(t))$ as solutions to (4) are well-defined on the expanded invariant set $\Delta$. □

**Remark 1.** It is well-known that the process $(Z(t) := (S_H(t), I_H(t), Y(t), \xi_t))$ is a Markov-Feller process on $\Delta \times \mathcal{M}$, i.e., for each bounded and continuous function $f : \Delta \times \mathcal{M} \mapsto \mathbb{R}$, the function: $z \mapsto \mathbb{E}_z f(Z(t))$ is a continuous function in $z$ for each fixed $t$. See, e.g., [5, 9, 23].

2.3. **Disease-free process.** Although (3) and (4) are not equivalent when $I_H = 0$, that does not affect our consideration of $(S_H(t), I_H(t), Y(t))$ on $\Delta$, because we only apply the transformation $(S_H, I_H, I_V) \mapsto (S_H, I_H, Y := I_V/I_H)$ when $I_H > 0$. We consider (4) on $\Delta_0 := \{(S_H, I_H, Y) \in \Delta : I_H = 0\}$ to learn about the dynamics of (4) on $\Delta^\circ$ where the transformation is well-defined. Since $I_H(t) = 0$ for all $t \ge 0$ if $I_H(0) = 0$, $\Delta_0$ is also an invariant set of (4). To determine the threshold of the model, we follow the approach in [4, 14] and consider the system when $I_H(t) = 0$. In the absence of disease, we have

$$\lim_{t\to\infty} S_H(t) = \frac{\omega_1}{\mu_1}, \text{ given } I_H(0) = 0. \tag{5}$$

If $I_H(0) = 0$, then $S_H(0) = \frac{\omega_1}{\mu_1}$, and the equation of $\frac{dY}{dt}$ in (4) is reduced to the following:

$$\begin{aligned} \frac{d\widetilde{Y}}{dt} &:= -\left(g\left(\xi_t, \frac{\omega_1}{\mu_1}, 0\right)\widetilde{Y} - \gamma_1(\xi_t)\right)\widetilde{Y} + \left(\widetilde{g}\left(\xi_t, \frac{\omega_2}{\mu_2}, 0\right) - \mu_2\widetilde{Y}\right) \\ &= -A(\xi_t)\widetilde{Y}^2 + B(\xi_t)\widetilde{Y} + C(\xi_t), \end{aligned} \tag{6}$$

where

$$A(\xi_t) = g\left(\xi_t, \frac{\omega_1}{\mu_1}, 0\right), \quad B(\xi_t) = \big(-\mu_2 + \gamma_1(\xi_t)\big), \quad C(\xi_t) = \widetilde{g}\left(\xi_t, \frac{\omega_2}{\mu_2}, 0\right). \tag{7}$$

Factor the equation (6), and for each fixed $\ell \in \mathcal{M}$ define

$$h(\ell, \widetilde{Y}) := \frac{d\widetilde{Y}}{dt} = -(\widetilde{Y} - q(\ell))(A(\ell)\widetilde{Y} + d(\ell)), \tag{8}$$

where $q(\ell)$ and $d(\ell)$ are given by

$$q(\ell) = \frac{C(\ell)}{d(\ell)} \quad \text{and} \quad d(\ell) = \frac{-B(\ell) + \sqrt{B(\ell)^2 + 4A(\ell)C(\ell)}}{2}. \tag{9}$$

**Proposition 2.1.** *Let $\hat{q} = \min\{q(\ell) : \ell \in \mathcal{M}\}, \check{q} = \max\{q(\ell) : \ell \in \mathcal{M}\}$. Then the process $(\widetilde{Y}(t), \xi_t)$ satisfying (6) has a unique invariant measure $\pi_Y$ on $[\hat{q}, \check{q}] \times \mathcal{M}$.*

*Proof.* For each fixed $\ell$, we have $A(\ell) > 0, C(\ell) > 0$, so there exists a unique root $q(\ell) > 0$ of $h(\ell, \widetilde{Y})$, and

$$\begin{cases} h(\ell, \widetilde{Y}) > 0, & 0 \le \widetilde{Y} < q(\ell), \\ h(\ell, \widetilde{Y}) < 0, & \widetilde{Y} > q(\ell). \end{cases} \tag{10}$$

If $\hat{q} = \check{q} = q^*$ then (10) implies that $\lim_{t\to\infty} \widetilde{Y}(t) = q^*$ for any initial value $\widetilde{Y}(0) > 0$. That is, $\boldsymbol{\delta}_{q^*} \times \pi$ is the unique invariant probability measure of the Markov process $(\widetilde{Y}(t), \xi_t)$, where $\delta_{q^*}$ is the Dirac measure at $q^*$.

Now we consider the case when $\hat{q} < \check{q}$. We deduce from (10) that $[\hat{q}, \check{q}]$ is an attracting invariant set of $(\widetilde{Y}(t))$. Let $\tau_{\hat{q}} = \inf\{t > 0 : \widetilde{Y}(t) > \hat{q}\}$. We want to show that

$$\mathbb{P}_x\{\tau_{\hat{q}} < \infty\} = 1$$

for $\widetilde{y}_0 > 0$, where $x = (\widetilde{y}_0, \ell_0)$ indicates the initial value of $(\widetilde{Y}(t), \xi_t)$. It is clearly true if $\widetilde{y}_0 > \hat{q}$. If $\widetilde{y}_0 \le \hat{q}$ $\forall t > 0$ then $h(\xi_t, \widetilde{Y}(t)) \ge 0$, given $t < \tau_{\hat{q}}$. Moreover, there exists $m_h > 0$ such that $h(\check{\ell}, \widetilde{y}_0) \ge m_h$ $\forall$ $\widetilde{y}_0 \le \hat{q}$, where $\check{\ell}$ is the state such that $h(\check{\ell}, \check{q}) = 0$. We have

$$\widetilde{Y}(t \wedge \tau_{\hat{q}}) - \widetilde{Y}(0) = \int_0^{t\wedge\tau_{\hat{q}}} h(\xi_s, \widetilde{Y}(s))ds \ge \int_0^{t\wedge\tau_{\hat{q}}} m_h \mathbf{1}_{\{\xi_s = \ell_0\}} ds. \tag{11}$$

Due to the ergodicity of $\xi_t$,

$$\lim_{t\to\infty} \int_0^t \mathbf{1}_{\{\xi_s = \xi_0\}} ds = \infty, \text{ a.s.},$$

while the left hand side of (11) is bounded, so it is implied that $\tau_{\hat{q}} < \infty$ a.s. Similarly, we can show that $\inf\{t : \widetilde{Y}(t) < \check{q}\} < \infty$ a.s. Therefore $\widetilde{Y}(t)$ will eventually enter and stay on $[\hat{q}, \check{q}]$.

On the other hand, if $\hat{Y}$ and $\check{Y}$ respectively satisfy the equations $d\hat{Y} = h(\hat{\ell}, Y)dt$ and $d\check{Y} = h(\check{\ell}, Y)dt$, where $\hat{\ell}$ and $\check{\ell}$ are the states such that $q(\hat{\ell}) = \hat{q}$ and $q(\check{\ell}) = \check{q}$, respectively, then $\lim_{t\to\infty} \hat{Y}(t) = \hat{q}$ and $\lim_{t\to\infty} \check{Y}(t) = \check{q}$ $\forall$ $\hat{Y}(0)$, so $\check{Y}(0) > 0$. Using [14, Lemma 3.1] implies that the support of $\{\widetilde{Y}(t)\}_{t\ge 0}$ for sufficiently large $t$ contains $[\hat{q}, \check{q}]$. This, together with the invariance of $[\hat{q}, \check{q}]$ and the fact that $\widetilde{Y}(t)$ will eventually enter $[\hat{q}, \check{q}]$ imply the existence and uniqueness of an invariant probability measure of $(\widetilde{Y}(t), \xi_t)$ with support $[\hat{q}, \check{q}] \times \mathcal{M}$. □

## 3. **Persistence and extinction.**

### 3.1. **Definitions.**

For each $\ell \in \mathcal{M}$, define the vector fields

$$F_\ell(S_H, I_H, Y) = \begin{bmatrix} \omega_1 - g(\ell, S_H, I_H Y) I_H Y - \mu_1 S_H \\ g(\ell, S_H, I_H Y) I_H Y - \mu_1 I_H - \gamma_1(\ell) I_H \\ \widetilde{g}\left(\ell, \frac{\omega_2}{\mu_2} - I_H Y, I_H\right) I_H - \mu_2 I_H Y \end{bmatrix}.$$

Let $[F, G]$ be the Lie bracket of two vector fields $F$ and $G$ and $\mathcal{F}_0$ the set of vector fields $\{F_\ell : \ell \in \mathcal{M}\}$. For $k = 1, 2, ...$, define $\mathcal{F}_k = \mathcal{F}_{k-1} \cup \{[F_\ell, V] : \ell \in \mathcal{M}, V \in \mathcal{F}_{k-1}\}$, where $\mathcal{F}_k(S_H, I_H, Y)$ is the vector space spanned by $\{V(S_H, I_H, Y) : V \in \mathcal{F}_k\}$. Similarly, let $\mathcal{G}_0 = \{F_\ell - F_m : \ell \neq m \in \mathcal{M}\}$ and $\mathcal{G}_k = \mathcal{G}_{k-1} \cup \{[F_\ell, V] : \ell \in \mathcal{M}, V \in \mathcal{G}_{k-1}\}$. We recall two definitions in [5] before stating the main results.

**Definition 3.1.** The weak bracket condition (resp. strong bracket condition) is satisfied at $(S_H, I_H, Y) \in \Delta^\circ$ if there exists $k \geq 0$ such that $\mathcal{F}_k(S_H, I_H, Y) = \mathbb{R}^3$ (resp. $\mathcal{G}_k(S_H, I_H, Y) = \mathbb{R}^3$).

**Remark 2.** In general, the strong bracket condition will be satisfied when the vector fields in different switching states are not proportional and sufficiently "nonlinear," although verifying it for a general set of parameters may be very complicated because the calculations of Lie brakets are cumbersome. For a specific set of parameters, we can however verify that, which will be shown in an example.

**Definition 3.2.** Let $\phi_t^\ell = \phi_{u_n}^{i_{n-1}} \circ ... \circ \phi_{u_1}^{i_0}$ be the semi flow associated with $F_\ell$. The positive orbit of $(S_H, I_H, Y) \in \Delta$ is the set

$$\Gamma^+(S_H, I_H, Y) = \left\{\phi_t^\ell(S_H, I_H, Y) : n \in \mathcal{N}, i_0, \ldots, i_{n-1} \in \mathcal{M}, u_1, \ldots, u_n > 0\right\}.$$

The accessible set of $(Z(t))$ from $\Delta^\circ$ is the (possibly empty) compact set $\Gamma \subset \Delta \times \mathcal{M}$ defined as

$$\Gamma = \bigcap_{z \in \Delta^\circ \times \mathcal{M}} \overline{\Gamma^+(z)}.$$

### 3.2. **Threshold.**

The threshold for extinction and persistence of the disease is given by

$$\lambda = \sum_{\ell \in \mathcal{M}} \int_{\hat{q}}^{\tilde{q}} \left(g\left(\ell, \frac{\omega_1}{\mu_1}, 0\right) y - \gamma_1(\ell)\right) \pi_Y(dy, \ell). \tag{12}$$

When the coefficients do not depend on $\xi_t$, we have

$$\lambda = g\left(\frac{\omega_1}{\mu_1}, 0\right) \frac{2\widetilde{g}(\frac{\omega_2}{\mu_2}, 0)}{\mu_2 - \gamma_1 + \sqrt{(\mu_2 - \gamma_1)^2 + 4g(\frac{\omega_1}{\mu_1})\widetilde{g}(\frac{\omega_2}{\mu_2}, 0)}} - \gamma_1$$

The intuition for the introduction of $\lambda$ is that whether $I_H(T)$ goes extinct or persists depends on the sign of the growth rate $\frac{\ln I_H(T)}{T}$ given that $I_H(t), t \in [0, T]$ is small for a long time $T$. We have

$$\frac{\ln I_H(T)}{T} = \frac{1}{T} \int_0^T \left(g(\xi_t, S_H(t), I_H(t) Y(t)) Y(t) - \gamma_1(\xi_t) I_H(t)\right) dt \tag{13}$$

When $I_H(t)$ is small, $Y(t) \approx \widetilde{Y}(t), S_H(t) \approx \frac{\omega_1}{\mu_1}$, and by the ergodicity we have

$$\begin{aligned}\frac{1}{T}\int_0^T &(g(\xi_t, S_H(t), I_H(t)Y(t))Y(t) - \gamma_1(\xi_t)I_H(t))\,dt \\ &\approx \frac{1}{T}\int_0^T \left(g(\xi_t, S_H(t), 0)\widetilde{Y}(t) - \gamma_1(\xi_t)\right)dt \approx \lambda.\end{aligned}$$

Thus, $\lambda$ plays the threshold role in determining the long-term behavior of the system. A rigorous statement and its proof is provided in the following theorem.

**Theorem 3.3.** *Assume that the conditions on $g, \widetilde{g}$ in Theorem 2.1 hold. We have the following conclusions.*

1. *If $\lambda > 0$, the disease is persistent in the sense that for any $\varepsilon > 0$, there exists $\delta > 0$ such that*
$$\liminf_{t\to 0} \mathbb{P}_z\{I_H(t) < \delta\} \geq 1 - \varepsilon. \tag{14}$$
*If there exists $(s_0, i_0, y_0) \in \Gamma \cap \Delta^\circ$ such that the strong bracket condition is satisfied at $(s_0, i_0, y_0)$, then there exists uniquely an invariant probability measure $\pi^*$ on $\Delta^\circ \times \mathcal{M}$ and the transition probability of $Z(t)$ converges to $\pi^*$ exponentially fast in total variation.*
2. *If $\lambda < 0$, the disease-free equilibrium is locally asymptotically stable. If $\Gamma \cap \Delta_0 \times \mathcal{M} \neq \emptyset$, then for any $z \in \mathbb{R}^{3,\circ}_+ \times \mathcal{M}$,*
$$\mathbb{P}_z\left\{\lim_{t\to\infty} \frac{\ln I_H(t)}{t} = \lambda\right\} = 1.$$
3. *The conclusions of part 2 hold if $g(\ell, s, v) \leq g\left(\ell, \frac{\omega_1}{\mu_1}, 0\right)$ and $\widetilde{g}(\ell, v, i) \leq \widetilde{g}\left(\ell, \frac{\omega_2}{\mu_2}, 0\right)$ for $0 \leq s, i \leq \frac{\omega_1}{\mu_1}$ and $0 \leq v \leq \frac{\omega_2}{\mu_2}$.*

*Proof.* The proof uses the techniques in [4] and [6]. When $I_H(t) = 0$, $\frac{dS_H}{dt} = \omega_1 - \mu_1 S_H$ which implies $\lim_{t\to\infty} S_H(t) = \frac{\omega_1}{\mu_1}$. Moreover, when $S_H = \frac{\omega_1}{\mu_1}$, $Y(t) = \widetilde{Y}(t)$, and $(\xi_t, \widetilde{Y}(t))$ has a unique invariant measure $\pi_Y$. As a result there exists a probability measure $\pi^*$ on $\Delta \times \mathcal{M}$ defined by

$$\pi^*\left(\{z = (s, i, y, \ell) : (s, i) \in A, (\ell, y) \in B\}\right) = \mathbf{1}_{\left\{\left(\frac{\omega_1}{\mu_1}, 0\right) \in A\right\}} \pi_Y(B),$$

and $\pi^*$ is the unique invariant measure of $(Z(t))$ on $\Delta_0 \times \mathcal{M}$.

To ease the reading, we present here the main idea of the proof before giving a rigorous proof. The idea is as follows. Using the fact that weak-limits of occupation measures must be invariant measures of the process $\{Z(t)\}$, the uniqueness of an invariant measure on $\Delta_0$ and the Feller property, we can show that for sufficiently large $T$ and sufficiently small $\delta$, we have from (13) that

$$\mathbb{E}_z[\ln I_H(T) - \ln i] \approx \lambda T$$

if $\operatorname{dist}(z, \Delta_0) \leq \delta$. This approximation gives us an idea about the long-term growth rate of $I_H$ when $Z(t)$ is close to $\Delta$. However, we can not draw a mathematical conclusion from it. We need to interchange the order of expectation and logarithm to obtain a Lyapunov-type estimate, which can be done by utilizing some properties

of the log-Laplace transformation. To be precise, we can show that for some $\theta > 0$, we have

$$\mathbb{E}_z \frac{I_H^\theta(T)}{I_H^\theta(0)} \leq \exp\left\{\frac{\lambda T\theta}{4}\right\}, \text{ if } \lambda < 0$$

and

$$\mathbb{E}_z \frac{I_H^\theta(0)}{I_H^\theta(T)} =\leq \exp\left\{\frac{-\lambda T\theta}{4}\right\}, \text{ if } \lambda > 0$$

With these standard Lyapunov estimate, we can easily obtain desired results in two cases: $\lambda > 0$ and $\lambda < 0$.

Now, we proceed with rigorous arguments. Define the occupation measure $\Pi_z^t(\cdot) = \frac{1}{t}\mathbb{E}_z \int_0^t \mathbf{1}_{\{Z(s)\in\cdot\}} ds$. Since $\Delta$ is compact, $\Pi_z^t$ is tight and any weak limit of $\Pi_t$ is an invariant measure of $(Z(t))$. If $z \in \Delta_0 \times \mathcal{M}$ then $\Pi_z^t(\Delta_0 \times \mathcal{M}) = 1$. This implies the weak limit of $\Pi_z^t$ is $\pi^*$ because $\pi^*$ is the unique invariant probability measure on $\Delta_0 \times \mathcal{M}$. As a result, for any $z \in \Delta_0 \times \mathcal{M}$, we have

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}_z \frac{1}{t}\int_0^t (g(\xi_u, S_H(u), I_H(u)Y(u))Y(u) - \gamma_1(\xi_u))\, du \\ = \int_{\Delta\times\mathcal{M}} (g(\ell', s', i'y')y' - \gamma(\ell'))\pi^*(dz') = \lambda.\end{aligned}$$

Since $\Delta_0 \times \mathcal{M}$ is compact and $(Z(t))$ has the Feller property, we can find a $T > 0$ that does not depend on $z \in \Delta_0 \times \mathcal{M}$ satisfying

$$\left|\mathbb{E}_z \frac{1}{T}\int_0^T (g(\xi_u, S_H(u), I_H(u)Y(u))Y(u) - \gamma_1(\xi_u))\, du - \lambda\right| < \frac{|\lambda|}{4}, \ \forall\ z \in \Delta_0 \times \mathcal{M}.$$

By the Feller property of $(Z(t))$, there exists $\delta > 0$ such that if $\text{dist}(z, \Delta_0 \times \mathcal{M}) \leq \delta$, then

$$\left|\mathbb{E}_z\left[\int_0^T (g(\xi_u, S_H(u), I_H(u)Y(u))Y(u) - \gamma_1(\xi_u))\, du\right] - \lambda\right| \leq T\frac{|\lambda|}{2},$$

for any $z \in \Delta \times \mathcal{M}$ such that $\text{dist}(z, \Delta_0 \times \mathcal{M}) \leq \delta$. As a result of [14, Lemma 3.5], there exists $\theta > 0$ such that,

$$\begin{aligned}\mathbb{E}_z \frac{I_H^\theta(T)}{I_H^\theta(0)} &= \mathbb{E}_z \exp\left\{\theta\int_0^T (g(\xi_u, S_H(u), I_H(u)Y(u))Y(u) - \gamma_1(\xi_u))du\right\} \\ &\leq \exp\left\{\frac{\lambda T\theta}{4}\right\}\end{aligned} \tag{15}$$

if $\lambda < 0$, and

$$\begin{aligned}\mathbb{E}_z \frac{I_H^\theta(0)}{I_H^\theta(T)} &= \mathbb{E}_z \exp\left\{-\theta\int_0^T (g(\xi_u, S_H(u), I_H(u))Y(u) - \gamma_1(\xi_u))du\right\} \\ &\leq \exp\left\{\frac{-\lambda T\theta}{4}\right\}\end{aligned} \tag{16}$$

if $\lambda > 0$, provided $\text{dist}(z, \Delta_0 \times \mathcal{M}) \leq \delta$.

**Proof of part 1:** $\lambda > 0$.

We have from (16) that

$$\mathbb{E}_z I_H^{-\theta}(T) \leq i^{-\theta} e^{\frac{-\lambda T\theta}{4}}, \tag{17}$$

if $\text{dist}(z, \Delta_0 \times \mathcal{M}) \leq \delta$. Since $dI_H^{-1}(t) \leq -H_{-1} I_H^{-1}(t)dt$ where

$$H_{-1} = \sup_{z \in \Delta \times \mathcal{M}} \{\gamma_1(\ell) - g(\ell, s, \ell y) y\},$$

we have

$$I_H^{-\theta}(t) \leq i^{-\theta} \exp\{\theta H_{-1} t\}, \text{ if } i \geq \delta. \tag{18}$$

From (17) and (18), we have

$$\mathbb{E}_z I_H^{-\theta}(T) \leq q i^{-\theta} + C_\theta \text{ for } z \in \Delta \times \mathcal{M}, q = e^{\frac{-\lambda T \theta}{4}}, C_\theta = \delta^\theta \exp\{\theta H_{-1} T\}.$$

By the Markov property, we deduce that

$$\mathbb{E}_z I_H^{-\theta}((k+1)T) \leq q \mathbb{E}_z I_H^{-\theta}(kT) + C_\theta \text{ for } z \in \Delta \times \mathcal{M}, k \in \mathcal{N}.$$

Using this recursively we obtain

$$\mathbb{E}_z I_H^{-\theta}(nT) \leq q^n i^{-\theta} + \frac{C_\theta(1-q^n)}{1-q} \text{ for } z \in \Delta \times \mathcal{M}, n \in \mathcal{N}.$$

This estimate together with (18) leads to

$$\mathbb{E}_z I_H^{-\theta}(t) \leq \left(q^n i^{-\theta} + \frac{C(1-q^n)}{1-q}\right) \exp(\theta H_{-1} T) \text{ for } t \in [nT, nT+T]. \tag{19}$$

Letting $n \to \infty$ we obtain $\limsup_{t\to\infty} \mathbb{E}_z I_H^{-\theta}(t) = \dfrac{C_\theta}{1-q} \exp(H_{-\theta} T)$, which implies (14).

Due to [5, Theorem 4.6] and (19), we obtain the exponentially fast convergence of the transition probability of $Z(t)$ to an invariant probability measure on $\Delta^\circ \times \mathcal{M}$ if there exists $(s_0, i_0, y_0) \in \Gamma \cap \Delta^\circ$ at which the strong bracket condition is satisfied.

**Proof of part 2:** $\lambda < 0$.

We have from (15) that

$$\mathbb{E}_z I_H^\theta(T) \leq i^\theta e^{\frac{\lambda T \theta}{4}}. \tag{20}$$

Moreover, if $\text{dist}(z, \Delta_0 \times \mathcal{M}) \leq \delta$, as a result

$$\mathbb{E}_z(I_H^\theta(T) \vee \delta^\theta) \leq i \vee^\theta \delta^\theta \ \ \forall z \in \Delta \times \mathcal{M}.$$

This and the Markov property of $(Z(t))$ imply that $U(k) = I_H^\theta(T) \vee \delta^\theta$ is a supermartingale. Defining the stopping time $\eta = \inf\{k : U(k) \geq \delta\}$, we have for all $\varepsilon > 0$,

$$\mathbb{E}_z U(\eta \wedge k) \leq I_H^\theta(0) \vee \delta^\theta I_H^\theta \leq (\varepsilon\delta)^\theta, \text{ if } i < \varepsilon\delta.$$

Then $\mathbb{P}\{\eta < k\} \leq \frac{\mathbb{E}_z U(\eta_\varepsilon \wedge k)}{\delta^\theta} \leq \varepsilon^\theta$ if $i < \varepsilon\delta$. Let $k \to \infty$, we have $\mathbb{P}\{\eta < \infty\} \leq \varepsilon^\theta$, where $\eta = \inf\{k : U(k) \geq \delta\}$.

Now, pick $\rho_1$ satisfying $1 > \rho_1 > e^{\frac{\lambda T \theta}{4}} := \rho_0$, we have from the Markov property of $(Z(t))$ and (20) that

$$\mathbb{E}_z \mathbf{1}_{\{\eta<k\}} I_H^\theta(kT) \leq \rho_0 \mathbb{E}_z \mathbf{1}_{\{\eta<k-1\}} I_H^\theta((k-1)T) \leq \cdots \leq \rho_0^k i^\theta, \text{ for } i \leq \varepsilon\delta, k \in \mathcal{N}$$

and

$$\mathbb{P}_z\left\{\mathbf{1}_{\{\eta<k+1\}} I_H^\theta(kT+T) \geq \rho_1^{k+1} i^\theta\right\} \leq \frac{\mathbb{E}_z[\mathbf{1}_{\{\eta<k\}} I_H^\theta(kT)]\rho_0}{\rho_1^{k+1}} \leq \left(\frac{\rho_0}{\rho_1}\right)^{k+1}$$

for $i \le \varepsilon\delta, k \in \mathcal{N}$. Since $\sum_{k=0}^{\infty}\left(\frac{\rho_0}{\rho_1}\right)^{k+1} < \infty$, from an application of Borel-Cantelli lemma, we obtain

$$\limsup_{T\to\infty} \mathbf{1}_{\{\eta<k\}} I_H^{\theta}(kT) = 0 \ \text{ a.s. if } i \le \varepsilon\theta$$

which, in view of $\mathbb{P}_z\{\eta < \infty\} \le \varepsilon^{\theta}$, implies

$$\mathbb{P}_z\left\{\lim_{k\to\infty} I_H(kT) = 0\right\} \ge 1 - \varepsilon^{\theta} \ \text{ if } \ i \le \varepsilon\theta.$$

Since

$$I_H(kT+t) \le \exp\left\{t \times \sup_{z\in\Delta,\ell\in\mathcal{M}} \{g(\ell,s,i,y)y - \gamma_1(\ell)\}\right\} I_H(kT) \text{ a.s.}, t \ge 0,$$

for some $k > 0$, we derive that $\sup_{0\le t\le T} I_H(kT+t) \le K_T I_H(kT)$ for some constant $K_T > 0$. As a result,

$$\mathbb{P}_z\left\{\lim_{k\to\infty} I_H(t) = 0\right\} \le 1 - \varepsilon^{\theta} \ \text{ if } \ i \le \varepsilon; \tag{21}$$

that means the disease-free equilibrium is locally asymptotically stable. If the closure of $\Gamma^+(s,i,y)$ in $\Delta^\circ$ has non-empty intersection with $\Delta_0$, it follows from [5, Lemma 3.1] that

$$\mathbb{P}_z\{I_H(t) < \varepsilon \text{ for some } t > 0\} > 0, \ \text{for any } z \in \Delta \times \mathcal{M},$$

which together with (21) and the strong Markov property of $(Z(t))$ implies

$$\mathbb{P}_z\{\lim_{t\to\infty} I_H(t) = 0\} > 0, z \in \Delta \times \mathcal{M}.$$

As a result, there is no invariant probability measure of $(Z(t))$ on $\Delta^\circ \times \mathcal{M}$, which leads to the claim that $\pi^*$ is the unique invariant probability measure of $(Z(t))$. Since $(Z(t))$ is a Markov-Feller process living inside a compact space, and $\pi^*$ is the unique invariant probability measure, we have that with probability 1, $\widetilde{\Pi}_z^t(.) = \frac{1}{t}\int_0^t \mathbf{1}_{\{Z(s)\in.\}} ds$ converges weakly to $\pi^*$; see [14, Lemma 5.3]. As a result,

$$\begin{aligned}
\lim_{T\to\infty} \frac{\ln I_H(T)}{T} &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(g(\xi_t, S_H(t), I_H(t)Y(t))Y(t) - \gamma_1(\xi_t)\right) dt \\
&= \lim_{T\to\infty} \int_{\Delta\times\mathcal{M}} \left(g(\ell', s', i'y')y' - \gamma_1(\ell)\right) \widetilde{\Pi}_z^T(dz') \\
&= \int_{\Delta\times\mathcal{M}} \left(g(\ell', s', i'y')y' - \gamma_1(\ell)\right) \widetilde{\pi}^*(dz') = \lambda < 0, \ \text{a.s.}
\end{aligned}$$

for any $z \in \Delta^\circ \times \mathcal{M}$. The proof for Part 2 is complete.

**Proof of part 3**

Under the hypothesis of Part 3, a standard comparison argument for systems of differential equations implies that $I_H(t) \le \widetilde{I}_H(t), I_V(t) \le \widetilde{I}_V(t)$ given they have the same initial values, where

$$\begin{aligned}
\frac{d\widetilde{I}_H}{dt} &= g(\xi_t, \frac{\omega_2}{\mu_2}, 0)\widetilde{I}_V - \gamma_1(\xi_t)\widetilde{I}_H, \\
\frac{d\widetilde{I}_V}{dt} &= \widetilde{g}\left(\xi_t, \frac{\omega_2}{\mu_2}, 0\right)\widetilde{I}_H - \mu_2\widetilde{I}_V.
\end{aligned} \tag{22}$$

Then with straightforward calculations, we have

$$\frac{d\ln(\widetilde{I}_H)}{dt} = g(\xi_t, \frac{\omega_1}{\mu_1}, 0)\widetilde{Y}(t) - \gamma_1(\xi_t),$$

where $\widetilde{Y}$ is the solution to (6). As a result,

$$\begin{aligned}\limsup_{t\to\infty} \frac{\ln I_H(t)}{t} &\leq \limsup_{t\to\infty} \frac{\ln \widetilde{I}_H(t)}{t} \\ &\leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(g\left(\xi_u, \frac{\omega_1}{\mu_1}, 0\right)\widetilde{Y}(t) - \gamma_1(\xi_u)\right) du = \lambda \\ &< 0.\end{aligned}$$

Then, combining with Part 2, we obtain that

$$\lim_{t\to\infty} \frac{\ln I_H(t)}{t} = \lambda, \text{ a.s. for any } z \in \Delta^\circ \times \mathcal{M}.$$

□

## 4. **Examples.**

4.1. **Average system.** Let us consider the case when the transmission rates are bilinear for the model (3), i.e.,

$$\begin{aligned}\frac{dS_H}{dt} &= \omega_1 - \beta(\xi_t)S_H I_V - \mu_1 S_H, \\ \frac{dI_H}{dt} &= \beta(\xi_t)S_H I_V - (\gamma(\xi_t) + \mu_1)I_H, \\ \frac{dI_V}{dt} &= \sigma(\xi_t)(\frac{\omega_2}{\mu_2} - I_V)I_H - \mu_2 I_V.\end{aligned} \tag{23}$$

For any probability measure $u = (u_k)_{k\in\mathcal{M}}$ on $\mathcal{M}$, consider the average deterministic system

$$\begin{aligned}\frac{dS_H}{dt} &= \omega_1 - \beta^u S_H I_V - \mu_1 S_H, \\ \frac{dI_H}{dt} &= \beta^u S_H I_V - (\gamma^u + \mu_1)I_H, \\ \frac{dI_V}{dt} &= \sigma^u(\frac{\omega_2}{\mu_2} - I_V)I_H - \mu_2 I_V,\end{aligned} \tag{24}$$

where $f^u = \sum f(k)u_k$ for $f \in \{\beta, \sigma, \gamma\}$.

The complete analysis of this deterministic model can be found in [32]. The basic reproduction number is given by

$$\mathcal{R}_0^u = \frac{\beta^u \sigma^u \omega_1 \omega_2}{\mu_1 \mu_2^2(\gamma^u + \mu_1)}.$$

The disease-free state $E_0 = (\frac{\omega_1}{\mu_1}, 0, 0)$ always exists and is found to be globally stable when $\mathcal{R}_0^u$ is less than one. The unique positive equilibrium

$$\left(S_H^* = \frac{\omega_1}{\mu_1 + \beta^u I_V^*}, I_H^* = \frac{(\mathcal{R}_0^u - 1)\mu_1\mu_2^2}{\sigma^u(\beta^u\omega_2 + \mu_1\mu_2)}, I_V^* = \frac{\sigma^u\omega_2 I_H^*}{\mu_2(\mu_2 + \sigma^u I_H^*)}\right) \tag{25}$$

is globally stable when $\mathcal{R}_0^u > 1$.

Let $E_*^u = \left(S_H^*, I_H^*, Y^* := \frac{I_V^*}{I_H^*}\right)$, and consider $(Z(t) = (S_H(t), I_H(t), Y(t), \xi_t))$, where the process $(S_H(t), I_H(t), I_V(t), \xi_t)$ is the solution to (23). In view of [9,

Lemma 3.5], we know that $E_*^u \in \Gamma^+(z)$ for any $z \in \Delta^\circ \times \mathcal{M}$. An application to Theorem 3.3 implies the following proposition.

**Proposition 4.1.**

1. *If $\lambda > 0$, the disease is persistent. Suppose that $\mathcal{R}_0^u > 0$ for some probability measure $u$ in $\mathcal{M}$, and that $\Gamma^+(E_*^u)$ contains a point satisfying the strong bracket condition. Then there exists a unique invariant probability measure $\pi^*$ on $\Delta^\circ \times \mathcal{M}$, and the transition probability of $Z(t)$ converges to $\pi^*$ exponentially fast in total variation. Moreover, $\Gamma^+(E_*^u)$ is the support of $\pi^*$.*
2. *If $\lambda < 0$, then for any $z = (s, i, y, \ell) \in \Delta^\circ \times \mathcal{M}$,*

$$\mathbb{P}_z\left\{\lim_{t\to\infty} \frac{\ln I_H(t)}{t} = \lambda\right\} = 1.$$

**Remark 3.** Our results will be applied to the special case when all the parameters are constant in each state. To be more specific, when there is no switching, the threshold for the fixed system in state $\ell$ is given in the terms of (12) by

$$\lambda_\ell = A(\ell)q(\ell) - \gamma_1(\ell), \tag{26}$$

where $q(\ell) = \dfrac{(\gamma_1(\ell) - \mu_2) + \sqrt{(\gamma_1(\ell) - \mu_2)^2 + 4A(\ell)C(\ell)}}{2A(\ell)}$, and $A, B$ and $C$ are given in (7). The system is disease free if $\lambda_\ell = A(\ell)q(\ell) - \gamma_1(\ell) < 0$. That condition is equivalent to $A(\ell)C(\ell) < \gamma_1(\ell)\mu_2$. Moreover, the disease persists if $\lambda_\ell > 0$, or equivalently, $A(\ell)C(\ell) > \mu_2\gamma_1(\ell)$.

4.2. **The case $\mathcal{M} = \{1, 2\}$.** When $m_0 = 2$, we can compute the density of the disease-free invariant measure $\pi_Y$ when $\hat{q} < \check{q}$. Without loss of generality, assume that $\hat{q} = q(1) < q(2) = \check{q}$. Using the formula in [10], the invariant measure has the density $\pi_Y$ given by,

$$\pi_Y(y, \ell) = \frac{\theta F(y)}{(y - q(\ell))(A(\ell)y + d(\ell))},\ \ell = 1, 2 \tag{27}$$

where

$$F(y) = \left|\frac{y - q(2)}{A(2)y + d(2)}\right|^{\frac{q_{21}}{d(2)+q(2)A(2)}} \left|\frac{y - q(1)}{A(1)y + d(1)}\right|^{\frac{q_{12}}{d(1)+q(1)A(1)}}, \tag{28}$$

and $\theta$ is given as

$$\theta^{-1} = \int_{q(1)}^{q(2)} \Big(\frac{F(y)}{(q(2) - y)(A(2)y + d(2))} + \frac{F(y)}{(y - q(1))(A(1)y + d(1))}\Big)dy. \tag{29}$$

**Proposition 4.2.** *Suppose that $\mathcal{M} = \{1, 2\}$. For $\varepsilon > 0$ sufficiently small, the switched system (23) with $(\xi_t)$ generated by $Q_\varepsilon$ exhibits the same long-term behavior as the average system (24), i.e., $\lambda$ has the same sign as $\mathcal{R}_0^u - 1$.*

*Proof.* We consider 2 cases.

*Case 1.* $\lambda_\ell < 0, \ell = 1, 2$
Suppose that

$$A(1)C(1) < \gamma_1(1)\mu_2, \quad A(2)C(2) < \gamma_1(2)\mu_2, \quad q(1) \neq q(2). \tag{30}$$

There are always sets of $A(1), A(2), B(1), B(2)$ satisfying (30) as well as

$$(uA(1) + (1 - u)A(2))\,(uB(1) + (1 - u)B(2)) > \mu_1\mu_2 \tag{31}$$

for some $u \in [0,1]$. Note that the left hand side of (31) has the same sign with $\mathcal{R}_0^u - 1$, where $\mathcal{R}_0^u$ is defined using the probability measure $(u, 1-u)$. We consider (23), where $(\xi_t)$ is the Markov chain on $\mathcal{M} = \{1,2\}$ with generator

$$Q_\varepsilon = \begin{pmatrix} \frac{u-1}{\varepsilon} & \frac{1-u}{\varepsilon} \\ \\ \frac{u}{\varepsilon} & \frac{-u}{\varepsilon} \end{pmatrix},$$

for sufficiently small $\varepsilon$. Let $\widetilde{Y}$ be as defined in and (6), with generator $Q_\varepsilon$ given above. It is well known (e.g. see [13]) that for any $T > 0$, there exists constant $K_T > 0$ such that

$$\mathbb{P}_{y,\ell}\left\{|\widetilde{Y}(t) - \bar{Y}(t)| < \varepsilon \ \forall t \in [0,T]\right\} > 1 - e^{-\varepsilon K_T}, \ \forall (y,\ell) \in [\hat{q}, \check{q}] \times \mathcal{M}, \tag{32}$$

and

$$\left|\mathbb{E}_\ell g(\xi_t) - \sum g(k) v_k\right| < \max_{k\in\mathcal{M}}\{|g(k)|\} e^{-\varepsilon K_T}, \ \forall (y,\ell) \in [\hat{q}, \check{q}] \times \mathcal{M}, \tag{33}$$

where $\bar{Y}(t)$ is the solution to

$$\frac{d\bar{Y}}{dt} = -\bar{A} + \bar{Y}^2 \bar{B} \bar{Y} + \bar{C},$$

and $\bar{f} = uf(1) + (1-u)f(2)$ for $f \in \{A, B, C\}$.

Consider function $\Lambda(y,\ell) = -\gamma_1(\ell) + A(\ell)y$ on $[\hat{q}, \check{q}]$. Let $\bar{q} = \dfrac{-\bar{B} + \sqrt{\bar{B}^2 + 4\bar{A}\bar{C}}}{\bar{A}}$ and $\bar{\gamma}_1 = u\gamma(1) + (1-u)\gamma(2) + \mu_1$. Then

$$\Lambda(y,\ell) = -\bar{\gamma}_1 + \bar{A}\,\bar{q} - (\gamma_1(\ell) - \bar{\gamma}_1) + \big(A(\ell) - \bar{A}\big)\,\bar{q} + A(\ell)(y - \bar{q}).$$

Since

$$\begin{cases} \dfrac{d\bar{Y}}{dt} < 0, \text{ if } \bar{Y} < \bar{q}, \\ \dfrac{d\bar{Y}}{dt} > 0, \text{ if } \bar{Y} > \bar{q}, \end{cases}$$

it is easy to show that for any $\delta > 0$, there exists $T > 0$ such that

$$|\bar{Y}(T) - \bar{q}| \le \frac{\delta}{\max_{\ell\in\mathcal{M}}\{A(\ell)\}}, \text{ for any } \bar{Y}(0) \in [\hat{q}, \check{q}]. \tag{34}$$

Due to (33), we can choose $\varepsilon_0 > 0$ such that

$$|\mathbb{E}_\ell A(\xi_T) - \bar{A}| \le \delta, \ |\mathbb{E}_\ell \gamma_1(\xi_T) - \bar{\gamma}_1| < \delta \, \forall \varepsilon < \varepsilon_0, \ell = 1,2. \tag{35}$$

From (32), (35) we have

$$\mathbb{E}_{y,\ell}|A(\ell)(\widetilde{Y} - \bar{q})| \le \delta + \max_{\ell\in\mathcal{M}}\{A(\ell)\}\varepsilon + M_Y e^{-\varepsilon K_T}, \tag{36}$$

where $M_Y = \max_{y\in[\hat{q},\check{q}],\ell\in\mathcal{M}}\{A(\ell)|y - \bar{q}| : y \in [\hat{q}, \check{q}]\}$. From (35) and (36)

$$|\mathbb{E}_{y,\ell}\Lambda(\widetilde{Y}(T), \xi_T) - \bar{A}\,\bar{q} + \bar{\gamma}_1| < 5\delta,$$

where $\varepsilon$ is sufficiently small. Due to the invariance of $\pi_Y$, we have

$$\lambda = \sum_{\ell=1}^{2} \int \Lambda(y,\ell)\pi_Y(dy,\ell) = \int [\mathbb{E}_{y,\ell}\Lambda(\widetilde{Y}(T), \xi_T)]\pi_Y(dy,\ell). \tag{37}$$

Then $|\lambda + \bar{\gamma} - \bar{A}\,\bar{q}| < 4\delta$. Note that if $\bar{A}\,\bar{q} - \bar{\gamma} > 0$ due to (31), we get that $\lambda > \dfrac{\bar{A}\,\bar{\gamma} - \bar{\gamma}}{2} > 0$ if $\delta$ is sufficiently small. As a result, the switching between

two disease-free systems makes the disease persist.

*Case 2.* $\lambda_\ell > 0, \ell = 1, 2$
Suppose that

$$A(1)C(1) > \gamma_1(1)\mu_2, \quad A(2)C(2) > \gamma_1(2)\mu_2, \quad q(1) \neq q(2) \tag{38}$$

but

$$(uA(1) + (1-u)A(2))(uC(1) + (1-u)C(2)) < \gamma_1\mu_2. \tag{39}$$

With similar arguments, when $\varepsilon > 0$ is sufficiently small and $(\xi_t)$ has generator

$$\begin{pmatrix} \frac{u-1}{\varepsilon} & \frac{u-1}{\varepsilon} \\ \\ \frac{u}{\varepsilon} & \frac{u}{\varepsilon} \end{pmatrix}$$

we have $\lambda < 0$. That is, switching between two systems exhibiting disease-persistence can make the disease disappear. □

### 4.3. **Simulations.**

**Example 4.1.** Dengue fever is transmitted by mosquitoes and has been modeled in the deterministic setting by many researchers, see [26], [25], [27], [30]. We apply dengue fever data from a scaled model [30, Table 2.1] to illustrate the effect of switching on the infection dynamics. Consider a community of 10,000 people and 30,000 mosquitoes, where initially 50 people and 600 mosquitoes are infectious with dengue, and 4,000 people are susceptible. The infection can be modeled in a fluctuating environment by (23), with initial conditions $S_H(0) = 0.4$, $I_H(0) = 0.005$, and $I_V(0) = 0.02$. Suppose that without intervention (state 1), the dynamics of the infection are described by the following set of parameters: $\omega_1 = 0.005, \omega_2 = 0.025, \beta(1) = 0.15, \sigma(1) = 0.15, \mu_1 = \omega_1, \mu_2 = \omega_2$, and $\gamma(1) = \gamma(2) = 0.3$. Now say, for example, some intervention like mosquito nets is introduced (state 2) at random points in time in such a way that the mosquito-to-human transmission rate is reduced to $\beta(2) = 0.05$. We can check that $\lambda_1 = 0.04018 > 0$ and $\lambda_2 = -0.00038 < 0$ as defined in (26), so the fixed system witnesses the persistence of the disease in state 1 but eradicates the disease with constant intervention (state 2). The equilibrium point for the average system is calculated under the uniform probability $u = (0.5, 0.5)$ by (25) as $(S_H^*, I_H^*, I_V^*) = (0.5317, 0.0077, 0.0440)$ with reproduction number $\mathcal{R}_0^u = 1.9672 > 1$. The threshold for the switched system with generator $Q_\varepsilon, \varepsilon = 25$, is calculated via (37) as $\lambda = 0.0500 > 0$ using the disease-free invariant measure given by (27). Accordingly, the switched system witnesses the persistence of the disease; see Figure 1.

**Example 4.2.** Consider the initial conditions $S_H(0) = 0.4$, $I_H(0) = 0.005$, and $I_V(0) = 0.02$, as in Example 4.1. Suppose that without intervention the dynamics in state 1 are described by the following set of parameters: $\omega_1 = 0.005, \omega_2 = 0.025, \beta(1) = 0.15, \sigma(1) = 0.05, \mu_1 = \omega_1, \mu_2 = \omega_2$, and $\gamma(1) = 0.3$. Then $\lambda_1 = -3.7922 \times 10^{-4} < 0$ and the disease eventually dies out. Now suppose that, for example, to speed up the eradication of the disease, some intervention is introduced (state 2) at random points in time in such a way that the mosquito-to-human transmission rate is reduced to $\beta(2) = 0.05$, but the human-to-mosquito transmission rate is raised to $\sigma(2) = 0.15$. Then $\lambda_2 = -3.7922 \times 10^{-4} < 0$. These theoretical parameters have been selected from observed ranges in the literature for related models

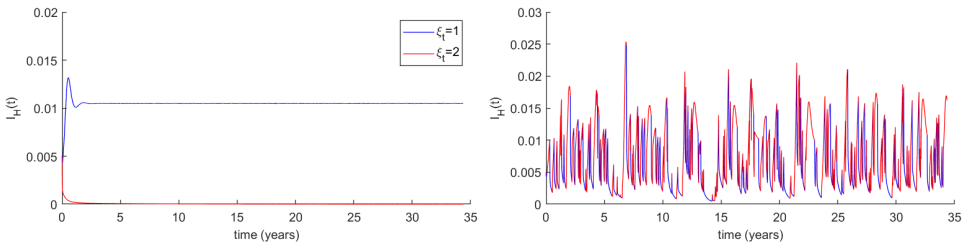


FIGURE 1. Sample paths of $I_H(t)$ (Example 4.1). In the deterministic systems (**LEFT**) there is persistence in state 1 and extinction in state 2. In the switched system (**RIGHT**), the infection persists.

[30] to satisfy (30), i.e., both fixed systems exhibit the extinction of the disease; see Figure 2.

However, the basic reproduction number for the average system as defined in (24) under the uniform probability measure $u = (0.5, 0.5)$ is calculated as $\mathcal{R}_0^u = 1.967 > 1$. That is, (31) holds. The positive equilibrium point of (24) is calculated as $(S_H^*, I_H^*, I_V^*) = (0.5317, 0.0077, 0.04403)$. To verify the strong bracket condition in a neighborhood of $(S_H^*, I_H^*, I_V^*)$, we do as follows: Let

$$F_\ell(S_H, I_H, I_V) = \begin{bmatrix} \omega_1 - \beta(\ell) S_H I_V - \mu_1 S_H \\ \beta(\ell) S_H I_V - \mu_1 I_H - \gamma_1 I_H \\ \sigma(\ell)\left(\frac{\omega_2}{\mu_2} - I_V\right) I - \mu_2 I_V \end{bmatrix}, \ell = 1, 2.$$

and

$$G_0 = F_1(x) - F_2(x), G_1 = [G_0, F_1(x)], G_k = [G_{k-1}, F_1(x)], k \geq 1.$$

Using the MATLAB Symbolic Toolbox, we show that $(S_H^*, I_H^*, I_V^*)$ does not solve the system of equations

$$\begin{cases} \det\left(\begin{bmatrix} G_0 & G_1 & G_2 \end{bmatrix}\right) = 0, \\ \det\left(\begin{bmatrix} G_0 & G_1 & G_3 \end{bmatrix}\right) = 0, \\ \det\left(\begin{bmatrix} G_0 & G_1 & G_4 \end{bmatrix}\right) = 0. \end{cases}$$

Then $\mathcal{G}_5 := \text{span}\{G_0, ..., G_4\} = \mathcal{F}^3$, and the strong bracket condition is satisfied at $(S_H^*, I_H^*, I_V^*)$. By proposition 4.1 there exists a unique invariant measure $\pi^*$ in the interior $\mathcal{F}_+^{3,\circ} \times \mathcal{M}$. The approximate joint density of $(S_H(t), I_H(t), \xi_t)$ is given in Figure 3 using 1000 simulations of the occupation measure over the period $[0, 1500000]$. As a result of proposition 4.2, for $\varepsilon$ sufficiently small, the disease in the switched system (23) will persist; see Figure 2 with $\varepsilon = 25$.

**Example 4.3.** Consider the following parameters: $\omega_1 = 1, \beta(1) = 1.1, \beta(2) = 2.1, \sigma(1) = 1, \sigma(2) = 2, \mu_1 = 1, \gamma(1) = 0.01, \gamma(2) = 3, \mu_2 = 1, \omega_2 = 1$, These parameters are unrealistic and have been chosen to satisfy the conditions (38) and (39). Each fixed system witnesses the persistence of the disease. However, fast switching ($u = 0.5$, $\varepsilon = 0.05$) makes the disease die out eventually, as illustrated in Figure 4.

5. **Conclusion.** In this paper, the vector-host model has been studied under switching environments. The effect of telegraph noise has been considered in the model parameters. The threshold that is crucial for determining the disease persistence

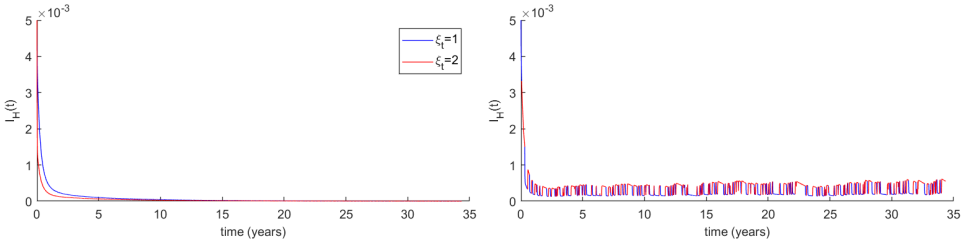


FIGURE 2. Sample paths of $I_H(t)$ (Example 4.2). In both deterministic systems (**LEFT**), $I_H(t)$ converges exponentially fast to 0. Switching makes the disease persist (**RIGHT**).

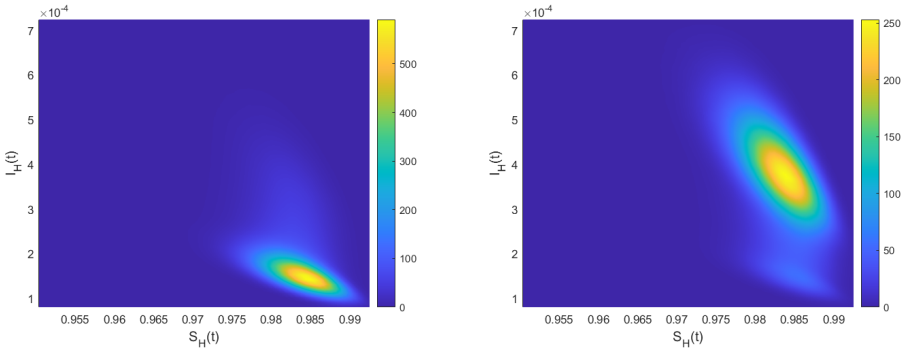


FIGURE 3. Joint density of $(S_H(t), I_H(t), \xi_t)$ in state 1 (**LEFT**) and state 2 (**RIGHT**), according to the invariant measure (Example 4.2).

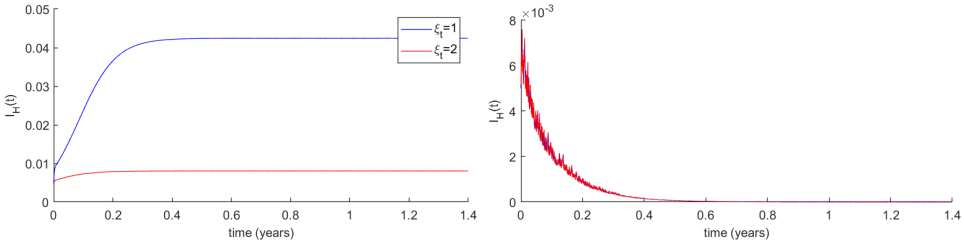


FIGURE 4. Sample paths of $I_H(t)$ (Example 4.3). In both deterministic systems, $I_H(t)$ converges to a positive equilibrium (**LEFT**). Switching allows for extinction (**RIGHT**).

and extinction for the stochastic model has been obtained. Conditions for the persistence and extinction of infection have been derived for the piecewise-deterministic model. In a case study with constant parameters, it has been found that in two deterministic systems where the disease persists, switching between them can allow for extinction, and vice-versa. Numerical simulations have been performed by taking dengue fever data from the literature to present the importance of switching in a real scenario.

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