MOMENTS OF WEIL REPRESENTATIONS OF FINITE SPECIAL UNITARY GROUPS

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Dedicated to the memory of Kay Magaard

ABSTRACT. We prove an "n^{th} moment $= 1$" result for irreducible Weil representations of degree $(q^n+1)/(q+1)$ of special unitary groups $\mathrm{SU}_n(q)$ for any odd $n \geq 3$ and any prime power q.

1. Introduction

For an **odd** integer $n \geq 3$, and a prime power $q \geq 2$, the irreducible representations (over $\mathbb{C}$) of lowest degree after the trivial representation of the group $\mathrm{SU}_n(q)$ are a symplectic representation of dimension $\frac{q^n+1}{q+1} - 1 = \frac{q^n-q}{q+1}$, and q representations of dimension $\frac{q^n+1}{q+1}$. When q is odd, exactly one of these q representations is orthogonal, otherwise none is. The direct sum of these $q+1$ representations is called the (big, or reducible) *Weil representation* of $\mathrm{SU}_n(q)$, and the $q+1$ individual representations are referred to as (irreducible) *Weil representations*, see e.g. [TZ1, Theorem 4.1] and [TZ2, §4].

In the paper [KT1], we wrote down $q+1$ rigid local systems on the affine line $\mathbb{A}^1/\overline{\mathbb{F}_p}$ whose geometric monodromy groups we conjectured to be the images of $\mathrm{SU}_n(q)$ in these $q+1$ representations. We were able to prove this only in the case when $n = 3$ and $\gcd(n, q+1) = 1$. In the sequel [KT3], we used a completely different method, which starts with results of Gross [Gr] and relies on [KT2], to prove these conjectures for any odd $n \geq 3$ and for any odd prime power q.

In the course of thinking about these questions, we stumbled upon a striking representation-theoretic fact about the q Weil representations of $\mathrm{SU}_n(q)$ ($n \geq 3$ odd) of dimension $\frac{q^n+1}{q+1}$. For each of them, their n^{th} moment (i.e. the dimension of the space of invariants in the n^{th} tensor power of the representation in question) is exactly one. For the irreducible representation of dimension $\frac{q^n+1}{q+1} - 1$, the n^{th} moment vanishes. At present we do not have a conceptual explanation for this phenomenon.

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Theorem 1. *Let q be a prime power, $n \geq 3$ any odd integer, and let $G = \mathrm{SU}_n(q)$. Suppose in addition that $(n,q) \neq (3,2)$. Let V be one of the $q+1$ complex irreducible Weil modules of G, of dimension $(q^n+1)/(q+1)$ or $(q^n-q)/(q+1)$. Then the subspace of G-invariants on $V^{\otimes n}$ has dimension 1 if $\dim(V) = (q^n+1)/(q+1)$, and 0 if $\dim(V) = (q^n-q)/(q+1)$.*

As stated in Theorem 1, each of the Weil modules of $\mathrm{SU}_n(q)$ of dimension $(q^n + 1)/(q+1)$ has a unique (up to scalar) polynomial invariant of degree n. It would be interesting to know what is the geometric significance of this polynomial invariant, and to find an explicit construction of it.

Given this result about n^{th} moments for $\mathrm{SU}_n(q)$ when n is odd, it is natural to wonder about the situation for n^{th} moments when n is even. [For n even and $q \geq 3$ a prime power, the irreducible representations (over $\mathbb{C}$) of lowest degree after the trivial representation of the group $\mathrm{SU}_n(q)$ are an orthogonal representation of dimension $\frac{q^n-1}{q+1}+1 = \frac{q^n+q}{q+1}$, and q representations of dimension $\frac{q^n-1}{q+1}$.] Already for $n=4$, the result is not so nice, cf. Theorem 4.1.

For the Weil representations of finite special linear groups $\mathrm{SL}_n(q)$ and symplectic groups $\mathrm{Sp}_{2n}(q)$, the latter with q odd, one also does not expect any nice regularity about the n^{th} moments. We record however a curious fact about the 4^{th} moments of Weil representations of $\mathrm{Sp}_{2n}(3)$, see Proposition 4.2.

2. Preliminaries

Let $q = p^f$ be any prime power and $n \geq 2$. It is well known, see e.g. [TZ2, §4], that the function

$$\zeta_{n,q} = \zeta_n : g \mapsto (-1)^n(-q)^{\dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(g-1_W)}$$

defines a complex character, called the (reducible) *Weil character*, of the general unitary group $\mathrm{GU}_n(q) = \mathrm{GU}(W)$, where $W = \mathbb{F}_{q^2}^n$ is a non-degenerate Hermitian space with Hermitian product $\circ$. Note that the $\mathbb{F}_q$-bilinear form

$$(u|v) = \mathrm{Trace}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\theta u \circ u)$$

on W, for a fixed $\theta \in \mathbb{F}_{q^2}^\times$ with $\theta^{q-1} = -1$, is non-degenerate symplectic. This leads to an embedding

$$\tilde{G} := \mathrm{GU}_n(q) \hookrightarrow \mathrm{Sp}_{2n}(q).$$

Moreover, if q is odd then the restriction of any of the two big Weil characters (of degree q^n, and denoted $\mathsf{Weil}_{1,2}$ in [KT2]) of $\mathrm{Sp}_{2n}(q)$ to $\mathrm{GU}_n(q)$ is exactly $\chi_2\zeta_n$, where χ_2 is the unique linear character of order 2 of $\tilde{G}$, cf. [TZ2, §4]. We will also denote by ζ_n the restriction of ζ_n to the special unitary group $G := \mathrm{SU}_n(q)$.

Fix a generator σ of $\mathbb{F}_{q^2}^\times$ and set $\rho := \sigma^{q-1}$. We also fix a primitive $(q^2-1)^{\mathrm{th}}$ root of unity $\boldsymbol{\sigma} \in \mathbb{C}^\times$ and let $\boldsymbol{\rho} = \boldsymbol{\sigma}^{q-1}$. Then

$$\zeta_n = \sum_{i=0}^{q} \tilde{\zeta}_{i,n} \tag{2.0.1}$$

decomposes as the sum of $q+1$ characters of $\tilde{G}$, where

$$\tilde{\zeta}_{i,n}(g) = \frac{(-1)^n}{q+1}\sum_{l=0}^{q} \boldsymbol{\rho}^{il}(-q)^{\dim \mathrm{Ker}(g-\rho^l\cdot 1_W)}; \tag{2.0.2}$$

see [TZ2, Lemma 4.1]. In particular, $\tilde{\zeta}_{i,n}$ has degree $(q^n-(-1)^n)/(q+1)$ if $i>0$ and $(q^n+(-1)^n q)/(q+1)$ if $i=0$.

We will let $\zeta_{i,n}$ denote the restriction of $\tilde{\zeta}_{i,n}$ to $G = \mathrm{SU}_n(q)$, for $0 \le i \le q$. If $n \ge 3$, then these $q+1$ characters are all irreducible and distinct. If $n=2$, then $\zeta_{i,n}$ is irreducible, unless q is odd and $i=(q+1)/2$, in which case it is a sum of two irreducible characters of degree $(q-1)/2$, see [TZ2, Lemma 4.7]. Formula (2.0.2) implies that Weil characters $\zeta_{i,n}$ enjoy the following branching rule while restricting to the natural subgroup $H := \mathrm{Stab}_G(w) \cong \mathrm{SU}_{n-1}(q)$ ($w \in W$ any anisotropic vector):

$$\zeta_{i,n}|_H = \sum_{j=0,\ j\neq i}^{q} \zeta_{j,n-1}. \tag{2.0.3}$$

Furthermore, the complex conjugation fixes $\tilde{\zeta}_{0,n}$ and sends $\tilde{\zeta}_{j,n}$ to $\tilde{\zeta}_{q+1-j,n}$ when $1 \le j \le q$. As $n \ge 3$ is odd, it is also known that $\tilde{\zeta}_{0,n}$ is of symplectic type; let $\Psi_0 : \tilde{G} \to \mathrm{Sp}(V)$ be a complex representation affording this character. If $2 \nmid q$, then $\tilde{\zeta}_{(q+1)/2,n}$ is of orthogonal type; let $\Psi_{(q+1)/2} : \tilde{G} \to \mathrm{O}(V)$ be a complex representation affording this character. In the remaining cases, let $\Psi_i : \tilde{G} \to \mathrm{GL}(V)$ be a complex representation affording the character $\tilde{\zeta}_{i,n}$.

3. Odd-dimensional unitary groups

In this section, we will consider special unitary groups $G := \mathrm{SU}_n(q) = \mathrm{SU}(W)$ where q is any prime power and $n \ge 3$ is odd. In fact, up until Theorem 3.11 we will assume that $n = 2k+1 \ge 5$, and fix a basis $(e_1,\ldots,e_k,f_1,\ldots,f_k,w)$ of the Hermitian space $W = \mathbb{F}_{q^2}^n$, in which the Hermitian form $\circ$ takes values

$$e_i \circ e_j = f_i \circ f_j = e_i \circ w = f_i \circ w = 0,\ e_i \circ f_j = \delta_{i,j},\ w \circ w = 1. \tag{3.0.1}$$

We also fix the notation

$$P_1 := \mathrm{Stab}_G(\langle e_1\rangle_{\mathbb{F}_{q^2}}) = Q_1L_1,\ P_k := \mathrm{Stab}_G(\langle e_1,\ldots,e_k\rangle_{\mathbb{F}_{q^2}}) = Q_kL_k,$$

where $Q_1 = \mathbf{O}_p(P_1)$, $Q_k = \mathbf{O}_p(P_k)$, $L_k \cong \mathrm{GL}_k(q^2)$. The action of any $X \in L_k = \mathrm{GL}_k(q^2)$ in the indicated basis of W is given by $\mathrm{diag}(X, {}^tX^{-q}, \det(X)^{q-1})$, see [ST, §5.1].

As shown in [GMST, Lemmas 12.5, 12.6], the Levi subgroup L has a unique orbit $\mathcal{O}$ on $\mathrm{Irr}(\mathbf{Z}(Q_k)) \smallsetminus \{1_{\mathbf{Z}(Q_k)}\}$ of smallest length $(q^{2k}-1)/(q+1)$, which then occurs in the restriction of any Weil character $\zeta_{i,n}$. Moreover, any $\lambda \in \mathcal{O}$ can only lie under an irreducible character of degree q of Q_k. In particular, this shows that

Lemma 3.1. *Suppose $n = 2k+1 \ge 5$. Then $\zeta_{0,n}$ is irreducible over P_k. If $1 \le i \le q$, then $\zeta_{i,n}|_{P_k} = \nu_i + \theta_i$, where $\theta_i \in \mathrm{Irr}(P_k)$ affords the orbit $\mathcal{O}$, and ν_i is a linear character of P_k trivial at $\mathbf{Z}(Q_k)$.*

Lemma 3.2. *In the notation of Lemma 3.1, assume that* $1 \leq i \leq q$. *Then* $\mathrm{Ker}(\nu_i) \geq Q_k$, *and if* $X \in L_k$ *has determinant* σ^t *as an element in* $\mathrm{GL}_k(q^2)$ *with* $t \in \mathbb{Z}$, *then* $\nu_i(X) = \boldsymbol{\sigma}^{(q-1)it}$.

Proof. As noted in Lemma 3.1, ν_i is trivial at $\mathbf{Z}(Q_k)$, and it is P_k-invariant. But L_k acts transitively on the $q^{2k}-1$ nontrivial linear characters of $Q_k/\mathbf{Z}(Q_k)$, so $\mathrm{Ker}(\nu_i) \geq Q_k$. Next, $[L_k, L_k] \cong \mathrm{SL}_k(q^2)$ is perfect, so ν_i is trivial at $[L_k, L_k]$. Thus there is some $0 \leq s \leq q^2-2$ such that $\nu_i(X) = \boldsymbol{\sigma}^{ts}$ for the listed $X \in L_k$. To find s, it suffices to evaluate $\nu_i(X)$ for some X_0 that generates L_k modulo $[L_k, L_k]$. Let γ be a generator of $\mathbb{F}_{q^{2k}}^{\times}$ such that $\gamma^{(q^{2k}-1)/(q^2-1)} = \sigma$, and choose $X_0 \in L_k$ conjugate to

$$\mathrm{diag}(\gamma, \gamma^{q^2}, \ldots, \gamma^{q^{2k-2}})$$

over $\overline{\mathbb{F}}_q$, so that $\det(X_0) = \sigma$. Since no eigenvalue of X_0 belongs to $\mathbb{F}_{q^2}$, X_0 cannot fix any $\lambda \in \mathcal{O}$, see formula (20) of [ST]), and so $\theta_i(X_0) = 0$ and $\nu_i(X_0) = \zeta_{i,n}(X_0)$. The absence of eigenvalues in $\mathbb{F}_{q^2}$ and the equality $\det(X_0)^{q-1} = \rho$ imply by (2.0.2) that $\zeta_{i,n}(X_0) = \boldsymbol{\rho}^i = \boldsymbol{\sigma}^{(q-1)i}$, i.e. $s = (q-1)i$ as stated. $\square$

Proposition 3.3. *Suppose* $n = 2k+1 \geq 5$. *Then* $(\zeta_n)^{n-1}$ *contains* $\zeta_{i,n}$ *with multiplicity one if* $i > 0$, *and zero if* $i = 0$.

Proof. Note that $(\zeta_n)^2$ is just the permutation character of G acting on the point set of W. Hence $(\zeta_n)^{n-1}$ is the permutation character of G acting on the set Ω of ordered k-tuples $\omega = (v_1, \ldots, v_k)$, $v_i \in W$. Let $\pi_\omega = \mathrm{Ind}_{G_\omega}^G(1_{G_w})$ denote the permutation character of G acting on the G-orbit of $\omega = (v_1, \ldots, v_k)$, where $G_\omega = \mathrm{Stab}_G(\omega)$, and suppose that $\zeta_{i,n}$ is an irreducible constituent of π_ω. Then

$$0 < [\pi_\omega, \zeta_{i,n}]_G = [1_{G_\omega}, \zeta_{i,n}|_{G_\omega}]_{G_\omega}; \tag{3.3.1}$$

in particular, 1_{G_ω} is an irreducible constituent of $\zeta_{i,n}|_{G_\omega}$.

(i) First we consider the case where $X := \langle v_1, \ldots, v_k \rangle_{\mathbb{F}_{q^2}}$ is contained in a non-degenerate subspace Y of W of codimension ≥ 2. Without loss we may assume that $e_1, f_1 \in Y^\perp$. Then G_ω contains a natural subgroup $M := \mathrm{SU}(\langle e_1, f_1 \rangle_{\mathbb{F}_{q^2}}) \cong \mathrm{SU}_2(q)$ (that acts trivially on Y). The branching rule (2.0.3) then shows that $\zeta_{i,n}|_M$ is a sum of Weil characters $\zeta_{j,2}$ of M. As mentioned above, an irreducible constituent λ of $\zeta_{j,2}$ can have degree 1 only when $(q,j) = (2, \neq 0)$ or $(q,j) = (3, (q+1)/2)$. In the former case, one can check that λ is actually the sign character of $M = \mathrm{SU}_2(2) \cong \mathrm{Sym}_3$. In the latter case, $\lambda(z) \neq 1$ for some element z of $M \cong \mathrm{SU}_2(3)$ of order 3. Thus λ can never be equal to 1_M, contradicting (3.3.1).

In particular, we have shown that X cannot be non-degenerate.

(ii) Suppose now that $0 \neq X \cap X^\perp$ has dimension $j \leq k-1$. By Witt's lemma, we may then assume that $X = \langle e_1, \ldots, e_j, w_1, \ldots, w_{k-j} \rangle_{\mathbb{F}_{q^2}}$, where $\langle w_1, \ldots, w_{k-j} \rangle_{\mathbb{F}_{q^2}}$ is a non-degenerate subspace of

$$\langle e_{j+1}, \ldots, e_k, f_{j+1}, \ldots, f_k \rangle_{\mathbb{F}_{q^2}}.$$

But then X is contained in the non-degenerate subspace

$$Y := \langle e_1, \ldots, e_j, f_1, \ldots, f_j, w_1, \ldots, w_{k-j} \rangle_{\mathbb{F}_{q^2}}$$

of codimension $n-(k+j) \geq 2$, contradicting (i).

(iii) We have shown that $\dim(X \cap X^\perp) = k$, i.e. X is totally singular of dimension k. There is only one G-orbit of such ω, and we may assume that $\omega = (e_1, \ldots, e_k)$. The description of P_k given in [ST, §5.1] shows that $G_\omega = Q_k$. Now Lemmas 3.1, 3.2, and (3.3.1) show that $[\pi_\omega, \zeta_{i,n}]_G = 1 - \delta_{0,i}$, as stated. $\square$

Next we define the following linear characters λ_i of the parabolic subgroup $P_1 = \mathrm{Stab}_G(\langle e_1 \rangle_{\mathbb{F}_{q^2}})$ for $1 \leq i \leq q$: if $g \in P_1$ sends e_1 to σ^t for $0 \leq t \leq q^2-2$, then $\lambda_i(g) = \boldsymbol{\sigma}^{-(q-1)it}$, and set

$$\Lambda_i := \mathrm{Ind}_{P_1}^G(\lambda_i).$$

Proposition 3.4. *Suppose $n = 2k+1 \geq 5$, $(n,q) \neq (5,2)$, and $1 \leq i \leq q$. Then Λ_i enters the character $(\zeta_n)^2$, and $[(\zeta_{i,n})^2, \Lambda_i] \geq 1$.*

Proof. (i) As discussed in [GMST, §11], $P_1' := \mathrm{Stab}_G(e_1) = Q_1 \rtimes L_1'$, where $L_1' = \mathrm{Stab}_G(e_1) \cap \mathrm{Stab}_G(f_1) \cong \mathrm{SU}_{n-2}(q)$. Note that Λ_i enters the character $\mathrm{Ind}_{P_1'}^{P_1}(1_{P_1'})$, which in turn enters the character $(\zeta_n)^2$. Furthermore, L_1 acts transitively on the $q-1$ nontrivial linear characters of $\mathbf{Z}(Q_1)$ (which has order q), and for each such character α there is a unique irreducible character of Q_1 of degree q^{n-2}, which then extends to a unique character M_α of P_1'. We fix some nontrivial $\alpha \in \mathrm{Irr}(\mathbf{Z}(Q_1))$ and let $K := \mathrm{Stab}_{P_1}(\alpha) = P_1' \cdot C_{q+1}$. By its uniqueness, M_α extends to K. Note that

$$\zeta_{i,n}(1) = (q^n+1)/(q+1) < 2q^{n-2}(q-1) = 2(q-1)M_\alpha(1).$$

It follows by Clifford's theorem that

$$\zeta_{i,n}|_{P_1} = \beta_i + \mathrm{Ind}_K^{P_1}(M_\alpha), \tag{3.4.1}$$

for some extension to K of M_α which we also denote by M_α, and for some character β_i of P_1 of degree $(q^{n-2}+1)/(q+1)$, with $\mathbf{Z}(Q_1) \leq \mathrm{Ker}(\beta_i)$. Next, $M_\alpha|_{L_1'} = \zeta_{n-2}$. Applying (2.0.3) to the standard subgroup L_1' and using (3.4.1), we get

$$\beta_i|_{L_1'} = \zeta_{i,n}|_{L_1'} - (q-1)\zeta_{n-2} = \sum_{j \neq i,\ j' \neq j} \zeta_{n-2,j'} - (q-1)\sum_{j'=0}^{q} \zeta_{n-2,j'} = \zeta_{n-2,i}.$$

In particular, $\beta_i \in \mathrm{Irr}(P_1)$.

(ii) As usual, $\bar{\chi}$ denotes the complex conjugate of any character χ. Note that $\mathrm{Stab}_{P_1}(\bar{\alpha}) = K$. Hence, (3.4.1) implies that

$$\overline{\zeta}_{i,n}|_{P_1} = \overline{\beta}_i + \mathrm{Ind}_K^{P_1}(\overline{M}_\alpha). \tag{3.4.2}$$

Observe that $\overline{M}_\alpha$ affords the $\mathbf{Z}(Q_1)$-character $q^{n-2}\bar{\alpha}$ and is irreducible over P_1'. By the aforementioned uniqueness, $\overline{M}_\alpha$ agrees with $M_{\bar{\alpha}}$ on P_1', where $M_{\bar{\alpha}}$ is the K-character of the $\bar{\alpha}$-isotypic component in $\zeta_{i,n}|_{P_1}$. As $K/P_1 \cong C_{q+1}$, these two characters differ from each other by a linear character of K/P_1', which extends to a linear character δ of $P_1/P_1' \cong C_{q^2-1}$. We have shown that

$$\mathrm{Ind}_K^{P_1}(\overline{M}_\alpha) = \mathrm{Ind}_K^{P_1}(M_{\bar{\alpha}} \cdot \delta|_K) = \mathrm{Ind}_K^{P_1}(M_{\bar{\alpha}}) \cdot \delta. \tag{3.4.3}$$

and

$$\zeta_{i,n}|_{P_1} = \beta_i + \mathrm{Ind}_K^{P_1}(M_{\bar{\alpha}}), \tag{3.4.4}$$

(iii) We aim to show that we one can take $\delta = \overline{\lambda}_i$ in (3.4.3). Let τ be an element of $\mathbb{F}_{q^{4k-2}}^\times$ of order $q^{2k-1}+1$ chosen such that $\tau^{(q^{2k-1}+1)/(q+1)} = \rho$. Then we can find an element $h \in K$ such that $h(e_1) = \rho e_1$ and h is conjugate to

$$\mathrm{diag}(\rho, \rho, \tau^{-2}, \tau^{2q}, \tau^{-2q^2}, \ldots, \tau^{-2(-q)^{2k-2}})$$

over $\overline{\mathbb{F}}_{q^2}$. Since $k \geq 2$ and $(k,q) \neq (2,2)$, by [Zs] there is a prime divisor ℓ of $q^{4k-2}-1$ that does not divide $\prod_{j=1}^{4k-3}(q^j-1)$. In particular, ℓ divides $(q^{2k-1}+1)$, and moreover the ℓ-part of $|P_1|$ is equal to the ℓ-part of $\beta_i(1)$, whence β_i is an irreducible character of P_1 of ℓ-defect zero. On the other hand, for any $1 \leq t \leq q$, ℓ divides $|h^t|$, whence $\beta_i(t) = 0$, and so we obtain by using (2.0.2), (3.4.2), (3.4.4) that

$$\mathrm{Ind}_K^{P_1}(M_{\bar{\alpha}})(h^t) = \zeta_{i,n}(h^t) = -(q-1)\boldsymbol{\rho}^{it},$$
$$\mathrm{Ind}_K^{P_1}(\overline{M}_{\alpha})(h^t) = \overline{\zeta}_{i,n}(h^t) = -(q-1)\boldsymbol{\rho}^{-it}.$$

It now follows from (3.4.3) that

$$\delta(h^t) = \boldsymbol{\rho}^{-2it} = \boldsymbol{\rho}^{(q-1)it} = \overline{\lambda}_i(h^t),$$

whence $\delta(g) = \overline{\lambda}_i(g)$ for all $g \in K$, since the choice of h ensures that h generates K modulo P_1'. Together with (3.4.3), we have shown that

$$(\mathrm{Ind}_K^{P_1}(M_{\bar{\alpha}}) \cdot \delta)(g) = (\mathrm{Ind}_K^{P_1}(M_{\bar{\alpha}}) \cdot \overline{\lambda}_i)(g) \tag{3.4.5}$$

for all $g \in K$. If $g \in P_1 \smallsetminus K$ then $\mathrm{Ind}_K^{P_1}(M_{\bar{\alpha}})(g) = 0$ since $K \lhd P_1$, and so (3.4.5) holds for g as well. Consequently,

$$\mathrm{Ind}_K^{P_1}(\overline{M}_{\alpha}) = \mathrm{Ind}_K^{P_1}(M_{\bar{\alpha}}) \cdot \overline{\lambda}_i.$$

This identity, together with (3.4.2) and (3.4.4), implies by Frobenius' reciprocity that

$$\begin{aligned}
[(\zeta_{i,n})^2, \Lambda_i]_G &= [\zeta_{i,n}\overline{\Lambda}_i, \overline{\zeta}_{i,n}]_G = [\zeta_{i,n} \cdot \mathrm{Ind}_{P_1}^G(\overline{\lambda}_i), \overline{\zeta}_{i,n}]_G \\
&= [\mathrm{Ind}_{P_1}^G(\zeta_{i,n}|_{P_1} \cdot \overline{\lambda}_i), \overline{\zeta}_{i,n}]_G = [\zeta_{i,n}|_{P_1} \cdot \overline{\lambda}_i, \overline{\zeta}_{i,n}]_{P_1} \\
&\geq [\mathrm{Ind}_K^{P_1}(M_{\bar{\alpha}}) \cdot \overline{\lambda}_i, \mathrm{Ind}_K^{P_1}(\overline{M}_{\alpha})]_{P_1} = 1,
\end{aligned}$$

as stated. □

Proposition 3.5. *Suppose $n = 2k+1 \geq 5$ and $0 < i \leq q$. Then $[(\Lambda_i)^k, \overline{\zeta}_{i,n}] = 1$.*

Proof. Recall G acts transitively on the set Ξ of isotropic 1-spaces in $W = \mathbb{F}_{q^2}^n$, with $P_1 = \mathrm{Stab}_G(\pi_1)$, where we set $\pi_j := \langle e_j \rangle_{\mathbb{F}_{q^2}}$ for $1 \leq j \leq k$. Hence the character Λ_i is afforded by a $\mathbb{C}G$-module

$$V = \mathrm{Ind}_{P_1}^G(V_{\pi_1}) = \oplus_{gP_1 \in G/P_1} V_{g(\pi_1)},$$

where $V_{\pi_1} = \langle v_{\pi_1} \rangle_{\mathbb{C}}$ is a one-dimensional P_1-module with character λ_i, and G permutes the summands via $h(V_{g(\pi_1)}) = V_{hg(\pi_1)}$. It follows that $(\Lambda_i)^k$ is afforded by the G-module

$$V^{\otimes k} = \langle v_\xi \mid \xi \in \Xi^k \rangle_{\mathbb{C}},$$

where $v_\xi = v_{\xi_1} \otimes v_{\xi_2} \otimes \ldots \otimes v_{\xi_k}$ for $\xi = (\xi_1, \xi_2, \ldots, \xi_k)$.

Consider the G-orbit Π of the k-tuple $\pi := (\pi_1, \pi_2, \ldots, \pi_k) \in \Xi^k$. Then the G-submodule

$$V(\Pi) := \langle v_\xi \mid \xi \in \Pi \rangle_{\mathbb{C}}$$

of $V^{\otimes k}$ affords the character $\mathrm{Ind}_R^G(\mu)$, where $R := \cap_{j=1}^k \mathrm{Stab}_G(\langle e_j \rangle_{\mathbb{F}_{q^2}})$, and

$$\mu(h) = \boldsymbol{\sigma}^{-(q-1)i\sum_{j=1}^k t_j}$$

if $h(e_j) = \sigma^{t_j}$ for $0 \leq t_j \leq q^2 - 2$ and $1 \leq j \leq k$.

Note that $Q_k \lhd R < P_k$ and $Q_k \leq \mathrm{Ker}(\mu)$. Furthermore, if $h \in L_k$ belongs to R and $h(e_j) = \sigma^{t_j}$, then $\det(h)$ (as an element in $\mathrm{GL}_k(q^2)$ is $\sigma^{\sum_{j=1}^k t_j}$, and so

$$\overline{\nu}_i(h) = \boldsymbol{\sigma}^{-(q-1)i\sum_{j=1}^k t_j} = \mu(h)$$

for the character ν_i considered in Lemma 3.2, i.e. $\overline{\nu}_i|_R = \mu$. By Lemma 3.1, we have therefore shown that

$$0 < [\mu, \overline{\zeta}_{i,n}|_R]_R = [\mathrm{Ind}_R^G(\mu), \overline{\zeta}_{i,n}]_G \leq [(\Lambda_i)^k, \overline{\zeta}_{i,n}]_G.$$

On the other hand, $(\Lambda_i)^k$ enters the character $(\zeta_n)^{n-1}$ by Proposition 3.4, whence the upper bound $[(\Lambda_i)^k, \overline{\zeta}_{i,n}] \leq 1$ follows from Proposition 3.3. $\square$

Next we will study some *see-saw dual pairs* (cf. [Ku]) to determine various branching rules. Our consideration is based on the following well-known formula [LBST, Lemma 5.5]:

Lemma 3.6. *Let ω be a character of the direct product $S \times G$ of finite groups S and G. Then*

$$\omega = \sum_{\alpha \in \mathrm{Irr}(S)} D_\alpha \otimes \alpha,$$

where

$$D_\alpha : g \mapsto \frac{1}{|S|} \sum_{x \in S} \overline{\alpha(x)} \omega(xg)$$

is either zero, or a character of G.

We will work with a finite group Γ that contains two dual pairs $S_1 \times G_1$ and $S_2 \times G_2$, where $G_1 \geq G_2$ and $S_2 \geq S_1$.

Lemma 3.7. *Let ω be a character of Γ, and decompose*

$$\omega|_{G_1 \times S_1} = \sum_{\alpha \in \mathrm{Irr}(S_1)} D_\alpha \otimes \alpha, \; \omega|_{G_2 \times S_2} = \sum_{\gamma \in \mathrm{Irr}(G_2)} \gamma \otimes E_\gamma$$

as in Lemma 3.6. Then, for any $\alpha \in \mathrm{Irr}(S_1)$ and any $\gamma \in \mathrm{Irr}(G_2)$ we have that

$$[D_\alpha|_{G_2}, \gamma]_{G_2} = [\alpha, E_\gamma|_{S_1}]_{S_1},$$

and hence

$$D_\alpha|_{G_2} = \sum_{\gamma \in \mathrm{Irr}(G_2)} [E_\gamma|_{S_1}, \alpha]_{S_1} \cdot \gamma.$$

Proof. Write $a_{\alpha,\gamma} := [D_\alpha|_{G_2}, \gamma]_{G_2}$, so that

$$D_\alpha|_{G_2} = \sum_{\gamma\in \mathrm{Irr}(G_2)} a_{\alpha,\gamma}\gamma.$$

Then

$$\begin{aligned}\omega|_{G_2\times S_1} &= \sum_{\alpha\in\mathrm{Irr}(S_1),\ \gamma\in\mathrm{Irr}(G_2)} a_{\alpha,\gamma}\gamma\otimes\alpha \\ &= \sum_{\gamma\in\mathrm{Irr}(G_2)}\gamma\otimes\sum_{\alpha\in\mathrm{Irr}(S_1)} a_{\alpha,\gamma}\alpha.\end{aligned}$$

Thus $E_\gamma|_{S_1} = \sum_{\alpha\in\mathrm{Irr}(S_1)} a_{\alpha,\gamma}\alpha$, and the statements follow. $\square$

First we consider the dual pair

$$G_2 \times S_2 \tag{3.7.1}$$

inside $\Gamma := \mathrm{GU}_{2n}(q)$, where $S_2 = \mathrm{GU}_2(q)$ and $G_2 = \mathrm{SU}_n(q)$, and $\omega = \zeta_{2n} = \zeta_{2n,q}$. More precisely, we view S_2 as $\mathrm{GU}(U)$, where $U = \langle v_1, v_2\rangle_{\mathbb{F}_{q^2}}$ is endowed with the Hermitian form $\circ$, with an orthonormal basis (v_1, v_2). Next, $G_2 = \mathrm{SU}_n(q)$ is $\mathrm{SU}(W)$, where $W = \mathbb{F}_{q^2}^n$ is endowed with the Hermitian form $\circ$ defined in (3.0.1). Now we consider $V = U \otimes_{\mathbb{F}_{q^2}} W$ with the Hermitian form $\circ$ defined via

$$(u\otimes w)\circ(u'\otimes w') = (u\circ u')(w\circ w')$$

for $u \in U$ and $w \in W$. The action of $G_2\times S_2$ on V induces a homomorphism $G_2\times S_2 \to \Gamma := \mathrm{GU}(V)$.

Now V is the orthogonal sum $V_1 \oplus V_2$, where $V_i := v_i \otimes W$. This gives us a subgroup

$$G_1 := \mathrm{SU}(V_1)\times\mathrm{SU}(V_2) \cong \mathrm{SU}_n(q)\times\mathrm{SU}_n(q)$$

of Γ that contains (the image of) G_2. In fact, G_2 embeds diagonally in G_1: $g \mapsto \mathrm{diag}(g, g)$. Next,

$$S_1 := \mathrm{GU}(\langle v_1\rangle_{\mathbb{F}_{q^2}})\times\mathrm{GU}(\langle v_2\rangle_{\mathbb{F}_{q^2}}) \cong \mathrm{GU}_1(q)\times\mathrm{GU}_1(q)$$

is just the non-split diagonal torus of S_2.

In the above basis (v_1, v_2) of U and for $0 \le i, j \le q$, we consider the character

$$\lambda_{i,j} : \mathrm{diag}(\rho^a, \rho^b) \mapsto \boldsymbol{\rho}^{ia+jb}$$

of S_1. Then, as explained in [TZ2, §4], $\zeta_{i,n}$ corresponds to the $\boldsymbol{\rho}^i$-eigenspace of the generator $\rho\cdot 1_W$ of $\mathbf{Z}(\mathrm{GU}_n(q))$, so that

$$D_{\lambda_{ij}} = \zeta_{i,n}\otimes\zeta_{j,n} \tag{3.7.2}$$

for the dual pair $G_1 \times S_1$.

We use the notation of [E] for the irreducible characters of $S_2 = \mathrm{GU}_2(q)$ (with the parameter $q + 1$ in the superscripts of characters changed to 0). For instance

$$\chi_1^{(t)}|_{S_1} = \lambda_{t,t}.$$

The decomposition

$$\omega|_{S_2\times G_2} = \sum_{\alpha\in\mathrm{Irr}(S_2)} \alpha\otimes C_\alpha \tag{3.7.3}$$

was described in [LBST, Proposition 6.3]. In particular, the G_2-characters

$$C^\circ_\alpha := C_\alpha - k_\alpha\cdot 1_{G_2}, \tag{3.7.4}$$

where $\alpha\in\mathrm{Irr}(S_2)$, are irreducible and pairwise distinct, and $k_\alpha\in\{0,1\}$ is listed in Table I.

TABLE I. Degrees of C°_α for $G_2 = \mathrm{SU}_n(q)$

α	$\alpha(1)$	$C^\circ_\alpha(1)$	k_α
$\chi_1^{(0)}$	1	$(q^n-(-1)^n)(q^{n-1}+(-1)^nq^2)/(q+1)(q^2-1)$	1
$\chi_1^{(t)}$, $t\neq 0$	1	$(q^n-(-1)^n)(q^{n-1}+(-1)^n)/(q+1)(q^2-1)$	0
$\chi_q^{(0)}$	q	$(q^n+(-1)^nq)(q^n-(-1)^nq^2)/(q+1)(q^2-1)$	1
$\chi_q^{(t)}$, $t\neq 0$	q	$(q^n-(-1)^n)(q^n+(-1)^nq)/(q+1)(q^2-1)$	0
$\chi_{q-1}^{(0,u)}$, $u\neq 0$	$q-1$	$(q^n-(-1)^n)(q^{n-1}-(-1)^nq)/(q+1)^2$	0
$\chi_{q-1}^{(t,u)}$, $t,u\neq 0$	$q-1$	$(q^n-(-1)^n)(q^{n-1}+(-1)^n)/(q+1)^2$	0
$\chi_{q+1}^{(t)}$	$q+1$	$(q^n-(-1)^n)(q^{n-1}+(-1)^n)/(q^2-1)$	0

This implies

Corollary 3.8. *For the decomposition*

$$\omega|_{G_2\times S_2} = \sum_{\gamma\in\mathrm{Irr}(G_2)} \gamma\otimes E_\gamma,$$

we have that

$$E_\gamma = \begin{cases} \alpha, & \gamma = C^\circ_\alpha \text{ for some } \alpha\in\mathrm{Irr}(S_2),\\ \chi_1^{(0)}+\chi_q^{(0)}, & \gamma = 1_{G_2},\\ 0, & \text{otherwise.}\end{cases}$$

Proposition 3.9. *Suppose $n = 2k+1\geq 5$ and $(n,q)\neq(5,2)$. For $0<i\leq q$, and in the notation of (3.7.3)–(3.7.4) we have*

$$\Lambda_i = C_{\chi_1^{(i)}} + C_{\chi_q^{(i)}}.$$

Among these two irreducible constituents, only $C_{\chi_1^{(i)}}$ enters $(\zeta_{i,n})^2$.

Proof. (i) First, an application of Mackey's formula reveals that Λ_i is the sum of two distinct irreducible characters of $G_2 = \mathrm{SU}_n(q)$. Clearly, $[\Lambda_i, 1_{G_2}] = 0$. By Proposition 3.5, Λ_i enters $(\zeta_n)^2 = \omega|_{G_2}$, so

$$\Lambda_i = C^\circ_{\beta_1} + C^\circ_{\beta_2}$$

for some $\beta_1\neq\beta_2\in\mathrm{Irr}(S_2)$. Next,

$$\Lambda_i(1) = (q^n-(-1)^n)(q^{n-1}+(-1)^n)/(q^2-1),$$

so $\beta_1, \beta_2 \neq \chi^{(t)}_{q+1}$, see Table I.

By Proposition 3.4, at least one of $\gamma_j := C^\circ_{\beta_j}$, $j = 1, 2$, is an irreducible constituent of

$$(\zeta_{i,n})^2 = D_{\lambda_{i,i}}|_{G_2},$$

see (3.7.2). As $\gamma_j \neq 1_{G_2}$, by Lemma 3.6 and Corollary 3.8 we have

$$[D_{\lambda_{i,i}}|_{G_2}, \gamma_j]_{G_2} = [\lambda_{i,i}, E_{\gamma_j}|_{S_1}]_{S_1} = [\lambda_{i,i}, \beta_j|_{S_1}]_{S_1}.$$

We have shown that $C^\circ_{\beta_j}$, is an irreducible constituent of $(\zeta_{i,n})^2$ precisely when $\lambda_{i,i}$ is an irreducible constituent of $\beta_j|_{S_1}$.

(ii) As in the proof of Proposition 3.4, let τ be an element of $\mathbb{F}^\times_{q^{4k-2}}$ of order $q^{2k-1}+1$ chosen such that $\tau^{(q^{2k-1}+1)/(q+1)} = \rho$. Then we fix an element $g \in L_1$ such that $g(e_1) = \sigma e_1$, $g(f_1) = \sigma^{-q} f_1$, and g is conjugate to

$$\mathrm{diag}(\sigma, \sigma^{-q}, \tau, \tau^{-q}, \tau^{q^2}, \ldots, \tau^{(-q)^{2k-2}})$$

over $\overline{\mathbb{F}}_{q^2}$. By [Zs] there is a prime divisor ℓ of $q^{4k-2}-1$ that does not divide $\prod_{j=1}^{4k-3}(q^j-1)$. In particular, ℓ divides $|\tau|$. It follows that σ and σ^{-q} are the only eigenvalues of g that belong to $\mathbb{F}_{q^2}$.

Assume in addition that $q > 2$; in particular, $\sigma \neq \sigma^{-q}$. Then, $\langle e_1\rangle_{\mathbb{F}_{q^2}}$ and $\langle f_1\rangle_{\mathbb{F}_{q^2}}$ are the only two g-invariant isotropic 1-spaces in W, and so

$$\Lambda_i(g) = 2\boldsymbol{\rho}^{-i}. \tag{3.9.1}$$

Next, for any $x \in S_2 = \mathrm{GU}_2(q)$, $\omega(gx) = 1$, unless x has, at least one, and therefore both, of σ^{-1} and σ^q as its eigenvalues. In this exceptional case, x belongs to class $C_4^{(-1)}$ in the notation of [E], and $\omega(gx) = q^2$. It follows from Lemma 3.6 that

$$C^\circ_\alpha(g) = \begin{cases} \boldsymbol{\rho}^{-t}, & \alpha = \chi^{(t)}_1,\ 0 < t \leq q, \\ 2, & \alpha = \chi^{(0)}_1, \\ \boldsymbol{\rho}^{-t}, & \alpha = \chi^{(t)}_q,\ 0 < t \leq q, \\ 0, & \alpha = \chi^{(0)}_q, \\ 0, & \alpha = \chi^{(t,u)}_{q-1},\ 0 \leq t, u \leq q. \end{cases}$$

Together with (3.9.1), this readily implies that $\{\beta_1, \beta_2\} = \{\chi^{(i)}_1, \chi^{(i)}_q\}$. Note that $\chi^{(i)}_1|_{S_1} = \lambda_{i,i}$, but $\chi^{(i)}_q|_{S_1}$ does not contain $\lambda_{i,i}$, so we are done.

(iii) Now we consider the case $q = 2$. As shown in (i), we may assume that $\beta_1|_{S_1}$ contains $\lambda_{i,i}$. It follows that $\beta_1 \in \{\chi^{(i)}_1, \chi^{(2i,0)}_{q-1}\}$. However degree consideration using Table I rules out $\chi^{(2i,0)}_{q-1}$ and shows that $\beta_1 = \chi^{(i)}_1$. Again by degree consideration we now see that $\beta_2 = \chi^{(t)}_q$ for some $t \in \{1, 2\}$. Furthermore, g fixes exactly three isotropic 1-spaces in W (namely, the ones spanned by e_1, f_1, and $e_1 + f_1$), so $\Lambda_i(g) = 3\boldsymbol{\rho}^{-i}$.

Arguing as in (ii), we see that

$$C^\circ_\alpha(g) = \begin{cases} \boldsymbol{\rho}^{-t}, & \alpha = \chi_1^{(t)},\ 0 < t \leq q, \\ 2, & \alpha = \chi_1^{(0)}, \\ 2\boldsymbol{\rho}^{-t}, & \alpha = \chi_q^{(t)},\ 0 < t \leq q, \\ 0, & \alpha = \chi_q^{(0)}. \end{cases}$$

Hence $\beta_2 = \chi_q^{(i)}$, and we are done since $\chi_q^{(i)}|_{S_1}$ does not contain $\lambda_{i,i}$. □

We will now work with three new dual pairs. First, we consider the dual pair $G_3 \times S_3$ inside $\Gamma := \mathrm{GU}_{2kn}(q)$, where $S_3 = \mathrm{GU}_{2k}(q)$ and $G_3 = \mathrm{SU}_n(q)$, and $\omega = \zeta_{2nk} = \zeta_{2nk,q}$. More precisely, we view S_3 as $\mathrm{GU}(U)$, where $U = \langle v_1, \ldots, v_{2k}\rangle_{\mathbb{F}_{q^2}}$ is endowed with the Hermitian form $\circ$, with an orthonormal basis $(v_1, \ldots, v_{2k})$. Next, $G_3 = \mathrm{SU}_n(q)$ is $\mathrm{SU}(W)$, where $W = \mathbb{F}_{q^2}^n$ is endowed with the Hermitian form $\circ$ defined in (3.0.1). Now we consider $V = U \otimes_{\mathbb{F}_{q^2}} W$ with the Hermitian form $\cdot$ defined via

$$(u \otimes w) \circ (u' \otimes w') = (u \circ u')(w \circ w')$$

for $u \in U$ and $w \in W$. The action of $G_3 \times S_3$ on V induces a homomorphism $G_3 \times S_3 \to \Gamma := \mathrm{GU}(V)$.

Now V is the orthogonal sum $\oplus_{i=1}^{2k} V_i$, where $V_i := v_i \otimes W$. This gives us a subgroup

$$G_1 := \mathrm{SU}(V_1) \times \mathrm{SU}(V_2) \times \ldots \times \mathrm{SU}(V_{2k}) \cong \mathrm{SU}_n(q)^{2k}$$

of Γ that contains (the image of) G_3. In fact, G_3 embeds diagonally in G_1: $g \mapsto \mathrm{diag}(g, g, \ldots, g)$. Next,

$$S_1 := \mathrm{GU}(\langle v_1\rangle_{\mathbb{F}_{q^2}}) \times \mathrm{GU}(\langle v_2\rangle_{\mathbb{F}_{q^2}}) \times \ldots \times \mathrm{GU}(\langle v_{2k}\rangle_{\mathbb{F}_{q^2}}) \cong \mathrm{GU}_1(q)^{2k}$$

is just the non-split diagonal torus of S_3. In the above basis $(v_1, v_2, \ldots, v_{2k})$ of U and for $1 \leq i \leq q$, we consider the character

$$\mu_i : \mathrm{diag}(\rho^{a_1}, \rho^{a_2}, \ldots, \rho^{a_{2k}}) \mapsto \boldsymbol{\rho}^{i(\sum_{j=1}^{2k} a_j)} \tag{3.9.2}$$

of S_1.

Next, for each $1 \leq j \leq k$ we embed one copy of $\mathrm{SU}(W)$ in

$$\mathrm{SU}(\langle v_{2j-1}, v_{2j}\rangle_{\mathbb{F}_{q^2}} \otimes W)$$

(by letting it act only on W). This gives an embedding of $G_2 := \mathrm{SU}_n(q)^k$ in G_1 via

$$\mathrm{diag}(g_1, g_2, \ldots, g_k) \mapsto \mathrm{diag}(g_1, g_1, g_2, g_2, \ldots, g_k, g_k).$$

At the same times, G_3 embeds diagonally in G_2 via $g \mapsto \mathrm{diag}(g, g, \ldots, g)$. The action of G_2 is centralized by

$$S_2 := \mathrm{GU}(\langle v_1, v_2\rangle_{\mathbb{F}_{q^2}}) \times \mathrm{GU}(\langle v_3, v_4\rangle_{\mathbb{F}_{q^2}}) \times \ldots \times \mathrm{GU}(\langle v_{2k-1}, v_{2k}\rangle_{\mathbb{F}_{q^2}}) \cong \mathrm{GU}_2(q)^k.$$

Recall the characters C_α of $\mathrm{SU}_n(q)$ introduced in (3.7.3).

Proposition 3.10. *Suppose $n = 2k+1 \geq 5$, $(n, q) \neq (5, 2)$, and $0 < i \leq q$. Then both $(C_{\chi_1^{(i)}})^k$ and $(\zeta_{i,n})^{n-1}$ contain $\overline{\zeta}_{i,n}$.*

Proof. (i) First we decompose

$$\omega|_{G_3\times S_3} = \sum_{\gamma\in \mathrm{Irr}(G_3)} \gamma \otimes E_\gamma$$

for the dual pair $G_3 \times S_3$. By Proposition 3.3, $\omega|_{G_3} = (\zeta_n)^{n-1}$ contains $\overline{\zeta}_{i,n}$ with multiplicity one. It follows that the G_3-character $E_{\overline{\zeta}_{i,n}}$ has degree 1, so there is some $0 \le m = m_i \le q$ such that

$$E_{\overline{\zeta}_{i,n}}(X) = \boldsymbol{\rho}^{mt}$$

whenever $X \in \mathrm{GU}_{2k}(q)$ has determinant equal to ρ^t.

(ii) Next we decompose

$$\omega|_{S_2\times G_2} = \sum_{\beta\in \mathrm{Irr}(S_2)} \beta \otimes F_\beta$$

for the dual pair $S_2 \times G_2$. Note by (3.7.3) that if

$$\beta = \beta_1 \otimes \beta_2 \otimes \ldots \otimes \beta_k,$$

then

$$F_\beta = C_{\beta_1} \otimes C_{\beta_2} \otimes \ldots \otimes C_{\beta_k}. \tag{3.10.1}$$

By Lemma 3.7,

$$[F_\beta|_{G_3}, \overline{\zeta}_{i,n}]_{G_3} = [\beta, E_{\overline{\zeta}_{i,n}}|_{S_2}]_{S_2}.$$

Since $E_{\overline{\zeta}_{i,n}}$ has degree 1, we see that $\overline{\zeta}_{i,n}$ is an irreducible constituent of $F_\beta|_{G_3}$ precisely when $\beta = E_{\overline{\zeta}_{i,n}}|_{S_2}$, that is when

$$\beta(X_1, X_2, \ldots, X_k) = \boldsymbol{\rho}^{m\sum_{j=1}^k t_j}$$

whenever $X_j \in \mathrm{GU}_2(q)$ has determinant equal to ρ^{t_j} for $1 \le j \le k$. In the notation of [E] we then have

$$\beta = \underbrace{\chi_1^{(m)} \otimes \chi_1^{(m)} \otimes \ldots \otimes \chi_1^{(m)}}_{k}. \tag{3.10.2}$$

(iii) Recall by Proposition 3.4 that Λ_i enters $(\zeta_n)^2$. It follows that $\Lambda_i^{\otimes k} = \underbrace{\Lambda_i \otimes \Lambda_i \otimes \ldots \otimes \Lambda_i}_{k}$ enters $\omega|_{G_2}$. Next, by Proposition 3.5, $\overline{\zeta}_{i,n}$ is an irreducible constituent of $(\Lambda_i)^k = \Lambda_i^{\otimes k}|_{G_3}$. Furthermore, by Proposition 3.9, $\Lambda_i = C_{\chi_1^{(i)}} + C_{\chi_q^{(i)}}$. Hence, using (3.10.1) we see that

$$\begin{aligned}\Lambda_i^{\otimes k} &= \sum_{1\le j\le k,\ \beta_j\in\{\chi_1^{(i)},\chi_q^{(i)}\}} C_{\beta_1} \otimes C_{\beta_2} \otimes \ldots \otimes C_{\beta_k}\\ &= \sum_{1\le j\le k,\ \beta_j\in\{\chi_1^{(i)},\chi_q^{(i)}\}} F_{\beta_1\otimes\beta_2\otimes\ldots\otimes\beta_k}.\end{aligned}$$

Applying the result (3.10.2) of (ii), we conclude that $m = i$ and $\overline{\zeta}_{i,n}$ is an irreducible constituent of

$$F_{\chi_1^{(m)}\otimes\chi_1^{(m)}\otimes\ldots\otimes\chi_1^{(m)}}|_{G_3} = (C_{\chi_1^{(i)}})^k.$$

(iv) The same argument as in (ii), but applied to the decomposition

$$\omega|_{S_1\times G_1} = \sum_{\alpha\in\mathrm{Irr}(S_1)} \alpha\otimes D_\alpha$$

for the dual pair $S_1\times G_1$ implies that $\overline{\zeta}_{i,n}$ is an irreducible constituent of $D_\alpha|_{G_3}$ precisely when $\alpha = E_{\overline{\zeta}_{i,n}}|_{S_1}$, that is when $\alpha=\mu_m$ as introduced in (3.9.2). As m was shown to be equal to i in (iii), we now have that $\overline{\zeta}_{i,n}$ is an irreducible constituent of

$$D_\alpha|_{G_3} = D_{\mu_i}|_{G_3} = (\zeta_{i,n})^{n-1}.$$

□

We can now prove Theorem 1, which we restate:

Theorem 3.11. *Let q be a prime power and let $G=\mathrm{SU}_n(q)$ with $n=2k+1\geq 3$. Suppose in addition that $(n,q)\neq(3,2)$. Then $(\zeta_{i,n})^n$ contains 1_G with multiplicity exactly one if $1\leq i\leq q$ and zero if $i=0$.*

Proof. For $n=3$, the statement was checked by A. Schaeffer Fry using the package Chevie [GHLMP]. Likewise, the case $(n,q)=(5,2)$ was checked using the package GAP [GAP]. So we may assume that $n\geq 5$ and $(n,q)\neq(5,2)$. Now for $i=0$ the statement follows from Proposition 3.3. For $1\leq i\leq q$ we have

$$[(\zeta_{i,n})^{n-1},\overline{\zeta}_{i,n}]_G = [(\zeta_{i,n})^n, 1_G]$$

is at most 1 by Proposition 3.3 and at least 1 by Proposition 3.10. □

4. Moments of Weil representations of $\mathrm{SU}_4(q)$

Theorem 1 naturally brings up the question: what are the n^{th} moments of Weil representations of $\mathrm{SU}_n(q)$ when $2|n$? Preliminary analysis indicates that the even-dimensional case does not behave as nicely as in the odd-dimensional case (particularly because real-valued characters usually have large even moments). We restrict ourselves to record the following result:

Theorem 4.1. *Consider the irreducible Weil characters $\zeta_{i,n}$, $0\leq i\leq q$, of $G:=\mathrm{SU}_n(q)$ as given in (2.0.2), and suppose $n=4$. Then*

$$[(\zeta_{i,4})^4, 1_G] = \begin{cases} q+1, & i=0,\\ q+2, & 2\nmid q,\ i=(q+1)/2,\\ q-1, & 4|(q+1),\ i=(q+1)/4,\ 3(q+1)/4,\\ 1, & \textit{otherwise.}\end{cases}$$

Proof. (i) We will use the dual pairs $G_1\times S_1=\mathrm{SU}_n(q)^2\times\mathrm{GU}_1(q)^2$ and $G_2\times S_2=\mathrm{SU}_n(q)\times\mathrm{GU}_2(q)$ as in (3.7.1). By [LBST, Proposition 6.3],

$$\begin{aligned}\omega|_{G_2\times S_2} &= \sum_{\alpha\in\mathrm{Irr}(S_2)} C_\alpha\otimes\alpha = \sum_{\gamma\in\mathrm{Irr}(G_2)}\gamma\otimes E_\gamma\\ &= \sum_{\alpha\in\mathrm{Irr}(S_2)} C^\circ_\alpha\otimes\alpha + 1_{G_2}\otimes\left(\chi_1^{(0)}+\chi_q^{(0)}\right),\end{aligned}$$

where $C^\circ_\alpha(1)$ are listed in Table I. The only new feature that arises in the case $n=4$ is that, according to [LBST, Proposition 6.5],

(a) If $\alpha \neq \beta$, then $C^\circ_\alpha = C^\circ_\beta$ precisely when $\{\alpha,\beta\} = \{\chi_1^{(t)}, \chi_1^{(q+1-t)}\}$ for some $t \in \{1,2,\ldots,q\} \smallsetminus \{(q+1)/2\}$; and

(b) All C°_α are irreducible, except when $2 \nmid q$ and $\alpha = \chi_1^{(q+1)/2}$, in which case C°_α is a sum of two distinct irreducible characters (of degree $(q^2+1)(q^2-q+1)/2$).
Hence, instead of Corollary 3.8 now we have

$$E_\gamma = \begin{cases} \alpha, & \text{if } \gamma \text{ is an irreducible constituent} \\ & \text{of } C^\circ_\alpha \text{ for some } \alpha \in \mathrm{Irr}(\mathrm{GU}_2(q)), \\ \chi_1^{(0)} + \chi_q^{(0)}, & \text{if } \gamma = 1_{G_2}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.1}$$

On the other hand,

$$\omega|_{G_1 \times S_1} = \sum_{\alpha \in \mathrm{Irr}(S_1)} D_\alpha \otimes \alpha,$$

where D_α is given in (3.7.2) for $\alpha = \lambda_{i,j} \in \mathrm{Irr}(\mathrm{GU}_1(q)^2)$. Applying Lemma 3.7 we then get

$$(\zeta_{i,4})^2|_{\mathrm{SU}_4(q)} = D_{\lambda_{i,i}}|_{G_2} = \sum_{\gamma \in \mathrm{Irr}(G_2} [E_\gamma|_{\mathrm{GU}_1(q)^2}, \lambda_{i,i}]_{\mathrm{GU}_1(q)^2} \cdot \gamma. \tag{4.1.2}$$

Direct computations show for $\alpha \in \mathrm{Irr}(\mathrm{GU}_2(q))$ that

$$[\alpha|_{\mathrm{GU}_1(q)^2}, \lambda_{i,i}]_{\mathrm{GU}_1(q)^2} = \begin{cases} \delta_{t,i}, & \alpha = \chi_1^{(t)}, \\ \delta_{t,2i}, & \alpha = \chi_{q+1}^{(t)}, \\ \delta_{t+u,2i}, & \alpha = \chi_{q-1}^{(t,u)}, \\ \delta_{t,i+(q+1)/2}, & \alpha = \chi_q^{(t)},\ 2 \nmid q, \\ 0, & \alpha = \chi_q^{(t)},\ 2|q, \end{cases} \tag{4.1.3}$$

and $\delta_{i,j}$ is defined to be 1 if $i \equiv j (\mathrm{mod}\, q+1)$ and 0 otherwise. Recall that in the notation for $\alpha \in \mathrm{Irr}(\mathrm{GU}_2(q))$, the superscripts are viewed as elements of $\mathbb{Z}/(q+1)\mathbb{Z}$ if $\alpha(1) \leq q$, and as elements of $\mathbb{Z}/(q^2-1)\mathbb{Z}$ if $\alpha(1) = q+1$. Moreover, $\chi_{q-1}^{(t,u)} = \chi_{q-1}^{(u,t)}$ and $\chi_{q+1}^{(t)} = \chi_{q+1}^{(-tq)}$.

(ii) Consider the case $2|q$. Then (4.1.1)–(4.1.3) imply that

$$(\zeta_{0,4})^2 = 1_G + C^\circ_{\chi_1^{(0)}} + \sum_{1 \leq t \leq q/2} C^\circ_{\chi_{q-1}^{(t,-t)}} + \sum_{1 \leq s \leq (q-2)/2} C^\circ_{\chi_{q+1}^{(s(q+1))}}.$$

As $\zeta_{0,4}$ is real-valued, it follows that $[(\zeta_{0,4})^4, 1_G]_G = q+1$.

Likewise, if $i \neq 0$, then the irreducible summands of $(\zeta_{i,4})^2$ are $C^\circ_{\chi_1^{(i)}}$, $C^\circ_{\chi_{q-1}^{(t,2i-t)}}$ with $t \neq i$, and $C^\circ_{\chi_{q+1}^{(s)}}$ with $s \equiv 2i (\mathrm{mod}\, q+1)$ (and $s \not\equiv 0 (\mathrm{mod}\, q-1)$); all with multiplicity one. It follows that the only common irreducible constituent of $(\zeta_{i,4})^2$ and $(\overline{\zeta}_{i,4})^2 = (\zeta_{q+1-i,4})^2$ is $C^\circ_{\chi_1^{(i)}} = C^\circ_{\chi_1^{(q+1-i)}}$, cf. (a) above. Thus $[(\zeta_{i,4})^4, 1_G]_G = 1$. In fact, this argument also applies to the case where $2 \nmid q$ and $(q+1) \nmid 4i$, where there is an extra irreducible summand $C^\circ_{\chi_q^{(i+(q+1)/2)}}$ (also with multiplicity 1) in $(\zeta_{i,4})^2$.

(iii) Assume now that $2 \nmid q$. Then (4.1.1)–(4.1.3) imply that

$$(\zeta_{0,4})^2 = 1_G + C^\circ_{\chi_1^{(0)}} + \sum_{1 \le t \le \frac{q-1}{2}} C^\circ_{\chi_{q-1}^{(t,-t)}} + C^\circ_{\chi_q^{(\frac{q+1}{2})}} + \sum_{1 \le s \le \frac{q-3}{2}} C^\circ_{\chi_{q+1}^{(s(q+1))}},$$

yielding $[(\zeta_{0,4})^4, 1_G]_G = q+1$. Likewise,

$$(\zeta_{\frac{q+1}{2},4})^2 = 1_G + C^\circ_{\chi_1^{(\frac{q+1}{2})}} + \sum_{1 \le t \le \frac{q-1}{2}} C^\circ_{\chi_{q-1}^{(t,-t)}} + C^\circ_{\chi_q^{(0)}} + \sum_{1 \le s \le \frac{q-3}{2}} C^\circ_{\chi_{q+1}^{(s(q+1))}}.$$

Since $\zeta_{\frac{q+1}{2},4}$ is real-valued and $C^\circ_{\chi_1^{(\frac{q+1}{2})}}$ is the sum of two distinct irreducible summands, $[(\zeta_{\frac{q+1}{2},4})^4, 1_G]_G = q+2$.

Finally, the irreducible summands of $(\zeta_{\frac{q+1}{4},4})^2$ are $C^\circ_{\chi_q^{(-\frac{q+1}{4})}}$, $C^\circ_{\chi_1^{(\frac{q+1}{4})}}$, $C^\circ_{\chi_{q-1}^{(t,\frac{q+1}{2}-t)}}$ with $t \neq \pm(q+1)/4$, and $C^\circ_{\chi_{q+1}^{(2s+1)(q+1)/2}}$; all with multiplicity one. As mentioned in (a), $C^\circ_{\chi_1^{(\frac{q+1}{4})}} = C^\circ_{\chi_1^{-(\frac{q+1}{4})}}$. Thus all of these characters, except for the first one, are common irreducible summands between $(\zeta_{\frac{q+1}{4},4})^2$ and $(\overline{\zeta}_{\frac{q+1}{4},4})^2 = (\zeta_{\frac{3(q+1)}{4},4})^2$. It follows that $[(\zeta_{\frac{q+1}{4},4})^4, 1_G]_G = q-1$. $\square$

We also record a curious fact about 4^{th} moments of Weil representations of $\mathrm{Sp}_{2n}(q)$, which holds specifically in the case $q = 3$.

Proposition 4.2. *Let $n \ge 2$ and let ξ, η denote an irreducible Weil character of $G = \mathrm{Sp}_{2n}(3)$ of degree $(3^n+1)/2$ and $(3^n-1)/2$, respectively. Then*

$$[\xi^4, 1_G]_G = 1 = [\eta^4, 1_G]_G.$$

Proof. It was shown in [MT, Proposition 5.4] that if $\chi \in \{\xi, \eta\}$ then $\mathrm{Sym}^2(\chi)$ and $\wedge^2(\chi)$ are irreducible, of distinct degrees. Furthermore, Lemma 3.3(ii) and formula (3.5) of [GMT] show that

$$\mathrm{Sym}^2(\xi) = \mathrm{Sym}^2(\bar{\xi}),\ \mathrm{Sym}^2(\eta) \neq \mathrm{Sym}^2(\bar{\eta}),\ \wedge^2(\xi) \neq \wedge^2(\bar{\xi}),\ \wedge^2(\eta) = \wedge^2(\bar{\eta}).$$

Since $\chi^2 = \mathrm{Sym}^2(\chi) + \wedge^2(\chi)$, the statement follows. $\square$

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