SECTIONAL RANK AND COHOMOLOGY

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Dedicated to our good friend Michel Broué

Abstract. Donovan's conjecture implies a bound on the dimensions of cohomology groups in terms of the size of a Sylow p-subgroup and we give a proof of a stronger bound (in terms of sectional p-rank) for $\dim H^1(G, V)$. We also prove a reduction theorem for higher cohomology.

1. Introduction

Let G be a finite group, p a prime and k an algebraically closed field of characteristic p. Donovan's conjecture (cf. [Ke]) asserts that for a fixed p-group D, there are only finitely many blocks B of any group algebra kG with defect group D up to Morita equivalence.

A trivial consequence of this conjecture is that there is a bound on the dimension of Ext-groups between irreducible modules (depending only on the defect group of the block containing the irreducibles).

Our main result considers what happens for the projective cover of the trivial module k and H^1 under a weaker condition, where we do not fix the (isomorphism type of) Sylow p-subgroups but only their sectional p-rank. Recall that the *sectional p-rank* of a finite group G is the maximal rank of an elementary abelian group isomorphic to L/K for some subgroups $K \lhd L$ of G. Even considering the case that $G = P$ is cyclic, one sees that there is no upper bound on the composition length of the projective cover of k. We do prove:

Theorem 1.1. *Let G be a finite group, p a prime and k an algebraically closed field of characteristic p. Let r be the sectional p-rank of G. There exists a constant $C = C(p, r)$ such that if J is the radical of the projective cover of the trivial G-module k, then J/J^2 is a direct sum of at most C irreducible kG-modules.*

We conjecture that the constant can be chosen to depend only on the sectional p-rank and not on the prime p. The proof we give shows that the only obstruction to proving this is the case of simple groups. We first prove a reduction to the case of simple groups. The sectional rank assumption implies that (for a fixed p) aside from finitely many simple groups, it suffices to consider cross characteristic modules of finite simple groups of Lie type and of bounded rank. We then use the main result of [GT] which essentially proves the theorem in that case. The results of [GT] show that the constant C can be chosen to be $|W| + e$ where W is the Weyl group and e is the twisted rank of G. We improve this result in Section 4.

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We also conjecture that this is true for all projective indecomposable modules for G (assuming bounded sectional p-rank of the defect group of the block). Some evidence for this follows from results of Gruber and Hiss [GHi] about classical groups (but with restrictions on the primes).

There are also some related results of Malle and Robinson [MR] aimed towards proving their conjecture that the number of simple modules in a given p-block is at most p^r where r is the sectional rank of the defect groups of the block. One cannot hope to bound this number independently of p.

A restatement of Theorem 1.1 is the following:

Corollary 1.2. *Let G be a finite group, p a prime and k an algebraically closed field of characteristic p. Let r be the sectional p-rank of G. Then there exist constants $A(p,r)$ and $B(p,r)$ such that*

(i) *the number of irreducible kG-modules V with $H^1(G,V) \neq 0$ is at most $A(p,r)$; and*
(ii) *if V is an irreducible kG-module, then $\dim H^1(G,V) \leq B(p,r)$.*

If one works with indecomposable modules, it is easy to see, using the Green correspondence, that the problem reduces to the case that the Sylow p-subgroups are normal. However, there are indecomposable P-modules V with arbitrarily large $\dim H^1(P,V)$ for most p-groups P. The one case where this does yield information is when G has a cyclic Sylow p-subgroup (i.e. the sectional p-rank is 1). It is well known (using the Green correspondence) that $\dim H^n(G,V) \leq 1$ for an *indecomposable* module V in characteristic p (cf. [GKKL, Lemma 3.5]).

In [Gu1], the first author asked whether there is a universal constant C such that $\dim H^1(G,V) < C$ for V any *faithful* absolutely irreducible G-module with G a finite group. This is still open but likely false (see [Lu] for examples with very large $\dim H^1(G,V)$). The existence of absolutely irreducible modules for simple groups with large first cohomology group depends on the validity of Lusztig's conjecture and on knowing that certain coefficients of Kazhdan-Lusztig polynomials can be very large (this gives examples for groups of Lie type and modules in the natural characteristic). In particular, there are no known examples in small characteristic.

Of course, $\dim H^1(G,k)$ can be arbitrarily large but is bounded if the sectional rank of the Sylow p-subgroups is bounded (indeed, it is bounded in terms of the Frattini quotient of a Sylow p-subroup). Thus, one needs to assume faithfulness or some condition on the Sylow p-subgroups to get upper bounds. For faithful absolutely irreducible modules, the upper bounds for $\dim H^1$ reduce to the case of finite simple groups. It is known [CPS, GT] that for G a finite simple group of Lie type of bounded rank s, there is a bound $\dim H^1(G,V) < C(s)$ for V any absolutely irreducible kG-module.

There are recent papers [EL, EEL] giving a reduction in the case of abelian defect groups and proving Donovan's conjecture when $p=2$ and the defect group is abelian. There are also reductions to quasisimple groups [Du] (in the case of nonabelian defect groups, the reduction is in terms of Cartan matrices rather than Morita equivalence).

Consider the following question, where $|G|_p$ denotes the p-part of the order $|G|$.

Question 1.3. *Does there exist a constant $C = C(r,n)$ such that $\dim H^n(G,V) \leq C$ for any finite group G with $|G|_p \leq p^r$ and V an irreducible kG-module ?*

One can ask whether there is such a bound in terms of sectional rank (and perhaps the constant depends on p as well). Likely this can be reduced to two questions. The first is whether this holds for finite simple groups. The second is whether there is a bound on $\dim H^n(P,k)$ for a p-group P (in terms of sectional rank). There is a result of Quillen (see [AE] for a generalization to any module) showing that the growth rate of $H^n(P,k)$ is determined by the maximal rank of an elementary abelian subgroup of P.

We do reduce Question 1.3 to the case of simple groups.

Theorem 1.4. *Let k be an algebraically closed field of characteristic p and let $n, r \geq 0$ be any integers. Suppose that there exists a a constant $C = C(p,r,n)$ such that $\dim H^j(S,V) \leq C$ for every finite simple group S with $|S|_p \leq p^r$, any irreducible kS-module, and for all $0 \leq j \leq n$. Then the same statement is true for all finite groups G with $|G|_p \leq p^r$, but with possibly a different constant $C' = C'(p,r,n)$.*

One can also raise a similar question about Ext. There should be a reduction to the case of simple groups. We give an example showing that $\dim \mathrm{Ext}^1_G(V,W)$ can be arbitrarily large even for V, W absolutely irreducible faithful modules. See [GKKL] for a similar example for H^2.

We also improve our H^1 results from [GT] giving bounds in terms of the sectional rank but also depending upon the prime. Here are some of the results in this direction:

Theorem 1.5. *Let G be a finite simple group of Lie type in characteristic ℓ. Let W be the Weyl group of G. Let $p \neq \ell$ be a prime and k an algebraically closed field of characteristic p. Then the following statements hold.*

(i) *The number of irreducible kG-modules V with $H^1(G,V) \neq 0$ is at most $|\mathrm{Irr}(W)|+3$.*
(ii) *Suppose that $p \nmid [G_i : B]$ for any minimal parabolic subgroup G_i properly containing a fixed Borel subgroup B of G. Then the number of irreducible kG-modules V with $H^1(G,V) \neq 0$ is less than $|\mathrm{Irr}(W)|$.*

Theorem 1.6. *Let G be a finite simple group of Lie type in characteristic ℓ of twisted rank e. Let W be the Weyl group of G. Let $p \neq \ell$ be a prime and k an algebraically closed field of characteristic p. Let V be an irreducible kG-module. Let $G_1, \dots, G_e$ denote the minimal parabolic subgroups properly containing a fixed Borel subgroup B of G.*

(i) *If $p \nmid |G_i|$ for $1 \leq i \leq e$, then $\dim H^1(G,V) \leq \dim V^B < |W|^{1/2}$.*
(ii) *If $p \nmid |B|$, then $\dim H^1(G,V) < |W|^{1/2}$.*
(iii) *In general, $\dim H^1(G,V) < e - 1 + |W|^{1/2}$. If $V^B = 0$, then $\dim H^1(G,V) \leq 1$.*
(iv) *Moreover, if $V_1, \dots, V_m$ are pairwise non-isomorphic representatives of isomorphism classes of irreducible kG-modules with $V_i^B \neq 0$ and $V_i \not\cong k$, then*

$$\sum_{i=1}^{m} (\dim H^1(G,V_i) + 1 - e/2)^2 \leq me^2/4 + |W| + e + 1.$$

Theorem 1.6(iii) shows that the sum of squares of $\dim H^1(G,V)$, adjusted suitably, with V running over all isomorphism classes of irreducible kG-modules, is also bounded roughly by $|W|$, and $m \leq |\mathrm{Irr}(W)| - 1$ by Theorem 1.5. See Section 4 for details and other related results.

2. Sectional Rank and H^1

Fix a prime p and k an algebraically closed field of characteristic p. If G is a finite group, let $s(G) = s_p(G)$ be the sectional p-rank of G. If V is a kX-module for a group X, we let $\mathbf{C}_X(V)$ be the kernel of the representation sending X to $\mathrm{GL}(V)$ and let V^X denote the submodule of X-fixed points in V.

A key result is the following easy consequence of the main result of [GT]. Again, we conjecture that the constants can be chosen independently of p.

Theorem 2.1. *Let S be a finite nonabelian simple group with $s(S) = s$ fixed. Then there exist constants $A(p,s)$ and $B(p,s)$ such that:*

(i) *if V is an irreducible kS-module, then* $\dim H^1(S,V) \le B(p,s)$*; and*
(ii) *there are at most $A(p,s)$ irreducible kS-modules V with $H^1(S,V) \neq 0$.*

Proof. Excluding only finitely many simple groups (depending upon s and p), we see that it suffices to prove the statement in the case where S is a finite simple group of Lie type in characteristic other than p. The result now follows by [GT] (see also Section 4 below for better results) where it was shown that $\dim J/J^2 \le |W| + e$, where J is the radical of the projective cover of k, G is a finite simple group of Lie type of twisted rank e with Weyl group W and p is not the characteristic of G. □

We first give a quick proof of Corollary 1.2(ii).

Corollary 2.2. *There exists a constant $C(p,s)$ such that*

$$\dim H^1(G,V) \le C(p,s)$$

for any finite group G with $s_p(G) = s$ and any irreducible kG-module V.

Proof. Let $Q := \mathbf{C}_G(V)$. By the inflation-restriction sequence in cohomology (or by Lemma 3.2),

$$\dim H^1(G,V) \le \dim H^0(G/Q, H^1(Q,V)) + \dim H^1(G/Q,V).$$

Since Q acts trivially on V, $H^0(G/Q,H^1(Q,V)) \cong \mathrm{Hom}_G(Q,V) = \mathrm{Hom}_G(Q/Q_1,V)$, where Q/Q_1 is the largest elementary abelian p-group quotient of Q. Since $|Q/Q_1| \le p^s$, we can view Q/Q_1 as an $\mathbb{F}_pG$-module of dimension $\le s$. Note that $\mathrm{Hom}_G(Q/Q_1,V)$ is a vector space over k.

As V is irreducible, it follows that $\dim_k \mathrm{Hom}_G(Q/Q_1,V) \le s/(\dim V)$. So it suffices to assume that $Q = 1$ and V is faithful.

In that case, it follows from [Gu2] that $\dim H^1(G,V) \le \dim H^1(S,W)$ where S is a subnormal simple subgroup of G and W is an S-submodule of V that is irreducible. Now apply Theorem 2.1. □

We now turn towards the proof of Corollary 1.2(i). We essentially split the problem into two cases. The first is when the module occurs as a split chief factor in the group and the second is when $H^1(G/\mathbf{C}_G(V),V) \neq 0$. Recall that a *split chief p-factor* of a finite group G is a chief factor H/K with $K \lhd H$ and H/K a p-group such that H/K has a complement in G/K.

Lemma 2.3. *Let G be a finite group of sectional p-rank s. In any minimal normal series of G, there is a bound $D(s)$ on the number of split chief factors of G that are p-groups.*

Proof. Consider a minimal counterexample. We may assume that the Frattini subgroup $\Phi(G) = 1$, whence $\Phi(F^*(G)) = 1$. We may also assume that $\mathbf{O}_{p'}(G) = 1$.

By induction, we may also assume that $E(G) = 1$. Thus, $F^*(G)$ is an elementary abelian p-group and is a semisimple G-module. Thus, $|G| \le p^s|\mathrm{GL}_s(p)|$. Thus, it suffices to consider the problem for completely reducible subgroups of $\mathrm{GL}_s(p)$. We just make the trivial observation that since the Sylow p-subgroup of G has order at most $p^{s(s+1)/2}$, the result is now clear. □

Note that in Lemma 2.3, we do need to consider split chief factors; indeed, in a cyclic group of order p^a, the sectional rank is 1 but the number of chief factors is a. If one only wanted a bound on the number of p-chief factors up to G-isomorphism, the proof above can be modified to obtain this (and this is all we need). We reiterate that the bound above does not depend on p. One could prove a much stronger statement using results in [GMP].

It is convenient to introduce $s'_p(G)$ which we define to be the maximal sectional p-rank of a section H/K of G that is a direct product of non-abelian simple groups.

Lemma 2.4. *Let G be a finite group with $s' := s'_p(G)$. There exist constants $C_i(p, s')$ such that:*

(i) *The number of irreducible kG-modules V such that $H^1(G/\mathbf{C}_G(V), V) \neq 0$ is at most $C_1(p, s')$.*
(ii) $\dim H^1(G/\mathbf{C}_G(V), V) \leq C_2(p, s')$.

Proof. Certainly, $s'_p(G/\mathbf{C}_G(V)) \leq s'$, so by induction on $|G|$ we may assume that V is faithful. Since $\mathbf{O}_p(G)$ acts trivially on any irreducible kG-module, we may assume that $\mathbf{O}_p(G) = 1$. If $\mathbf{O}_{p'}(G)$ acts nontrivially on V, then by the restriction-inflation sequence, we see that $H^1(G/\mathbf{C}_G(V), V) = 0$. Thus, we may also assume that $\mathbf{O}_{p'}(G) = 1$.

So $F^*(G) = S_1 \times \ldots \times S_t$ where the S_i's are non-abelian simple groups (and clearly $t \leq s'$). By the above, $F^*(G)$ acts nontrivially on V, and so $H^0(F^*(G), V) = 0$. Decompose $V|_{F^*(G)} = c \oplus_{i=1}^d W_i$, where the W_i are G-conjugate, pairwise non-isomorphic irreducible $kF^*(G)$-modules and $c \geq 1$. Also write $W_i = W_{i,1} \otimes \ldots \otimes W_{i,t}$, where $W_{i,j}$ is a simple S_j-module. If at least two of the $W_{1,j}$'s are nontrivial, then by the Künneth formula and the inflation-restriction sequence, $H^1(F^*(G), W_i) = 0$ and so $H^1(G/\mathbf{C}_G(V), V) = 0$. Thus we may assume that $W_{1,1} \ncong k$ but $W_{1,j} \cong k$ for all $j > 1$. Now G permutes the S_j's. Assume this action is intransitive, say S_t is not G-conjugate to S_1. Then the described shape of W_1 implies that $W_{i,t} \cong k$ for all i and so $S_t \leq \mathbf{C}_G(V)$, contrary to our assumption. Hence, we may assume that $F^*(G)$ is the unique minimal normal subgroup of G and all the S_i's are G-conjugate.

The argument above shows that $H^1(G, V) \neq 0$ implies that $V = \mathrm{Ind}^G_{\mathbf{N}_G(S_1)}(W)$ for some irreducible kN-module W with $N := \mathbf{N}_G(S_1)$ (in fact, $W|_{F^*(G)} \cong cW_1$). Note that modding out by $\mathbf{C}_G(S_1) \geq S_2 \times \ldots \times S_t$ does not change the computation for H^1. Thus, it suffices to consider the case that $F^*(G) = S$ is a simple group (with bounded $s'_p(S)$).

Using Shapiro's Lemma once more, we may assume that S acts homogeneously on V and so (passing to a central p'-cover if necessary), we may assume that $V = W \otimes U$ where U is a G/S module.

Applying the inflation-restriction sequence again, we see that

$$H^1(G, V) = H^0(G/S, H^1(S, V)).$$

By taking a G-resolution and restricting to S, we see that $H^j(N, V) \cong H^j(S, W) \otimes U$ as a G/S-module. By Theorem 2.1, this gives the bound on $\dim H^1$ and also shows there is a bound on the number of possible modules W so that $H^1(S, W) \neq 0$ (and so also $H^1(N, V) \neq 0$). Thus, there are only finitely many simple modules of N that we need to consider and so we may fix this.

Now $H^0(N/S, H^1(S, W) \otimes U))$ is nonzero if and only if U is a quotient (as an N/S-module) of $H^1(S, W)$ and so there are only finitely many possibilities for U, whence the result. □

We can now prove Corollary 1.2 (which is equivalent to Theorem 1.1).

We have already shown in Corollary 2.2 that there is a bound on $\dim H^1(G, V)$.

Next we show there is a bound on the number of irreducible kG-modules V with $H^1(G, V) \neq 0$. By Lemma 2.3 there are only finitely many such modules which occur as split chief factors of G and so we may assume that V is not a chief factor of G. Thus, by [AG, 2.10], we have that $H^1(G, V) = H^1(G/\mathbf{C}_G(V), V)$ and then Lemma 2.4 applies.

3. Higher Cohomology

We fix a prime p and an algebraically closed field k of characteristic p.

We first note the trivial result:

Lemma 3.1. *Let H be a finite group and $V = W_1 \otimes W_2$ a tensor product of kH-modules with W_2 irreducible. Then $\dim H^0(H, V) \le (\dim W_1)/(\dim W_2)$.*

We need the following result that follows from an easy spectral sequence argument. See [Ho] or [GKKL, Lemma 3.7].

Lemma 3.2. *Let G be a finite group, N a normal subgroup of G and V a kG-module. Then $\dim H^n(G, V) \le \sum_{i=0}^{n} \dim H^i(G/N, H^{n-i}(N, V))$.*

Proof of Theorem 1.4. We induct on $n+r+|G|$. If $r = 0$ or $n = 0$, then the result is clear.

Let V be an irreducible kG-module. If $V \cong k$, then the cohomology ring $H^*(G, k)$ embeds in $H^*(P, k)$ for P a Sylow p-group and the result holds. So assume that V is nontrivial.

By Shapiro's Lemma, we may assume that V is a primitive kG-module (otherwise $V = \mathrm{Ind}_H^G(W)$ for some proper subgroup H).

Let N be a maximal (proper) normal subgroup of G. Set $S = G/N$. Then N acts homogeneously on V by primitivity. Passing to a p'-central cover of G if necessary, $V \cong W_1 \otimes W_2$ where W_1 is a kG-module that is N-irreducible and W_2 is an irreducible G/N-module. In this bigger group, the quotient need no longer be simple but modulo a center of p'-order, it is (but that can only reduce the size of the cohomology groups). By Lemma 3.2,

$$\dim H^n(G, V) \le \sum_{i=0}^{n} \dim H^i(G/N, H^{n-i}(N, V)).$$

As we observed earlier, we see that $H^j(N, V) \cong H^j(N, W_1) \otimes W_2$ as a G-module.

If p does not divide $|S|$, then we see by irreducibility of W_2 that

$$\dim H^n(G, V) \le \dim H^0(G/N, H^n(N, W_1) \otimes W_2) \le \dim H^n(N, W_1).$$

Thus we may assume that G/N has no nontrivial p'-quotients; in particular, $|N|_p < |G|_p$.

If G/N has order p, then as we noted $\dim H^j(G/N, k) \le 1$ and so $\dim H^j(G/N, W) \le \dim W$. Also, in this case $W_1 = V$ and using Lemma 3.2 we have

$$\dim H^n(G, V) \le \sum_{i=0}^{n} \dim H^i(N, V),$$

and this is at most $\sum_{j=0}^{n} C(p, r-1, j)$ and the result holds.

More generally, if G/N has order at most some integer e, we can pass to G_0 where G_0/N is a Sylow p-subgroup of G/N. Then the restriction map on cohomology from G to G_0 is injective. Note that V restricted to G_0 has at most $\dim W_2 \le e^{1/2}$ composition factors (all isomorphic to W_1 as N-modules). Using the previous case and induction, we get a bound for $\dim H^n(G, V)$.

The remaining case is when $S \cong G/N$ is a nonabelian simple group of sufficiently large order e. Let

$$M := \max\{\dim H^j(N, W_1) \mid 0 \le j \le n\}.$$

As $|S|_p \le p^r$ is bounded, we may choose e sufficiently large so that S is a simple group of Lie type of rank r_0 and defined over a field $\mathbb{F}_q$ in characteristic $\neq p$. (Indeed, if S is alternating group of degree m then $m \le pr$ and so $|S| < (pr)!$. If S is a simple group of Lie type in characteristic p, then $|S| < p^{3r}$.) Now, the Landazuri-Seitz-Zalesskii bound [KL, Theorem 5.3.9] implies that, if the smallest dimension of nontrivial simple kS-modules is at most M then both the rank r_0 and the size q of the definition field of S are at most some constant M_1, whence $|S|$ is at most some constant M_2 (depending on M). Choosing e sufficiently large, we can ensure that any simple S-module of dimension at most M is trivial. Then

$$\dim H^n(G,V) \le C_1 \cdot \sum_{j=0}^{n} \dim H^j(S,W_2),$$

where C_1 is an upper bound for $\dim H^j(N,W_1)$ and the result follows. □

4. Cross Characteristic H^1

In this section, we take G to be a finite simple group of Lie type of twisted rank e over the field of size q. Fix a Borel subgroup B of G with unipotent radical Q. Let $G_1,\dots,G_e$ denote the minimal parabolic subgroups properly containing B. Let p be a prime not dividing q and k an algebraically closed field of characteristic p.

Our goal is to improve the bounds from [GT] on $\dim H^1(G,V)$ with V an irreducible kG-module.

We first prove Theorem 1.5 that improves the bound for the number of irreducible kG-modules with nontrivial H^1. This critically depends on results of Geck and Rouquier (see [GP]) as well as results from [GT]. The original bound from [GT] was of the magnitude of the order of the Weyl group W.

Theorem 4.1. *The number of irreducible kG-modules with nontrivial H^1 is at most $|\mathrm{Irr}(W)|+3$. If $p \nmid [G_i : B]$ for $1 \le i \le e$, then this number is less than $|\mathrm{Irr}(W)|$.*

Proof. It follows from results of Geck and Rouquier (see [GP, 7.5.6, 8.2.5]) that the number of distinct simple kG-modules V with $V^B \ne 0$ is at most $|\mathrm{Irr}(W)|$. By [GT, Theorem 1.3(ii)] and Corollary 4.5 (below), there are at most 4 irreducible kG-modules with $V^B = 0 \ne H^1(G,V)$ and there are none if $p \nmid [G_i : B|$ for all i. Also note that $H^1(G,k)=0$ as G is perfect. Hence both statements follow. □

Next we derive upper bounds on $\dim H^1(G,V)$. Note that if S is a simple kG-module, then, by Frobenius reciprocity, the multiplicity of S in the socle of $M = \mathrm{Ind}_B^G(k) = k_B^G$ is $\dim S^B$. We also recall, see Lemma 2.1 and Proposition 3.1(ii)] of [GT], that $S^Q = S^B$ if S is a submodule of M and that $S^B \ne 0$ if and only $S^{\mathbf{O}_{p'}(B)} \ne 0$.

In particular, this gives that

$$\sum_S (\dim S^B)^2 \le |W| = \dim \mathrm{End}_G(k_B^G),$$

where the sum is over all (isomorphism classes of) simple kG-modules.

First we record an elementary result.

Lemma 4.2. *Let H be a finite group. Assume that H is generated by subgroups H_1 and H_2 and set $A := H_1 \cap H_2$. Let V be a kH-module and assume that*

$$H^1(H_1,V) = H^1(H_2,V) = 0.$$

Then $\dim H^1(H,V) \le \dim V^A$.

Proof. Let $D := \mathrm{Der}(H,V)$ and consider the restriction map

$$\pi : D \to \mathrm{Der}(H_1,V) \times \mathrm{Der}(H_2,V).$$

Since the H_i generate H, π is injective. For $\delta \in D$, let δ_i be the image of δ in $\mathrm{Der}(H_i,V)$ with $i = 1,2$. By assumption, δ_i is the inner derivation $\delta(v_i)$ corresponding to some

$v_i \in V$. Since $\delta_1 - \delta_2$ vanishes on A, we see that $\dim \pi(D) \le \dim V + \dim V^A$, whence the result. □

This has the following corollary.

Corollary 4.3. *Assume that* $p \nmid |G_i|$ *for all* i. *If* V *is any* kG*-module, then* $\dim H^1(G,V) \le \dim V^B$. *If* V *is irreducible, then* $\dim H^1(G,V) < |W|^{1/2}$.

Proof. Split the set Δ of positive simple roots into two subsets Δ_j, $j = 1, 2$, so that the root subgroups in each subset commute (this is easy to do). Let H_j be the subgroup of G generated by B and the roots subgroups corresponding to $\pm\Delta_j$ for $j = 1, 2$. The construction of Δ_j and the assumption that $p \nmid |G_i|$ for all i imply that $p \nmid |H_j|$. In particular, $H^1(H_j, V) = 0$. Clearly, $G = \langle H_1, H_2 \rangle$ as it contains all (positive simple) root subgroups, and $H_1 \cap H_2 = B$. Now the first statement follows by applying Lemma 4.2. The second statement also follows, since $\dim V^B < |W|^{1/2}$ as noted above. □

We generalize the previous results.

Lemma 4.4. *Let* H *be a finite group. Assume that* H *is generated by subgroups* H_1 *and* H_2 *and set* $A := H_1 \cap H_2$. *Let* V *be a* kH*-module. Assume that the restriction maps from* $H^1(H_i, V)$ *to* $H^1(H, V)$ *are injective. Then* $\dim H^1(H, V) \le \dim H^1(A, V) + \dim V^A$.

Proof. This is very similar to the proof of Lemma 4.2. Let $\delta \in \mathrm{Der}(H, V)$ and let δ_i be the restriction of δ to H_i.

Consider the restriction map π_1 from $\mathrm{Der}(H,V) \to \mathrm{Der}(H_1,V)$. Since G is generated by H_1 and H_2, $\mathrm{Ker}(\pi_1)$ embeds into $\mathrm{Der}_H(H_2, V)$, the space of the derivations on H_2 that are 0 on A. Now, if $\delta \in \mathrm{Der}_A(H_2, V)$, then δ is inner on H and so also on H_2 (since the restriction map is injective on H^1). Thus, $\mathrm{Der}_A(H_2, V)$ can be identified with V^A, and the result follows. □

This gives:

Corollary 4.5. *Let* V *be a* kG*-module. Assume that* $p \nmid |G_i : B|$ *for all* $1 \le i \le e$. *If* V *is any* kG*-module, then the following statements hold.*

(i) $\dim H^1(G,V) \le \dim H^1(B,V) + \dim V^B$.
(ii) *If* V *is irreducible, then* $\dim H^1(G,V) \le (e+1)\dim V^B < (e+1)|W|^{1/2}$.
(iii) *If* V *is any* kG*-module, then* $\dim H^1(G,V) \le (e+1)\dim V^Q$.
(iv) *If* V *is a submodule of* k_B^G, *then* $\dim H^1(G,V) \le (e+1)\dim V^B$.

Proof. Let H_1 and H_2 be constructed as in the proof of Corollary 4.3. Then the H_j, $j = 1, 2$, are parabolic subgroups and $p \nmid [H_j : B]$. Now apply Lemma 4.4 to see that (i) holds.

We next prove (ii) and so assume that V is irreducible. If $V^B = 0$, then $H^1(G,V) = 0$ by [GT, Theorem 6.1], and we are done. So we may assume that $V^B \ne 0$.

Setting $R := \mathbf{O}_{p'}(B)$, we have $V = [R,V] \oplus V^R$ and $H^1(B,[R,V]) = 0$. Also, $V^R = V^B$ by [GT, Proposition 3.1], and so $H^1(B,V) = H^1(B,V^R) = H^1(B,V^B)$. As B/R has rank $\le e$ as an abelian group, $\dim H^1(B,V^B) \le e \dim V^B$ (and is in fact 0 if $p \nmid |B|$). As noted previously, $\dim V^B < |W|^{1/2}$, whence (ii) follows.

Now (iii) follows from (ii) by the long exact sequence in cohomology since $\dim V^Q$ is additive over composition factors. Finally (iv) follows from (iii), since $\dim V^B = \dim V^Q$ for any submodule V of k_B^G by [GT, Proposition 3.1(ii)]. □

So we have obtained an upper bound of the magnitude of $|W|^{1/2}$ unless p divides $[G_i : B]$ for some G_i. There cannot be a result in general that bounds $\dim H^1(G,V)$ in terms of

$\dim V^B$ since there are (albeit very few) examples with $V^B = 0$ and $\dim H^1(G, V) = 1$, see [GT, §6].

We will give another bound in all cases, using the property that

$$\dim H^1(G, V) = \dim H^1(G, V^*) \tag{4.1}$$

for any irreducible kG-module V *with* $V^B \neq 0$ (in cross characteristic, and G is a finite simple group of Lie type as before). For classical groups, any irreducible module V is quasi-equivalent to its dual [DGPS, 2.1, 2.4] and [TZ] (i.e. V^* is a twist of V by an automorphism) whence (4.1) holds (without the extra assumption that $V^B \neq 0$). The equality (4.1) for exceptional groups of Lie type follows from the following results about the socle and the head of indecomposable summands of $M = k_B^G$.

Lemma 4.6. *Let $M = k_B^G$.*

(i) *M is a direct sum of indecomposable modules with simple socle and simple head which are isomorphic.*
(ii) *If Y is an indecomposable summand of M, then the isomorphism class of Y is determined by its socle (or head).*
(iii) *If Y is an indecomposable summand of M, then its socle and head are self-dual kG-modules.*

Proof. (i) and (ii) follow by [CE, 1.20, 1.25, 1.28]. Next we prove (iii). Let Y be an indecomposable summand of M with simple socle S. We claim that $Y \cong Y^*$. If we prove this, then by (i) and (ii), $S \cong S^*$ and the result follows. Note that since M is self-dual, Y^* is a summand of M as well.

Let $E := \mathrm{End}_G(M)$ and let X be the projective indecomposable E-module given by the Morita correspondence (i.e. $X = \mathrm{Hom}_G(Y, M)$ and X^* the corresponding projective module). Let $\mathcal{O}$ be a discrete valuation ring in characteristic 0 with residue field contained in k and let K be the quotient field of $\mathcal{O}$. Let $M' := \mathrm{Ind}_B^G(\mathcal{O})$ be the corresponding induced module over $\mathcal{O}$ and let $E' := \mathrm{End}_G(M')$ be the corresponding endomorphism ring. Then there is a bijection between the indecomposable projective summands of E and E'.

Then $L := \mathrm{End}_{KG}(\mathrm{Ind}_B^G(K)) \cong K \otimes E'$ is a Hecke algebra and by a result of Lusztig (see [GP, 8.4.7, 9.3.9]) is split semisimple over $\mathbb{Q}[q^{1/2}, q^{-1/2}]$, whence all its projective modules are defined over $\mathbb{R}$. In particular, we can take $\mathcal{O}$ to be contained in $\mathbb{R}$. Thus, the projective indecomposable summands of E' are defined over $\mathcal{O}$ and so by (i) and (ii), the same is true for the socle (and head) of each summand. It follows by the Morita correspondence that the Brauer character of every indecomposable summands of M is real, whence (iii) follows. □

Note that if $p \nmid |B|$, then M is projective and the first two statements of Lemma 4.6 hold trivially (since they hold for any projective indecomposable kG-module). The self-duality statement 4.6(iii) critically requires Lusztig's result. For G classical, one has a much easier proof of the fact that every simple kG-module is quasiequivalent to its dual (i.e. equivalent via an automorphism) and so the cohomology groups are isomorphic (which is the result we use below).

Corollary 4.7. *Let V be an irreducible kG-module with $V^B \neq 0$.*

(i) *Then V has multiplicity $\dim V^B$ in the socle of k_B^G, and $V \cong V^*$.*
(ii) *Let V_2 be an irreducible kG-module with $V_2^B \neq 0$ and $V_2 \not\cong V$. Suppose that L is a kG-module with $\mathrm{soc}(L) = V_1 \cong V$ and $L/V_1 \cong V_2^{\oplus m}$ for some $m \geq 0$, and that $L \cong V_1 \oplus V_2^{\oplus m}$ as B-module. Let X_i denote an indecomposable direct summand of k_B^G with socle V_i for $i = 1, 2$. Then L embeds in X_1.*

Proof. (i) As previously noted, the first statement follows by Frobenius reciprocity. Next, V embeds in an indecomposable summand Y of k_B^G. By Lemma 4.6(i) and (iii), we now have $V \cong \mathrm{soc}(Y)$ and $V \cong V^*$.

(ii) Let $f_i := \dim V_i^B > 0$ for $i = 1, 2$. Then

$$\dim \mathrm{Hom}_G(L, k_B^G) = \dim \mathrm{Hom}_B(L, k) = f_1 + mf_2,$$

whereas

$$\dim \mathrm{Hom}_G(L/V_1, k_B^G) = \dim \mathrm{Hom}_B(L/V_1, k) = mf_2.$$

Now we apply Lemma 4.6 to decompose k_B^G into a direct sum of its indecomposable direct summands. Let Y be such a summand with $\mathrm{soc}(Y) = W$. Note that $\mathrm{Hom}_G(L, Y) = 0$ if $W \ncong V_1, V_2$. (Otherwise a nonzero quotient L' of L embeds in Y, and so either V_1 or V_2 embeds in $\mathrm{soc}(Y) = W$.) A similar argument shows that

$$\dim \mathrm{Hom}_G(L, X_2) = \dim \mathrm{Hom}_G(L/V_1, X_2) = m \cdot \dim \mathrm{Hom}_G(V_2, X_2) = m.$$

As X_i has multiplicity f_i in k_B^G, it follows that

$$\begin{aligned} f_1 + mf_2 = \dim \mathrm{Hom}_G(L, k_B^G) &= f_1 \cdot \dim \mathrm{Hom}_G(L, X_1) + f_2 \cdot \dim \mathrm{Hom}_G(L, X_2) \\ &= f_1 \cdot \dim \mathrm{Hom}_G(L, X_1) + mf_2, \end{aligned}$$

and so $\dim \mathrm{Hom}_G(L, X_1) = 1$. Also note that $\dim \mathrm{Hom}_G(L/V_1, X_1) = 0$ as $\mathrm{soc}(X_1) \ncong V_2$. Hence L embeds in X_1, as stated. □

Next we need to relate $\dim H^1(G, V)$ with the multiplicity of V not in the socle of $M = k_B^G$ but in $\mathrm{soc}(M/k)$. Note that if p does not divide $|B|$, the projective cover $P(k)$ of the trivial module is a direct summand of k_B^G and so the multiplicity of any irreducible module V in $P(k)/k$ is precisely $\dim \mathrm{Ext}_G^1(V, k) = \dim H^1(G, V^*)$.

We start by computing a related quantity.

Lemma 4.8. *Let V be an irreducible kG-module with $\dim V^B = f > 0$ and $V \ncong k$, and set $h := \dim H^1(G, V)$. Let a be the dimension of the image of $\mathrm{Res}_B^G : H^1(G, V) \to H^1(B, V)$ in $H^1(B, V)$. Let X be an indecomposable summand of k_B^G with socle V, and let J be the indecomposable summand of k_B^G with trivial socle.*

(i) $\dim(X/V)^G = h - a$ *and* $a \leq e/f$.
(ii) *The image of* $\mathrm{Res}_B^G : \mathrm{Ext}_G^1(V, k) \to \mathrm{Ext}_B^1(V, k)$ *has dimension a in* $\mathrm{Ext}_B^1(V, k)$.
(iii) *There exists a submodule N of J with N/k a direct sum of $h - a$ copies of V.*

Proof. (a) Let $D := \mathrm{Der}(G, V)$. Then D is the (unique) module with socle V and trivial head of dimension h. By the definition of a, there is a subspace D_0 with $V \subseteq D_0 \subseteq D$, $\dim D/D_0 = a$, such that $D_0 \cong V \oplus k^{\oplus(h-a)}$ as B-modules. Of course V is still the socle of D_0. By Corollary 4.7(ii), D_0 embeds in X. We may then identify D_0 with a submodule of X, and $\mathrm{soc}(D_0)$ with $\mathrm{soc}(X) = V$, and then have

$$\dim(X/V)^G \geq \dim(D_0/V)^G = h - a.$$

Conversely, if $\dim(X/V)^G = h - b$, then there exists a submodule $Y \subseteq X$ with socle V and $Y/V \cong k^{\oplus(h-b)}$. Since $V, Y \subseteq X \subseteq k_B^G$, we know by [GT, Proposition 3.1(ii)] that

$$\dim Y^B = \dim Y^Q = (h - b) + \dim V^Q = (h - b) + \dim V^B = (h - b) + f.$$

Next, the B-module Y decomposes as $[Y, Q] \oplus Y^Q$, and likewise $V = [V, Q] \oplus V^Q$. Counting the dimensions, we see that $[V, Q] = [Y, Q]$, and thus

$$Y = [V, Q] \oplus Y^Q,$$

as B-module, with B acting trivially on $Y^Q \supseteq V^Q$. We have therefore shown that Y splits as $V \oplus k^{\oplus(h-b)}$ as a B-module. Since Y embeds in D, this implies by the definition of a that $h - b \leq h - a$, whence $\dim(X/V)^G = h - a$ as stated.

By Corollary 4.7(i), V occurs in the socle of k_B^G with multiplicity f. It follows by Lemma 4.6 that X occurs as a direct summand of k_B^G with multiplicity f. Hence,

$$f \cdot \dim H^1(G, X) \leq \dim H^1(G, k_B^G) = \dim H^1(B, k) \leq e,$$

the latter inequality because the abelian group B/Q has rank $\leq e$. We have shown that

$$\dim H^1(G, X) \leq e/f. \tag{4.2}$$

(b) We again look at the above constructed submodule D_0 of X, and consider the short exact sequence

$$0 \to D_0 \to X \to X/D_0 \to 0.$$

This gives rise to the sequence

$$0 \to H^0(G, X/D_0) \to H^1(G, D_0) \to H^1(G, X). \tag{4.3}$$

Recall that $D_0/V \cong k^{\oplus(h-a)}$ and $\dim(X/V)^G = h - a$ as shown in (a). Together with $H^1(G, k) = 0$, this implies that $H^0(G, X/D_0) = 0$. Using (4.2) and (4.3), we now see that

$$\dim H^1(G, D_0) \leq \dim H^1(G, X) \leq e/f. \tag{4.4}$$

We now claim

$$\dim H^1(G, D_0) = a \tag{4.5}$$

Consider the short exact sequence $0 \to V \to D_0 \to k^{\oplus(h-a)} \to 0$. Thus, we have

$$0 = H^0(G, D_0) \to H^0(G, k^{\oplus(h-a)}) \to H^1(G, V) \to H^1(G, D_0) \to H^1(G, k^{\oplus(h-a)}) = 0,$$

and the claim follows. Thus, $a \leq e/f$ as stated in (i).

(c) Note that the natural isomorphism from $H^1(G, V) = \mathrm{Ext}^1_G(k, V)$ to $\mathrm{Ext}^1_G(V^*, k)$ gives an isomorphism of the subspaces of them which are trivial on B, since if a short exact sequence splits for B, so does its dual. It follows that the image of

$$\mathrm{Res}^G_B : \mathrm{Ext}^1_G(V^*, k) \to \mathrm{Ext}^1_B(V^*, k)$$

has dimension $a = h - \dim(X/V)^G$. Now (ii) follows since $V \cong V^*$ by Corollary 4.7(i).

(d) By (ii), there exists a G-module N with socle k and $N/k \cong V^{\oplus(h-a)}$, such that $N \cong k \oplus V^{\oplus(h-a)}$ as B-module. By Corollary 4.7(ii), N embeds in J. □

Proof of Theorem 1.6. (i) is established in Corollary 4.3. We now prove (iii) and (iv). By [GT, Corollary 6.5], it suffices to consider $V_1, \ldots, V_m$, pairwise non-isomorphic representatives of isomorphism classes of irreducible kG-modules V with $V^B \neq 0$ and $V \not\cong k$. Note that $H^1(G, k) = 0$.

Keep the notation as in Lemma 4.8, but with the index i attached to the objects defined for V_i. So V_i is an irreducible kG-module, with $\dim V_i^B = f_i > 0$, $h_i = \dim H^1(G, V_i)$, J is the indecomposable summand of k_B^G with trivial socle, and N_i is a submodule of J with $N_i/k \cong V_i^{\oplus(h_i - a_i)}$, and $a_i \leq e/f_i$. Working in J/k, we obtain a G-submodule $N \supseteq k = \mathrm{soc}(J)$ with

$$N/k \cong \oplus_{i=1}^m V_i^{\oplus(h_i - a_i)}.$$

Clearly, $\dim H^1(G, N/k) = \sum_i h_i(h_i - a_i)$. By considering

$$0 \to k \to N \to N/k \to 0,$$

we have

$$0 = H^1(G, k) \to H^1(G, N) \to H^1(G, N/k) \to H^2(G, k).$$

Let $\kappa := \dim H^2(G,k)$. Note that $\kappa = 0$ if p does not divide $|B|$. If $p||B|$, then $\kappa \leq 1$ unless $p = 2$ and G is of type D_m with m even in which case $\kappa = 2$. Thus,

$$\dim H^1(G,N) \geq \sum_i h_i(h_i - a_i) - \kappa. \tag{4.6}$$

On the other hand, using

$$0 \to N \to J \to J/N \to 0,$$

we see that

$$\dim H^1(G,N) \leq \dim H^1(G,J) + \dim H^0(G,J/N). \tag{4.7}$$

We do not have a very good control over the last term. But note that

$$\dim H^0(G,J/N) \leq \dim H^0(Q,J/N) = \dim H^0(Q,J) - \dim H^0(Q,N).$$

By [GT, Proposition 3.1(ii)], $H^0(Q,k_B^G) = H^0(B,k_B^G)$ has dimension $|W|$. Let X_i denote an indecomposable summand of k_B^G with socle V_i. We have seen in the proof of Lemma 4.8 that $\dim H^0(Q,X_i) \geq f_i + (h_i - a_i)$ and X_i occurs with multiplicity f_i as a summand of k_B^G. As $V_i \not\cong k$, we get

$$\dim H^0(Q,J) \leq |W| - \sum_i (f_i^2 + f_i(h_i - a_i)).$$

On the other hand, $\dim H^0(Q,N) = 1 + \sum_i f_i(h_i - a_i)$, and so

$$\dim H^0(Q,J/N) \leq |W| - 1 - \sum_i (f_i^2 + 2f_i(h_i - a_i)).$$

Also we have that

$$\dim H^1(G,J) + \sum_i f_i \cdot \dim H^1(G,X_i) \leq \dim H^1(G,k_B^G) = \dim H^1(B,k) \leq e,$$

and $\dim H^1(G,X_i) \geq \dim H^1(G,D_{0,i}) = a_i$ by (4.4) and (4.5). Hence

$$\dim H^1(G,J) \leq e - \sum_i a_i f_i.$$

Putting all this in (4.6) and (4.7) yields:

$$\sum_i h_i(h_i - a_i) \leq (e - \sum_i a_i f_i) + |W| + \kappa - 1 - \sum_i (f_i^2 + 2f_i(h_i - a_i)).$$

Equivalently,

$$\sum_i (h_i + f_i)(h_i + f_i - a_i) \leq |W| + e + \kappa - 1. \tag{4.8}$$

In particular, for any i we have

$$(h_i + f_i)(h_i + f_i - a_i) \leq |W| + e + \kappa - 1.$$

As $a_i \leq e/f_i$ by Lemma 4.8 and $f_i \geq 1$, we obtain for each i that

$$h_i + f_i \leq \frac{a_i}{2} + \sqrt{\frac{a_i^2}{4} + |W| + e + \kappa - 1} \leq \frac{e}{2} + \sqrt{\frac{e^2}{4} + |W| + e + \kappa - 1} < e + |W|^{1/2}.$$

Thus $h_i < e + |W|^{1/2} - 1$, as stated in (iii).

In general,

$$(h_i + f_i)(h_i + f_i - a_i) \geq (h_i + f_i)(h_i + f_i - e/f_i) \geq (h_i + 1)(h_i + 1 - e),$$

and so (4.8) implies (iv).

A much simpler version of the previous proof gives a slightly better bound if $p \nmid |B|$. In that case $H^1(B,V) = 0 = H^1(B,k)$ and $\kappa = 0$. This yields Theorem 1.6(ii):

Theorem 4.9. *If p does not divide $|B|$, then $\dim H^1(G,V) < |W|^{1/2}$.*

Thus we have completed the proof of Theorem 1.6. □

We point out some easy corollaries.

Corollary 4.10. *Let L be a kG-submodule of k_B^G. Then* $\dim H^1(G,L) < |W| + \dim H^1(B,k)$.

Proof. This follows from the long exact sequence in cohomology applied to

$$0 \to L \to k_B^G \to X \to 0.$$

If $L^G \neq 0$, then this gives $\dim H^1(G,L) \leq \dim H^0(G,X) + \dim H^1(G,k_B^G)$, and the result holds since

$$\dim H^0(G,X) \leq \dim H^0(Q,X) < \dim H^0(Q,k_B^G) = |W|.$$

If $L^G = 0$, then replace k_B^G by the sum Z of all indecomposable summands of k_B^G not containing the G-fixed space and argue similarly (noting that $\dim H^1(G,Z) \leq \dim H^1(G,k_B^G) = \dim H^1(B,k)$). □

If we assume that p does not $|G_i|$ for any i, we can get some stronger results.

Corollary 4.11. *Assume that $p \nmid |G_i|$ for all i. Let $L = X/Y$ with $Y < X$ kG-submodules of k_B^G. Then* $\dim H^1(G,L) \leq \dim L^B = \dim X^B - \dim Y^B \leq |W|$. *Moreover,* $\dim H^1(G,X) \leq |W|/2$.

Proof. The first statement follows by Corollary 4.5(i) since $p \nmid |B|$. Let $M = k_B^G$ as above, and suppose that X is a kG-submodule of M. Arguing as in the proof of Corollary 4.10, we see that $\dim H^0(G,M/X) = \dim H^1(G,X)$. We also have that

$$\dim H^1(G,X) \leq \dim X^B = \dim X^Q.$$

Thus,

$$\begin{aligned}\dim H^1(G,X) &= (1/2)(\dim H^1(G,X) + \dim H^0(G,M/X)) \\ &\leq (1/2)(\dim X^Q + \dim(M/X)^Q) \\ &= (1/2)\dim M^Q = |W|/2.\end{aligned}$$

□

This allows us to say something about H^2 (but only for submodules of k_B^G).

Corollary 4.12. *Assume that $p \nmid |G_i|$ for all i. Let L be a kG-submodule of k_B^G. Then* $\dim H^2(G,L) \leq |W| - \dim L^B$.

Proof. Consider the short exact sequence $0 \to L \to k_B^G \to X \to 0$. This yields

$$0 = H^1(G,k_B^G) \to H^1(G,X) \to H^2(G,L) \to H^2(G,k_B^G) = 0.$$

Thus, $H^2(G,L) \cong H^1(G,X)$ and the previous corollary applies. □

We can do a bit better for irreducible kG-modules with nontrivial B-fixed points.

Corollary 4.13. *Assume that $p \nmid |G_i|$ for all i. Let $\mathcal{X}$ be the set of isomorphism classes of the nontrivial irreducible kG-modules and let $f_V := \dim V^G$. Then*

$$\sum_{V\in\mathcal{X}} f_V \cdot \dim H^2(G,V) \leq |W| - \sum_{V\in\mathcal{X}} f_V^2.$$

Proof. Let L be the complement to k in the socle of k_B^G. Then L is the direct sum of f_V copies of each $V \in \mathcal{X}$. Now apply Corollary 4.12, noting that $\dim L^B = \sum_{V\in\mathcal{X}} f_V^2$. □

In particular, this implies that if $f := \dim V^B > 0$ and V is an irreducible kG-module, then $\dim H^2(G, V) < |W|/f$.

One can weaken the assumption that p does not divide $|G_i|$ and obtain some weaker results. Unfortunately, these results do not yield any information about modules with no B-fixed points.

5. An Example

Here we give an easy example showing that one cannot in general bound $\dim \mathrm{Ext}^1_G(V, W)$ for V, W faithful absolutely irreducible G-modules. There are examples known as well using Kazhdan-Lusztig polynomials for G a simple finite group of Lie type and V, W modules in the defining characteristic. There are no such examples known for cross characteristic modules. We give a trivial example for semisimple groups.

Let $G = S_1 \times \ldots \times S_t$ be a direct product of t finite non-abelian simple groups. Let V_i be an absolutely irreducible S_i-module with $\dim \mathrm{Ext}^1_{S_i}(V_i, V_i) = e_i$. Let $V = V_1 \otimes \ldots \otimes V_t$. Then by the Künneth formula, we see that $\dim \mathrm{Ext}^1_G(V, V) = \sum_{i=1}^t e_i$. Since there are examples with $e_i > 0$, we see that $\dim \mathrm{Ext}^1_G(V, V)$ can grow arbitrarily large with t (but if the sectional rank of the Sylow p-groups is bounded, then so is t).

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