UNITARY t-GROUPS

EIICHI BANNAI, GABRIEL NAVARRO, NOELIA RIZO, AND PHAM HUU TIEP

Abstract. Relying on the main results of [GT], we classify all unitary t-groups for $t \geq 2$ in any dimension $d \geq 2$. We also show that there is essentially a unique unitary 4-group, which is also a unitary 5-group, but not a unitary t-group for any $t \geq 6$.

1. Introduction

Unitary t-designs have recently attracted a lot of interest in quantum information theory. The concept of unitary t-design was first conceived in physics community as a finite set that approximates the unitary group $\mathrm{U}_d(\mathbb{C})$, like any other design concept. It seems that works of Gross–Audenaert–Eisert [GAE] and Scott [Sc] marked the start of the research on unitary t-designs. Roy–Scott [RS] gives a comprehensive study of unitary t-designs from a mathematical viewpoint.

It is known that unitary t-designs in $\mathrm{U}_d(\mathbb{C})$ always exist for any t and d, but explicit constructions are not so easy in general. A special interesting case is the case where a unitary t-design itself forms a *group*. Such a finite group in $\mathrm{U}_d(\mathbb{C})$ is called a *unitary t-group*. Some examples of unitary 5-groups are known in $\mathrm{U}_2(\mathbb{C})$. For $d \geq 3$, some unitary 3-groups have been known in $\mathrm{U}_d(\mathbb{C})$. But no example of unitary 4-groups in dimensions $d \geq 3$ was known. It seems that the difficulty of finding 4-groups in $\mathrm{U}_d(\mathbb{C})$ for $d \geq 3$ has been noticed by many researchers (see e.g. Section 1.2 of [ZKGG]). The purpose of this paper is to clarify this situation. Namely, we point out that this problem in dimensions ≥ 4 is essentially solved in the context of finite group theory by Guralnick–Tiep [GT]. We also show that the classification of unitary 2-groups in $\mathrm{U}_d(\mathbb{C})$ for $d \geq 5$ is derived from [GT] as well. Building on this, we provide a complete description of unitary t-groups in $\mathrm{U}_d(\mathbb{C})$ for all $t, d \geq 2$.

2. Unitary t-groups in dimension $d \geq 5$

We now recall the notion of unitary t-groups, following [RS, Corollary 8]. Let $V = \mathbb{C}^d$ be endowed with standard Hermitian form and let $\mathcal{H} = \mathrm{U}(V) = \mathrm{U}_d(\mathbb{C})$

2010 *Mathematics Subject Classification.* 05B30, 20C15, 81P45.

Key words and phrases. Unitary t-designs, Unitary t-groups.

The research of the second and third authors is partially supported by the Spanish Ministerio de Educación y Ciencia proyecto MTM2016-76196-P and Prometeo Generalitat Valenciana. The fourth author gratefully acknowledges the support of the NSF (grant DMS-1840702) and the Joshua Barlaz Chair in Mathematics. The paper is partially based upon work supported by the NSF under grant DMS-1440140 while the second, third, and fourth authors were in residence at MSRI (Berkeley, CA), during the Spring 2018 semester. It is a pleasure to thank the Institute for the hospitality and support.

denote the corresponding unitary group. Then a finite subgroup $G < \mathcal{H}$ is called a *unitary t-group* for some integer $t \geq 1$, if

$$\frac{1}{|G|}\sum_{g\in G}|\mathrm{tr}(g)|^{2t} = \int_{X\in\mathcal{H}}|\mathrm{tr}(X)|^{2t}dX. \tag{1}$$

Note that the right-hand-side in (1) is exactly the $2t$-*moment* $M_{2t}(\mathcal{H}, V)$ as defined in [GT], whereas the left-hand-side is the $2t$-moment $M_{2t}(G, V)$. Recall, see e.g. [FH, §26.1], that the complex irreducible representations of the real Lie algebra $\mathfrak{su}_d$ and the complex Lie algebra $\mathfrak{sl}_d$ are the same. It follows that $M_{2t}(\mathcal{H}, V) = M_{2t}(\mathcal{G}, V)$ for $\mathcal{G} = \mathrm{GL}(V)$. Given these basic observations, we can recast the main results of [GT] in the finite setting as follows.

Theorem 1. *Let $V = \mathbb{C}^d$ with $d \geq 5$ and $\mathcal{G} = \mathrm{GL}(V)$. Assume that $G < \mathcal{G}$ is a finite subgroup. Then $M_8(G, V) > M_8(\mathcal{G}, V)$. In particular, if $d \geq 5$ and $t \geq 4$, then there does not exist any unitary t-group in $\mathrm{U}_d(\mathbb{C})$.*

Proof. The first statement is precisely [GT, Theorem 1.4]. The second statement then follows from the first and [GT, Lemma 3.1]. □

We note that [GT, Theorem 1.4] also considers any Zariski closed subgroups G of $\mathcal{G}$ with the connected component G° being reductive. Then the only extra possibility with $M_8(G, V) = M_8(\mathcal{G}, V)$ is when $G \geq [\mathcal{G}, \mathcal{G}] = \mathrm{SL}(V)$. In fact, [GT] also considers the problem in the modular setting.

Combined with Theorem 10 (below), Theorem 1 yields the following consequence, which gives the complete classification of unitary t-groups for any $t \geq 4$:

Corollary 2. *Let $G < \mathrm{U}_d(\mathbb{C})$ be a finite group and $d \geq 2$. Then G is a unitary t-group for some $t \geq 4$ if and only if $d = 2$, $t = 4$ or 5, and $G = \mathbf{Z}(G)\mathrm{SL}_2(5)$.*

Next, we obtain the following consequences of [GT, Theorems 1.5, 1.6], where $F^*(G) = F(G)E(G)$ denotes the generalized Fitting subgroup of any finite group G (respectively, $F(G)$ is the Fitting subgroup and $E(G)$ is the layer of G); furthermore, we follow the notation of [Atlas] for various simple groups. If G is a finite group and V is a $\mathbb{C}G$-module, then $V\downarrow_H$ denotes the restriction of V to a subgroup $H \leq G$. We also refer the reader to [GMST] and [TZ2] for the definition and basic properties of *Weil representations* of (certain) finite classical groups.

Theorem 3. *Let $V = \mathbb{C}^d$ with $d \geq 5$ and let $\mathcal{G} = \mathrm{GL}(V)$. For any finite subgroup $G < \mathcal{G}$, set $\bar{S} = S/\mathbf{Z}(S)$ for $S := F^*(G)$. Then $M_4(G, V) = M_4(\mathcal{G}, V)$ if and only if one of the following conditions holds.*

(i) (Lie-type case) *One of the following holds.*
  (a) *$\bar{S} = \mathrm{PSp}_{2n}(3)$, $n \geq 2$, $G = S$, and $V\downarrow_S$ is a Weil module of dimension $(3^n \pm 1)/2$.*
  (b) *$\bar{S} = \mathrm{U}_n(2)$, $n \geq 4$, $[G : S] = 1$ or 3, and $V\downarrow_S$ is a Weil module of dimension $(2^n - (-1)^n)/3$.*

(ii) (Extraspecial case) *$d = p^a$ for some prime p and $F^*(G) = F(G) = \mathbf{Z}(G)E$, where $E = p_+^{1+2a}$ is an extraspecial p-group of order p^{1+2a} and type $+$. Furthermore, $G/\mathbf{Z}(G)E$ is a subgroup of $\mathrm{Sp}(W) \cong \mathrm{Sp}_{2a}(p)$ that acts transitively on $W \smallsetminus \{0\}$ for $W = E/\mathbf{Z}(E)$, and so is listed in Theorem 5 (below). If $p > 2$ then $E \lhd G$; if $p = 2$ then $F^*(G)$ contains a normal subgroup $E_1 \lhd G$,* *where $E_1 = C_4 * E$ is a central product of order 2^{2a+2} of $\mathbf{Z}(E_1) = C_4 \le \mathbf{Z}(G)$ with E.*

(iii) (Exceptional cases) *$S = \mathbf{Z}(G)[G^*, G^*]$, and $(\dim(V), \bar{S}, G^*)$ is as listed in Table* I. *Furthermore, in all but lines* 2–6 *of Table* I, *$G = \mathbf{Z}(G)G^*$. In lines* 2–6, *either $G = S$ or $[G : S] = 2$ and G induces on $\bar{S}$ the outer automorphism listed in the fourth column of the table.*

In particular, $G < \mathcal{H} = \mathrm{U}(V)$ is a unitary 2-group if and only if G is as described in (i)–(iii).

TABLE I. Exceptional examples in $\mathcal{G} = \mathrm{GL}_d(\mathbb{C})$ with $d \ge 5$

d	$\bar{S}$	G^*	Outer	The largest $2k$ with $M_{2k}(G,V) = M_{2k}(\mathcal{G},V)$	$M_{2k+2}(G,V)$ vs. $M_{2k+2}(\mathcal{G},V)$
6	A_7	$6\mathsf{A}_7$		4	21 vs. 6
6	$\mathrm{L}_3(4)^{(\star)}$	$6\mathrm{L}_3(4) \cdot 2_1$	2_1	6	56 vs. 24
6	$\mathrm{U}_4(3)^{(\star)}$	$6_1 \cdot \mathrm{U}_4(3)$	2_2	6	25 vs. 24
8	$\mathrm{L}_3(4)$	$4_1 \cdot \mathrm{L}_3(4)$	2_3	4	17 vs. 6
10	M_{12}	$2M_{12}$	2	4	15 vs. 6
10	M_{22}	$2M_{22}$	2	4	7 vs. 6
12	$Suz^{(\star)}$	$6Suz$		6	25 vs. 24
14	${}^2B_2(8)$	${}^2B_2(8) \cdot 3$		4	90 vs. 6
18	$J_3^{(\star)}$	$3J_3$		6	238 vs. 24
26	${}^2F_4(2)'$	${}^2F_4(2)'$		4	26 vs. 6
28	Ru	$2Ru$		4	7 vs. 6
45	M_{23}	M_{23}		4	817 vs. 6
45	M_{24}	M_{24}		4	42 vs. 6
342	$O'N$	$3O'N$		4	3480 vs. 6
1333	J_4	J_4		4	8 vs. 6

Note that in Table I, the data in the sixth column is given when we take $G = G^*$.

Proof. We apply [GT, Theorem 1.5] to $(G, \mathcal{G})$. Then case (A) of the theorem is impossible as G is finite, and case (D) leads to case (iii) as $\mathcal{G} = \mathrm{GL}(V)$.

In case (B) of [GT, Theorem 1.5], we have that $\bar{S} = \mathrm{PSp}_{2n}(q)$ with $n \ge 2$ and $q = 3, 5$, or $\bar{S} = \mathrm{PSU}_n(2)$ with $n \ge 4$, and $V\downarrow_S$ is irreducible. It is easy to see that the latter condition implies that G/S has order 1 or 3. Next, $L = E(G)$ is a quotient of $\mathrm{Sp}_{2n}(q)$ or $\mathrm{SU}_n(2)$ by a central subgroup, and $S = \mathbf{Z}(S)L$. Let χ denote the character of the G-module V. As $d > 4$, the condition $M_4(G,V) = M_4(\mathcal{G},V)$ is equivalent to that G act irreducibly on both $\mathrm{Sym}^2(V)$ and $\wedge^2(\chi)$ (see the discussion in [GT, §2]). Hence, if $\chi\downarrow_L$ is real-valued, then either $\mathrm{Sym}^2(\chi\downarrow_L)$ or $\wedge^2(\chi\downarrow_L)$ contains 1_L, whence either $\mathrm{Sym}^2(\chi\downarrow_S)$ or $\wedge^2(\chi\downarrow_S)$ contains a linear character. But both $\mathrm{Sym}^2(V)$ and $\wedge^2(V)$ have dimension at least $d(d-1)/2 \ge 10$ and $[G : S] \le 3$, so G cannot act irreducibly on them, a contradiction. We have shown that $\chi\downarrow_L$ is not real-valued. Now using Theorems 4.1 and 5.2 of [TZ1], we can rule out the case $\bar{S} = \mathrm{PSp}_{2n}(5)$ and the case $(\bar{S}, \dim(V)) = (\mathrm{PSU}_n(2), (2^n + 2(-1)^n)/3)$, as $\chi\downarrow_L$ is real-valued in those cases.

Case (C), together with [GT, Lemma 5.1], leads to case (ii) listed above, except for the explicit description of E and E_1. Suppose $p > 2$. Then at least one element in $E \smallsetminus \mathbf{Z}(E)$ has order p, whence all elements in $E \smallsetminus \mathbf{Z}(E)$ have order p by the transitivity of $G/\mathbf{Z}(G)E$ on $W \smallsetminus \{0\}$, i.e. E has type $+$. Also, note that E is generated by all elements of order p in $\mathbf{Z}(G)E$, and so $E \lhd G$. Next suppose that $p = 2$ and let $E_1 \lhd G$ be generated by all elements of order at most 4 in $\mathbf{Z}(G)E$. If $|\mathbf{Z}(G)| < 4$, then $F^*(G) = E_1 = E$ is an extraspecial 2-group of order 2^{1+2a} of type ϵ for some $\epsilon = \pm$. In this case, $G/\mathbf{Z}(G)E \hookrightarrow O^{\epsilon}_{2a}(2)$ and so cannot be transitive on $W \smallsetminus \{0\}$ (as $a \geq 2$), a contradiction. So $|\mathbf{Z}(G)| \geq 4$. In this case, one can show that $E_1 = C_4 * E$ with $\mathbf{Z}(E) < C_4 \leq \mathbf{Z}(G)$, and since $C_4 * 2^{1+2a}_+ \cong C_4 * 2^{1+2a}_-$, we may choose E to have type $+$. □

We note that the case of Theorem 3 where G is almost quasisimple was also treated in [M]. More generally, the classification of subgroups of a classical group $\mathsf{Cl}(V)$ in characteristic p that act irreducibly on the heart of the tensor square, symmetric square, or alternating square of $V \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$, is of particular importance to the Aschbacher-Scott program [A] of classifying maximal groups of finite classical groups. See [Mag], [MM], [MMT] for results on this problem in the modular case.

Theorem 4. *Let $V = \mathbb{C}^d$ with $d \geq 5$ and let $\mathcal{G} = \mathrm{GL}(V)$. Assume G is a finite subgroup of $\mathcal{G}$. Then $M_6(G, V) = M_6(\mathcal{G}, V)$ if and only if one of the following two conditions holds.*

(i) (Extraspecial case) *$d = 2^a$ for some $a > 2$, and $G = \mathbf{Z}(G)E_1 \cdot Sp_{2a}(2)$, where $E \cong 2^{1+2a}_+$ is extraspecial and of type $+$ and $E_1 = C_4 * E$ with $C_4 \leq \mathbf{Z}(G)$.*

(ii) (Exceptional cases) *Let $\bar{S} = S/\mathbf{Z}(S)$ for $S = F^*(G)$. Then*
$$\bar{S} \in \{\mathrm{L}_3(4), \mathrm{U}_4(3), Suz, J_3\},$$
and $(\dim(V), \bar{S}, G^*)$ is as listed in the lines marked by $^{(\star)}$ in Table I. Furthermore, either $G = \mathbf{Z}(G)G^*$, or $\bar{S} = \mathrm{U}_4(3)$ and $S = \mathbf{Z}(G)G^*$.

In particular, $G < \mathcal{H} = \mathrm{U}(V)$ is a unitary 3-group if and only if G is as described in (i), (ii).

Proof. Apply [GT, Theorem 1.6] and also Theorem 3(ii) to $(G, \mathcal{G})$. □

The transitive subgroups of $\mathrm{GL}_n(p)$ are determined by Hering's theorem [He] (see also [L, Appendix 1]), which however is not easy to use in the solvable case. For the complete determination of unitary 2-groups in Theorem 3(ii), we give a complete classification of such groups in the symplectic case that is needed for us. The notations such as `SmallGroup`$(48, 28)$ are taken from the `SmallGroups` library in [GAP].

Theorem 5. *Let p be a prime and let $W = \mathbb{F}_p^{2n}$ be endowed with a non-degenerate symplectic form. Assume that a subgroup $H \leq \mathrm{Sp}(W)$ acts transitively on $W \smallsetminus \{0\}$. Then $(H, p, 2n)$ is as in one of the following cases.*

(A) (Infinite classes)*:*
  (i) *$n = bs$ for some integers $b, s \geq 1$, and $\mathrm{Sp}_{2b}(p^s)' \lhd H \leq \mathrm{Sp}_{2b}(p^s) \rtimes C_s$.*
  (ii) *$p = 2$, $n = 3s$ for some integer $s \geq 2$; and $G_2(2^s) \lhd H \leq G_2(2^s) \rtimes C_s$.*

(B) (Small cases)*:*
  (i) *$(2n, p) = (2, 3)$, and $H = Q_8$.*

(ii) $(2n, p) = (2, 5)$, *and* $H = \mathrm{SL}_2(3)$.
(iii) $(2n, p) = (2, 7)$, *and* $H = \mathrm{SL}_2(3).C_2 = \texttt{SmallGroup}(48, 28)$.
(iv) $(2n, p) = (2, 11)$, *and* $H = \mathrm{SL}_2(5)$.
(v) $(2n, p) = (4, 3)$, *and* $H = \texttt{SmallGroup}(160, 199)$, $\texttt{SmallGroup}(320, 1581)$, $2.\mathsf{S}_5$, $\mathrm{SL}_2(9)$, $\mathrm{SL}_2(9) \rtimes C_2 = \texttt{SmallGroup}(1440, 4591)$, *or* $C_2.((C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{A}_5) = \texttt{SmallGroup}(1920, 241003)$.
(vi) $(2n, p) = (6, 2)$, *and* $H = \mathrm{SL}_2(8)$, $\mathrm{SL}_2(8) \rtimes C_3$, $\mathrm{SU}_3(3)$, $\mathrm{SU}_3(3) \rtimes C_2$.
(vii) $(2n, p) = (6, 3)$ *and* $H = \mathrm{SL}_2(13)$.

Proof. We may assume that $(2n, p)$ is not in one of the small cases listed in (B), which are computed using [GAP]. We have that $[H : \mathbf{C}_H(v)] = p^{2n} - 1$, for every $v \in W \smallsetminus \{0\}$. Now we apply Hering's theorem, as given in [L, Appendix 1] and analyze possible classes for H.

(a) Suppose that $H \leq \Gamma\mathrm{L}_1(p^{2n})$, which is the semidirect product of Γ_0 (the multiplicative field of $\mathbb{F}_{p^{2n}}$) and the Galois automorphism σ of order $2n$. If $n = 1$, then $H \leq \mathrm{SL}_2(p)$, which has order $p(p-1)(p+1)$, and we may assume that $p \geq 13$. As the smallest index of proper subgroups of $\mathrm{SL}_2(p)$ is $p+1$ (see e.g. [TZ1, Table VI]), we conclude that $H = \mathrm{SL}_2(p)$. So we may assume that $n > 1$. We may also assume that $(2n, p) \neq (2, 6)$. Hence, we can consider a Zsigmondy (odd) prime divisor r of $p^{2n} - 1$ [Zs], and have that the order of p mod r is $2n$. Thus $2n$ divides $r - 1$. Let $C = H \cap \Gamma_0$. Note that r divides $|C|$ (because r does not divide $2n$), and hence C acts irreducibly on W. Since $C < \mathrm{Sp}(W)$, by [Hu, Satz II.9.23] we have that $|C|$ divides $p^n + 1$. Hence, $|H|$ divides $2n(p^n + 1)$, and thus $p^n - 1$ divides $2n$. This is not possible.

(b) Aside from the possibilities listed in (A) and (B), we need only consider the possibility $2n = as$ with $a \geq 3$, $p^n \neq 2^2, 3^2, 2^3, 3^3$, and $H \rhd \mathrm{SL}_a(p^s)$. Let $\mathfrak{d}(X)$ denote the smallest degree of faithful complex representations of a finite group X. Since $H \leq \mathrm{Sp}_{2n}(p)$, by [TZ1, Theorem 5.2] we have that

$$\mathfrak{d}(X) \leq (p^n + 1)/2 = (p^{as/2} + 1)/2.$$

On the other hand, since $H \rhd \mathrm{SL}_a(p^s)$, by [TZ1, Theorem 3.1] we also have that

$$\mathfrak{d}(X) \geq (p^{as} - p^s)/(p^s - 1) > p^{s(a-1)}.$$

As $a \geq 3$, this is impossible. $\square$

3. An infinite family of "almost" unitary 3-groups in high dimensions

As follows from Theorem 4, the Weil representations $\Phi : G \to \mathrm{GL}(V)$ of dimensions $(3^m \pm 1)/2$ of the symplectic group $\mathrm{Sp}_{2m}(3)$, do not give rise to unitary 3-groups, even though they yield unitary 2-groups (see Theorem 3(i)). However, we record the following result, which shows that the failure is minimal: $M_6(G/\mathrm{Ker}(\Phi), V) = 7$ whereas $M_6(\mathrm{GL}(V), V) = 6$, and thus the Weil representations lead to "almost" unitary 3-groups.

Theorem 6. *Let $m \geq 3$ be an integer, and let $\Phi : G \to \mathrm{GL}(V)$ be an irreducible Weil representation for $G = \mathrm{Sp}_{2m}(3)$ of degree $(3^m \pm 1)/2$. Then $M_6(G/\mathrm{Ker}(\Phi), V) = 7$.*

Proof. Recall, see [GMT, §3], that G has four (distinct) irreducible Weil characters, $\xi, \bar{\xi}$ of degree $(3^m + 1)/2$, and $\eta, \bar{\eta}$ of degree $(3^m - 1)/2$. Now, by [GMT, Theorem

1.3] and its proof,

$$\xi^3 = (\mathrm{Sym}^3(\xi) - \bar{\xi}) + 2\mathsf{S}_{2,1}(\xi) + \wedge^3(\xi) + \bar{\xi}$$

is a decomposition of ξ^3 into irreducible summands, and the listed irreducible summands are pairwise distinct. It follows that $[\xi^3, \xi^3]_G = 7$, and so $M_6(G/\mathrm{Ker}(\Phi), V) = 7$ if Φ affords the character ξ or $\bar{\xi}$. (Here, $\mathsf{S}_{2,1}$ denotes the Schur functor labeled by the partition $(2,1)$ of 3, see [FH, (6.8), (6.9)].) Similarly,

$$\eta^3 = \mathrm{Sym}^3(\eta) + 2\mathsf{S}_{2,1}(\eta) + (\wedge^3(\eta) - \bar{\eta}) + \bar{\eta}$$

is a decomposition of η^3 into irreducible summands, and the listed irreducible summands are pairwise distinct. It follows that $[\eta^3, \eta^3]_G = 7$, and so $M_6(G/\mathrm{Ker}(\Phi), V) = 7$ if Φ affords the character η or $\bar{\eta}$. □

Note that $\mathrm{Ker}(\Phi) = 1$ if $\dim V$ is even, and $\mathrm{Ker}(\Phi) = \mathbf{Z}(G) \cong C_2$ if $\dim V$ is odd.

4. Unitary t-groups in dimensions at most 4

In this section we complete the classification of unitary t-groups in dimension ≤ 4. First we introduce some key groups for this classification, where we use the notation of [GAP] for $\texttt{SmallGroup}(64, 266)$ and $\texttt{PerfectGroup}(23040, 2)$.

Proposition 7. *Consider an irreducible subgroup*

$$E_4 = C_4 * 2_+^{1+4} = \texttt{SmallGroup}(64, 266)$$

of order 2^6 of $\mathrm{GL}(V)$, where $V = \mathbb{C}^4$, and let $\Gamma_4 := \mathbf{N}_{\mathrm{GL}(V)}(E_4)$. Then the following statements hold.

(i) *Γ_4 induces the subgroup $A^+ \cong C_2^4 \cdot \mathsf{S}_6$ of all automorphisms of E_4 that act trivially on $\mathbf{Z}(E_4) = C_4$.*

(ii) *The last term $\Gamma_4^{(\infty)}$ of the derived series of Γ_4 is $L = \texttt{PerfectGroup}(23040, 2)$, a perfect group of order 23040 and of shape $E_4 \cdot \mathsf{A}_6$. Furthermore, $\Gamma_4^{(\infty)}$ is a unitary 3-group.*

Proof. (i) It is well known, see e.g. [Gr, p. 404], that $A^+ \cong \mathrm{Inn}(E_4) \cdot \mathsf{S}_6$ with $\mathrm{Inn}(E_4) \cong C_2^4$. Certainly, $\Gamma_4/\mathbf{C}_{\Gamma_4}(E_4) \hookrightarrow A^+$. Let ψ denote the character of E_4 afforded by V, and note that ψ and $\overline{\psi}$ are the only two irreducible characters of degree 4 of E_4, and they differ by their restrictions to $\mathbf{Z}(E_4)$. Now for any $\alpha \in A^+$, $\psi^\alpha = \psi$. It follows that there is some $g \in \mathrm{GL}(V)$ such that $gxg^{-1} = \alpha(x)$ for all $x \in E_4$; in particular, $g \in \Gamma_4$. We have therefore shown that $\Gamma_4/\mathbf{C}_{\Gamma_4}(E_4) \cong A^+$.

(ii) Using [GAP], one can check that $L := \texttt{PerfectGroup}(23040, 2)$ embeds in $\mathrm{GL}(V)$, with a character say χ, and $F^*(L) \cong E_4$. So without loss we may identify $F^*(L)$ with E_4 and obtain that $L < \Gamma_4$. Again using [GAP] we can check that $[\chi^3, \chi^3]_L = 6 = M_6(\mathrm{GL}(V))$, which means that L is a unitary 3-group. As L is perfect, we have that $L \leq \Gamma_4^{(\infty)}$. Next, L acting on E_4 induces the perfect subgroup $A^{++} \cong C_2^4 \cdot \mathsf{A}_6$ of index 2 in A^+, and the same also holds for $\Gamma_4^{(\infty)}$. Hence, for any $g \in \Gamma_4^{(\infty)}$, we can find $h \in L$ such that the conjugations by g and by h induce the same automorphism of E_4. By Schur's Lemma, $gh^{-1} \in \mathbf{Z}(\Gamma_4)$, whence $\Gamma_4^{(\infty)} \leq \mathbf{Z}(\Gamma_4)L$. Taking the derived subgroup, we see that $\Gamma_4^{(\infty)} \leq L$, and so $\Gamma_4^{(\infty)} = L$, as stated. □

Next, we recall three *complex reflection groups* G_{29}, G_{31}, and G_{32} in dimension 4, namely, the ones listed on lines 29, 31, and 32 of [ST, Table VII]. A direct calculation using the computer packages GAP3 [Mi], [S+], and Chevie [GHMP], shows that each of these 3 groups G, being embedded in $\mathcal{H} = \mathrm{U}_4(\mathbb{C})$, is a unitary 2-group. Also,

$$F(G_{29}) \cong F(G_{31}) \cong \texttt{SmallGroup}(64, 266),\ F(G_{32}) = \mathbf{Z}(G_{32}) \cong C_6,$$

and

$$G_{29}/F(G_{29}) \cong \mathsf{S}_5,\ G_{31}/F(G_{31}) \cong \mathsf{S}_6,\ G_{32} \cong C_3 \times \mathrm{Sp}_4(3).$$

In what follows, we will identify $F(G_{29})$ and $F(G_{31})$ with the subgroup E_4 defined in Proposition 7. Let us denote the derived subgroup of G_k by G'_k for $k \in \{29, 31, 32\}$. With this notation, we can give a complete classification of unitary 2-groups and unitary 3-groups in the following statement.

Theorem 8. *Let $V = \mathbb{C}^4$, $\mathcal{G} = \mathrm{GL}(V)$, and let $G < \mathcal{G}$ be any finite subgroup. Then the following statements hold.*

- (A) *With E_4, Γ_4 and L as defined in Proposition 7, we have that $[\Gamma_4, \Gamma_4] = L = G'_{31}$ and $\Gamma_4 = \mathbf{Z}(\Gamma_4)G_{31}$. Furthermore, $M_4(G, V) = M_4(\mathcal{G}, V)$ if and only if one of the following conditions holds*
  - (A1) *$G = \mathbf{Z}(G)H$, where $H \cong 2\mathsf{A}_7$ or $H \cong \mathrm{Sp}_4(3) \cong G'_{32}$.*
  - (A2) *$L = [G, G] \le G < \Gamma_4$.*
  - (A3) *$E_4 \lhd G < \Gamma_4$, and, after a suitable conjugation in Γ_4,*
  $$G'_{29} = [G, G] \le G \le \mathbf{Z}(\Gamma_4)G_{29}.$$
  In particular, $G < \mathcal{H} = \mathrm{U}(V)$ is a unitary 2-group if and only if G is as described in (A1)–(A3).
- (B) *$M_6(G, V) = M_6(\mathcal{G}, V)$ if and only if G is as described in* (A1)–(A2). *In particular, $G < \mathrm{U}(V)$ is a unitary 3-group if and only if G is as described in* (A1)–(A2).
- (C) *$M_8(G, V) > M_8(\mathcal{G}, V)$. In particular, no finite subgroup of $\mathrm{U}_4(\mathbb{C})$ can be a unitary 4-group.*

Proof. (A) First we assume that $M_4(G, V) = M_4(\mathcal{G}, V)$, and let χ denote the character of G afforded by V. The same proof as of [GT, Theorem 1.5] and Theorem 3 shows that one of the following two possibilities must occur.

• **Almost quasisimple case**: $S \lhd G/\mathbf{Z}(G) \le \mathrm{Aut}(S)$ for some finite non-abelian simple group S. By the results of [M], we have that $S \cong \mathsf{A}_7$ or $\mathrm{PSp}_4(3)$. It is straightforward to check that $E(G) \cong 2\mathsf{A}_7$, respectively $\mathrm{Sp}_4(3)$, and furthermore G cannot induce a nontrivial outer automorphism on S. Recall that in this case we have $F^*(G) = \mathbf{Z}(G)E(G)$ and so $\mathbf{C}_G(E(G)) = \mathbf{C}_G(F^*(G)) = \mathbf{Z}(G)$. It follows that $G = \mathbf{Z}(G)E(G)$, and (A1) holds. Moreover, using [GAP] we can check that $[\alpha^2, \alpha^2] = 2$, $[\alpha^3, \alpha^3] = 6$, but $[\alpha^4, \alpha^4] = 38$, respectively 25, for $\alpha := \chi\downarrow_{E(G)}$. Thus we have checked in the case of (A1) that $M_{2t}(G, V) = M_{2t}(\mathcal{G}, V)$ for $t \le 3$, but $M_8(G, V) > M_8(\mathcal{G}, V)$, since $M_8(\mathcal{G}, V) = 24$ by [GT, Lemma 3.2].

• **Extraspecial case**: $F^*(G) = F(G) = \mathbf{Z}(G)E_4$ and $E_4 \lhd G$, in particular, $G \le \Gamma_4$; furthermore, $G/\mathbf{Z}(G)E_4 \le \mathrm{Sp}(W)$ satisfies conclusion (A)(i) of Theorem 5 for $W = E_4/\mathbf{Z}(E_4) \cong \mathbb{F}_2^4$. Suppose first that $G/\mathbf{Z}(G)E_4 \ge \mathrm{Sp}_4(2)' \cong \mathsf{A}_6$. In this case, G induces (at least) all the automorphisms of E_4 that belong to the subgroup

A^{++} in the proof of Proposition 7. As in that proof, this implies that $\mathbf{Z}(\Gamma_4)G \geq L$. Taking the derived subgroup, we see that

$$[G, G] \geq L, \tag{2}$$

i.e. we are in the case of (A2). Moreover,

$$6 = M_6(\mathcal{G}, V) \leq M_6(G, V) \leq M_6(L, V),$$

and $M_6(L, V) = 6$ as shown above. Hence $M_{2t}(G, V) = M_{2t}(\mathcal{G}, V)$ for $t \leq 3$. Applying (2) to $G = G_{31}$ and recalling that $|L| = |G'_{31}|$, we see that $L = G'_{31}$. Next, G_{31} and Γ_4 induce the same subgroup A^+ of automorphisms of E_4, hence $\Gamma_4 = \mathbf{Z}(\Gamma_4)G_{31}$. Taking the derived subgroup, we obtain that $L = [\Gamma_4, \Gamma_4]$, and so (2) implies that $[G, G] = L$.

Next we consider the case where $G/\mathbf{Z}(G)E_4 = \mathrm{SL}_2(4) \cong \mathsf{A}_5$ or $\mathrm{SL}_2(4) \rtimes C_2 \cong \mathsf{S}_5$. Using [Atlas], it is easy to check that $\mathrm{Sp}(W) \cong \mathsf{S}_6$ has two conjugacy classes $\mathcal{C}_{1,2}$ of (maximal) subgroups that are isomorphic to S_5, and two conjugacy classes $\mathcal{C}'_{1,2}$ of subgroups that are isomorphic to A_5. Any member of one class, say $\mathcal{C}'_1$, is irreducible, but not absolutely irreducible on W, that is, preserves an $\mathbb{F}_4$-structure on W, and is contained in a member of, say $\mathcal{C}_1$. Any member of the other class $\mathcal{C}_2$ is absolutely irreducible on W and preserves a quadratic form Q of type $-$ on W; in particular, it has two orbits of length 5 and 10 on $W \smallsetminus \{0\}$ (corresponding to singular vectors, respectively non-singular vectors, in W with respect to Q), and is contained in a member of $\mathcal{C}_2$. On the other hand, since G is transitive on $W \smallsetminus \{0\}$ by [GT, Lemma 5.1], the last term $G^{(\infty)}$ of the derived series of G must have orbits of only one size on $W \smallsetminus \{0\}$. Applying this analysis to $K := G_{29}$, we see that K/E_4 must belong to $\mathcal{C}_1$ and the derived subgroup of $K/\mathbf{Z}(K)E_4$ as well as $[K, K]/E_4$ belong to $\mathcal{C}'_1$. Hence, after a suitable conjugation in Γ_4, we may assume that

$$G_{29}/E_4 \geq G/\mathbf{Z}(G)E_4 \geq G'_{29}/E_4;$$

in particular, the subgroup of automorphisms of E_4 induced by G is either the one induced by G_{29}, or the one induced by G'_{29}. In either case, we have that

$$G \leq \mathbf{Z}(\Gamma_4)G_{29},\ G'_{29} \leq \mathbf{Z}(\Gamma_4)[G, G].$$

As G'_{29} is perfect, taking the derived subgroup we obtain that $[G, G] = G'_{29}$, i.e. (A3) holds.

(B) We have already mentioned above that $M_6(G, V) = M_6(\mathcal{G}, V)$ for the groups G satisfying (A1) or (A2). By [GT, Lemma 3.1], it remains to show that for the groups G satisfying (A3), $M_6(G, V) \neq M_6(\mathcal{G}, V)$. Assume the contrary: $M_6(G, V) = M_6(\mathcal{G}, V)$. By [GT, Remark 2.3], this equality implies that G is irreducible on all the simple $\mathcal{G}$-submodules of $V \otimes V \otimes V^*$, which can be seen using [Lu, Appendix A.7] to decompose as the direct sum of simple summands of dimension 4 (with multiplicity 2), 20, and 36. Let θ denote the character of G afforded by the simple $\mathcal{G}$-summand of dimension 36. Note that χ vanishes on $F(G) \smallsetminus \mathbf{Z}(G)$ and faithful on $\mathbf{Z}(G)$. It follows that

$$\chi^2\overline{\chi} \downarrow_{F(G)} = 16\chi \downarrow_{F(G)} .$$

As $\chi \downarrow_{F(G)}$ is irreducible, we see that $\theta \downarrow_{F(G)} = 9(\chi \downarrow_{F(G)})$. But $\chi \downarrow_{F(G)}$ obviously extends to $G \rhd F(G)$. It follows by Gallagher's theorem [Is, (6.17)] that $G/F(G)$ admits an irreducible character β of degree 9 (such that $\theta \downarrow_G = (\chi \downarrow_G)\beta$). This is a contradiction, since $G/F(G) \cong \mathsf{A}_5$ or S_5.

(C) Assume the contrary: $M_8(G,V) = M_8(\mathcal{G},V)$. Then $M_6(G,V) = M_6(\mathcal{G},V)$ by [GT, Lemma 3.1]. By (B), we may assume that G satisfies (A1) or (A2). By [GT, Remark 2.3], the equality $M_8(G,V) = M_8(\mathcal{G},V)$ implies that G is irreducible on the simple $\mathcal{G}$-submodule $\mathrm{Sym}^4(V)$ (of dimension 35) of $V^{\otimes 4}$. This in turn implies, for instance by Ito's theorem [Is, (6.15)] that 35 divides $|G/\mathbf{Z}(G)|$. The latter condition rules out (A2) since $|G/\mathbf{Z}(G)|$ divides $2^4 \cdot |\mathrm{Sp}_4(2)|$ in that case. Finally, we already mentioned above that $M_8(G,V) > M_8(\mathcal{G},V)$ in the case of (A1). $\square$

To handle the remaining cases $d = 2, 3$, we first note:

Lemma 9. *Let $\mathcal{G} = \mathrm{SL}(V)$ for $V = \mathbb{C}^2$. Then the following statements hold.*

(i) $M_6(\mathcal{G},V) = 5$, $M_8(\mathcal{G},V) = 14$, *and* $M_{10}(\mathcal{G},V) = 42$.

(ii) *Suppose* $M_{2t}(G,V) = M_{2t}(\mathcal{G},V)$ *for a finite group* $G < \mathcal{G}$. *If* $t \geq 4$ *then* 5 *divides* $|G/\mathbf{Z}(G)|$. *If* $t \geq 6$ *then* 7 *divides* $|G/\mathbf{Z}(G)|$.

(iii) *Suppose* $\mathrm{SL}_2(5) \cong G < \mathcal{G}$. *Then* $M_{2t}(G,V) = M_{2t}(\mathcal{G},V)$ *for* $1 \leq t \leq 5$ *but* $M_{2t}(G,V) > M_{2t}(\mathcal{G},V)$ *for* $t \geq 6$.

Proof. Note that the symmetric powers $\mathrm{Sym}^k(V)$, $k \geq 0$, are pairwise non-isomorphic irreducible $\mathbb{C}\mathcal{G}$-modules, with $\mathrm{Sym}^0(V) \cong \mathbb{C} \cong \wedge^2(V)$, and $V \otimes V \cong \mathrm{Sym}^2(V) \oplus \mathbb{C}$. Now using [FH, Exercise 11.11] we obtain for all $a \geq 1$ that

$$\mathrm{Sym}^a(V) \oplus V \cong \mathrm{Sym}^{a+1}(V) \oplus \mathrm{Sym}^{a-1}(V)$$

as $\mathbb{C}\mathcal{G}$-modules. It follows that

$$\begin{aligned} V^{\otimes 3} &\cong \mathrm{Sym}^3(V) \oplus V^{\oplus 2}, \\ V^{\otimes 4} &\cong \mathrm{Sym}^4(V) \oplus (\mathrm{Sym}^2(V))^{\oplus 3} \oplus \mathbb{C}^{\oplus 2}, \\ V^{\otimes 5} &\cong \mathrm{Sym}^5(V) \oplus (\mathrm{Sym}^3(V))^{\oplus 4} \oplus V^{\oplus 5} \end{aligned}$$

as $\mathbb{C}\mathcal{G}$-modules (with the superscripts indicating the multiplicities), implying (i).

For (ii), note by Remark 2.3 and Lemma 3.1 of [GT] that the assumption implies that G is irreducible on $\mathrm{Sym}^4(V)$ of dimension 5 if $t \geq 4$, and on $\mathrm{Sym}^6(V)$ of dimension 7 if $t \geq 6$.

The first assertion in (iii) can be checked using (i) and [GAP], and the second assertion follows from (ii). $\square$

Now we recall three complex reflection groups $G_4 \cong \mathrm{SL}_2(3)$, $G_{12} \cong \mathrm{GL}_2(3)$, and $G_{16} \cong C_5 \times \mathrm{SL}_2(5)$ in dimension $d = 2$, listed on lines 4, 12, and 16 of [ST, Table VII], and three complex reflection groups $G_{24} \cong C_2 \times \mathrm{SL}_3(2)$, $G_{25} \cong 3^{1+2}_+ \rtimes \mathrm{SL}_2(3)$, and $G_{27} \cong C_2 \times 3\mathsf{A}_6$ in dimension $d = 3$, listed on lines 24, 25, and 27 of [ST, Table VII]. As above, for any of these 6 groups G_k, G'_k denotes its derived subgroup. A direct calculation using the computer packages GAP3 [Mi], [S+], and Chevie [GHMP], shows that each of these 6 groups G, being embedded in $\mathcal{H} = \mathrm{U}_d(\mathbb{C})$, is a unitary 2-group; furthermore, G_{12}, G'_{16}, and G'_{27} are unitary 3-groups. One can check that $F(G_4) \cong F(G_{12})$ is a quaternion group $Q_8 = 2^{1+2}_-$, and we will identify them with an irreducible subgroup $E_2 \cong Q_8$ of $\mathrm{GL}_2(\mathbb{C})$. Also, $E_3 := F(G_{25}) \cong 3^{1+2}_+$ is an extraspecial 3-group of order 27 and exponent 3, which is an irreducible subgroup of $\mathrm{GL}_3(\mathbb{C})$. Let $\Gamma_d := \mathbf{N}_{\mathrm{GL}_d(\mathbb{C})}(E_d)$ for $d = 2, 3$. Now we can give a complete classification of unitary t-groups in dimensions 2 and 3.

Theorem 10. *Let $V = \mathbb{C}^d$ with $d = 2$ or 3, $\mathcal{G} = \mathrm{GL}(V)$, and let $G < \mathcal{G}$ be any finite subgroup. Then the following statements hold.*

(A) *Suppose* $d = 2$. *Then* $M_4(G, V) = M_4(\mathcal{G}, V)$ *if and only if one of the following conditions holds*
   (A1) $G = \mathbf{Z}(G)H$, *where* $H = G'_{16} \cong \mathrm{SL}_2(5)$.
   (A2) $E_2 \lhd G < \Gamma_2$ *and* $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})H$, *where* $H = G_{12} \cong \mathrm{GL}_2(3)$.
   (A3) $E_2 \lhd G < \Gamma_2$ *and* $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})H$, *where* $H = G_4 \cong \mathrm{SL}_2(3)$.
   In particular, $G < \mathcal{H} = \mathrm{U}(V)$ *is a unitary* 2*-group if and only if* G *is as described in* (A1)–(A3). *Furthermore,* $G < \mathcal{H} = \mathrm{U}(V)$ *is a unitary* 3*-group if and only if* G *is as described in* (A1)–(A2). *Moreover, such a subgroup* G *can be a unitary* t*-group for some* $t \geq 4$ *if and only if* $4 \leq t \leq 5$ *and* G *is as described in* (A1).

(B) *Suppose* $d = 3$. *Then* $M_4(G, V) = M_4(\mathcal{G}, V)$ *if and only if one of the following conditions holds*
   (B1) $G = \mathbf{Z}(G)H$, *where* $H = G'_{27} \cong 3\mathsf{A}_6$.
   (B2) $G = \mathbf{Z}(G)H$, *where* $H = G'_{24} \cong \mathrm{SL}_3(2)$.
   (B3) $E_3 \lhd G < \Gamma_3$. *Moreover, either* $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})G'_{25}$, *or* $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})G_{25}$.
   In particular, $G < \mathcal{H} = \mathrm{U}(V)$ *is a unitary* 3*-group if and only if* G *is as described in* (B1), *and no finite subgroup of* $\mathrm{U}(V)$ *can be a unitary* 4*-group.*

Proof. Let $G < \mathcal{G}$ be any finite subgroup such that $M_{2t}(G, V) = M_{2t}(\mathcal{G}, V)$ for some $t \geq 2$; in particular,

$$M_4(G, V) = M_4(\mathcal{G}, V). \tag{3}$$

First we note that if $K < \mathcal{G}$ is any finite subgroup that is *equal to* G *up to scalars*, i.e. $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})K$, then by [GT, Remark 2.3] we see that $M_{2t}(K, V) = M_{2t}(\mathcal{G}, V)$. So, instead of working with G, we will work with the following finite subgroup

$$K := \{\lambda g \mid g \in G, \lambda \in \mathbb{C}^\times, \det(\lambda g) = 1\} < \mathrm{SL}(V).$$

Next, we observe that G acts primitively on V. (Otherwise G contains a normal abelian subgroup A with $G/A \hookrightarrow \mathsf{S}_d$. In this case, by Ito's theorem G cannot act irreducibly on the irreducible $\mathcal{G}$-submodule of dimension $d^2 - 1$ of $V \otimes V^*$, and so G violates (3) by [GT, Remark 2.3].) Now, using the fact that $d = \dim(V) \leq 3$ is a prime number, it is straightforward to show that one of the following two possibilities must occur.

• **Almost quasisimple case**: $S \lhd G/\mathbf{Z}(G) \leq \mathrm{Aut}(S)$ for some finite non-abelian simple group S. By the results of [M], we have that $S \cong \mathrm{PSL}_2(5)$ if $d = 2$, and $S \cong \mathrm{SL}_3(2)$ or A_6 if $d = 3$. Arguing as in the proof of Theorem 8, we see that (A1), (B1), or (B2) holds. In the case of (A1), $M_{2t}(G, V) = M_{2t}(\mathcal{G}, V)$ if and only if $2 \leq t \leq 5$ by Lemma 9. In the case of (B2), G cannot act irreducibly on $\mathrm{Sym}^3(V)$ of dimension 10, whence $M_{2t}(G, V) = M_{2t}(\mathcal{G}, V)$ if and only if $t = 2$. Assume we are in the case of (B1). As mentioned above, then we have $M_{2t}(G, V) = M_{2t}(\mathcal{G}, V)$ for $t = 2, 3$. However, if ϖ_1 and ϖ_2 denote the two fundamental weights of $[\mathcal{G}, \mathcal{G}] \cong \mathrm{SL}_3(\mathbb{C})$, then $V^{\otimes 2} \otimes (V^*)^{\otimes 2}$ contains an irreducible $[\mathcal{G}, \mathcal{G}]$-submodule with highest weight $2\varpi_1 + 2\varpi_2$ of dimension 27 (see [Lu, Appendix A.6]). Clearly, G cannot act irreducibly on this submodule, and so $M_8(G, V) > M_8(\mathcal{G}, V)$ by [GT, Remark 2.3].

• **Extraspecial case**: $F^*(G) = F(G) = \mathbf{Z}(G)E_d$ and $E_d \lhd G$, in particular, $G \leq \Gamma_d$; furthermore, $G/\mathbf{Z}(G)E_d \leq \mathrm{Sp}(W)$ satisfies conclusion (A)(i) of Theorem 5 for $W = E_d/\mathbf{Z}(E_d) \cong \mathbb{F}_d^2$. The latter condition is equivalent to require $G/\mathbf{Z}(G)E_d$ to contain the unique subgroup C_3 of $\mathrm{Sp}_2(2) \cong \mathsf{S}_3$ when $d = 2$ and the unique subgroup Q_8 of $\mathrm{Sp}_2(3) \cong \mathrm{SL}_2(3)$ when $d = 3$. Note that $G_4 \cong \mathrm{SL}_2(3)$, respectively $G_{12} \cong \mathrm{GL}_2(3)$, induces the subgroup C_3, respectively S_3, of outer automorphisms of $E_2 \cong Q_8$. Similarly, $G'_{25} \cong 3_+^{1+2} \rtimes Q_8$, respectively $G_{25} \cong 3_+^{1+2} \rtimes \mathrm{SL}_2(3)$, induces the subgroup Q_8, respectively $\mathrm{SL}_2(3)$, of outer automorphisms of $E_3 \cong 3_+^{1+2}$ that act trivially on $\mathbf{Z}(E_3)$. Now arguing as in the proof of Theorem 8, we see that (A2), (A3), or (B3) holds. In the case of (A3), $M_8(G, V) > M_8(\mathcal{G}, V)$ by Lemma 9, and we already mentioned above that $M_6(G, V) = M_6(\mathcal{G}, V)$. In the case of (A2), G cannot act irreducibly on $\mathrm{Sym}^3(V)$ of dimension 4, so $M_{2t}(G, V) = M_{2t}(\mathcal{G}, V)$ if and only if $t = 2$. In the case of (B3), G cannot act irreducibly on $\mathrm{Sym}^3(V)$ of dimension 10, so $M_{2t}(G, V) = M_{2t}(\mathcal{G}, V)$ if and only if $t = 2$. □

References

[A] M. Aschbacher, On the maximal subgroups of the finite classical groups, *Invent. Math.* **76** (1984), 469–514.

[Atlas] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, '*An ATLAS of Finite Groups*', Clarendon Press, Oxford, 1985.

[FH] W. Fulton and J. Harris, '*Representation Theory*', Springer-Verlag, New York, 1991.

[GAP] The GAP group, 'GAP - *groups, algorithms, and programming*', Version 4.4, 2004, http://www.gap-system.org.

[GHMP] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer, CHEVIE – a system for computing and processing generic character tables, *Appl. Algebra Eng. Comm. Comput.* **7** (1996), 175–210.

[Gr] R. L. Griess, Automorphisms of extra special groups and nonvanishing degree 2 cohomology, *Pacif. J. Math.* **48** (1973), 403–422.

[GAE] D. Gross, K. Audenaert, J. Eisert, Evenly distributed unitaries: On the structure of unitary designs, *J. Math. Physics* **48** (2007), 052104.

[GMST] R. M. Guralnick, K. Magaard, J. Saxl, and Pham Huu Tiep, Cross characteristic representations of symplectic groups and unitary groups, *J. Algebra* **257** (2002), 291–347.

[GMT] R. M. Guralnick, K. Magaard, and Pham Huu Tiep, Symmetric and alternating powers of Weil representations of finite symplectic groups, *Bull. Inst. Math. Acad. Sinica* **13** (2018), 443–461.

[GT] R. M. Guralnick and Pham Huu Tiep, Decompositions of small tensor powers and Larsen's conjecture, *Represent. Theory* **9** (2005), 138–208.

[He] C. Hering, Transitive linear groups and linear groups which contain irreducible subgroups of prime order, II, *J. Algebra* **93** (1985), 151–164.

[Hu] B. Huppert, '*Endliche Gruppen I*', Springer-Verlag, 1967.

[Is] I. M. Isaacs, '*Character Theory of Finite Groups*', AMS-Chelsea, Providence, 2006.

[L] M. W. Liebeck, The affine permutation groups of rank three, *Proc. London Math. Soc.* **54** (1987), 477–516.

[Lu] F. Lübeck, Small degree representations of finite Chevalley groups in defining characteristic, *LMS J. Comput. Math.* **4** (2001), 135–169.

[Mag] K. Magaard, On the irreducibility of alternating powers and symmetric squares, *Arch. Math.* **63** (1994), 211–215.

[MM] K. Magaard and G. Malle, Irreducibility of alternating and symmetric squares, *Manuscripta Math.* **95** (1998), 169–180.

[MMT] K. Magaard, G. Malle and Pham Huu Tiep, Irreducibility of tensor squares, symmetric squares, and alternating squares, *Pacific J. Math.* **202** (2002), 379–427.

[M] G. Malle, Almost irreducible tensor squares, *Comm. Algebra* **27** (1999), 1033–1051.

[Mi] J. Michel, The development version of the CHEVIE package of GAP3, *J. Algebra* **435** (2015), 308–336.

[RS] A. Roy and A. J. Scott, Unitary designs and codes, *Designs Codes Crypt.* **53** (2009), 13–31.

[S+] M. Schönert et al, '*GAP – Groups, Algorithms, and Programming*', Lehrstuhl D für Mathematik, RWTH Aachen, Germany, sixth edition, 1997.

[Sc] A. J. Scott, Optimizing quantum process tomography with unitary 2-designs, *J. Phys. A* **41** (2008), 055308.

[ST] G. C. Shephard and J. A. Todd, Finite unitary reflection groups, *Can. J. Math.* **6** (1954), 274 - 304.

[TZ1] Pham Huu Tiep and A. E. Zalesskii, Minimal characters of the finite classical groups, *Commun. Algebra* **24** (1996), 2093–2167.

[TZ2] Pham Huu Tiep and A. E. Zalesskii, Some characterizations of the Weil representations of the symplectic and unitary groups, *J. Algebra* **192** (1997), 130–165.

[ZKGG] H. Zhu, R. Kueng, M. Grassl, and D. Gross, The Clifford group fails gracefully to be a unitary 4-design, arXiv:1609.18172v1.

[Zs] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math. Phys.* **3** (1892), 265–284.

E. Bannai, Professor Emeritus, Kyushu University, Fukuoka 819-0395, Japan

E-mail address: `bannai@math.kyushu-u.ac.jp`

G. Navarro, Department of Mathematics, Universitat de València, Dr. Moliner 50, 46100 Burjassot, Spain.

E-mail address: `gabriel.navarro@uv.es`

N. Rizo, Department of Mathematics, Universitat de València, Dr. Moliner 50, 46100 Burjassot, Spain.

E-mail address: `noelia.rizo@uv.es`

P. H. Tiep, Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA

E-mail address: `tiep@math.rutgers.edu`