

The Einstein–Maxwell Equations and Conformally Kähler Geometry

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The Einstein–Maxwell Equations and Conformally Kähler Geometry

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Abstract: Page’s Einstein metric on $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ is conformally related to an extremal Kähler metric. Here we construct a family of conformally Kähler solutions of the Einstein–Maxwell equations that deforms the Page metric, while sweeping out the entire Kähler cone of $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$. The same method also yields analogous solutions on every Hirzebruch surface. This allows us to display infinitely many geometrically distinct families of solutions of the Einstein–Maxwell equations on the smooth 4-manifolds $S^2 \times S^2$ and $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$.

Let (M, h) be a connected, oriented Riemannian 4-manifold. We will say that h is an *Einstein–Maxwell metric* if there is a 2-form F on M such that the pair (h, F) satisfies the *Einstein–Maxwell equations*

$$dF = 0 \tag{1}$$

$$d \star F = 0 \tag{2}$$

$$\left[r + F \circ F \right]_0 = 0, \tag{3}$$

where r is the Ricci tensor of h , the subscript $[\]_0$ indicates the trace-free part with respect to h , and the symmetric tensor $(F \circ F)_{jk} = F_j{}^\ell F_{\ell k}$ is obtained by composing F with itself as an endomorphism of TM . In physics terminology, Eqs. (1) to (3) are sometimes called the *Euclidean Einstein–Maxwell equations with cosmological constant*. This terminology emphasizes two important points: we are taking h to be a Riemannian metric rather than a Lorentzian one; and, while these equations imply that the scalar curvature s of h must be constant, this constant is allowed to be non-zero.

These equations turn out to naturally arise in connection with many interesting geometric questions, including some of the most active current research topics in Kähler

geometry. For example, if M admits a complex structure, and if h is a constant-scalar-curvature Kähler (cscK) metric on M , then h is Einstein–Maxwell. Indeed, if we set $F = \frac{2-s}{4}\omega + \rho$, where ω and ρ are respectively the Kähler and Ricci forms of the Kähler metric h , then (h, F) solves the Einstein–Maxwell equations.

The existence of cscK metrics for a fixed complex structure and fixed Kähler class is actually a difficult open problem, and is the subject of a great deal of current cutting-edge research [14]. However, the problem becomes much more tractable [2, 19, 35] if one instead just asks whether or not there is *some* complex structure in a given deformation class and *some* compatible Kähler class for which a solution exists. Using this observation in tandem with another recent development [11] in Kähler geometry, it is then relatively easy to prove the following [26]:

Theorem. *Let M be the underlying smooth compact 4-manifold of a compact complex surface. If M is of Kähler type—i.e., if $b_1(M)$ is even—then M admits Einstein–Maxwell metrics. By contrast, if M is not of Kähler type and has vanishing geometric genus, then M does not admit Einstein–Maxwell metrics.*

This surprising relationship between Einstein–Maxwell metrics and the Kähler condition immediately raises the following question: If M is the underlying smooth 4-manifold of a compact complex surface, is every Einstein–Maxwell metric on M actually a Kähler metric? However, the answer turns out to be *no*. In fact, the proof of the above theorem depends in part on the fact that the one-point blow-up of the complex projective plane admits a non-Kähler Einstein metric discovered by Page [34]. On the other hand, as pointed out by Derdziński [12], the Page metric, while not Kähler, is nonetheless *conformal* to a Kähler metric. Are there other Einstein–Maxwell metrics on this same space that are conformally Kähler? The answer is *yes*!

Theorem A. *Let $M \approx \mathbb{CP}_2 \# \overline{\mathbb{CP}_2}$ be the blow-up of the complex projective plane at a point, equipped with its standard complex structure, and let Ω be any Kähler class on M . Then Ω contains a Kähler metric g which is conformal to a (non-Kähler) Einstein–Maxwell metric h .*

However, in contrast to the situation for cscK metrics [10, 13], conformally Einstein–Maxwell metrics are generally not uniquely determined, up to complex automorphism, by their Kähler classes. Indeed, the present author has elsewhere [29] shown that this non-uniqueness phenomenon occurs on $M = \mathbb{CP}_1 \times \mathbb{CP}_1$, where certain Kähler classes contain both a cscK metric and a Kähler metric of non-constant scalar curvature that is conformally Einstein–Maxwell. For the one-point blow-up of the complex projective plane, the situation is analogous:

Theorem B. *The metric g in Theorem A is not always unique. To make this more precise, express an arbitrary Kähler class as*

$$\Omega = u\mathcal{L} - v\mathcal{E},$$

where $\mathcal{L}$ and $\mathcal{E}$ are respectively the Poincaré duals of a projective line and the exceptional curve, and where u and v are real numbers with $u > v > 0$. If $u/v > 9$, then Ω contains two geometrically distinct, $U(2)$ -invariant Kähler metrics g that are conformal to Einstein–Maxwell metrics h . For one of these metrics g , there are, moreover, two distinct $U(2)$ -invariant functions $f > 0$, which give rise to two different Einstein–Maxwell metrics $h = f^{-2}g$ of the same volume; however, the two different h thus arising from this g are actually isometric, in an orientation-reversing manner. By contrast,

when $u/v \leq 9$, there is, up to complex automorphisms, a unique Kähler metric g in Ω which is conformal to a $U(2)$ -invariant Einstein–Maxwell metric h .

We emphasize that our proof of the uniqueness assertion in Theorem B is entirely dependent on the assumption of $U(2)$ -invariance. An intriguing problem, which we leave for the interested reader, is to determine if any such unicity persists in the absence of this assumption. However, symmetry assumptions certainly could play a decisive role here. For example, exactly one of the Kähler metrics g that we will construct in each Kähler class Ω engenders an Einstein–Maxwell metric h which has an orientation-reversing isometry in addition to its $U(2)$ symmetry. If we had required h to also have such an isometry from the outset, the bifurcation phenomenon described by Theorem B would therefore have been eliminated, and we would then be left with exactly one solution g in every Kähler class.

Of course, our definition of an Einstein–Maxwell metric allows for the possibility that the metric might actually be Einstein, corresponding to the possibility that the 2-form might vanish. This does indeed occur on $\mathbb{CP}_2 \# \mathbb{CP}_2$, as the Page metric is certainly an example. On the other hand, the other solutions under discussion here are definitely *not* Einstein:

Theorem C. *There is a unique value of u/v , given by*

$$\begin{aligned} \frac{u}{v} = & \left[\frac{1}{2} \left(\sqrt[3]{1+\sqrt{2}} - \frac{1}{\sqrt[3]{1+\sqrt{2}}} \right) \right]^{-1/2} \\ & + 2 \sqrt{ \left[\frac{1}{2} \left(\sqrt[3]{1+\sqrt{2}} - \frac{1}{\sqrt[3]{1+\sqrt{2}}} \right) \right]^{1/2} - \left[\frac{1}{2} \left(\sqrt[3]{1+\sqrt{2}} - \frac{1}{\sqrt[3]{1+\sqrt{2}}} \right) \right]^2 } \\ \approx & 3.1839334, \end{aligned}$$

for which the Einstein–Maxwell metric of Theorem B becomes a constant times Page's Einstein metric [34]. For other values of u/v , these Einstein–Maxwell metrics are not Bach-flat, and so are not even conformally Einstein.

The same framework used to prove the above results also produces solutions on every Hirzebruch surface. Recall [4] that a Hirzebruch surface is a compact complex surface which is a holomorphic $\mathbb{CP}_1$ -bundle over $\mathbb{CP}_1$. Every Hirzebruch can be expressed as

$$\Sigma_k := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))$$

for a unique non-negative integer k , where $\mathbb{P}$ indicates the fiber-wise projectivization of a holomorphic rank-2, and, by a standard abuse of notation, $\mathcal{O}$ and $\mathcal{O}(k)$ respectively denotes the trivial line bundle and the holomorphic line bundle of degree k over $\mathbb{CP}_1$. The Σ_k are mutually non-isomorphic as complex manifolds, but there are only two diffeomorphism types:

$$\Sigma_k \approx \begin{cases} \mathbb{CP}_2 \# \overline{\mathbb{CP}}_2, & \text{if } k \text{ is odd; or} \\ S^2 \times S^2, & \text{if } k \text{ is even.} \end{cases}$$

In fact, up to biholomorphism, the Hirzebruch surfaces are the *only* complex surfaces diffeomorphic to $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ or $S^2 \times S^2$.

For the Einstein–Maxwell metrics considered here, the behavior observed on most Hirzebruch surfaces is simpler than that seen on $\Sigma_0 = \mathbb{CP}_1 \times \mathbb{CP}_1$ or on the one-point blow-up Σ_1 of $\mathbb{CP}_2$:

Theorem D. *Let $M = \Sigma_k$ be the k th Hirzebruch surface, with its fixed complex structure, and let Ω be any Kähler class on M . Then Ω contains a Kähler metric g which is conformal to an Einstein–Maxwell metric h . Moreover, if $k \geq 2$, there is a unique such Kähler metric g which is invariant under the standard action of $U(2)$ on Σ_k .*

However, every Hirzebruch surface is diffeomorphic to either $S^2 \times S^2$ or $\mathbb{CP}_2 \# \overline{\mathbb{CP}_2}$, so Theorem D asserts the existence of an infinite number of families of solutions on both of these smooth compact 4-manifolds. It seems plausible that these families may actually belong to different connected components of the moduli space of solutions. In any case, our construction certainly does imply an interesting result in this direction:

Theorem E. *Let the smooth oriented 4-manifold M be either $\mathbb{CP}_2 \# \overline{\mathbb{CP}_2}$ or $S^2 \times S^2$, and, for any $\Omega \in H^2(M, \mathbb{R})$ with $\Omega^2 > 0$, let*

$$\mathcal{M}_\Omega = \{\text{solutions } (h, F) \text{ of (1–3) on } M \mid F^+ \in \Omega\} / [\text{Diff}_H(M) \times \mathbb{R}^+]$$

be the moduli space of Ω -compatible solutions of the Einstein–Maxwell equations on M ; here $\text{Diff}_H(M)$ denotes the group of diffeomorphisms of M which act trivially on $H^2(M)$, and $\mathbb{R}^+$ acts by compatibly rescaling h and F^- . Then, for every positive integer N , there is a choice of Ω such that $\mathcal{M}_\Omega$ has at least N connected components.

1. Generalities

While the physical interest of the Einstein–Maxwell equations may seem self-evident, these equations are also inherently interesting for reasons that are intrinsic to Riemannian geometry. For example, there are several remarkable scalar-curvature estimates [18, 24, 26] in 4-dimensional Riemannian geometry that depend on the cohomology class of a harmonic self-dual 2-form. Such estimates typically amount to assertions about the volume-normalized Einstein–Hilbert functional

$$h \mapsto \frac{\mathfrak{S}}{V_h^{-1/2}} \int_M s_h d\mu_h$$

on the space $\mathcal{G}$ of smooth Riemannian metrics on M , where s_h denotes the scalar curvature of h , $d\mu_h$ is the metric volume measure, and V_h is the total volume of (M, h) . If M is a smooth compact oriented 4-manifold, and if $\Omega \in H^2(M, \mathbb{R})$ is a fixed cohomology class with $\Omega^2 > 0$, it is therefore natural to consider the set $\mathcal{G}_\Omega$ of smooth Riemannian metrics h on M for which the harmonic representative ω of Ω is self-dual. One can then show that $\mathcal{G}_\Omega$ is a Fréchet manifold, and indeed is a closed submanifold of the space of smooth Riemannian metrics, of finite codimension $b_-(M)$. This allows us to consider the variational problem arising from the restriction

$$\begin{aligned} \mathfrak{S}|_{\mathcal{G}_\Omega} : \mathcal{G}_\Omega &\longrightarrow \mathbb{R} \\ h &\longmapsto \frac{\int_M s_h d\mu_h}{\sqrt{\int_M d\mu_h}} \end{aligned} \quad (4)$$

of the normalized Einstein–Hilbert functional to the Ω -adapted metrics. The critical points of this variational problems are then [26, 29] just the solutions of the Einstein–Maxwell equations which are appropriately related to Ω :

Proposition 1. A metric $h \in \mathcal{G}_\Omega$ is a critical point of the variational problem (4) if and only if there is a harmonic 2-form F with self-dual part $F^+ \in \Omega$ such that the pair (h, F) solves (1)–(3).

One corollary is that an Einstein–Maxwell metric h on a 4-manifold must have constant scalar curvature; indeed, if h belongs to $\mathcal{G}_\Omega$, so does its entire conformal class, and the restriction of $\mathfrak{S}$ to a conformal class is exactly the variational problem used by Yamabe to characterize metrics of constant scalar curvature [6]. At the other end of things, the Einstein metrics are exactly the critical points of $\mathfrak{S}$ on the space of all Riemannian metrics, and this provides further explanation for the fact that Einstein metrics are (rather special) solutions of the Einstein–Maxwell equations. Yet another interesting consequence is the following:

Proposition 2. Let M and Ω be as above, and let

$$\mathcal{M}_\Omega = \{\text{solutions } (h, F) \text{ of (1)–(3) on } M \mid F^+ \in \Omega\} / [\text{Diff}_H(M) \times \mathbb{R}^+]$$

be the moduli space of Ω -compatible solutions of the Einstein–Maxwell equations. Here $\text{Diff}_H(M)$ denotes the group of diffeomorphisms of M which act trivially on $H^2(M, \mathbb{R})$, and $\mathbb{R}^+$ acts by compatibly rescaling h and F^- . If (h, F) and $(\tilde{h}, \tilde{F})$ are solutions such that $s_h V_h^{1/2} \neq s_{\tilde{h}} V_{\tilde{h}}^{1/2}$, then these solutions belong to different connected components of $\mathcal{M}_\Omega$.

Proof. If $(h, F) \in \mathcal{G} \times \Gamma(\Lambda^2)$ is any smooth solution of (1)–(3), there is a finite-dimensional smooth submanifold $\mathcal{V} \subset \mathcal{G} \times \Gamma(\Lambda^2)$ such that any other solution in a neighborhood $\mathcal{V}$ of (h, F) is the pull-back of an element of $\mathcal{V}$ via some diffeomorphism. To see this, let $\mathcal{S}$ be the set of metrics of the form $\hat{h} = h + \hat{h}$, where $\hat{h}$ is a symmetric and transverse traceless with respect to h , and consider the smooth map

$$\begin{aligned} \mathcal{S} \times \Gamma(\Lambda^2) &\xrightarrow{\mathcal{E}} \Gamma(\odot_0^2 \Lambda^1) \times \Gamma(\Lambda^2) \\ (\hat{h}, \hat{F}) &\mapsto \left(\hat{F}_{\hat{h}} + [\hat{F} \circ \hat{F}]_{0, \hat{h}}, (d + \delta_{\hat{h}})^2 \hat{F} \right) \end{aligned}$$

where $\hat{F} = F + \hat{F} \in \Gamma(\Lambda^2)$. Because of our assumption that $\hat{h}$ is transverse-traceless, we can modify the first term by adding $(\delta^* \delta_h)_0$ without altering the answer, but with the effect [5] that the linearization of the equation at h becomes elliptic. Let $\mathcal{T}$ denote image of the linearization $\mathcal{E}_{*(h, F)}$, and let $\wp : \Gamma(\odot_0^2 \Lambda^1) \times \Gamma(\Lambda^2) \rightarrow \mathcal{T}$ be the L^2 -orthogonal projection. Then $\wp \circ \mathcal{E}$ is a submersion on some neighborhood (h, F) , and $\mathcal{V} = (\wp \circ \mathcal{E})^{-1}(0)$ is a finite-dimensional manifold that contains a gauge-fixed version of any solution of (1)–(3) in a neighborhood $\mathcal{V}$ of (h, F) .

Now choose a basis $\mathbf{a}_1, \dots, \mathbf{a}_{b_2}$ for $H^2(M)$, and, for each subset $\mathbf{I} \subset \{1, \dots, b_2(M)\}$, let $\pi_{\mathbf{I}} : H^2(M) \rightarrow \mathbb{R}^{|\mathbf{I}|}$ denote the map that sends the deRham class $\mathbf{a}$ to the relevant $|\mathbf{I}|$ components of $\mathbf{a} - \Omega$ relative to the chosen basis. Let $\Pi : \mathcal{V} \rightarrow H^2(M)$ be the smooth map defined by $(\hat{h}, \hat{F}) \mapsto [\hat{F}_{\hat{h}}^+]$, let $\mathcal{V}_{\mathbf{I}} \subset \mathcal{V}$ be the open subset of $\mathcal{V}$ on which $\pi_{\mathbf{I}} \circ \Pi$ has derivative of rank $|\mathbf{I}|$, and let $\mathcal{Z}_{\mathbf{I}} \subset \mathcal{V}_{\mathbf{I}}$ be the smooth submanifold defined by $(\pi_{\mathbf{I}} \circ \Pi)^{-1}(0)$. If $(\hat{h}, \hat{F}) \in \mathcal{V}$ then happens both to be a genuine solution of (1)–(3) and to have the property that $\hat{h} \in \mathcal{G}_\Omega$, then at least one of the smooth manifolds $\mathcal{Z}_{\mathbf{I}}$ will then have the property that its image in $\mathcal{G}$ passes through $\hat{h}$, with $T_{\hat{h}} \mathcal{Z}_{\mathbf{I}} \subset T_{\hat{h}} \mathcal{G}_\Omega$; the value of $|\mathbf{I}|$ for which this occurs is just the rank of the derivative of Π at $(\hat{h}, \hat{F})$. It follows

that, for any Einstein–Maxwell metric $\tilde{h} \in \mathcal{V} \cap \mathcal{G}_\Omega$, the real number $s_{\tilde{h}} V_{\tilde{h}}^{1/2} = \mathfrak{S}(\tilde{h})$ is a critical value of the pull-back of $\mathfrak{S}$ to one of the manifolds $\mathcal{Z}_1 \subset \mathcal{G} \times \Gamma(\Lambda^2)$.

Now since $\mathcal{G} \times \Gamma(\Lambda^2)$ is second countable, there is a countable collection $\mathcal{V}_j$ of such open sets, and an associated countable collection of finite-dimensional manifolds $\mathcal{Z}_{j,1}$, such that every Einstein–Maxwell metric $\tilde{h} \in \mathcal{G}_\Omega$ is the pull-back via some diffeomorphism of an Einstein–Maxwell metric $\hat{h} \in \mathcal{G}_\Omega$ through which passes some $\mathcal{Z}_{j,1}$ with $T_{\hat{h}} \mathcal{Z}_{j,1} \subset T_{\hat{h}} \mathcal{G}_\Omega$. This expresses the set of real numbers occurring as $s V^{1/2}$ for Ω -compatible Einstein–Maxwell metrics as a subset of a countable union of the critical values of specific smooth maps $\mathcal{Z}_{j,1} \rightarrow \mathbb{R}$, where each $\mathcal{Z}_{j,1}$ is a finite-dimensional smooth manifold. But Sard's theorem tells us that, for each $(j, 1)$, the set of these critical values has measure zero in $\mathbb{R}$, and the countable union of these over all $(j, 1)$ therefore has measure zero, too. In particular, the values of $s V^{1/2}$ occurring for Ω -compatible Einstein–Maxwell metrics cannot contain an open interval. Thus, if t_1 and t_2 are distinct values of $s V^{1/2}$ that occur for two different Ω -compatible Einstein–Maxwell metrics, there is a number t_0 between t_1 and t_2 which is *not* a value of $s V^{1/2}$ for any Ω -compatible Einstein–Maxwell metric. The open sets $(\mathfrak{S}|_{\mathcal{G}_\Omega})^{-1}(-\infty, t_0)$ and $(\mathfrak{S}|_{\mathcal{G}_\Omega})^{-1}(t_0, \infty)$ thus provide a separation of $\mathcal{M}_\Omega$ into two open disjoint sets, each of which contains just one of the given metrics. This shows that the two given metrics must belong to different connected components, as claimed. $\square$

The above argument avoids having to show that the moduli space $\mathcal{M}_\Omega$ is locally smoothly path-wise connected. However, the latter does appear to be true, and to follow from a modification of the argument given by Koiso [21] in the Einstein case. The idea is to show that the manifold $\mathcal{V}$ appearing in the proof of Proposition 2 carries a natural real-analytic structure, and that the Einstein–Maxwell equations then cut out a real-analytic subset of $\mathcal{V}$, thereby providing a real-analytic Kuranishi-type model for the local pre-moduli space; cf. [6, Corollary 12.50]. Ultimately, this is related to the fact that solutions of the Einstein–Maxwell equations are real-analytic in harmonic coordinates. The proof of the latter, which we leave to the interested reader, merely consists of combining [32, Theorem 6.7.6] with [29, Proposition 4].

The physical motivation for the Einstein–Maxwell equations (1)–(3) assigns a central role to the 2-form F , which represents an electromagnetic field. It may therefore seem strange that we will often simply refer to h as an *Einstein–Maxwell metric*, without explicitly mentioning F . There is good reason for this terminology, however. Indeed, in our Riemannian setting, the metric h essentially determines the relevant 2-form:

Proposition 3. *Suppose that (M, h) is an Einstein–Maxwell manifold, where the 4-manifold is connected and oriented. If h is not actually Einstein, then the 2-form F needed to make (h, F) solve the Einstein–Maxwell equations (1)–(3) is completely determined by h , modulo substitutions of the type*

$$F^+ \rightsquigarrow c F^+, \quad F^- \rightsquigarrow c^{-1} F^-$$

where $c \neq 0$ is a real constant.

Proof. First notice that (3) can be rewritten as

$$\mathring{F} = -2F^+ \circ F^-, \quad (5)$$

where $\mathring{F} = r - (s/4)h$ is the trace-free part of the Ricci tensor, and where $F^\pm = (F \pm \star F)/2$ are the self-dual and anti-self-dual parts of F . If we assume that h is not

Einstein, then there exists an open ball $\mathcal{B}$ on which $\hat{r} \neq 0$, and (5) then algebraically determines F on $\mathcal{B}$ up to substitutions of the form

$$F^+ \rightsquigarrow uF^+, \quad F^- \rightsquigarrow u^{-1}F^-$$

for a smooth non-vanishing function u defined on $\mathcal{B}$. However, Eqs. (1) and (2) imply that F^+ is closed, and requiring that $\tilde{F}^+ := uF^+$ also be closed then results in the condition that

$$0 = d(uF^+) = du \wedge F^+ + u dF^+ = du \wedge F^+.$$

However, a self-dual form is non-degenerate on the set where it is non-zero, and F^+ is non-zero on $\mathcal{B}$ by hypothesis. It therefore follows that $du = 0$ on $\mathcal{B}$, and hence that $u = c$ on this open ball. Thus, if h is not actually an Einstein metric, any other candidate $\tilde{F}^+$ for F^+ would have to coincide with cF^+ , for some constant c , on a non-empty open set $\mathcal{B} \subset M$. But since $\tilde{F}^+ - cF^+$ then belongs to the kernel of $d + d^*$ and vanishes on an open set of our connected 4-manifold M , unique continuation for harmonic forms [3] then implies that $\tilde{F}^+ \equiv cF^+$ on all of M . Applying the same argument to F^- , we thus see that if the Einstein–Maxwell metric h is not actually Einstein, then the metric h determines the closed and co-closed 2-form $F = F^+ + F^-$ modulo changes of the type

$$F^+ \rightsquigarrow cF^+, \quad F^- \rightsquigarrow c^{-1}F^-$$

for a non-zero constant c . $\square$

Because the physical electromagnetic field is known to actually be the curvature of a principal $U(1)$ bundle, it might seem reasonable to also demand that F be chosen so that $[F/2\pi] \in H_{dR}^2(M)$ is the Chern class of a circle bundle. Fortunately, however, this condition can be satisfied, essentially “free of charge,” in the context that will concern us here:

Proposition 4. *Let h be an Einstein–Maxwell metric on a smooth compact 4-manifold M with $b_+(M) = b_-(M) = 1$. Then there is a constant multiple $\hat{h}$ of h and a 2-form F such that $(\hat{h}, F)$ solves the Einstein–Maxwell equations, and such that $F/2\pi$ represents an integer class in deRham cohomology.*

Proof. Since one could simply take $F = 0$ in the Einstein case, we may assume that h is not Einstein. Now observe that if a is any positive constant, $h \rightsquigarrow a^2h$ leaves the Ricci tensor unchanged, but results in $F \circ F \rightsquigarrow a^{-2}F \circ F$. Thus, if (h, F) solves the Einstein–Maxwell equations (1)–(3), then (a^2h, aF) is also a solution. But if h is not Einstein, the harmonic forms $(aF^+ + a^{-1}F^-)/2\pi$ then sweep out an open cone in $H^2(M) \cong \mathbb{R}^2$, and, because $H^2(M, \mathbb{Q}) \subset H^2(M, \mathbb{R})$ is dense, they thereby realize infinitely many rational classes. By clearing denominators, this cone therefore also includes elements of $H^2(M, \mathbb{Z})/\text{torsion}$. The claim therefore follows. $\square$

The present investigation will actually focus on a special class of solutions of the Einstein–Maxwell equations, first introduced in [29]:

Definition 1. Let (M^4, J) be a complex surface. A solution (h, F) of the Einstein–Maxwell equations (1)–(3) on (M, J) will be called *strongly Hermitian* if both h and F are invariant under the action of the integrable almost-complex structure J :

$$\begin{aligned} h &= h(J\cdot, J\cdot), \\ F &= F(J\cdot, J\cdot). \end{aligned}$$

When this happens, we will then say that h is a strongly Hermitian Einstein–Maxwell metric on (M, J) .

The following is one of the principal results of [29]:

Proposition 5. Let (M^4, J) be a compact complex surface. Then h is a strongly Hermitian Einstein–Maxwell metric on (M, J) iff there is a Kähler metric g on (M, J) and a real holomorphy potential $f \neq 0$ on (M, g, J) such that $h = f^{-2}g$ has constant scalar curvature.

Here a real holomorphy potential f is a real-valued function whose gradient with respect to the Kähler metric g is the real part of a holomorphic vector field; this is equivalent to requiring either that $J \operatorname{grad} f$ be a Killing field of g , or that the Riemannian Hessian ∇df of f be J -invariant. The harmonic 2-form F needed to solve (1)–(3) in conjunction with $h = f^{-2}g$ can then be taken to be

$$F = \frac{\omega}{2} + f^{-2} \left(\rho + 2if^{-1} \partial \bar{\partial} f \right)^{-}$$

where $\omega = g(J\cdot, \cdot)$ and $\rho = r_g(J\cdot, \cdot)$ are respectively the Kähler and Ricci forms of g and where the final superscript indicates projection to the anti-self-dual part of a 2-form. However, this choice of F is of course not quite unique, and can be modified in the manner described by Proposition 3.

2. Solutions on Spherical Shells

In this section, we describe an essentially local construction of Einstein–Maxwell metrics on a spherical shell $S^3 \times I$, where $I \subset \mathbb{R}$ is an open interval. We assume from the outset that the metrics in question are *cohomogeneity one* [17], with an isometric action of $U(2)$ whose generic orbit is the 3-sphere $S^3 = U(2)/U(1)$. The metric on such an orbit then submerses onto a homogeneous metric on S^2 , which must be some number $\varrho^2/4$ times the standard unit-sphere metric. At least generically, we can then use the positive function ϱ as a coordinate, and so write our metric in so-called *Bianchi IX form*

$$g = \frac{d\varrho^2}{\Phi(\varrho)} + \varrho^2 \left[\sigma_1^2 + \sigma_2^2 + \tilde{\Phi}(\varrho)\sigma_3^2 \right]$$

where $\sigma_1, \sigma_2, \sigma_3$ is a left-invariant co-frame on S^3 which is orthonormal with respect to the usual metric on $S^3 = \mathrm{SU}(2) = \mathrm{Sp}(1)$. The fact that the metric coefficients of σ_1 and σ_2 are equal reflects the fact that $U(2)$ acts on S^3 with isotropy subgroup $U(1)$, and the latter acts on a cotangent space by rotations in σ_1 and σ_2 . On the other hand, Φ and $\tilde{\Phi}$ are for the moment completely arbitrary positive functions.

The structure equations for $\mathrm{SU}(2)$ tell us that our co-frame satisfies

$$d\sigma_1 = 2\sigma_2 \wedge \sigma_3, \quad d\sigma_2 = 2\sigma_3 \wedge \sigma_1, \quad d\sigma_3 = 2\sigma_1 \wedge \sigma_2.$$

Notice, in particular, that $\sigma_1 \wedge \sigma_2$ is closed. In fact, this closed 2-form is simply the pull-back of the area 2-form of the curvature-4 metric on S^2 .

This picture automatically provides us with a g -compatible almost-complex structure J , characterized by

$$\frac{d\varrho}{\sqrt{\Phi(\varrho)\tilde{\Phi}(\varrho)}} \mapsto \sigma_3, \quad \sigma_1 \mapsto \sigma_2.$$

This actually turns our spherical shell into a complex manifold.

Lemma 1. *The almost-complex structure J is integrable, and so makes g into a Hermitian metric.*

Proof. With respect to the given J , the $(1, 0)$ -forms are spanned by

$$\frac{d\varrho}{\sqrt{\Phi(\varrho)\tilde{\Phi}(\varrho)}} + i\sigma_3 \quad \text{and} \quad \sigma_1 + i\sigma_2.$$

Let $\mathcal{J}$ denote the differential ideal generated by these two 1-forms. Because

$$d\left(\frac{d\varrho}{\sqrt{\Phi(\varrho)\tilde{\Phi}(\varrho)}} + i\sigma_3\right) = 2i\sigma_1 \wedge \sigma_2 = (\sigma_1 + i\sigma_2) \wedge (2i\sigma_2)$$

and

$$d(\sigma_1 + i\sigma_2) = 2\sigma_2 \wedge \sigma_3 + 2i\sigma_3 \wedge \sigma_1 = (\sigma_1 + i\sigma_2) \wedge (-2i\sigma_3),$$

we have $d\mathcal{J} \subset \mathcal{J}$. Thus $\mathcal{J}$ is a closed differential ideal, and $[T^{0,1}, T^{0,1}] \subset T^{0,1}$. Hence J is integrable, in the sense of the Newlander–Nirenberg theorem [33]. $\square$

Imposing the Kähler condition now gives us a simple constraint.

Lemma 2. *The metric g is Kähler with respect to J if and only if*

$$\Phi \equiv \tilde{\Phi}.$$

Proof. The associated 2-form of (g, J) is

$$\omega = \sqrt{\frac{\tilde{\Phi}}{\Phi}} \varrho \, d\varrho \wedge \sigma_3 + \varrho^2 \sigma_1 \wedge \sigma_2.$$

Thus

$$\begin{aligned} d\omega &= -\sqrt{\frac{\tilde{\Phi}}{\Phi}} \varrho \, d\varrho \wedge d\sigma_3 + 2\varrho d\varrho \wedge \sigma_1 \wedge \sigma_2 + \varrho^2 d(\sigma_1 \wedge \sigma_2) \\ &= \left(1 - \sqrt{\frac{\tilde{\Phi}}{\Phi}}\right) 2\varrho \, d\varrho \wedge \sigma_1 \wedge \sigma_2, \end{aligned}$$

showing that (g, J) is Kähler iff $\Phi \equiv \tilde{\Phi}$. $\square$

We will henceforth assume that g is Kähler, and so given by

$$g = \frac{d\varrho^2}{\Phi(\varrho)} + \varrho^2 \left[\sigma_1^2 + \sigma_2^2 + \Phi(\varrho)\sigma_3^2 \right] \quad (6)$$

with Kähler form

$$\omega = \varrho \, d\varrho \wedge \sigma_3 + \varrho^2 \sigma_1 \wedge \sigma_2.$$

In this context, the Killing field ξ defined by

$$\sigma_3(\xi) = 1, \quad d\varrho(\xi) = \sigma_1(\xi) = \sigma_2(\xi) = 0$$

acquires considerable interest, as it preserves ω , and is generated by the Hamiltonian

$$t = \frac{\varrho^2}{2}.$$

This observation immediately gives ϱ a more global and intrinsic meaning than our previous provisional definition might indicate. In particular, the fact that the symplectic reductions have area $2\pi t$ now becomes a special case of the Duistermaat–Heckman formula [15].

Treating the Hamiltonian t as a coordinate now allows us to put our Kähler metric g into the standard form [23]

$$g = w \, \check{g} + w \, dt^2 + w^{-1} \theta^2,$$

where $\check{g}$ is w^{-1} times the usual metric on the symplectic reduction, and where $\theta(\xi) = 1$. Comparing this with (6), we now immediately see that $\theta = \sigma_3$ and that $w^{-1} = \varrho^2 \Phi$. It then follows that

$$\check{g} = w^{-1} \left[\varrho^2 (\sigma_1^2 + \sigma_2^2) \right] = \varrho^4 \Phi (\sigma_1^2 + \sigma_2^2)$$

is a metric of Gauss curvature

$$\check{K} = \frac{4}{\varrho^4 \Phi} = \frac{1}{t^2 \Phi}$$

and Kähler form

$$\check{\omega} = \varrho^4 \Phi \, \sigma_1 \wedge \sigma_2 = 4t^2 \Phi \, \sigma_1 \wedge \sigma_2$$

on the 2-sphere S^2 . The formalism of [23] therefore allows us to calculate the scalar curvature s or g , using the general formula [23, 28, 30]

$$s \, d\mu = \left[2\check{K}\check{\omega} - \frac{d^2}{dt^2} \check{\omega} \right] \wedge dt \wedge \theta,$$

where $d\mu = \omega^2/2$ is the volume form of g . In our case, the latter is explicitly given by

$$d\mu = \varrho^3 \, d\varrho \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = 2t \, dt \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3,$$

so we have

$$2t \, s \, dt \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \left[\frac{2}{t^2 \Phi} 4t^2 \Phi - \frac{d^2}{dt^2} (4t^2 \Phi) \right] \sigma_1 \wedge \sigma_2 \wedge dt \wedge \sigma_3,$$

and hence

$$s = \frac{4}{t} - \frac{2}{t} \frac{d^2}{dt^2} (t^2 \Phi) = \frac{2}{t} \frac{d^2}{dt^2} [t^2(1 - \Phi)]. \quad (7)$$

As a simple illustration of this formula, let us now use this to determine when g is an *extremal* Kähler metric, in the sense of Calabi [8, 9].

Lemma 3. *The Kähler metric (g, J) defined by (6) is extremal iff*

$$s = at + b$$

for real constants a and b , where $t = \varrho^2/2$.

Proof. Let $\eta = J\nabla s$, where s is the scalar curvature of g . Then g is extremal iff η is a Killing field. However, s is invariant under the isometry group, and is therefore a function of ϱ , or equivalently, a function of t . Thus, we automatically have $\eta = u\xi$, where $u = ds/dt$, and where $\xi = J\nabla t$ is already known to be a Killing field. If u is constant, then η is a Killing field, and s is an affine-linear function of t . Conversely, if η is a Killing field, then

$$0 = \nabla^{(a} \eta^{b)} = \nabla^{(a} u \xi^{b)} = u \nabla^{(a} \xi^{b)} + \xi^{(b} \nabla^{a)} u = \xi^{(a} \nabla^{b)} u,$$

and it follows that ∇u must vanish, since the symmetric tensor product of two non-zero vectors is always non-zero. This shows that u must be constant, and that s must therefore be an affine-linear function of t , as claimed. $\square$

Proposition 6. *The Kähler metric (g, J) defined by (6) is extremal iff*

$$\begin{aligned} \Phi &= At^2 + Bt + 1 + Ct^{-1} + Dt^{-2} \\ &= \frac{A}{4}\varrho^4 + \frac{B}{2}\varrho^2 + 1 + \frac{2C}{\varrho^2} + \frac{4D}{\varrho^4} \end{aligned}$$

for real constants A, B, C , and D , subject only to the constraint that $\Phi > 0$ in the region of interest. Moreover, the scalar curvature of g is then given by

$$s = -12(2At + B) = -12(A\varrho^2 + B).$$

Proof. By Lemma 3 and Eq. (7), the extremal condition is equivalent to

$$\frac{d^2}{dt^2} [t^2(\Phi - 1)] = -\frac{t}{2}(at + b)$$

and integrating twice therefore yields

$$t^2(\Phi - 1) = -\frac{a}{24}t^4 - \frac{b}{12}t^3 + Ct + D.$$

Setting $a = -24A$ and $b = -12B$ then yields the result. $\square$

These metrics coincide, in principle, with those discovered by Eugenio Calabi in his remarkable first paper on extremal Kähler metrics [8]. Calabi's approach to the problem, however, took the independent variable to be the radius in complex coordinates, and took the unknown to be a Kähler potential; by contrast, we have used the Hamiltonian t as our independent variable, and solved directly for the metric rather than for a potential. We emphasize that our Bianchi IX approach allows one to display the general solution *explicitly*, whereas Calabi was instead forced to prove his transformative results using only an indirect understanding of the solutions. Notice that g has constant scalar curvature iff $A = 0$, and is scalar-flat iff $A = B = 0$. ALE metrics of this type were first introduced in [22], where the reader can also find a direct comparison between the Bianchi IX formalism and Calabi's approach. The metric is Ricci-flat if $A = B = C = 0$, in which case g becomes the Eguchi–Hanson metric [16], unless $A = B = C = D = 0$, when it is flat.

We now turn to the problem of constructing strongly Hermitian solutions of the Einstein–Maxwell equations. By [29, Theorem A], such metrics are exactly those of the form $h = f^{-2}g$, where g is a Kähler metric, $f > 0$ is a real holomorphy potential, and $h = f^{-2}g$ has constant scalar curvature, except in the exceptional case that h is anti-self-dual and Einstein. Here a real holomorphy potential means a real-valued function f such that $J\nabla f$ is a Killing field.

Our eventual goal is to construct strongly Hermitian Einstein–Maxwell metrics which are $U(2)$ -invariant and live on a compact complex surface (M^4, J) . In this setting, the Kähler form ω of g will be harmonic with respect to h , and will necessarily be the unique harmonic form in its deRham class. This means that ω will be $U(2)$ -invariant, and hence that $f = 2^{-1/4}|\omega|_h^{1/2}$ will also necessarily be $U(2)$ -invariant. The Kähler metric $g = f^2h$ will therefore be $U(2)$ -invariant, too, and so can locally be expressed in the form (6). Since f is a function on the space of $U(2)$ -orbits, it must, in our local picture, be a function of t . However, both t and f are then Hamiltonians whose symplectic gradients are Killing fields, and the same argument used to prove Lemma 3 therefore implies that f must be an affine function of t . Since we are only interested in solutions which are not cscK, this means that f must take the form $ct + d$, where $c \neq 0$. However, multiplying f by a non-zero constant just rescales h into another solution of the same problem, so we can henceforth take f to be of the form

$$f = t - \alpha$$

for some real constant α .

Requiring that $h = f^{-2}g$ have constant scalar curvature then amounts to saying that

$$(6\Delta + s)f^{-1} = \kappa f^{-3},$$

or equivalently as

$$s = \kappa f^{-2} - 6f\Delta f^{-1}, \quad (8)$$

where $\Delta = -\nabla \cdot \nabla = -\star d \star d$ is the geometer's Laplacian, and κ is some real constant. Rewriting (6) as

$$g = \frac{dt^2}{2t\Phi} + 2t \left[\sigma_1^2 + \sigma_2^2 + \Phi \sigma_3^2 \right],$$

and thereby re-expressing the associated 2-form as

$$\omega = dt \wedge \sigma_3 + 2t\sigma_1 \wedge \sigma_2,$$

the fact that

$$\frac{dt}{\sqrt{2t\Phi}}, \sqrt{2t\Phi}\sigma_3, \sqrt{2t}\sigma_1, \sqrt{2t}\sigma_2$$

is an oriented orthonormal basis tells us that

$$\star dt = 4t^2 \Phi \sigma_1 \wedge \sigma_2 \wedge \sigma_3,$$

so that, for any function $\varphi(t)$,

$$\star d\varphi(t) = 4t^2 \Phi \varphi'(t) \sigma_1 \wedge \sigma_2 \wedge \sigma_3.$$

Thus

$$d \star d\varphi = 4[t^2 \Phi \sigma_1 \varphi']' dt \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3,$$

and

$$\Delta\varphi = -\star d \star d\varphi = -\frac{2}{t}[t^2 \Phi \varphi']'$$

for any function of t , where primes denote derivatives with respect to t . Setting

$$\Psi := t^2 \Phi,$$

and setting $f = t - \alpha$, we can thus rewrite (8) as

$$s = \frac{\kappa}{(t - \alpha)^2} - \frac{12}{t}(t - \alpha) \left[\frac{\Psi}{(t - \alpha)^2} \right]'$$

However, (7) tells us that

$$s = \frac{2}{t} [2 - \Psi''],$$

and equating these two expressions thus tells us that

$$2 - \Psi'' = \frac{\kappa t}{2(t - \alpha)^2} - 6(t - \alpha) \left[\frac{\Psi}{(t - \alpha)^2} \right]'$$

In other words, we will obtain a conformally Kähler solution of the Einstein–Maxwell equations iff Ψ solves the linear inhomogeneous equation

$$(t - \alpha)^2 \Psi'' - 6(t - \alpha) \Psi' + 12\Psi = 2(t - \alpha)^2 - \frac{\kappa}{2}(t - \alpha) - \frac{\kappa\alpha}{2}. \quad (9)$$

However, the linear operator

$$\Psi \mapsto (t - \alpha)^2 \Psi'' - 6(t - \alpha) \Psi' + 12\Psi$$

acts on powers of $(t - \alpha)$ by

$$(t - \alpha)^\ell \mapsto [\ell(\ell - 1) - 6\ell + 12](t - \alpha)^\ell = (\ell - 4)(\ell - 3)(t - \alpha)^\ell,$$

so the general solution of (9) is

$$\Psi = \mathfrak{A}(t - \alpha)^4 + \mathfrak{B}(t - \alpha)^3 + (t - \alpha)^2 - \frac{\kappa}{12}(t - \alpha) - \frac{\kappa\alpha}{24} \quad (10)$$

$$= \mathfrak{A}x^4 + \mathfrak{B}x^3 + x^2 + \mathfrak{C}x + \frac{\mathfrak{C}\alpha}{2} \quad (11)$$

where, for clarity, we have set $x = t - \alpha$ and $\mathfrak{C} = -\kappa/12$. Notice that (10) is not quite the general quartic function of t , because the five coefficients only depend on the four constants $\mathfrak{A}$, $\mathfrak{B}$, α , and κ . Also notice that the polynomial (11) in x completely determines α when $\mathfrak{C} \neq 0$, thereby allowing one to reconstruct (10) from a generic quartic polynomial in x for which the coefficient of x^2 is 1.

Making systematic use of the new variable $x = f = t - \alpha$ now allows us to put the above results in a simple, concise form. In these terms, the general solution of our problem is provided by the Kähler metric

$$g = (x + \alpha) \left[\frac{dx^2}{2\Psi} + 2(\sigma_1^2 + \sigma_2^2) \right] + \frac{2\Psi}{x + \alpha} \sigma_3^2 \quad (12)$$

associated to a quartic polynomial Ψ of the special form (11). The associated Einstein–Maxwell metric is then given by

$$h = \frac{g}{x^2}. \quad (13)$$

Proposition 7. *Let Ψ be a quartic polynomial in x of the special form (11), with $\mathfrak{A}$ and α both non-zero. Let g and h be the corresponding Kähler and Einstein–Maxwell metrics defined by (12) and (13) on a spherical shell where x , $x + \alpha$, and Ψ are all positive. Then the following are equivalent:*

- (i) *The Hermitian metric h is Einstein.*
- (ii) *The conformal class $[g] = [h]$ is Bach-flat.*
- (iii) *The Kähler metric g is extremal.*
- (iv) $\mathfrak{B} = 2\mathfrak{A}\alpha$.

Proof. By Proposition 6, with $\Psi = t^2\Phi$, the metric g is extremal iff

$$\Psi = \mathfrak{A}x^4 + \mathfrak{B}x^3 + x^2 + \dots = At^4 + Bt^3 + t^2 + \dots$$

After making the substitution $x = t - \alpha$ and then comparing the coefficients of t^2 , we obtain

$$\mathfrak{A}(6\alpha^2) + \mathfrak{B}(-3\alpha) = 0,$$

and, since we have assumed that $\mathfrak{A}$ and α are non-zero, this happens iff

$$\alpha = \frac{\mathfrak{B}}{2\mathfrak{A}}, \quad (14)$$

thus showing that (iii) $\iff$ (iv). Similarly, by comparing the coefficients of t^4 and t^3 , we also have $A = \mathfrak{A}$ and $B = \mathfrak{B} - 4A\alpha$. In the extremal case, Eq. (14) and Proposition 6 therefore tell us that

$$s = 12(At + B) = 24A \left(t + \frac{B}{2A} \right) = 24A \left(t + \frac{\mathfrak{B}}{2\mathfrak{A}} - 2\alpha \right) = 24A(t - \alpha),$$

which is to say that s is a non-zero constant times $x = t - \alpha$. However, a result of Derdziński [12, Proposition 4] asserts that if g is an extremal Kähler metric in real dimension 4, with non-constant scalar curvature s , then $s^{-2}g$ is Einstein iff the latter metric has constant scalar curvature. But $h = x^{-2}g$ has constant scalar curvature $s_h = \kappa$ by construction, so this shows that (iii) $\Rightarrow$ (i). On the other hand, in real dimension 4, any Einstein metric is Bach-flat, and any Bach-flat Kähler metric is extremal. Hence (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and we are done. $\square$

Proposition 8. *Let h be any Einstein–Maxwell metric on a spherical shell arising by rescaling the Kähler metric associated with a quartic polynomial $\Psi(t)$. Then h is also obtained by rescaling a second Kähler metric $\hat{g}$, which is instead compatible with an oppositely oriented complex structure on the shell. Moreover, in inverted coordinates, $\hat{g}$ is associated with the quartic polynomial*

$$\tilde{\Psi}(t) = t^4 \Psi(t^{-1}).$$

Proof. If a $U(2)$ -invariant Kähler metric g is expressed as

$$g = t \left[\frac{dt^2}{2\Psi(t)} + 2(\sigma_1^2 + \sigma_2^2) \right] + \frac{2\Psi(t)}{t} \sigma_3^2$$

then the substitution $t = 1/\tau$ yields

$$\begin{aligned} g &= \frac{1}{t^2} \left(\tau \left[\frac{d\tau^2}{2\tau^4 \Psi(\tau^{-1})} + 2(\sigma_1^2 + \sigma_2^2) \right] + \frac{2\tau^4 \Psi(\tau^{-1})}{\tau} \sigma_3^2 \right) \\ &= \frac{1}{t^2} \left(\tau \left[\frac{d\tau^2}{2\tilde{\Psi}(\tau)} + 2(\sigma_1^2 + \sigma_2^2) \right] + \frac{2\tilde{\Psi}(\tau)}{\tau} \sigma_3^2 \right) \end{aligned}$$

where $\tilde{\Psi}(\tau) := \tau^4 \Psi(\tau^{-1})$. This shows that the metric $\hat{g} = t^2 g = g/t^2$ is also Kähler, although instead compatible with an oppositely oriented complex structure. Moreover, when Ψ is a quartic polynomial, $\tilde{\Psi}(\tau) = \tau^4 \Psi(\tau^{-1})$ is once again a quartic polynomial. $\square$

In other words, in the terminology of [1], these metrics are *ambi-Kähler*. We note in passing that if $h = (t - \alpha)^{-2}g$ for some $\alpha > 0$, then $h = (1 - \alpha/t)^{-2}\hat{g} = \hat{\alpha}^2(t - \hat{\alpha})^{-2}\hat{g}$, where $\hat{\alpha} := 1/\alpha$. Requiring that h have constant scalar curvature is thus equivalent either to stipulating that Ψ take the form (10), or to requiring that the expansion of $\tilde{\Psi}$ in $(t - \hat{\alpha})$ be analogously constrained. We leave it as an exercise for the interested reader to verify by direct calculation that these algebraic constraints on the quartics Ψ and $\tilde{\Psi}$ are indeed equivalent.

3. Solutions on Compact 4-Manifolds

In the previous section, we produced a family of Einstein–Maxwell metrics on spherical shells $(a, b) \times S^3$. For simplicity, we will henceforth systematically use $x = t - \alpha$ as the “radial” variable on our shell, so that g is given by (12), with $x \in (a, b)$, and h is given by (13). We will now next seek to ascertain when some $\mathbb{Z}_k$ -quotient of $(a, b) \times (S^3/\mathbb{Z}_k)$ of such a shell has a metric-space completion which is a compact Riemannian manifold, where the $\mathbb{Z}_k$ -action is generated by $\exp(2\pi\xi/k)$.

Proposition 9. Let $\Psi(x)$ be a quartic polynomial of the form (11). Suppose that $\mathfrak{A} \neq 0$, that a and b have the same sign, that $x + \alpha > 0$ for $x \in [a, b]$, and that $\Psi(x) > 0$ for $x \in (a, b)$. If

$$\Psi(a) = \Psi(b) = 0, \quad \Psi'|_{x=a} = k(a + \alpha), \quad \text{and} \quad \Psi'|_{x=b} = -k(b + \alpha),$$

then the metric g defined on $(a, b) \times (S^3/\mathbb{Z}_k)$ by (12) extends to a Kähler metric on a compact complex manifold (M, J) obtained by adding two copies of $\mathbb{CP}_1$, one at $x = a$ and one at $x = b$. This metric is invariant under an isometric action of $U(2)$ on (M, g) , and (13) defines a strongly Hermitian Einstein–Maxwell metric h on (M, J) .

Proof. Under the composition of the Hopf map $S^3/\mathbb{Z}_k \rightarrow \mathbb{CP}_1$ and the factor projection $(a, b) \times (S^3/\mathbb{Z}_k) \rightarrow (S^3/\mathbb{Z}_k)$, our metric g lives on an annulus bundle over $\mathbb{CP}_1$. However, if we now choose to view each fiber annulus as a twice-punctured 2-sphere, our hypotheses then guarantee that fiber-wise metric extends smoothly to a smooth metric on this S^2 . Indeed, the fiber-wise metric takes the form

$$g|_{\text{fiber}} = \frac{dx^2}{\Upsilon(x)} + \Upsilon(x)d\vartheta^2 = \frac{dx^2}{\Upsilon(x)} + \frac{\Upsilon(x)}{k^2}d\hat{\vartheta}^2$$

where $\Upsilon = 2\Psi/(x + \alpha)$, $d\vartheta = \sigma_3|_{\text{fiber}}$, and $\hat{\vartheta} = k\vartheta$. Our hypotheses guarantee that $\Upsilon(x) = 2k(x - a) + O((x - a)^2)$ and that $\Upsilon' = 2k + O(x - a) = 2k + O(\Upsilon)$, where the error terms are rational functions of $x - a$ which are regular at $x - a = 0$, and so are real-analytic functions of Υ in a neighborhood of 0. On an interval $x \in (a, a + \varepsilon)$ where Υ is increasing, let us therefore choose

$$\mathfrak{x} = \frac{\sqrt{\Upsilon}}{k}$$

as a new “radial” coordinate. We then have $d\mathfrak{x} = [\Upsilon' dt]/(2k\sqrt{\Upsilon})$, so the fiber metric becomes

$$g|_{\text{fiber}} = \left(\frac{2k}{\Upsilon'}\right)^2 d\mathfrak{x}^2 + \mathfrak{x}^2 d\hat{\vartheta}^2 = (1 + \mathbb{I}(\mathfrak{x}^2))d\mathfrak{x}^2 + \mathfrak{x}^2 d\hat{\vartheta}^2$$

for some real-analytic function $\mathbb{I}(u)$ which vanishes at $u = 0$. It follows that the fiber metric is a real-analytic Riemannian metric in a neighborhood of the puncture $x = a$. The same argument with $x - a$ replaced by $b - x$, similarly shows that the fiber metric extends real-analytically across the puncture $x = b$.

Regularity of the remaining components of the metric g is now straightforward. Indeed, notice that x is a real-analytic function of $\mathfrak{x}^2$, where $\mathfrak{x}$ is the local radial fiber coordinate introduced above. Since $\sigma_1^2 + \sigma_2^2$ is the standard curvature-4 metric on $\mathbb{CP}_1$, the terms in the metric gotten by multiplying $\sigma_1^2 + \sigma_2^2$ by $2(x + \alpha)$ are therefore real-analytic. The rest of the metric is then just obtained by extending the fiber metric to TM by taking it to annihilate the horizontal space of the standard homogeneous connection on the Chern class $\pm k$ disk bundle over $\mathbb{CP}_1$, so the resulting metric g is actually real-analytic on M . By the same argument, the Kähler form

$$\omega = dx \wedge \sigma_3 + 2(x + \alpha)\sigma_1 \wedge \sigma_2,$$

of g also extends real-analytically to M ; and since $\nabla\omega = 0$ on an open set of M , it follows that ω is a parallel form on (M, g) . Thus g is actually a Kähler metric on M .

Moreover, the fiber 2-spheres are holomorphic curves on an open dense set, and hence everywhere by continuity. Their Riemannian normal bundles are therefore J -invariant, and we therefore see that the two copies of $\mathbb{CP}_1$ we have added at $x = a$ and $x = b$ are now actually holomorphic curves. The original isometric action of $U(2)$ on the shell also acts isometrically along these added curves, and so defines a global isometric action on (M, g) . Finally, the real-analytic function x on M is a holomorphy potential on a dense set, and hence everywhere, while the globally defined Hermitian metric $h = x^{-2}g$ has constant scalar curvature on an open dense set, and hence everywhere. It follows that h is a strongly Hermitian Einstein–Maxwell metric on (M, J) . $\square$

Let us now construct the most general quartic polynomial $\Psi(x)$ with the required properties. We begin by choosing two real numbers $b > a$. We will then try to arrange for $x = a$ and $x = b$ to be two successive zeros of Ψ by setting

$$\Psi = (b - x)(x - a)Q(x)$$

for some quadratic polynomial Q which is positive on $[a, b]$. In order to ensure that these zeros merely correspond to coordinate singularities, though, Proposition 9 insists that we stipulate that

$$\Psi'|_{x=a} = k(a + \alpha), \quad \Psi'|_{x=b} = -k(b + \alpha)$$

and this now becomes the requirement that

$$Q(x) = \frac{k(x + \alpha) + E(b - x)(x - a)}{b - a}$$

for some positive integer k and some real constant E . Since $h = g/x^2$, we must also, in keeping with Proposition 9, require that a and b have the *same sign*, so as to guarantee that x^2 will be positive on $[a, b]$. Our challenge is now to arrange for the resulting quartic

$$\Psi = \frac{(b - x)(x - a)}{b - a} [k(x + \alpha) + E(b - x)(x - a)] \quad (15)$$

to take the form

$$\Psi = \mathfrak{A}x^4 + \mathfrak{B}x^3 + x^2 + \mathfrak{C}x + \frac{\mathfrak{C}\alpha}{2}$$

required by (11), while still remaining positive on the interval (a, b) . Along the way, we must also remember to verify that $t = x + \alpha$ is strictly positive on $[a, b]$, so that Eqs. (12) and (13) will actually give rise to a positive definite metric g .

Expanding (15) in powers of x and setting the coefficient of x^2 equal to 1, as in (11), we obtain the equation

$$\frac{k(a + b - \alpha) + E(a^2 + 4ab + b^2)}{b - a} = 1,$$

which is equivalent to the requirement that

$$E = \frac{k\alpha - (k + 1)a - (k - 1)b}{a^2 + 4ab + b^2}. \quad (16)$$

The other constraint imposed by requiring that (15) take the form (11) is that the ratio between the constant term and the coefficient of x should be $\alpha/2$:

$$\frac{\alpha}{2} = \frac{abE - \alpha k}{\alpha k(a^{-1} + b^{-1}) - (2(a+b)E + k)}$$

This is equivalent to

$$k(a^{-1} + b^{-1})\alpha^2 - (2(a+b)E - k)\alpha - 2abE = 0.$$

After using the linear substitution (16) for E , this becomes the quadratic equation

$$\frac{[(a+b)\alpha + ab][(a+b)^2k\alpha + 2a^2b(k+1) + 2ab^2(k-1)]}{ab(a^2 + 4ab + b^2)} = 0$$

for α . Thus, there are precisely two possible choices for α ; either

$$\alpha = -\frac{ab}{a+b} \quad (17)$$

or else

$$\alpha = -\frac{2ab[a(k+1) + b(k-1)]}{k(a+b)^2}. \quad (18)$$

Each of these choices then determines a value for E via (16), and thus a polynomial $\Psi(x)$ via (15).

Of course, $t = x + \alpha$ must be positive on $[a, b]$ for (12) to give rise to a metric on a compact manifold. But this requirement is simply equivalent to the condition that $a + \alpha > 0$. If (17) holds,

$$a + \alpha = a - \frac{ab}{a+b} = \frac{a^2}{a+b} > 0,$$

and, since a and b have the same sign, this condition is satisfied if and only if $b > a > 0$, independent of the value of k . By contrast, if (18) holds, we then have

$$a + \alpha = a - \frac{2ab[a(k+1) + b(k-1)]}{k(a+b)^2} = -\frac{(b-a)[(k-2)ab + ka^2]}{k(a+b)^2},$$

and, because a and b have the same sign, this is positive if and only if $k = 1$ and $b > a > 0$. From now on, we may thus assume that $b > a > 0$. When $k \geq 2$, we will also only need to consider the choice of α given by (17). On the other hand, if $k = 1$, both (17) and (18) remain viable candidates.

The final condition required for (12) to yield a solution on a compact manifold is that Ψ must be positive on the open interval (a, b) . This will happen if and only if

$(b-a)Q(x)$ is positive on the closed interval $[a, b] \subset \mathbb{R}^+$. For the choice (17), this function can be expressed in terms of the variable $y = x - a$ as

$$\begin{aligned}(b-a)Q(x) &= k(x+\alpha) + E(b-x)(x-a) \\ &= k\left(x - \frac{ab}{a+b}\right) + \frac{-\frac{kab}{a+b} - (k+1)a - (k-1)b}{a^2 + 4ab + b^2}(b-x)(a-x) \\ &= k\left(y + a - \frac{ab}{a+b}\right) + \frac{-\frac{kab}{a+b} - (k+1)a - (k-1)b}{a^2 + 4ab + b^2}(b-a-y)y \\ &= \frac{ka^2}{a+b} + \frac{b^3 + (3k-1)ab^2 + (7k-1)a^2b + (2k+1)a^3}{(a+b)(a^2 + 4ab + b^2)}y \\ &\quad + \frac{(k-1)b^2 + 3kab + (k+1)a^2}{(a+b)(a^2 + 4ab + b^2)}y^2,\end{aligned}$$

which is strictly positive when $y = x - a > 0$, and so is positive for $x \in [a, b]$, for any $k \geq 1$. For the choice (18), with $k = 1$, we instead have

$$\begin{aligned}(b-a)Q(x) &= (x+\alpha) + E(b-x)(x-a) \\ &= \left(x - \frac{4a^2b}{(a+b)^2}\right) + \frac{-\frac{4a^2b}{(a+b)^2} - 2a}{a^2 + 4ab + b^2}(b-x)(x-a) \\ &= \left(y + a - \frac{4a^2b}{(a+b)^2}\right) - \frac{4a^2b + 2a(a+b)^2}{(a+b)^2(a^2 + 4ab + b^2)}(b-a-y)y \\ &= \frac{a(b-a)^2 + (3a^2 + b^2)y + 2ay^2}{(a+b)^2},\end{aligned}$$

which is again positive for $y = x - a \geq 0$, and so, in particular, for $x \in [a, b]$. Thus, when $k = 1$, both (17) and (18) give us a compact solution for each choice of $b > a > 0$. When $k \geq 2$, only (17) works, but this choice in any case provides us with a compact solution for each $b > a > 0$. To summarize:

Proposition 10. *Equations (12) and (13) give rise to a strongly Hermitian Einstein–Maxwell metric h on a compact complex surface (M, J) if, for some $b > a > 0$, Ψ is given by (15), (16), and either (17) or the $k = 1$ case of (18). Moreover, any metric thus constructed is invariant under an action of $U(2)$ on M such that the generic orbit of $SU(2)$ has fundamental group $\mathbb{Z}_k$, where $k > 0$ is the integer occurring in the expression for Ψ .*

4. Geometry of the Solutions

We have now constructed some interesting families of Einstein–Maxwell metrics on compact complex surfaces. It remains to completely understand the differential and algebraic geometry of these solutions. We begin with the following global characterization:

Theorem 1. *Let h be a strongly Hermitian Einstein–Maxwell metric on a compact complex surface (M^4, J) with $b_- \neq 0$. Suppose that h is not a Kähler metric, and is invariant under an $SU(2)$ -action on M which has a 3-dimensional orbit. Then (M, J) is*

the k th Hirzebruch surface Σ_k for some $k > 0$, and (M, h) contains an open dense set $\mathcal{U}$ which is isometric to a shell $(a, b) \times (S^3/\mathbb{Z}_k)$, equipped with a metric given by (12) and (13), for Ψ defined by (15), (16), and either (17) or the $k = 1$ case of (18). The set $\mathcal{U} \subset \Sigma_k = \mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O})$ is exactly the complement of the two holomorphic sections of $\Sigma_k \rightarrow \mathbb{CP}_1$ arising from the two sub-bundles $\mathcal{O}(k)$ and $\mathcal{O}$ of $\mathcal{O}(k) \oplus \mathcal{O}$, while the restriction of the projection $\Sigma_k \rightarrow \mathbb{CP}_1$ to $\mathcal{U} \approx (a, b) \times (S^3/\mathbb{Z}_k)$ is just the composition of the factor projection $(a, b) \times (S^3/\mathbb{Z}_k) \rightarrow (S^3/\mathbb{Z}_k)$ and the Hopf map $(S^3/\mathbb{Z}_k) \rightarrow \mathbb{CP}_1$.

Proof. Recall that h is said to be a strongly Hermitian Einstein–Maxwell metric on (M, J) iff there is a 2-form F such that both h and F are both J -invariant, and such that (h, F) is a solution of the Einstein–Maxwell equations (1)–(3). Since (M, J) is a compact complex surface, this is equivalent [29, Theorem B] to saying that h has constant scalar curvature, and can be expressed as $h = f^{-2}g$, where g is a Kähler metric on (M, J) , and $f \neq 0$ is a real holomorphy potential. In particular, this implies that (M, J) is of Kähler type. Since h is assumed to be non-Kähler, f must be non-constant, thereby making $\xi := J \operatorname{grad}_g f$ a nontrivial Killing field of g . Moreover, since $\xi f = 0$, it follows that ξ is also a Killing field of $h = f^{-2}g$.

Let $\zeta_1, \zeta_2, \zeta_3$ be infinitesimal generators of the $SU(2)$ -action, where $[\zeta_1, \zeta_2] = \zeta_3$ and its cyclic permutations all hold. Since $SU(2)$ acts by isometries of h which are homotopic to the identity, it preserves any 2-form which is harmonic with respect to h , and therefore preserves the Kähler form ω of g . Consequently, it therefore preserves the holomorphy potential $f = \pm 2^{-1/4} |\omega|_h^{1/2}$, and therefore preserves $g = f^2 h$. Since the action preserves both ω and g , it follows that it also preserves J . Thus, the real vector fields ζ_j represent infinitesimal symplectomorphisms of (M, ω) , and the complex vector fields $Z_j = \eta_j - iJ\eta_j$, $j = 1, 2, 3$, are holomorphic vector fields on (M, J) . Moreover, the commutation relation $[\zeta_1, \zeta_2] = \zeta_3$ guarantees that $\omega(\zeta_1, \zeta_2)$ is a Hamiltonian for ζ_3 , and so, by taking cyclic permutations, we thus see that the ζ_j are all globally Hamiltonian vector fields. However, a Hamiltonian vector field is zero at a minimum of the Hamiltonian, and, since M is compact by hypothesis, such minima must in fact exist. This shows that the Killing fields ζ_j and the associated holomorphic vector fields Z_j all have zeros somewhere on M .

If $\mathcal{X}$ is a 3-dimensional orbit of $SU(2)$, then $T_p M = T_p \mathcal{X} + J(T_p \mathcal{X})$ at any $p \in \mathcal{X}$, and some pair of the Z_j spans $T^{1,0} M$ in a neighborhood of any $p \in \mathcal{X}$; by renumbering, we may take these vector fields to be Z_1 and Z_2 . If α is a holomorphic 1-form on M , then α is completely determined in a neighborhood of p by the holomorphic functions $\alpha(Z_1)$ and $\alpha(Z_2)$, which are its components in the holomorphic co-frame dual to (Z_1, Z_2) . But $\alpha(Z_j)$ is a globally defined holomorphic function on M , and so is constant; and since Z_j has a zero somewhere on M , this constant must be zero. Thus $\alpha \equiv 0$ in a neighborhood of p , and therefore on all of M by uniqueness of analytic continuation. In other words, $h^{1,0}(M) = \dim H^0(M, \Omega^1) = 0$. But since (M, J) is of Kähler type, the Hodge decomposition therefore tells us that $b_1(M) = 2h^{1,0}(M) = 0$. In particular, M has Euler characteristic $\chi(M) = 2 - 2b_1 + b_2 = 2 + b_+ + b_- \geq 4$, since the non-triviality of the Kähler class $[\omega]$ shows that $b_+(M) \neq 0$, and we have $b_-(M) \neq 0$ by hypothesis.

Since $\xi = J \operatorname{grad} f$ is a Killing field, the zero set of ξ is a union of totally geodesic submanifolds [20]; moreover, every normal derivative of ξ at such a submanifold must be non-zero, because the restriction of ξ to any normal geodesic must be a Jacobi field which is not identically zero. This implies that f is a generalized Morse function in the

sense of Bott [7]. On the other hand, since $\xi - iJ\xi$ is holomorphic,

$$\nabla_{\bar{\mu}} \nabla_{\bar{\nu}} f = g_{\lambda\bar{\nu}} \nabla_{\bar{\mu}} \nabla^{\lambda} f = 0,$$

and since f is real, it therefore follows that the Riemannian Hessian ∇df , computed relative to g , is J -invariant. Consequently, the naïve Hessian of f is J -invariant at any critical point. Thus, the critical set of f is a union of totally geodesic holomorphic curves and isolated, non-degenerate critical points. However, since f is $SU(2)$ -invariant, any isolated critical point q would have to be fixed by the $SU(2)$ -action, and since the action, being isometric, commutes with the exponential map at q , the Hessian would also have to be invariant under a non-trivial representation of $SU(2)$ on $T_q M \cong \mathbb{C}^2$, and would therefore have to be a non-zero multiple of g . This shows that any isolated critical point must be a non-degenerate local maximum or local minimum. However, the J -invariance of the Hessian also implies that any critical submanifold of real dimension 2 is also a local maximum or minimum of f . In particular, there are no critical points where the Hessian has index 1. Since M is connected, the set of local minima must therefore be connected, because this excludes any way to join two components by passing a critical point. Similarly, the set of local maxima must also be connected. If we use C_- and C_+ to denote the sets of local minima and local maxima, respectively, it follows that each of these two sets is either a point or a compact connected Riemann surface. In particular, these sets have Euler characteristic $\chi(C_{\pm}) \leq 2$, with equality iff $C_{\pm} \cong \mathbb{CP}_1$. However, since ξ is a Killing field, the Euler characteristic of M coincides [20] with the Euler characteristic of its fixed point set. But, since we have already observed that $\chi(M) = 2 + b_+(M) + b_-(M) \geq 4$, we therefore have

$$4 \leq \chi(M) = \chi(C_+) + \chi(C_-) \leq 2 + 2 = 4,$$

and it therefore follows that $\chi(C_{\pm}) = 2$, that $C_+ \cong C_- \cong \mathbb{CP}_1$, and that $b_+(M) = b_-(M) = 1$.

Because f is invariant under the action of $SU(2)$, and because the level sets of f are all connected, any 3-dimensional orbit $\mathcal{X}$ must coincide with some non-critical level set of f . However, the flow of $\text{grad } f = -J\xi$ carries one such level set to any other, and because the action of this flow commutes with that of $SU(2)$, every non-empty non-critical level set is conversely an $SU(2)$ -orbit. On the other hand, since the action also preserves the Riemannian distance from either critical level set $C_{\pm}$, some, and hence any, 3-dimensional orbit $\mathcal{X}$ of $SU(2)$ is diffeomorphic to the unit normal bundle of C_+ or C_- . In particular, the unit normal bundle of $C_{\pm}$ has finite fundamental group, so the normal bundle of $C_{\pm}$ is necessarily non-trivial. Moreover, if we set $k = |\pi_1(\mathcal{X})|$ for some 3-dimensional orbit $\mathcal{X}$, then k coincides with the absolute value $|C_{\pm}^2|$ of the self-intersection numbers of these complex curves. However, $b_+(M) = b_-(M) = 1$, and $C_- \cdot C_+ = 0$ because $C_- \cap C_+ = \emptyset$; thus, only one of the curves C_+ , C_- can have positive self-intersection, and only one of them can have negative self-intersection. At the price of possibly replacing f with $-f$, we can thus arrange that $C_{\pm}^2 = \pm k$, where $k > 0$.

Because $SU(2)$ acts transitively and isometrically on each level set $\mathcal{X} = f^{-1}(t)$, the function $u = |\text{grad } f|$ is constant on each level set of f , and $\varphi = u^{-1}df$ is therefore a closed 1-form on the set $\mathcal{U}$ where it is defined. The unit vector field $\eta = u^{-1} \text{grad } f$ therefore satisfies

$$g_{ac} \nabla_{\eta} \eta^c = \eta^b \nabla_b \varphi_a = \eta^b \nabla_a \varphi_b = \eta^b \nabla_a g_{bc} \eta^c = \frac{1}{2} \nabla_a |\eta|^2 = 0,$$

and η is therefore a geodesic vector field. Hence $J\xi = -u\eta$ is tangent to the normal geodesic sprays of C_+ and C_- . On the other hand, we have already observed that ξ corresponds, under the normal exponential maps, to a rotation vector field in the fibers of both these normal bundles. If $\mathfrak{R}$ denotes the Riemannian distance from C_- to C_+ , the Morse-theoretic picture of $f : M \rightarrow \mathbb{R}$ thus amounts to saying that M is the union of the normal disk bundles of radius $\mathfrak{R}/2$, glued together along their boundaries in such a manner that the boundary of every fiber disk is sent to the boundary of a fiber disk on the opposite side via a reflection. This displays M as the total space of a smooth 2-sphere bundle $\varpi : M \rightarrow S^2$. However, the fiber 2-spheres of the submersion ϖ must be holomorphic curves, because their tangent spaces are spanned by ξ and $J\xi$ on a dense set. Moreover, the restriction of ϖ to $\mathcal{U} = M - (C_- \cup C_+)$ becomes a holomorphic submersion $\mathcal{U} \rightarrow \mathbb{CP}^1$ for a unique choice of complex structure on the target S^2 , since the fibers are the orbits of a free holomorphic $\mathbb{C}^\times$ -action on $\mathcal{U}$. Our submersion thus becomes a smooth map $\varpi : M \rightarrow \mathbb{CP}^1$ which is holomorphic on an open dense set, and therefore holomorphic everywhere. Thus (M, J) is the total space of a holomorphic $\mathbb{CP}^1$ -bundle over a complex curve. On the other hand, one can show [4, Proposition V.4.1] that any such $\mathbb{CP}^1$ -bundle is the projectivization $\mathbb{P}(\mathcal{V})$ of a rank-2 holomorphic vector bundle $\mathcal{V}$. The two curves $C_\pm$ in M now determine a pair of line sub-bundles $\mathcal{L}_\mp$ of $\mathcal{V} \rightarrow \mathbb{CP}^1$ such that $\mathcal{V} = \mathcal{L}_- \oplus \mathcal{L}_+$. Hence

$$M = \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) = \mathbb{P}(\mathcal{O} \oplus (\mathcal{L}_-^* \otimes \mathcal{L}_+)) = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(\ell))$$

for some integer ℓ . But since $\ell = C_+^2 = k$, this shows that (M, J) is biholomorphic to the k th Hirzebruch surface $\Sigma_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))$.

The exponential-map model shows that ξ is periodic, and generates a free circle action on $\mathcal{U} = M - (C_- \cup C_+)$. Moreover, this same model also reveals that any 3-dimensional $\mathrm{SU}(2)$ -orbit $\mathcal{X}$ is a non-critical level set $f^{-1}(x)$ of the Hamiltonian f of ξ , and that the circle bundle $\mathcal{X} \rightarrow \mathcal{X}/S^1$ over any symplectic quotient $f^{-1}(x)/S^1 \approx S^2$ is isomorphic to the unit normal bundle of C_- , which has Chern class $-k$. On the other hand, at the price of replacing g with c^2g , for a positive constant c , while simultaneously replacing f with cf , we can now replace ξ with $c^{-1}\xi$. We can thus arrange for ξ to have minimal period $2\pi/k$. On the universal cover $\tilde{\mathcal{U}}$ of $M - (C_- \cup C_+)$, this then implies that ξ has period 2π , and the symplectic reduction quotient $f^{-1}(x) \rightarrow S^2$ thus becomes the circle bundle of Chern class -1 , with ξ generating the standard action of S^1 . The Duistermaat-Heckman formula [15] thus asserts that the area of the symplectic reduction of $f^{-1}(x)$ must therefore be $2\pi(x + \alpha)$ for some real constant α . Moreover, if we adopt the convention that $\mathrm{SU}(2)$ acts from the left, the ξ becomes a left-invariant vector field on each $\mathrm{SU}(2)$ -orbit in $\tilde{\mathcal{U}}$, and so generates a right action of $\mathrm{U}(1)$ which enriches the $\mathrm{SU}(2)$ -action into a $\mathrm{U}(2)$ -action. This now puts g in the form (12) on $\mathcal{U}$, while simultaneously putting h in the form (13). But since $g(\xi, \xi) = 2\Psi/(x + \alpha)$ must tend to zero as we approach $C_\pm$, it follows that Ψ must vanish at $a = x(C_-)$ and $b = x(C_+)$. Moreover, since ξ has minimal period $2\pi/k$ on M , the derivative of $\|\xi\|$ along unit-speed geodesics orthogonal to $C_\pm$ must tend to $\mp k$ as we approach $C_\pm$, so

$$\left[\frac{2\Psi}{x + \alpha} \right]^{1/2} \frac{d}{dx} \left[\frac{2\Psi}{x + \alpha} \right]^{1/2} = \frac{1}{2} \frac{d}{dx} \left[\frac{2\Psi}{x + \alpha} \right] = \left[\frac{2\Psi}{x + \alpha} \right]' \rightarrow \pm k$$

as $x \rightarrow a^+$ or b^- , and hence

$$\Psi(a) = \Psi(b) = 0, \quad \Psi'(a) = k(a + \alpha), \quad \Psi'(b) = -k(b + \alpha).$$

Our previous discussion of the ODE for Ψ then shows that it must be defined by (15), (16), and either (17) or the $k = 1$ case of (18). $\square$

In particular, the constructed solutions give us Einstein–Maxwell metrics on all the Hirzebruch surfaces $\Sigma_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k)) \rightarrow \mathbb{CP}_1$ for $k > 0$. Of course, this list omits the Hirzebruch surface $\Sigma_0 = \mathbb{CP}_1 \times \mathbb{CP}_1$, but Σ_0 does carry obvious solutions provided by cscK metrics. Our construction therefore proves the following:

Theorem 2. *Let Σ_k be any Hirzebruch surface, and let $\Omega \in H^2(\Sigma_k, \mathbb{R})$ be any Kähler class. Then Ω can be represented by a Kähler metric g which is conformally related to an Einstein–Maxwell metric h . Moreover, if $k \geq 2$, there is, up to isometry, exactly one such g in Ω such that h is invariant under the standard action of $U(2)$ on Σ_k .*

Proof. If $k > 0$, we obtain such a metric on $\Sigma_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))$ for any $b > a > 0$ by letting α be given by (17). From the symplectic point of view, the resulting manifold is obtained by applying symplectic cutting [31] to $\mathbb{R}^+ \times (S^3/\mathbb{Z}_k)$, equipped with the symplectic form

$$\omega = dt \wedge \sigma_3 + 2t \sigma_1 \wedge \sigma_2$$

where the cut has been carried out at the level sets $t = a + \alpha$ and $t = b + \alpha$, where t is a Hamiltonian for the periodic vector field ξ . These level sets become the holomorphic curves $\mathcal{C}_\pm$ of self-intersection $\pm k$ arising from the two summands of $\mathcal{O} \oplus \mathcal{O}(k)$. The symplectic form on these curves is just $2t\sigma_1 \wedge \sigma_2$, so their areas are given by $\omega(\mathcal{C}_-) = 2\pi(a + \alpha)$ and $\omega(\mathcal{C}_+) = 2\pi(b + \alpha)$. Plugging in the value for α given by (17) therefore tells us that

$$\omega(\mathcal{C}_-) = \frac{2\pi a^2}{a+b} \quad \text{and} \quad \omega(\mathcal{C}_+) = \frac{2\pi b^2}{a+b}.$$

However, the fiber $\mathcal{F}$ of $\Sigma_k \rightarrow \mathbb{CP}_1$ is also a holomorphic curve, with homology class given by

$$\mathcal{F} = \frac{1}{k} (\mathcal{C}_+ - \mathcal{C}_-)$$

and $\mathcal{F}$ and $\mathcal{C}_-$ together generate $H_2(\Sigma_k, \mathbb{Z})$. Since

$$\omega(\mathcal{C}_-) = \frac{2\pi a^2}{a+b} \quad \text{and} \quad \omega(\mathcal{F}) = \frac{2\pi(b-a)}{k},$$

our construction allows us to take the areas of $\mathcal{C}_-$ and $\mathcal{F}$ to be any pair of positive numbers by choosing $b > a > 0$ appropriately. Since the area of any holomorphic curve is certainly positive, every Kähler class on Σ_k , $k > 0$, is swept out taking α to be given by (17).

On the other hand, when $k = 0$, every Kähler class on $\Sigma_0 = \mathbb{CP}_1 \times \mathbb{CP}_1$ contains a cscK metric obtained by taking the Riemannian product of two round metrics on S^2 of appropriate radii. The claim therefore follows. $\square$

Theorems A and D now follow, as specializations of Theorem 2.

Next, let us notice that the family of Kähler metrics g on Σ_1 arising from (18), with $k = 1$, behaves quite differently from the family arising from (17). As a matter of notation, recall that Σ_1 is also the one-point blow-up of $\mathbb{CP}_2$; the curves $\mathcal{C}_-$ and $\mathcal{C}_+$ are

therefore usually called $\mathcal{E}$ and $\mathcal{L}$, because $\mathcal{L} = \mathcal{C}_+$ corresponds to a generic projective line in $\mathbb{CP}_2$, while the curve $\mathcal{E} = \mathcal{C}_-$ is *exceptional*, in the sense that its embedding in Σ_1 is rigid. Since $\omega(\mathcal{C}_-) = 2\pi(a + \alpha)$ and $\omega(\mathcal{C}_+) = 2\pi(b + \alpha)$, the value of α provided by (18), with $k = 1$, yields

$$\begin{aligned}\omega(\mathcal{E}) = \omega(\mathcal{C}_-) &= 2\pi \left(a - \frac{4a^2b}{(a+b)^2} \right) = \frac{2\pi a(b-a)^2}{(b+a)^2} \\ \omega(\mathcal{L}) = \omega(\mathcal{C}_+) &= 2\pi \left(b - \frac{4a^2b}{(a+b)^2} \right) = \frac{2\pi b(b+3a)(b-a)}{(a+b)^2},\end{aligned}$$

so that the Kähler metric g arising from the data $b > a > 0$ then belongs to the Kähler class

$$\Omega = [\omega] = \frac{2\pi b(b+3a)(b-a)}{(a+b)^2} \mathcal{L} - \frac{2\pi a(b-a)^2}{(a+b)^2} \mathcal{E}.$$

Writing this schematically as

$$\Omega = u\mathcal{L} - v\mathcal{E},$$

we then have

$$u - v = 2\pi(b - a) \quad (19)$$

$$\frac{u}{v} = \frac{b(b+3a)}{a(b-a)} \quad (20)$$

and these two pieces of information of course completely determine (u, v) as a function of (a, b) . However, this function is neither injective nor surjective. To clarify this point, set

$$b/a = 1 + 2\mathfrak{z},$$

where $\mathfrak{z}$ is an arbitrary positive real number. Then (20) may be rewritten as

$$\frac{u}{v} = 5 + 2 \left(\mathfrak{z} + \frac{1}{\mathfrak{z}} \right).$$

Now notice that the right-hand side is invariant under $\mathfrak{z} \mapsto 1/\mathfrak{z}$, tends to $+\infty$ as $\mathfrak{z} \rightarrow +\infty$, and has positive $\mathfrak{z}$ -derivative when $\mathfrak{z} > 1$. It therefore follows that $u/v \geq 9$, and that any $u/v > 9$ arises from exactly two values of $\mathfrak{z}$, which are interchanged by $\mathfrak{z} \mapsto 1/\mathfrak{z}$. Also note that when $\mathfrak{z} = 1$, or in other words when $b/a = 3$, the $k = 1$ case of (18) yields

$$\alpha = - \frac{4a^2b}{(a+b)^2} \Big|_{b/a=3} = -\frac{3a}{4}$$

while plugging $b/a = 3$ into (17) similarly results in

$$\alpha = - \frac{ab}{a+b} \Big|_{b/a=3} = -\frac{3a}{4}.$$

Plugging either of these into (16) and (15), with $k = 1$, thus produces exactly the same polynomial Ψ for a given pair with $b = 3a > 0$. Thus, while we have three different

solutions Ψ for any given Kähler class when $u/v > 9$, these solutions actually merge into a single solution when $u/v = 9$.

While knowing a polynomial Ψ is equivalent to knowing a pair (g, f) consisting of a Kähler metric g and a holomorphy potential f such that $h = f^{-2}g$ has constant scalar curvature, it is not immediately clear whether the metrics g arising from different $\Psi(x)$ are actually different. Let us therefore prove that a Kähler metric g arising from (17) can never be isometric to one arising from (18), except in the case $u/v = 9$ discussed above. To do this, first observe that, for any of the Kähler metrics on $\mathbb{CP}_2 \# \mathbb{CP}_2$ under discussion, the identity component of the isometry group must be exactly $U(2)$, since $U(2) \subset \text{Iso}_0(M, g)$, $\text{Iso}_0(M, g) \subset \text{Aut}_0(M, J)$, and $U(2) \subset \text{Aut}_0(M, J)$ is a maximal compact subgroup. Thus $\text{Iso}_0(M, g)$ has exactly two 2-dimensional orbits, namely $\mathcal{C}_\pm$, and the unordered pair consisting of the values of the scalar curvature s at the exceptional orbits $\mathcal{C}_\pm$ therefore constitutes a Riemannian invariant of g . Now the scalar curvature of g is given by

$$s(x) = \frac{2}{x + \alpha} (2 - \Psi''(x)),$$

and therefore takes the values

$$s(a) = \frac{2}{a + \alpha} (2 - \Psi''(a)) \quad \text{and} \quad s(b) = \frac{2}{b + \alpha} (2 - \Psi''(b))$$

along $\mathcal{C}_-$ and $\mathcal{C}_+$, respectively. Substituting the $k = 1$ versions of (15) and (16), we thus have

$$\begin{aligned} s(a) &= \frac{12a(b^2 + a^2 + 2b\alpha)}{(b - a)(a^2 + 4ab + b^2)(a + \alpha)} \\ s(b) &= \frac{12b(b^2 - 3a^2 + 4ab + 2a\alpha)}{(b - a)(a^2 + 4ab + b^2)(b + \alpha)} \end{aligned}$$

for these metrics. Plugging in the value for α given by (18), with $k = 1$, we then obtain

$$s(a) = \frac{12}{b - a} = s(b),$$

so, for this family, the scalar curvature takes the same value along these two exceptional orbits. By contrast, substituting the value for α given by (17) instead results in the values

$$s(a) = \frac{12(b^3 - ab^2 + a^2b + a^3)}{a(b - a)(a^2 + 4ab + b^2)} \quad \text{and} \quad s(b) = \frac{12(b^3 + 5ab^2 - a^2b - 3a^3)}{b(b - a)(a^2 + 4ab + b^2)},$$

so that

$$s(b) - s(a) = \frac{12(3a - b)(a + b)^2}{ab(a^2 + 4ab + b^2)},$$

and hence $s(b) \neq s(a)$ except in the special case $b/a = 3$, $u/v = 9$ discussed above. When $u/v > 9$, the Kähler metrics g arising from (17) are thus never isometric to any metric arising from (18).

By contrast, as was pointed out to the author by Christina Tønnesen-Friedman, the pairs of metrics in a given Kähler class arising from (18) are actually isometric. To see

this most clearly, let us re-express the polynomial $\Psi(x)$ in terms of the more intrinsic variable $t = x + \alpha = \varrho^2/2$ we first introduced in the context of the Bianchi IX formalism; in these terms

$$g = t \left[\frac{dt^2}{2\Psi} + 2(\sigma_1^2 + \sigma_2^2) \right] + \frac{2\Psi}{t} \sigma_3^2$$

so it will therefore suffice that the two choices of Ψ actually become the same when expressed in terms of t . However, if we set $a = a + \alpha$ and $b = b + \alpha$, the $k = 1$ version of (15) then becomes

$$\Psi = \frac{(b-t)(t-a)}{b-a} [t + E(b-t)(t-a)]$$

where $a = v/2\pi$ and $b = u/2\pi$ encode the areas of the holomorphic curves $\mathcal{C}_\pm$, and so are obviously determined by the Kähler class; thus, we merely need to show that, for two cohomologous Kähler forms arising from (18), the values of E given by (16) and (18) actually agree. To see this, let us first parameterize the allowed values of a and b by

$$\begin{aligned} a &= \frac{\mathfrak{w}}{\mathfrak{z}} \\ b &= (1+2\mathfrak{z}) \frac{\mathfrak{w}}{\mathfrak{z}} \end{aligned}$$

for $(\mathfrak{w}, \mathfrak{z}) \in (\mathbb{R}^+)^2$. The $k = 1$ case of (18) then becomes

$$\alpha = -\frac{4a^2b}{a+b} = -\frac{\mathfrak{w}(1+2\mathfrak{z})}{(1+\mathfrak{z})^2}$$

and the $k = 1$ case of (16) therefore becomes

$$E = -\frac{2a}{(a+b)^2} = -\frac{\mathfrak{z}}{2\mathfrak{w}(1+\mathfrak{z})^2} = -\frac{1/\mathfrak{z}}{2\mathfrak{w}(1+1/\mathfrak{z})^2},$$

which is manifestly invariant under $\mathfrak{z} \mapsto 1/\mathfrak{z}$. However, since

$$\begin{aligned} a &= a + \alpha = \frac{\mathfrak{w}\mathfrak{z}}{(1+\mathfrak{z})^2} = \frac{\mathfrak{w}/\mathfrak{z}}{(1+1/\mathfrak{z})^2} \\ b &= b + \alpha = \frac{\mathfrak{w}(2+\mathfrak{z})(1+2\mathfrak{z})}{(1+\mathfrak{z})^2} = \frac{\mathfrak{w}(2+1/\mathfrak{z})(1+2/\mathfrak{z})}{(1+1/\mathfrak{z})^2} \end{aligned}$$

we also see that $(\mathfrak{w}, \mathfrak{z}) \mapsto (\mathfrak{w}, 1/\mathfrak{z})$ exactly interchanges the two solutions of type (18) in a fixed Kähler class. The corresponding metrics are therefore isometric. On the other hand, we emphasize that α is *not* invariant under $(\mathfrak{w}, \mathfrak{z}) \mapsto (\mathfrak{w}, 1/\mathfrak{z})$; indeed,

$$\alpha(\mathfrak{w}, \mathfrak{z}) - \alpha(\mathfrak{w}, \frac{1}{\mathfrak{z}}) = -\frac{\mathfrak{w}(1+2\mathfrak{z})}{(1+\mathfrak{z})^2} + \frac{\mathfrak{w}(1+2/\mathfrak{z})}{(1+1/\mathfrak{z})^2} = \mathfrak{w} \left(\frac{\mathfrak{z}-1}{\mathfrak{z}+1} \right),$$

so that the functions $f = x = t - \alpha$ associated with these two solutions are always different whenever $\mathfrak{z} \neq 1$, or in other words whenever $u/v > 9$. To summarize:

Theorem 3. *Let $M = \Sigma_1$ be the blow-up of $\mathbb{CP}_2$ at a point, and let $\Omega = u\mathcal{L} - v\mathcal{E} \in H^2(M, \mathbb{R})$ be a Kähler class. If $u/v \in (1, 9]$, then, up to isometry, Ω contains a unique $U(2)$ -invariant Kähler metric which is conformal to an Einstein–Maxwell metric. By contrast, when $u/v \in (9, \infty)$, there are, up to isometry, exactly two such Kähler metrics g , and these give rise to three distinct Einstein–Maxwell metrics h via conformal rescaling.*

So far, we have been concentrating on geometric properties of the Kähler metric g , with an emphasis on its Kähler class. However, Proposition 8 shows that different Kähler metrics g can determine isometric Einstein–Maxwell metrics h . In the present context, this means that, under the orientation-reversing diffeomorphisms isotopic to the fiber-wise antipodal map of $\Sigma_k \rightarrow \mathbb{CP}_1$, conformally Kähler Einstein–Maxwell metrics pull back to conformally Kähler Einstein–Maxwell metrics. Moreover, Theorem 1 guarantees that these pull-backs continue to belong to the constructed families.

To understand the specifics of this phenomenon, let us now calculate the areas $\mathcal{A}_h$ of certain holomorphic curves with respect to the constructed Hermitian metrics h . Since $h = g/x^2$, and since the Hamiltonian x is constant on C_- and C_+ , we always have

$$\mathcal{A}_h(C_-) = \frac{2\pi(a + \alpha)}{a^2} \quad \text{and} \quad \mathcal{A}_h(C_+) = \frac{2\pi(b + \alpha)}{b^2}.$$

If α is given by (17), this then implies that

$$\mathcal{A}_h(C_-) = \mathcal{A}_h(\tilde{C}) = \frac{2\pi}{a + b},$$

no matter the value of k . By contrast, if $k = 1$ and α is given by (18), then

$$\mathcal{A}_h(C_-) = \frac{2\pi(b - a)^2}{a(b + a)^2}, \quad \text{and} \quad \mathcal{A}_h(C_+) = \frac{2\pi(b + 3a)(b - a)}{b(a + b)^2},$$

and we therefore have

$$\frac{\mathcal{A}_h(C_+)}{\mathcal{A}_h(C_-)} = \frac{a^2}{b^2} \frac{u}{v}.$$

Now recall that, if we set $b/a = 1 + 2\mathfrak{z}$, the Kähler class, up to rescaling, is characterized by the number

$$\frac{u}{v} = \frac{b(b + 3a)}{a(b - a)} = (1 + 2\mathfrak{z})(1 + \frac{2}{\mathfrak{z}})$$

so that $\mathfrak{z}$ and $1/\mathfrak{z}$ give rise to the same ray $\mathbb{R}^+\Omega = \mathbb{R}^+(u\mathcal{L} - v\mathcal{E})$ in the Kähler cone. But we now see that

$$\frac{\mathcal{A}_h(C_+)}{\mathcal{A}_h(C_-)} = \frac{a^2}{b^2} \frac{u}{v} = \frac{1 + 2/\mathfrak{z}}{1 + 2\mathfrak{z}}$$

so that interchanging these two solutions inverts the ratio of the areas of the holomorphic curves C_+ and C_- with respect to h .

By Proposition 8, all the Einstein–Maxwell metrics h we have constructed are ambi-Kähler, meaning [1] that they are conformally related to both a Kähler metric g compatible with the given orientation, and with a Kähler metric $\tilde{g}$ compatible with the opposite

orientation. If ω and $\tilde{\omega}$ are the Kähler forms of g and $\tilde{g}$, then ω spans the self-dual h -harmonic 2-forms on $M = \Sigma_k$, while $\tilde{\omega}$ spans the anti-self-dual h -harmonic 2-forms. Because $b_2(M) = b_2(\Sigma_k) = 2$, the spans $\mathbb{R}[\omega]$ and $\mathbb{R}[\tilde{\omega}]$ therefore completely determine each other, since $[\omega] \cdot [\tilde{\omega}] = 0$ with respect to the intersection pairing. If $\mathcal{F}$ is the homology class of the fiber of $\Sigma_k \rightarrow \mathbb{CP}_1$, the rays $\mathbb{R}^+[\omega]$ and $\mathbb{R}^+[\tilde{\omega}]$ therefore similarly determine each other, via the requirements $[\omega] \cdot [\tilde{\omega}] = 0$, $[\omega](\mathcal{F}) > 0$, and $[\tilde{\omega}](\mathcal{F}) < 0$. However, there is an orientation reversing diffeomorphism $\tau : M \rightarrow M$ which interchanges the two relevant complex structures, and interchanges $\mathcal{C}_+$ and $\mathcal{C}_-$; in fact, τ is essentially the antipodal map on the fibers of $\varpi : M \rightarrow \mathbb{CP}_1$, and in particular satisfies $\tau^2 = \text{id}_M$. Now, since τ reverses orientation, $(\tau^*A) \cdot (\tau^*B) = -A \cdot B$ for any $A, B \in H^2(M)$. Hence

$$[\omega] \cdot (\tau^*[\omega]) = (\tau^*\tau^*[\omega]) \cdot (\tau^*[\omega]) = -(\tau^*[\omega]) \cdot [\omega] = -[\omega] \cdot (\tau^*[\omega])$$

and it therefore follows that $[\omega] \cdot (\tau^*[\omega]) = 0$. Since we also have $(\tau^*[\omega])(\mathcal{F}) < 0$, it follows that $\mathbb{R}^+[\tilde{\omega}] = \mathbb{R}^+(\tau^*[\omega])$, so that $[\tilde{\omega}]$ is a positive constant times $\tau^*[\omega]$. Thus, up to scale, $h \mapsto \tau^*h$ must simply permute the solutions g in a given Kähler class. Since the holomorphic curves $\mathcal{C}_\pm$ have equal areas with respect to the Einstein–Maxwell metrics h arising from (17), but different areas with respect to the Einstein–Maxwell metrics h arising from (18), we therefore deduce the following:

Proposition 11. *Each of the Einstein–Maxwell metrics h arising from (17) admits an orientation-reversing isometry which interchanges $\mathcal{C}_+$ and $\mathcal{C}_-$. On the other hand, the two different Einstein–Maxwell metrics h on Σ_1 arising via (18) from a Kähler class with $u/v > 9$ are interchanged, up to overall scale, by an orientation-reversing diffeomorphism of $M = \Sigma_1$.*

Theorem B now follows from Theorem 3 and Proposition 11.

Let us now re-examine the scalar curvature s_h of the metrics $h = x^{-2}g$. By Eqs. (8), (11), and (15),

$$s_h = \kappa = -\frac{24}{\alpha} \Psi|_{x=0} = \frac{24}{\alpha} \frac{ab}{b-a} [k\alpha - abE].$$

When α is defined by (17), this is given by

$$s_h = \frac{24ab(b^2 - a^2 + kab)}{(b-a)(a^2 + 4ab + b^2)},$$

whereas, when α is given by (18), with $k = 1$, we instead have

$$s_h = \frac{12ab}{b-a}.$$

In particular, the constant scalar curvature of any of the constructed metrics is necessarily positive.

Of course, the scalar curvature is not invariant under multiplying h by a constant, so it is far more interesting to instead compute the scale-invariant quantity

$$s_h V_h^{1/2} = \frac{\int_M s_h d\mu_h}{\sqrt{\int_M d\mu_h}}$$

where $V_h = \int d\mu_h$ is the total volume of (M, h) . However, since $h = x^{-2}g$, we always have

$$\begin{aligned} V_h &= \int_{[a,b] \times (S^3/\mathbb{Z}_k)} x^{-4} 2(x+\alpha) dx \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \\ &= \frac{2\pi^2}{k} \int_{x=a}^{x=b} \frac{2(x+\alpha)}{x^4} dx \\ &= \frac{4\pi^2}{k} \left[\frac{x^{-2}}{-2} + \alpha \frac{x^{-3}}{-3} \right]_a^b \\ &= \frac{4\pi^2}{k} \left[\frac{1}{2a^2} - \frac{1}{2b^2} + \alpha \left(\frac{1}{3a^3} - \frac{1}{3b^3} \right) \right]. \end{aligned}$$

If α is given by (17), we therefore have

$$V_h = \frac{2\pi^2}{k} \frac{(b-a)(a^2+4ab+b^2)}{3a^2b^2(a+b)}$$

and our previous calculation therefore tells us that

$$s_h V_h^{1/2} = 8\pi\sqrt{6} \frac{b^2 - a^2 + kab}{\sqrt{k(b^2 - a^2)(a^2 + 4ab + b^2)}}. \quad (21)$$

By contrast, when α is given by (18), with $k = 1$, we have

$$V_h = 2\pi^2 \frac{(b-a)^2[3b^2+4ab+5a^2]}{3a^2b^2(a+b)^2}$$

and hence

$$s_h V_h^{1/2} = 4\pi \frac{\sqrt{6(3b^2+4ab+5a^2)}}{(a+b)}. \quad (22)$$

The curvature of the metrics determined by (17) thus behaves rather differently from the curvature of those determined by (18). Indeed, when (22) holds, $\lim_{b/a \rightarrow \infty} s_h V_h^{1/2}$ coincides with the value of $sV^{1/2}$ for the standard Fubini-Study metric on $\mathbb{CP}_2$. By contrast, (21) tells us that the other families on the Σ_k have $\lim_{b/a \rightarrow \infty} s_h V_h^{1/2}$ equal to the value of $sV^{1/2}$ for the standard orbifold metric on $S^4/\mathbb{Z}_k$; and this in turn tends to 0 as $k \rightarrow \infty$, even though $\lim_{k \rightarrow \infty} s_h V_h^{1/2} = +\infty$ for any fixed value of b/a .

Let us now systematically compare the metrics h determined by (17) for a fixed cohomology class $\Omega \in H^2(M)$ on $M \approx S^2 \times S^2$ or $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$, as we change the complex structure by letting k vary. To do this, we will need a fixed basis for $H^2(M)$ that is independent of k . One such basis is provided by the classes

$$\begin{aligned} \mathcal{F} &= \frac{1}{k} (\mathcal{C}_+ - \mathcal{C}_-) \\ \mathcal{D} &= \mathcal{C}_- + \left\lfloor \frac{k}{2} \right\rfloor \mathcal{F}. \end{aligned}$$

When k is even, these then correspond to the two factors of $S^2 \times S^2$; for k odd, they instead correspond to the fiber class and the exceptional curve $\mathcal{E}$ in $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$. Now recall that, with α given by (17), the self-dual harmonic form ω then satisfies $\omega(\mathcal{F}) = 2\pi(b-a)$ and $\omega(\mathcal{C}_-) = 2\pi a^2/(b+a)$, so that $\Omega = [\omega]$ satisfies

$$\frac{\Omega(\mathcal{D})}{\Omega(\mathcal{F})} = \frac{1}{(b/a)^2 - 1} + \left\lfloor \frac{k}{2} \right\rfloor > \left\lfloor \frac{k}{2} \right\rfloor.$$

In particular, this shows that a given cohomology class is only adapted to finitely many of the constructed Einstein–Maxwell metrics h , although this number grows roughly linearly as $\Omega(\mathcal{D})/\Omega(\mathcal{F}) \rightarrow \infty$.

Now let some positive integer $N \geq 2$ be given. Let $\Omega \in H^2(M, \mathbb{R})$ be the cohomology class with $\Omega(\mathcal{D}) = 5N$ and $\Omega(\mathcal{F}) = 1$. Since

$$\frac{b}{a} = \sqrt{1 + \frac{1}{\frac{\Omega(\mathcal{D})}{\Omega(\mathcal{F})} - \left\lfloor \frac{k}{2} \right\rfloor}}$$

we then have $(b/a)^2 \in (1, 1 + 1/4N)$ for $k = 1, \dots, 2N$. Equation (21) now implies that

$$\frac{s_h^2 V_h}{64\pi^2} - k \left(5N - \left\lfloor \frac{k}{2} \right\rfloor \right) \in \left(0, \frac{5}{4}k + 2 \right)$$

and it thus follows that the values of $s_h V_h^{1/2}$ are increasing in both k even and k odd for $k \in \{1, \dots, 2N\}$. This shows that, for $M = \mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ or $S^2 \times S^2$, the restriction of the normalized Einstein–Hilbert functional to the Fréchet manifold $\mathcal{G}_\Omega(M)$ has N different critical levels for such a choice of Ω . But since Proposition 2 shows that the Einstein–Hilbert functional is constant on each component of the moduli space $\mathcal{M}_\Omega$ of Ω -compatible Einstein–Maxwell metrics, it follows that $\mathcal{M}_\Omega$ has at least N connected components. This proves Theorem E.

5. The Page Metric Revisited

While various known results [12, 25, 27] imply that Page's Einstein metric [34] on $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ and the product metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$ are the only conformally Kähler, Einstein metrics on compact 4-manifolds of signature zero, it is still interesting to see, in detail, how this broad assertion manifests itself within the narrower context of the present investigation. We will therefore wrap up our discussion by concretely locating the Page metric among the Einstein–Maxwell metrics constructed in this paper.

By Proposition 7, the metric h is Einstein iff Ψ satisfies

$$\mathfrak{B} = 2\alpha\mathfrak{A}.$$

Since we have

$$\Psi = \frac{(b-x)(x-a)}{b-a} \left[k(x+\alpha) + \frac{k\alpha - (k+1)a - (k-1)b}{a^2 + 4ab + b^2} (b-x)(x-a) \right]$$

it follows that

$$\begin{aligned}\mathfrak{A} &= \frac{k\alpha - (k+1)a - (k-1)b}{(b-a)(a^2 + 4ab + b^2)} \\ \mathfrak{B} &= \frac{-2k(a+b)\alpha + (k-2)b^2 + a^2(2+k)}{(b-a)(a^2 + 4ab + b^2)}.\end{aligned}$$

The Einstein–Maxwell metric h constructed from Ψ is therefore Einstein iff

$$2k\alpha^2 + 2(b-a)\alpha - (k+2)a^2 - (k-2)b^2 = 0. \quad (23)$$

When α is given by (18), with $k = 1$, Eq. (23) becomes

$$2\left(-\frac{4a^2b}{(a+b)^2}\right)^2 - 2(b-a)\frac{4a^2b}{(a+b)^2} - 3a^2 + b^2 = 0$$

and by dividing by a^2 and setting $z = b/a$, this can be rewritten as

$$\frac{(z-1)^3}{(z+1)^4} (z^3 + 7z^2 + 13z + 3) = 0.$$

Since we automatically have $z > 1$, this shows that no Einstein–Maxwell metric h in this family is Einstein—or even Bach-flat.

On the other hand, if α is instead given by (17), Eq. (23) becomes

$$2k\frac{a^2b^2}{(a+b)^2} - 2(b-a)ab - (k+2)a^2 - (k-2)b^2 = 0,$$

and, again dividing by a^2 and setting $z = b/a$, this can be rewritten as

$$-\frac{(k-2)z^4 + 2(k-1)z^3 + 2(k+1)z + (k+2)}{(1+z)^2} = 0.$$

The Einstein–Maxwell metric h is therefore Einstein if and only if $z = b/a > 1$ solves the quartic equation

$$(k-2)z^4 + 2(k-1)z^3 + 2(k+1)z + (k+2) = 0.$$

When $k \geq 2$, however, the coefficients are all non-negative, so such a solution cannot exist. On the other hand, when $k = 1$, the equation becomes

$$z^4 - 4z - 3 = 0,$$

and this actually has a unique solution $z > 1$, because $z^4 - 4z - 3$ is negative when $z = 1$, has positive derivative for $z > 1$, and tends to $+\infty$ for large z . In fact, this solution can be expressed in terms of radicals by Ferrari's method, and is explicitly given by

$$\begin{aligned}\frac{b}{a} = z &= \sqrt{\frac{1}{2}\left(\sqrt[3]{1+\sqrt{2}} - \frac{1}{\sqrt[3]{1+\sqrt{2}}}\right)} \\ &+ \sqrt{\left[\frac{1}{2}\left(\sqrt[3]{1+\sqrt{2}} - \frac{1}{\sqrt[3]{1+\sqrt{2}}}\right)\right]^{-1/2} - \frac{1}{2}\left(\sqrt[3]{1+\sqrt{2}} - \frac{1}{\sqrt[3]{1+\sqrt{2}}}\right)} \\ &\approx 1.784358\end{aligned} \quad (24)$$

Since $u/v = (b/a)^2$ for the Kähler metrics in the family associated to the choice of α given by (17), Theorem C therefore follows by squaring the right-hand side of (24) to obtain u/v . We leave it as an exercise for the interested reader to directly compare the resulting metric h , as defined by (12) and (13), with the expression discovered by Page.

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