

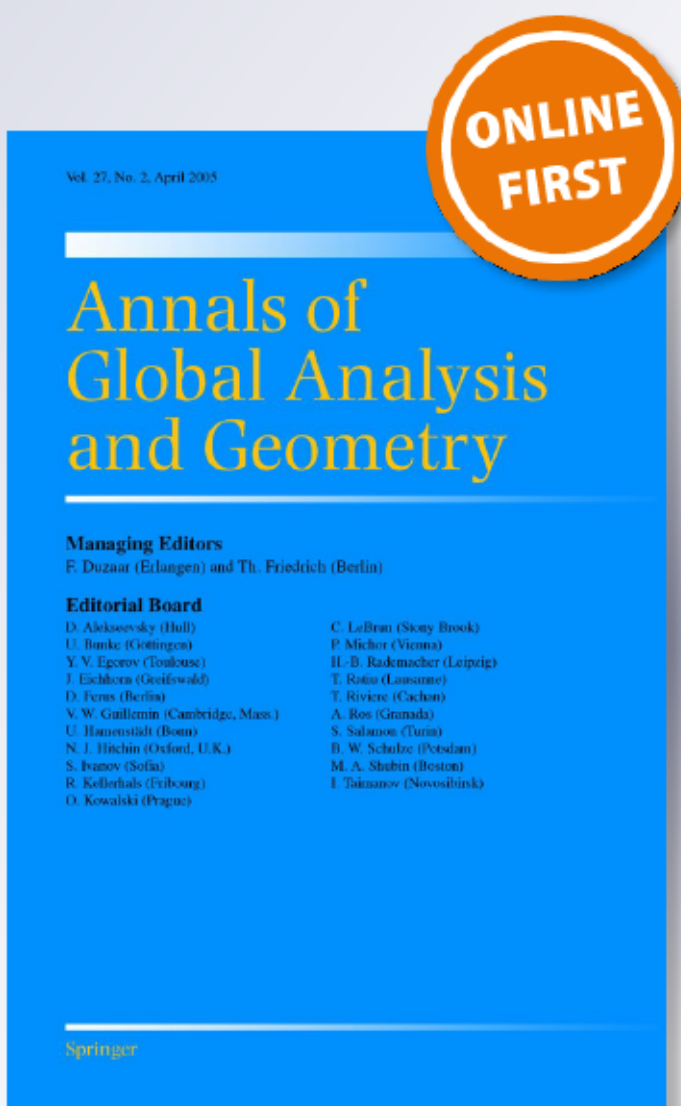
# *Mass, Kähler manifolds, and symplectic geometry*

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# Mass, Kähler manifolds, and symplectic geometry

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## Abstract

In the author's previous joint work with Hein (Commun Math Phys 347:183–221, 2016), a mass formula for asymptotically locally Euclidean Kähler manifolds was proved, assuming only relatively weak fall-off conditions on the metric. However, the case of real dimension 4 presented technical difficulties that led us to require fall-off conditions in this special dimension that are stronger than the Chruściel fall-off conditions that sufficed in higher dimensions. Nevertheless, the present article shows that techniques of four-dimensional symplectic geometry can be used to obtain all the major results of Hein-LeBrun (2016), assuming only Chruściel-type fall-off. In particular, the present article presents a new proof of our Penrose-type inequality for the mass of an asymptotically Euclidean Kähler manifold that only requires this very weak metric fall-off.

**Keywords** Mass · Asymptotically locally Euclidean · Kähler · Scalar curvature · Symplectic · Pseudo-holomorphic curve · Penrose inequality

A complete connected non-compact Riemannian manifold  $(M, g)$  of real dimension  $n \geq 3$  is said to be *asymptotically Euclidean* (or *AE*) if there is a compact subset  $K \subset M$  such that  $M - K$  consists of finitely many components, each of which is diffeomorphic to the complement of a closed ball  $D^n \subset \mathbb{R}^n$  in such a manner that  $g$  becomes the standard Euclidean metric plus terms that fall off sufficiently rapidly at infinity. More generally, a Riemannian  $n$ -manifold  $(M, g)$  is said to be *asymptotically locally Euclidean* (or *ALE*) if the complement of a compact set  $K$  consists of finitely many components, each of which is diffeomorphic to a quotient  $(\mathbb{R}^n - D^n)/\Gamma_i$ , where  $\Gamma_i \subset O(n)$  is a finite subgroup that acts freely on the unit sphere, in such a way that  $g$  again becomes the Euclidean metric plus error terms that fall off sufficiently rapidly at infinity. The components of  $M - K$  are called the *ends* of  $M$ ; their fundamental groups are the aforementioned groups  $\Gamma_i$ , and might in principle be different for different ends of the manifold.

There is no clear consensus regarding the exact fall-off conditions that should be imposed on the metric  $g$ , as various authors have in practice tweaked the definition to dovetail with the technical requirements demanded by their favorite techniques. However, the weakest standard hypotheses that seem to lead to compelling results are the ones introduced by Chruściel [5]:

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- (i) The metric  $g$  is of class  $C^2$ , with scalar curvature  $s$  in  $L^1$ ; and
- (ii) in some asymptotic chart at each end of  $M^n$ , and for some  $\varepsilon > 0$ , the components of the metric satisfy

$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon}), \quad g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}).$$

With these very weak hypotheses, Chruściel's argument shows that the *mass*

$$m(M, g) := \lim_{\varrho \rightarrow \infty} \frac{\Gamma(\frac{n}{2})}{4(n-1)\pi^{n/2}} \int_{S_\varrho/\Gamma_i} [g_{k\ell,k} - g_{kk,\ell}] \mathbf{n}^\ell d\mathbf{a}_E$$

of an ALE manifold  $(M, g)$  at any given end is both well defined and invariant under a large class of changes in asymptotic coordinate system. Here, commas indicate partial derivatives in the given asymptotic coordinates, summation over repeated indices is understood,  $S_\varrho$  is the Euclidean coordinate sphere of radius  $\varrho$ ,  $\Gamma_i$  is the fundamental group of the relevant end,  $\Gamma$  is the Euler Gamma function,  $d\mathbf{a}_E$  is the  $(n-1)$ -dimensional volume form induced on this sphere by the Euclidean metric, and  $\mathbf{n}$  is the outward-pointing Euclidean unit normal vector. While Chruściel's paper actually only discusses the AE case, his argument immediately extends to the more general ALE setting under discussion here. The fact that fall-off conditions on the metric are by no means a matter of widespread consensus is nicely illustrated by Bartnik's powerful and better-known theorem [2] on the coordinate-invariance of the mass, which was proved around the same time as Chruściel's work; while Bartnik's conclusion regarding the coordinate-invariance of the mass is markedly stronger than Chruściel's, it is obtained at the price of replacing hypothesis (ii) above with the stronger assumption that the  $g_{jk} - \delta_{jk}$  belong to a weighted Sobolev space  $W_{-\tau}^{2,q}$  for some  $q > n$  and some  $\tau > (n-2)/2$ . Bartnik's paper is also notable for showing, by counter-example, that any significant weakening of Chruściel's conditions (i) and (ii) would result in the mass being ill-defined and/or coordinate dependent.

In joint work with Hein [7], the present author has elsewhere shown that if  $(M, g, J)$  is an ALE Kähler manifold of complex dimension  $m$ , then  $M$  has only one end, and that the mass at this unique end is given by

$$m(M, g) = \frac{1}{(2m-1)\pi^{m-1}} \langle \clubsuit(-c_1), [\omega]^{m-1} \rangle + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g d\mu_g$$

where  $s_g$  and  $d\mu_g$  are, respectively, the scalar curvature and volume form of  $g$ ,  $c_1 = c_1^{\mathbb{R}}(M, J) \in H^2(M)$  is the first Chern class of the complex structure,  $[\omega] \in H^2(M)$  is the Kähler class of  $g$ ,  $\clubsuit : H^2(M) \rightarrow H_c^2(M)$  is the inverse of the natural morphism from compactly supported to ordinary de Rham cohomology, and  $\langle \cdot, \cdot \rangle$  is the duality pairing between  $H_c^2(M)$  and  $H^{2m-2}(M)$ . If one accepts it as *given* that  $M$  has only one end, our proof [7, Sect. 3] of the above mass formula only requires the Chruściel fall-off hypotheses (i) and (ii), and provides an entirely self-contained proof of the coordinate-invariance of the mass in the Kähler case. However, our proof that  $M$  can only have one end, merely assuming the metric fall-off condition (ii), works well only when  $m \geq 3$ ; when  $m = 2$ , our proof only managed to obtain the same conclusion from Chruściel's fall-off hypothesis if  $\varepsilon > 1/2$ . This and related phenomena led us, in [7], to instead insist on Bartnik-type metric fall-off in the special case of real dimension 4.

This note will provide a remedy for this state of affairs. Many of the analytic subtleties encountered in the four-dimensional case are subtly intertwined with the fact that the complex structure of an ALE Kähler surface need not be standard at infinity. By contrast, we will show here that the *symplectic structure* at infinity of such a manifold is always standard, even with extremely weak fall-off assumptions on the metric. By developing symplectic versions of



some of the previous proofs, we will thus be able to show that, even when  $m = 2$ , all the main results of [7] continue to hold even when the metric simply satisfies Chruściel's weak fall-off hypotheses (i) and (ii). In particular, we will see that our Penrose-type inequality [7, Theorem E] for the mass of an AE Kähler manifold remains valid even in real dimension 4, assuming only the mildest reasonable fall-off assumptions on the metric.

## 1 The asymptotic symplectic structure

For clarity and concreteness, we will restrict the following discussion to real dimension 4. However, most of what follows does work, *mutatis mutandis*, in higher dimensions, and indeed is actually far less delicate in that setting.

Let  $(M^4, g, J)$  be an ALE Kähler surface, which we hypothetically allow to perhaps have several ends. Throughout, we will simply assume that  $g$  satisfies the Chruściel fall-off hypothesis, and in this section we will actually only make use of hypothesis (ii) with  $n = 4$ . Thus, on any given end  $M_{\infty, i}$  of  $M$ , we assume that there are asymptotic coordinates  $(x^1, \dots, x^4)$  on the universal cover  $\tilde{M}_{\infty, i}$  of  $M_{\infty, i}$  in which the components of the metric satisfy

$$g_{jk} = \delta_{jk} + O(|x|^{-1-\varepsilon}), \quad g_{jk, \ell} = O(|x|^{-2-\varepsilon})$$

for some  $\varepsilon > 0$ , and such that the fundamental group  $\Gamma_i$  of the end acts by rotations of the coordinates  $(x^1, \dots, x^4)$  in a manner that preserves both the background-model Euclidean metric  $\delta$  and the given Kähler metric  $g$ .

Because  $g$  is Kähler by assumption, the associated complex structure  $J$  satisfies  $\nabla J = 0$ , where  $\nabla$  is the Levi-Civita connection of  $g$ . However, since our fall-off hypothesis implies that  $\nabla = \nabla + O(|x|^{-2-\varepsilon})$ , where  $\nabla$  is the flat Levi-Civita connection of  $\delta$ , the elementary argument presented in [7, Sect. 2] shows there is a  $\delta$ -compatible constant-coefficient almost-complex structure  $J_0$  on  $\mathbb{R}^4$  such that

$$J = J_0 + O(|x|^{-1-\varepsilon}), \quad \nabla J = O(|x|^{-2-\varepsilon}).$$

After rotating our coordinates  $(x^1, \dots, x^4)$  if necessary, we may moreover arrange for  $J_0$  to become the standard complex structure

$$dx^1 \otimes \frac{\partial}{\partial x^2} - dx^2 \otimes \frac{\partial}{\partial x^1} + dx^3 \otimes \frac{\partial}{\partial x^4} - dx^4 \otimes \frac{\partial}{\partial x^3}$$

on  $\mathbb{C}^2$ . Since the action of the fundamental group  $\Gamma_i$  preserves both  $J$  and  $\delta$ , it now follows that  $\Gamma_i \subset \mathrm{U}(2)$ . More importantly, we therefore automatically obtain fall-off conditions

$$\omega = \omega_0 + O(|x|^{-1-\varepsilon}), \quad \nabla \omega = O(|x|^{-2-\varepsilon}), \quad (1.1)$$

for the Kähler form  $\omega = g(J \cdot, \cdot)$  of  $g$ , where  $\omega_0 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$  is the standard symplectic form on  $\mathbb{R}^4 = \mathbb{C}^2$ .

**Proposition 1.1** *Let  $(M^4, g, J)$  be an ALE Kähler surface, let  $M_{\infty, i}$  be an end of  $M$ , let  $\tilde{M}_{\infty, i}$  be the universal cover of  $M_{\infty, i}$ , and let*

$$(x^1, \dots, x^4) : \tilde{M}_{\infty, i} \rightarrow \mathbb{R}^4 - B$$

*be a diffeomorphism, where  $B \subset \mathbb{R}^4$  is a standard closed ball of some large radius centered at the origin. Suppose, moreover, that these asymptotic coordinates have been chosen in accordance with the above discussion, so that the Kähler form  $\omega$  of  $(M, g, J)$  is  $\mathbb{C}^2$  and*

satisfies the fall-off conditions (1.1) in this coordinate system, while the action of  $\pi_1(M_{\infty,i})$  on  $M_{\infty,i}$  by deck transformations is represented in these coordinates by the action of a finite group  $\Gamma_i \subset U(2)$  of unitary transformations, acting on  $\mathbb{R}^4 = \mathbb{C}^2$  in the usual way. Then, there is  $\Gamma_i$ -equivariant  $C^2$ -diffeomorphism  $\Phi: \mathbb{R}^4 - C \rightarrow \mathbb{R}^4 - D$ , where  $C \subset \mathbb{R}^4$  is a standard closed ball centered at the origin, where  $D \subset \mathbb{R}^4$  is a smooth 4-ball whose boundary  $\partial D$  is a  $\Gamma_i$ -invariant differentiable  $S^3$ , and where  $B \subset C \cap D$ , such that

$$\Phi^* \omega = \omega_0,$$

with  $|\Phi(x) - x| = O(|x|^{-\varepsilon})$  and  $|\Phi_* - I| = (|x|^{-1-\varepsilon})$ .

**Proof** The following proof is largely a quantitative refinement of Moser's stability argument [17].

Let  $a$  denote the radius of the given closed ball  $B \subset \mathbb{R}^4$ , and notice that we can identify  $\mathbb{R}^4 - B$  with  $S^3 \times (a, \infty)$  by means of the smooth diffeomorphism  $x \mapsto (x/|x|, |x|)$ . Letting  $\varrho = |x| \in (a, \infty)$  denote the radial coordinate, and letting  $\eta = \frac{\partial}{\partial \varrho}$  denote the unit radial vector field in  $\mathbb{R}^4$ , we now define an  $r$ -dependent 1-form  $\varphi_r$  on  $S^3$  by restricting the 1-form  $\eta \lrcorner (\omega - \omega_0)$ , which in any case has vanishing radial component, to  $S^3 \times \{r\} \subset S^3 \times (a, \infty)$ :

$$\varphi_r := \eta \lrcorner (\omega - \omega_0) \Big|_{\varrho=r}, \quad r \in (a, \infty).$$

Because our fall-off conditions guarantee that  $\varphi_r = O(r^{-\varepsilon})$  as a 1-form on  $S^3$ , it follows that, for any choice of  $\varrho_0 \in (a, \infty)$ ,

$$\psi = \int_{\varrho_0}^{\varrho} \varphi_r dr$$

is a well-defined  $\varrho$ -dependent 1-form on  $S^3$  of growth  $O(\varrho^{1-\varepsilon})$ , with first partial derivatives on  $S^3$  of similar growth. Viewing  $\psi$  as a 1-form on  $S^3 \times (a, \infty)$  with vanishing component in the  $\varrho$ -direction, our assumptions thus not only guarantee it is a 1-form of class  $C^2$ , but also that its components in  $\mathbb{R}^4$  satisfy the fall-off conditions

$$\psi_k = O(|x|^{-\varepsilon}), \quad \psi_{k,\ell} = O(|x|^{-1-\varepsilon}).$$

However, Cartan's magic formula for the Lie derivative tells us that

$$\mathcal{L}_{\frac{\partial}{\partial \varrho}} (\omega - \omega_0) = \eta \lrcorner d(\omega - \omega_0) + d[\eta \lrcorner (\omega - \omega_0)] = d[\eta \lrcorner (\omega - \omega_0)]$$

because  $\omega$  and  $\omega_0$  are both closed; and since the Lie derivative commutes with  $d$  on  $C^2$  forms, we also have

$$\mathcal{L}_{\frac{\partial}{\partial \varrho}} d\psi = d[\mathcal{L}_{\frac{\partial}{\partial \varrho}} \psi] = d\varphi = d[\eta \lrcorner (\omega - \omega_0)],$$

too. It follows that  $\alpha := (\omega - \omega_0) - d\psi$  is a closed,  $\varrho$ -independent 2-form on  $S^3 \times (a, \infty)$ . Moreover, since

$$\eta \lrcorner \alpha = \eta \lrcorner [(\omega - \omega_0) - d\psi] = \varphi - \mathcal{L}_{\eta} \psi = \varphi - \varphi = 0,$$

it follows that  $\alpha$  is actually the pull-back of a closed 2-form on  $S^3$ . But since  $H^2(S^3) = 0$ , we therefore have  $\alpha = d\beta$  for some  $\varrho$ -independent 1-form  $\beta$  on  $S^3$ . Moreover, since  $\omega$  and  $\omega_0$  are both  $\Gamma_i$ -invariant, it follows that  $\varphi$ ,  $\psi$ , and  $\alpha$  are all  $\Gamma_i$ -invariant, too; by averaging, we

can therefore arrange for  $\beta$  to also be  $\Gamma_i$ -invariant, while still satisfying the equation  $\alpha = d\beta$ . Setting  $\theta := \psi + \beta$ , we then have

$$\omega - \omega_0 = d\theta$$

for a  $\Gamma_i$ -invariant 1-form  $\theta$  of class  $C^2$  on  $\mathbb{R}^4 - B$  with fall-off

$$\theta = O(|x|^{-\varepsilon}), \quad \nabla \theta = O(|x|^{-1-\varepsilon}).$$

Let us next consider the family of convex combinations

$$\omega_t = (1-t)\omega_0 + t\omega = \omega_0 + t(\omega - \omega_0), \quad t \in [0, 1], \quad (1.2)$$

of the given symplectic forms  $\omega$  and  $\omega_0$ . Because  $(\omega - \omega_0) = O(|x|^{-1-\varepsilon})$  as a 2-form on  $\mathbb{R}^4 - B$ , there is some  $b > a$  such that  $|\omega - \omega_0| < 1/\sqrt{2}$  for all  $\varrho \geq b$ , where the norm of a 2-form here is calculated with respect to the Euclidean metric  $\delta$ . This then implies that, for any vector  $v \in T_x \mathbb{R}^4 = \mathbb{R}^4$ , one has

$$|v \lrcorner \omega_t| > \frac{1}{2}|v| \quad \forall t \in [0, 1]$$

whenever  $\varrho = |x| > b$ ; here the vector norm is again measured with respect to the Euclidean metric  $\delta$ . Thus, when  $\varrho > b$  and  $t \in [0, 1]$ , the maps  $T_x \mathbb{R}^4 \rightarrow T_x^* \mathbb{R}^4$  defined by the contractions  $v \mapsto v \lrcorner \omega_t$  are not only invertible, but have inverses of operator norm  $< 2$  with respect to  $\delta$ . Defining a  $t$ -dependent  $C^2$  vector field  $X_t$  on the exterior region  $\varrho \geq b$  by

$$X_t \lrcorner \omega_t = -\theta, \quad t \in [0, 1], \quad (1.3)$$

our fall-off conditions therefore tell us that  $|X_t|_\delta = O(\varrho^{-\varepsilon})$  and  $|\nabla X_t|_\delta = O(\varrho^{-1-\varepsilon})$ . In particular, it follows that there is some  $c \geq b + 1$  such that  $|X_t|_\delta < 1$  on the entire region  $\varrho \geq c - 1$ , for every  $t \in [0, 1]$ . Also notice that we automatically have

$$X_t \lrcorner \theta = -\omega_t(X_t, X_t) = 0, \quad (1.4)$$

and that  $X_t$  is  $\Gamma_i$ -invariant, for every  $t \in [0, 1]$ .

Fixing coordinates  $(x^1, \dots, x^4, t)$  on  $\mathbb{R}^5 = \mathbb{R}^4 \times \mathbb{R}$ , we now consider the closed 2-form

$$\Omega = \omega_0 + d(t\theta)$$

on an open neighborhood of the region  $|x| \geq c - 1$ ,  $0 \leq t \leq 1$ , where the  $t$ -independent forms  $\omega_0$  and  $\theta$  are understood to denote the pull-backs of the corresponding forms on  $\mathbb{R}^4$ . Since  $d\theta = \omega - \omega_0$ , we may rewrite this as

$$\Omega = \omega_t + dt \wedge \theta, \quad (1.5)$$

so that restriction of  $\Omega$  to the various  $t = \text{constant}$  slices simply yields the 2-forms  $\omega_t$  of (1.2). The  $C^2$  vector field

$$\xi = \frac{\partial}{\partial t} + X_t$$

on our region of  $\mathbb{R}^5$  therefore satisfies

$$\xi \lrcorner \Omega = \left[ \frac{\partial}{\partial t} + X_t \right] \lrcorner [\omega_t + dt \wedge \theta] = \theta - \theta = 0$$

by dint of (1.3)–(1.5). Thus, Cartan's magic formula now yields

$$\mathcal{L}_\xi \Omega = \xi \lrcorner d\Omega + d[\xi \lrcorner \Omega] = 0.$$

The flow of  $\xi$ , which simply acts on  $t$  by “time translation,” therefore locally moves the 2-form  $\omega_t$  on any given time slice to the corresponding 2-form at a later time. However, because the vector field  $X_t$  always has Euclidean length  $|X_t| < 1$  in the region  $\varrho > c - 1$ , the flow-line of  $\xi$  starting at any  $(x, 0)$  with  $|x| \geq c$  is well defined for all  $t \in [0, 1]$ , and remains within  $B_1(x) \times [0, 1]$ . Thus, letting  $C$  denote the Euclidean ball  $\varrho \leq c$ , there is a family

$$\Phi_t : \mathbb{R}^4 - \dot{C} \rightarrow \mathbb{R}^4, \quad t \in [0, 1],$$

of  $C^2$  maps given by following the flow of  $\xi$  from  $(\mathbb{R}^4 - \dot{C}) \times \{0\}$  to  $\mathbb{R}^4 \times \{1\}$ . These maps are  $C^2$  diffeomorphisms between  $\mathbb{R}^4 - C$  and their images, and satisfy  $\Phi_t^* \omega_t = \omega_0$ . In particular,  $\Phi := \Phi_1$  provides a symplectomorphism between  $(\mathbb{R}^4 - C, \omega_0)$  and  $(\mathcal{U}, \omega)$ , for some open set  $\mathcal{U} \subset \mathbb{R}^4$ . But since a time-reversed version of our argument shows that backward trajectories of the flow from  $\varrho \geq c + 1$  are also defined and remain in the region  $\varrho \geq c$  for  $t \in [0, 1]$ , every point in the region  $\varrho \geq c + 1$  must belong to the image  $\mathcal{U}$  of  $\Phi$ . Moreover, we can now extend  $\Phi$  as a  $\Gamma_i$ -equivariant  $C^2$  diffeomorphism  $\mathbb{R}^4 \rightarrow \mathbb{R}^4$  by extending the vector fields  $X_t$  to  $\mathbb{R}^4$  while keeping  $|X_t| < 1$  by multiplying the fields defined by (1.4) by a cutoff function  $\phi(\varrho)$  which is  $\equiv 1$  for  $\varrho > c - 1$  and  $\equiv 0$  for  $\varrho < c - 1 - \epsilon$ . In particular, the closed set  $D = \mathbb{R}^4 - \mathcal{U}$  is actually diffeomorphic to a standard 4-ball, and its boundary is a  $\Gamma_i$ -invariant differentiable  $S^3$ . Finally, because  $|X_t| = O(\varrho^{-\epsilon})$  and  $|\nabla X_t|_\delta = O(\varrho^{-1-\epsilon})$ , we have  $|\Phi(x) - x| = O(|x|^{-\epsilon})$  and  $|\Phi_* - I| = (|x|^{-1-\epsilon})$ .  $\square$

## 2 Some useful symplectic orbifolds

If  $\Gamma \subset \mathrm{U}(2)$  is a finite subgroup, the standard action of  $\Gamma$  on  $\mathbb{C}^2$  extends to  $\mathbb{CP}_2 = \mathbb{C}^2 \sqcup \mathbb{CP}_1$  in an obvious way—namely, by letting a  $2 \times 2$  complex matrix  $A$  act on  $\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}$  by  $A \oplus 1$ , and then remembering that  $\mathbb{CP}_2 = (\mathbb{C}^3 - 0)/\mathbb{C}^\times$ . Since this construction gives us an inclusion  $\mathrm{U}(2) \hookrightarrow \mathrm{PSU}(3)$ , the induced action of  $\Gamma$  on  $\mathbb{CP}_2$  preserves the standard Fubini–Study metric; and since the action also preserves the complex structure of  $\mathbb{CP}_2$ , it also preserves the Fubini–Study Kähler form  $\omega$ , which we will choose to regard as a symplectic form on  $\mathbb{CP}_2$ . We may therefore choose to view the quotient  $(\mathbb{CP}_2, \omega)/\Gamma$  as a symplectic orbifold.

We will henceforth confine our discussion to those  $\Gamma$  that *act freely on the unit sphere*  $S^3 \subset \mathbb{C}^2$ . Our goal here will then be to construct preferred partial desingularizations of every symplectic orbifold  $(\mathbb{CP}_2, \omega)/\Gamma$  that arises in this way. Of course, if  $\Gamma = \{1\}$ , then  $\mathbb{CP}_2/\Gamma$  is smooth, so there is nothing to do in this regard. We may therefore assume from now on that  $\Gamma \neq \{1\}$ . With this assumption, the origin in  $\mathbb{C}^2 \subset \mathbb{CP}_2$  automatically projects to a singular point  $p \in \mathbb{CP}_2/\Gamma$ ; and since we have assumed that  $\Gamma$  acts freely on the unit sphere  $S^3$ , and hence on all of  $\mathbb{C}^2 - \{0\}$ , the singular point  $p$  is automatically isolated. More specifically, every other singular point arises from some element of the “line at infinity”  $\mathbb{CP}_1 \subset \mathbb{CP}_2$ . Our objective in this section will be to symplectically modify  $(\mathbb{CP}_2, \omega)/\Gamma$  in a manner that leaves the singularity at  $p$  unaltered, but eliminates all the other singularities.

In preparation for this, let us first notice that the center  $\mathbf{Z} \cong \mathrm{U}(1)$  of  $\mathrm{U}(2)$  consists of scalar multiples of the diagonal matrix, and acts trivially on the  $\mathbb{CP}_1$  at infinity. Moreover, since  $\mathbf{Z} = \mathrm{U}(1) \cong \mathbb{R}/\mathbb{Z}$ , the finite group  $\mathbf{Z} \cap \Gamma$  must be cyclic, and thus isomorphic to  $\mathbb{Z}_\ell$  for some positive integer  $\ell$ . Our first step is therefore to consider the quotient  $\mathbb{CP}_2/\mathbb{Z}_\ell$ . Away from the base-point  $\hat{p}$  arising from  $[0 : 0 : 1] \in \mathbb{CP}_2$ , this space is topologically non-singular, and can be given a smooth structure such that  $\omega$  descends to it as a symplectic form.



This is perhaps most easily seen via Lerman's theory of symplectic cuts [12]; namely, the Fubini–Study symplectic form on  $\mathbb{CP}_2$  is obtained by taking the symplectic cut at  $\mathcal{H} \leq 1/2$  of  $(\mathbb{C}^2, \omega_0)$  for the Hamiltonian  $\mathcal{H} = (|z_1|^2 + |z_2|^2)/2$ , which generates a free periodic action of period  $2\pi$  at and near the boundary. It follows that  $\mathbb{CP}_2/\mathbb{Z}_\ell$  is simply obtained from  $\mathbb{C}^2/\mathbb{Z}_\ell$  by taking the symplectic cut at  $\widehat{\mathcal{H}} \leq \ell/2$  for the Hamiltonian  $\widehat{\mathcal{H}} = \ell\mathcal{H}$ , which again generates a free periodic action of period  $2\pi$  at and near the boundary.<sup>1</sup> If  $\ell > 1$ , the global quotient  $(\mathbb{CP}_2, \omega)/\mathbb{Z}_\ell$  can thus be viewed as a symplectic orbifold  $(X_\ell, \omega)$  with exactly one single singular point  $\hat{p}$ , corresponding to the origin in  $\mathbb{C}^2$ . The symplectic cut construction gives us a symplectic 2-sphere  $\Sigma \subset X_\ell$  of self-intersection  $+\ell$  that corresponds to the line at infinity  $\mathbb{CP}_1 \subset \mathbb{CP}_2$ , and we note in passing that  $X_\ell - \{\hat{p}\}$  is actually diffeomorphic to the  $\mathcal{O}(\ell)$  line bundle over  $\mathbb{CP}_1$ . Since the symplectic condition on a submanifold is open, we can also obviously perturb this  $(+\ell)$ -sphere “at infinity” so as to produce an embedded 2-sphere  $\Sigma'$  that meets  $\Sigma$  in only one point, at which  $\Sigma$  and  $\Sigma'$  are tangent to order  $\ell - 1$ . Moreover, one can do this in such a manner that  $\Sigma \cap \Sigma'$  is any chosen point of  $\Sigma$ , and so that  $\Sigma'$  avoids any given small neighborhood of the singular base-point  $\hat{p}$ . Indeed, one can even do this explicitly in the present context, by just taking  $\Sigma'$  to be the image in  $\mathbb{CP}_2/\mathbb{Z}_\ell$  of a generic complex line in  $\mathbb{CP}_2$ . Of course, almost everything said here is also trivially true in the case of  $\ell = 1$ ; the only thing that is substantially different about the case of  $X_1 = \mathbb{CP}_2$  is that  $\hat{p}$  is a non-singular point when  $\ell = 1$ . Whatever the value of  $\ell$ , we also automatically have

$$\langle c_1(X_\ell), [\Sigma'] \rangle = \langle c_1(X_\ell), [\Sigma] \rangle = \chi(\Sigma) + \Sigma \cdot \Sigma = 2 + \ell \geq 3 \quad (2.1)$$

as an immediate consequence of the adjunction formula.

We now wish to treat the general  $\Gamma \subset \mathbf{U}(2)$  that acts freely on  $S^3$ . We do so by first noticing that  $\mathbb{CP}_2/\Gamma = X_\ell/\check{\Gamma}$ , where  $\check{\Gamma} := \Gamma/(\mathbb{Z} \cap \Gamma) = \Gamma/\mathbb{Z}_\ell$ . Of course, if  $\check{\Gamma} = \{1\}$ , we are already done. Otherwise, notice that since  $\Gamma$  acts freely on  $S^3$ , and hence on  $\mathbb{C}^2 - \{0\}$ , the fact that  $\mathbb{Z}_\ell \subset \Gamma$  is central implies that  $\check{\Gamma}$  also acts freely on  $(\mathbb{C}^2 - \{0\})/\mathbb{Z}_\ell$ , and hence on  $X_\ell - (\Sigma \sqcup \{\hat{p}\})$ . The singular points of  $(X_\ell - \{\hat{p}\})/\check{\Gamma}$  therefore all arise from points of  $\Sigma \approx S^2$  that are fixed by some non-trivial subgroup of  $\check{\Gamma}$ . However, since  $\mathbf{U}(2)/\mathbb{Z} = \mathbf{PSU}(2) \cong \mathbf{SO}(3)$ , our group  $\check{\Gamma}$  can be thought of as a finite subgroup of  $\mathbf{SO}(3)$ , in a way that simultaneously realizes the given action of  $\Gamma$  on  $\Sigma$  as the tautological action of  $\check{\Gamma} \subset \mathbf{SO}(3)$  on  $S^2 = \mathbf{SO}(3)/\mathbf{SO}(2)$ . But since the isotropy group  $\subset \mathbf{SO}(3)$  of any point in  $S^2$  is isomorphic to  $\mathbf{SO}(2) \cong \mathbb{R}/\mathbb{Z}$ , the stabilizer  $\subset \check{\Gamma}$  of any point of  $\Sigma$  is necessarily cyclic—and of course is actually trivial for all but a finite number of points!

While the above arguments in principle provide all the information we will need to prove the main result in this section, it is still worth mentioning the classical fact that the only possible finite groups  $\check{\Gamma} \subset \mathbf{SO}(3)$  are the oriented isometry groups of a polygon or regular polyhedron in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , and that this implies that the quotient  $\Sigma/\check{\Gamma}$  is always a topological 2-sphere with exactly two or three singular points [24, Chapter 13], which actually arise from the orbits of the vertices, edge-centers, and/or face-centers of the corresponding geometric figure. The non-trivial possibilities are tabulated below. In particular, the orbifold  $X_\ell/\check{\Gamma}$  will actually have exactly 0, 2, or 3 singularities other than the singular base-point  $p$  that is the image of  $\hat{p} \in X_\ell$ .

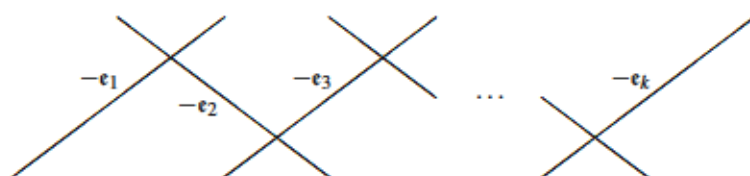
<sup>1</sup> More generally, symplectic orbifold singularities of codimension 2 are always symplectically invisible. The essential points are that the fixed-point set is automatically a symplectic submanifold, and that the area form on  $\mathbb{C}/\mathbb{Z}_\ell$  induced by the standard area form on  $\mathbb{C}$  becomes a constant times the standard area form on  $\mathbb{C}$  if one declares that the complex variable  $\zeta = z^\ell/|z|^{\ell-1}$  provides an admissible chart on the quotient.

| Group       | Figure                                                                            | Singularities                              |
|-------------|-----------------------------------------------------------------------------------|--------------------------------------------|
| Cyclic      |  | $\mathbb{Z}_n, \mathbb{Z}_n$               |
| Dihedral    |  | $\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_n$ |
| Tetrahedral |  | $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_3$ |
| Octahedral  |  | $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ |
| Icosahedral |  | $\mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_5$ |

While this classification does tell us the possible orders of the cyclic groups associated with each singularity, it does not actually completely describe the local action of these stabilizers  $\mathbb{Z}_q \subset \check{I}$  on  $X_\ell$ , since the action of  $\mathbb{Z}_q$  on  $\Sigma$  does not determine its action on the normal bundle of  $\Sigma$ . However, we do know that these stabilizer groups have only isolated fixed points, since  $\check{I}$  acts freely on  $X_\ell - (\Sigma \sqcup \{\hat{p}\})$ . Thus, if  $\mathbb{Z}_q \subset \check{I}$  is the stabilizer of some fixed point, its action is locally modeled on the action of  $\mathbb{Z}_q$  on  $\mathbb{C}^2$  generated by

$$(z_1, z_2) \mapsto (e^{2\pi i/q} z_1, e^{2\pi i p/q} z_2) \quad (2.2)$$

for a unique integer  $p$  with  $0 < p < q$  and  $\gcd(p, q) = 1$ . In complex geometry, there is a standard minimal resolution of any such singularity, obtained by replacing the singular point with a Hirzebruch–Jung string [1, 8], meaning a finite string



of copies of  $\mathbb{CP}^1$  whose self-intersection numbers  $-\epsilon_j$  are the negatives of the integers  $\epsilon_j \geq 2$  determined inductively by the algorithm

$$\mathfrak{d}_1 = \frac{q}{p}, \quad \epsilon_j = \lceil \mathfrak{d}_j \rceil, \quad \mathfrak{d}_{j+1} = (\epsilon_j - \mathfrak{d}_j)^{-1},$$

where the process terminates at the first  $j$  for which  $\mathfrak{d}_j$  is an integer.

While it should be feasible to carry out a symplectic version of Hirzebruch's construction via a sequence of symplectic cuts, we will instead remove these cyclic singularities by exploiting Sect. 1. Indeed, for each action (2.2), Calderbank and Singer [4] have constructed a family of ALE scalar-flat Kähler surfaces whose single end is diffeomorphic to  $L(q, p) \times \mathbb{R}^+$ , where  $L(q, p) = S^3/\mathbb{Z}_q$  is the lens space associated with the given action (2.2). The Calderbank–Singer manifolds are, by construction, diffeomorphic to Hirzebruch's minimal resolutions of  $\mathbb{C}^2/\mathbb{Z}_q$ , and satisfy Chruściel's fall-off hypotheses with  $\varepsilon = 1$ . For any chosen metric in the family, Proposition 1.1 thus tells us that the Calderbank–Singer manifold contains a compact set whose complement is symplectomorphic to  $(\mathbb{C}^2 - \overline{\mathcal{B}}_\pi, \omega_0)/\mathbb{Z}_q$ , for the specified action (2.2) on  $\mathbb{C}^2$ , where  $\overline{\mathcal{B}}_\pi \subset \mathbb{C}^2$  is the standard closed ball of some radius  $\pi > 0$  centered at the origin. By multiplying the Calderbank–Singer metric (and therefore its Kähler form) by

a sufficiently small positive constant, we may then arrange for this statement to actually hold with  $\pi$  replaced by any small radius  $r > 0$  we like. However, each orbifold singularity  $y$  we wish to eliminate has a neighborhood modeled on  $(\mathcal{B}_{\mathcal{R}}, \omega_0)/\mathbb{Z}_q$  for some radius  $\mathcal{R} > 0$ , and for some  $\mathbb{Z}_q$  action of type (2.2). Choosing our rescaling of the Calderbank–Singer manifold so that  $r < \mathcal{R}$  then allows us to delete a closed neighborhood  $(\overline{\mathcal{B}}_r, \omega_0)/\mathbb{Z}_q$  of the singular point  $y$ , remove the end  $(\mathbb{C}^2 - \mathcal{B}_{\mathcal{R}}, \omega_0)/\mathbb{Z}_q$  from the Calderbank–Singer manifold, and then glue the two resulting open manifolds together by identifying the two constructed copies of the annulus quotient  $(\mathcal{B}_{\mathcal{R}} - \overline{\mathcal{B}}_r, \omega_0)/\mathbb{Z}_q$  via the tautological symplectomorphism between them. Since we only need to eliminate a finite number of singular points this way, we may also take the radius  $\mathcal{R}$  of these surgery regions to all be small enough so that these surgeries take place in disjoint regions, and so do not interfere with each other. Similarly, after choosing some non-singular reference point  $z \in \Sigma/\check{\Gamma} \subset X_{\ell}/\check{\Gamma}$ , we also require that the surgery radius  $\mathcal{R}$  be small enough that neither  $z$  nor the singular base-point  $p = [\hat{p}]$  belongs to the closure of these surgery regions.

We will now verify that this construction proves the following result:

**Proposition 2.1** *Let  $\Gamma \subset U(2)$  be a finite subgroup  $\neq \{1\}$  that acts freely on the unit sphere  $S^3 \subset \mathbb{C}^2$ . Then, there is a four-dimensional compact connected symplectic orbifold  $(X_{\Gamma}, \omega_{\Gamma})$  such that*

- (I)  *$(X_{\Gamma}, \omega_{\Gamma})$  contains exactly one singular point  $p$ ;*
- (II)  *$p$  has a neighborhood symplectomorphic to  $(\mathcal{B}, \omega_0)/\Gamma$  for some standard open ball  $\mathcal{B} \subset \mathbb{C}^2$  centered at the origin, where  $\Gamma$  acts on  $(\mathbb{C}^2, \omega_0)$  in the tautological manner, as a subgroup of  $U(2)$ ; and*
- (III) *there is a symplectic immersion  $j : S^2 \looparrowright X_{\Gamma} - \{p\}$ , with at worst transverse positively oriented double points, such that*

$$\int_{S^2} j^*[c_1(X_{\Gamma} - \{p\}, J)] \geq 3$$

*for some, and hence any,  $\omega$ -compatible almost-complex structure  $J$ .*

**Proof** We need only check the last condition, since the first two are obviously satisfied as long as the surgery radius  $\mathcal{R}$  is small. To produce the immersed sphere promised by condition (III), recall that we can construct an embedded sphere  $\Sigma' \subset X_{\ell} - \hat{p}$  of self-intersection  $\ell$  that only touches  $\Sigma$  at a chosen point of the latter 2-sphere. Let us now take the point  $\Sigma \cap \Sigma'$  to be one whose stabilizer under the action of  $\check{\Gamma}$  is trivial, so that it projects to a non-singular point  $z \in \Sigma/\check{\Gamma} \subset X_{\ell}/\check{\Gamma}$ . By shrinking the surgery radius  $\mathcal{R}$ , we can then guarantee that  $z$  lies outside the closure of the surgery regions. Also recall that one can take  $\Sigma'$  to be the image in  $X_{\ell} = \mathbb{CP}_2/\mathbb{Z}_{\ell}$  of a projective line  $\mathbb{CP}_1 \subset \mathbb{CP}_2$  that avoids the origin  $[0:0:1]$ , is not the line at infinity, and passes through the point of  $\mathbb{CP}_2$  that maps to  $z$ . This construction in particular guarantees that  $\Sigma' - \{z\}$  is a holomorphic curve with respect to a complex structure on  $X_{\ell} - (\Sigma \sqcup \{\hat{p}\})$  that is invariant under the action on  $\Gamma \subset U(2)$ . Projecting  $\Sigma'$  to  $X_{\ell}/\check{\Gamma}$  therefore gives us an immersed symplectic 2-sphere whose self-intersections all belong to the open set  $\mathcal{V} := [X_{\ell} - (\Sigma \sqcup \{\hat{p}\})]/\check{\Gamma}$ . It follows that these self-intersections are all transverse and positive, because in this region our sphere is a totally geodesic holomorphic curve with respect to the metric and  $\omega$ -compatible complex structure induced of  $\mathcal{V}$  by the Fubini–Study metric and complex structure on  $\mathbb{CP}_2$ . If necessary, we can then smoothly perturb this immersed 2-sphere near any triple points in order to produce a symplectic immersion  $j : S^2 \looparrowright X_{\Gamma} - \{p\}$  that has at worst transverse, positively oriented double points. Since the image of  $j$  is closed and avoids all the singular points of  $X_{\ell}/\check{\Gamma}$ , we can also arrange that it is disjoint from the



surgery regions by shrinking the surgery radius  $\mathcal{R}$  if necessary. Finally, since  $f^*c_1$  coincides with the restriction of  $c_1(X_\ell)$  to  $\Sigma'$ , it follows that  $\int_{S^2} f^*c_1 = 2 + \ell \geq 3$  by (2.1).  $\square$

Since the orbifolds we have just constructed will play an essential role in the next section, it now seems appropriate to give them a name:

**Definition 2.2** Let  $\Gamma \subset \mathrm{U}(2)$  be any finite subgroup that acts freely on the unit sphere  $S^3 \subset \mathbb{C}^2$ . Then,

- If  $\Gamma \neq \{1\}$ , a  $\Gamma$ -capsule will mean one of the standard symplectic orbifolds  $(X_\Gamma, \omega_\Gamma)$  satisfying conditions (I)–(III) that we have constructed in this section. The unique singular point  $p \in X_\Gamma$  will then be called the *base-point* of the  $\Gamma$ -capsule.
- When  $\Gamma = \{1\}$ , we instead define the associated  $\Gamma$ -capsule  $(X_\Gamma, \omega_\Gamma)$  to be  $\mathbb{CP}_2$ , equipped with its standard Fubini–Study symplectic structure. In this case, the base-point  $p$  of  $X_\Gamma$  will simply mean  $[0:0:1] \in \mathbb{CP}_2$ .

Thus, Proposition 2.1 can be restated as saying that, whenever  $\Gamma \subset \mathrm{U}(2)$  is a finite subgroup that acts freely on  $S^3$ , there always exists a  $\Gamma$ -capsule.

### 3 Capping off the ends

Now, suppose that  $(M^4, g, J)$  is an ALE Kähler manifold of complex dimension 2, where the metric is merely assumed to satisfy the Chruściel fall-off conditions (i)–(ii) for some  $\varepsilon > 0$ , in some real coordinate system at each end. This definition does not obviously exclude the possibility that  $M$  might actually have several ends. However, our first main result is that such a scenario is actually impossible:

**Theorem 3.1** *Let  $(M^4, g, J)$  be an ALE Kähler surface, where the metric is merely assumed to satisfy the fall-off hypotheses (i)–(ii) for some  $\varepsilon > 0$ . Then,  $M$  has exactly one end.*

**Proof** For each end  $M_{\infty,i} \approx (S^3/\Gamma_i) \times \mathbb{R}^+$  of  $M$ , we may first choose some  $\Gamma_i$ -capsule  $(X_{\Gamma_i}, \omega_{\Gamma_i})$ , the existence of which is guaranteed by Proposition 2.1. By Proposition 1.1, each end  $M_{\infty,i}$  contains an asymptotic region symplectomorphic to  $(\mathbb{C}^2 - \overline{\mathcal{B}_{\mathfrak{R}}}, \omega_0)/\Gamma_i$  for some sufficiently large common radius  $\mathfrak{R}$ . On the other hand, the base-point of each  $\Gamma_i$ -capsule has a neighborhood symplectomorphic to  $(\mathcal{B}_\tau, \omega_0)/\Gamma_i$  for some small common radius  $\tau$ , and by shrinking this radius if necessary we can guarantee that this ball quotient in each  $\Gamma_i$ -capsule does not meet some chosen symplectically immersed 2-sphere satisfying (III). We now inflate the  $\Gamma_i$ -capsules by replacing their symplectic forms  $\omega_{\Gamma_i}$  by  $t^2\omega_{\Gamma_i}$  for some large  $t > 0$ . In the inflated  $\Gamma_i$ -capsules, the base-point now has a neighborhood symplectomorphic to  $(\mathcal{B}_R, \omega_0)/\Gamma_i$ , where  $R = t\tau$ . Thus, by taking  $t$  to be sufficiently large, we may arrange that  $R > \mathfrak{R}$ . By now removing  $\overline{\mathcal{B}_{\mathfrak{R}}}/\Gamma_i \ni p$  from each  $\Gamma_i$ -capsule and  $(\mathbb{C}^2 - \overline{\mathcal{B}_R})/\Gamma_i$  from each  $M_{\infty,i}$ , we are then left with pieces that may be glued together symplectically along copies of  $(\mathcal{B}_R - \overline{\mathcal{B}_{\mathfrak{R}}})/\Gamma_i$  to produce a compact symplectic 4-manifold  $(N, \hat{\omega})$ .

Now,  $(N, \hat{\omega})$  has been constructed so that it contains a symplectically immersed 2-sphere  $j_i : S^2 \looparrowright N$  in each capped-off end. Moreover, this 2-sphere has at worst positive transverse double points, and satisfies  $\int_{S^2} j_i^*c_1 \geq 3$ . If the sphere has any double points at all, a result of McDuff [15, Theorem 1.4] then tells us that  $N$  symplectomorphic to a rational complex surface, and so orientedly diffeomorphic to either  $S^2 \times S^2$  or  $\mathbb{CP}_2 \# k\overline{\mathbb{CP}_2}$  for some  $k \geq 0$ . On the other hand, if the sphere has no double points, it is then an embedded symplectic 2-sphere of self-intersection  $\geq 3 - 2 = 1 > 0$ , so an earlier result of McDuff [14, Corollary 1.6] once



again tells us that  $N$  is orientedly diffeomorphic to a rational complex surface. In particular, it follows that  $b_+(M) = 1$ , meaning that the intersection form  $H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \rightarrow \mathbb{R}$  is of type  $(+ \cdots -)$ .

Now, each of the immersed spheres  $f_i(S^2)$  we have constructed can be modified to yield a connected embedded symplectic surface  $\mathcal{S}_i$  by replacing a small neighborhood of each double point with a cylinder  $S^1 \times (-\epsilon, \epsilon)$ . This process increases the genus but does not change the homology class; moreover, it can be carried out while remaining completely inside the truncated  $\Gamma_i$ -capsule containing  $f_i(S^2)$ . We therefore have

$$\langle c_1(N), [\mathcal{S}_i] \rangle = \int_{S^2} f_i^* c_1 \geq 3,$$

and, since  $\mathcal{S}_i$  is symplectic and embedded, the adjunction formula allows us to rewrite this as

$$\chi(\mathcal{S}_i) + [\mathcal{S}_i] \cdot [\mathcal{S}_i] \geq 3.$$

But since  $\mathcal{S}_i$  certainly has Euler characteristic  $\chi(\mathcal{S}_i) \leq 2$ , it therefore follows that

$$[\mathcal{S}_i] \cdot [\mathcal{S}_i] \geq 3 - \chi(\mathcal{S}_i) \geq 3 - 2 = 1,$$

so each of these surfaces has positive homological self-intersection. However, notice that  $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$  when  $i \neq j$ , since the truncated  $\Gamma$ -capsules where they live are, by construction, disjoint. This has the homological consequence that

$$[\mathcal{S}_i] \cdot [\mathcal{S}_j] = 0 \quad \forall i \neq j.$$

It follows that  $b_+(N)$  is at least as large as the number of ends of  $M$ . But since we have also just seen that  $b_+(N) = 1$ , this means that there can be at most one end. As our definition of an ALE manifold moreover requires  $M$  to be non-compact, it therefore follows that  $M$  has exactly one end.  $\square$

**Remark** The regularity of the gluing maps used in the above construction depends, via Proposition 1.1, on the regularity of the given metric  $g$ . Thus, if  $g$  is merely  $C^2$ , our symplectic manifold  $(N, \hat{\omega})$  is ostensibly merely a symplectic manifold with  $C^2$  coordinate transformations between Darboux coordinate charts. This might lead one to worry, because many of the cited papers in symplectic topology implicitly assume that all objects under discussion are of class  $C^\infty$ . Fortunately, such fears are misplaced, for general reasons we will now explain. Indeed, by a celebrated result of Whitney [25], there exists a smooth structure on  $N$  which is compatible with the given  $C^2$  structure, and the  $C^\infty$  2-forms, defined relative to this chosen smooth structure, will then be dense among  $C^1$  closed forms in the cohomology class  $[\hat{\omega}]$ . However, if the smooth form  $\tilde{\omega} \in [\hat{\omega}]$  is sufficiently close to  $\hat{\omega}$  in the  $C^1$  topology, all the convex combinations  $(1-t)\hat{\omega} + t\tilde{\omega} \in [\omega]$ ,  $t \in [0, 1]$ , will be symplectic forms, and Moser's stability argument [17] will then produce a  $C^1$  symplectomorphism between the  $C^2$  symplectic manifold  $(N, \hat{\omega})$  and the smooth symplectic manifold  $(N, \tilde{\omega})$ . Thus, our use of classification results for smooth symplectic manifolds is entirely justified.

**Corollary 3.2** *The mass formula of [7] holds in all complex dimensions  $\geq 2$ , merely assuming Chruściel fall-off conditions on the metric. In particular, if  $(M^4, g, J)$  is an ALE Kähler surface that merely satisfies (i)–(ii) for some  $\varepsilon > 0$ , then its mass is given by*

$$m(M, g) = -\frac{1}{3\pi} \langle \clubsuit(c_1), [\omega] \rangle + \frac{1}{12\pi^2} \int_M s_g d\mu_g \quad (3.1)$$

where  $s_g$  and  $d\mu_g$  are the scalar curvature and metric volume form,  $c_1$  is the first Chern class of  $(M, J)$ ,  $\clubsuit$  is the inverse of the natural homomorphism  $H_c^2(M) \rightarrow H^2(M)$ , and  $\langle \cdot, \cdot \rangle$  is the natural duality pairing between  $H^2(M)$  and  $H_c^2(M)$ .

**Proof** The case of complex dimension  $\geq 3$  was already proved in [7]. In the case of complex dimension 2, the proof of [7, Theorem 5.1] now proves the claim as long as one replaces the citation of [7, Proposition 4.2] with a reference to the above Theorem 3.1.  $\square$

Because there are other plausible methods available for proving Theorem 3.1, some might wonder if all our work in Sect. 2 was worth the effort. Fortunately, the ideas we have described here have other consequences which provide further justification for the current project:

**Proposition 3.3** *If  $(M^4, g, J)$  is any ALE Kähler surface with Chruściel metric fall-off, then  $M$  is diffeomorphic to the complement of a tree of symplectically embedded 2-spheres in a rational complex surface.*

**Proof** By a tree of embedded 2-spheres, we mean a union of transversely intersecting embedded symplectic 2-spheres such that the dual graph representing their intersection pattern is connected and contains no loops. The tree we have in mind here is determined by  $\Gamma$ , and is specifically the subset of a  $\Gamma$ -capsule gotten by attaching the appropriate Hirzebruch–Jung string to each orbifold point of  $\Sigma/\tilde{\Gamma} \approx S^2$ . Since the proof of Theorem 3.1 shows that  $M$  can be diffeomorphically compactified into a rational symplectic manifold  $N$  by attaching a truncated  $\Gamma$ -capsule  $Y = X_\Gamma - \overline{\mathcal{B}}/\Gamma$ , the result follows from the fact that the complement of the obvious tree in  $Y$  is diffeomorphic to  $(S^3/\Gamma) \times (0, 1)$ .  $\square$

Here is another immediate consequence of the same ideas:

**Proposition 3.4** *For any ALE Kähler surface  $(M^4, g, J)$  with Chruściel metric fall-off, the fundamental group of  $M$  is finite.*

**Proof** Once again, the proof of Theorem 3.1 shows that compactifying  $M$  by adding a truncated  $\Gamma$ -capsule  $Y$  results in a symplectic 4-manifold  $N$  that is diffeomorphic to a rational complex surface. In particular, this assertion means that  $N$  is simply connected. However, we also have

$$N = M \cup Y, \quad M \cap Y \approx (S^3/\Gamma) \times (0, 1) \approx M_\infty,$$

where  $Y$  is obtained from a  $\Gamma$ -capsule  $X_\Gamma$  by removing a closed neighborhood  $\overline{\mathcal{B}}_\epsilon/\Gamma$  of the base-point  $p$ . However, since  $Y$  deformation retracts to the tree of 2-spheres obtained by attaching a Hirzebruch–Jung string to each orbifold singularity of  $\Sigma/\tilde{\Gamma} \approx S^2$ , it follows that  $Y$  is simply connected. The Seifert–van Kampen theorem therefore tells us that  $\pi_1(N)$  is the quotient of  $\pi_1(M)$  by the image of  $\pi_1(M \cap Y) \cong \Gamma$ . But since  $\pi_1(N) = \{1\}$ , this means that  $\Gamma \rightarrow \pi_1(M)$  is surjective. In particular,  $\pi_1(M)$  is necessarily finite.  $\square$

It is worth emphasizing that, despite persistent rumors to the contrary,  $M$  really might not be simply connected, even in the Ricci-flat case. For pertinent examples and classification results, see [21, 27].

Of course, the simplest case of the present story is when the manifold in question is *asymptotically Euclidean* (AE); these are the special ALE manifolds for which  $\Gamma = \{1\}$ . It is only in this setting that one can hope to prove a positive mass theorem [11, 19, 20, 26], asserting that nonnegative scalar curvature necessarily implies nonnegative mass; in the more general ALE setting, such statements are typically false [10]. But in the AE setting, one can

even sometimes prove Penrose-type inequalities [3,9,18], which offer lower bound for the mass in terms of the areas of suitable minimal submanifolds of the space in question. In the Kähler context, a sharp lower bound of this type was given by [7, Theorem E]. However, while our proof of this result only required Chruściel fall-off in complex dimensions  $\geq 3$ , we needed to assume stronger fall-off in hypotheses complex dimension 2.

Fortunately, the ideas developed here provide a way around this difficulty.

**Theorem 3.5** (Penrose inequality for Kähler manifolds) *Let  $(M^{2m}, g, J)$  be an AE Kähler manifold, where the metric merely satisfies the Chruściel fall-off hypotheses (i)–(ii) for some  $\varepsilon > 0$  in some real asymptotic coordinate system. If the scalar curvature  $s$  of  $g$  is everywhere nonnegative, then  $(M, J)$  carries a numerically canonical divisor  $D$  that is expressed as a sum  $\sum n_j D_j$  of compact complex hypersurfaces with positive integer coefficients, with the property that  $\bigcup_j D_j \neq \emptyset$  whenever  $(M, J)$  is not diffeomorphic to  $\mathbb{R}^{2m}$ . In terms of this divisor, the mass of the manifold then satisfies*

$$m(M, g) \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum_j n_j \text{Vol}(D_j)$$

and equality holds if and only if  $(M, g, J)$  is scalar-flat Kähler.

**Proof** Since this was already proved in [7] in complex dimension  $\geq 3$ , we may henceforth restrict ourselves to the case where  $(M^4, J)$  is a complex surface. In this case, the proof of Theorem 3.1 shows that we can produce a compact symplectic manifold  $(N, \omega)$  by removing a standard symplectic end  $(\mathbb{R}^4 - \mathcal{B}_{\mathbb{R}}, \omega_0)$  and replacing it  $\mathbb{CP}_2$  minus a ball, equipped with some multiple of the Fubini–Study symplectic form. In this setting, a projective line in  $\mathbb{CP}_2$  gives us a symplectic 2-sphere of self-intersection  $+1$  in  $(N, \omega)$ . A result of McDuff [14, Corollary 1.5] then tells us that  $(N, \omega)$  is symplectomorphic to a blow-up of  $\mathbb{CP}_2$ , equipped with some Kähler form, in a way that sends the given 2-sphere to a projective line  $\mathbb{CP}_1$  that avoids all the blown-up points. Removing this “line at infinity,” we thus see that  $M$  must be diffeomorphic to  $\mathbb{R}^4 \# k\mathbb{CP}_2$ , where  $k = b_2(M)$ , and  $H_2(M, \mathbb{Z})$  is moreover generated by the homology classes of  $k$  disjoint symplectic 2-spheres  $E_1, \dots, E_k \subset M$  of self-intersection  $-1$ . But then, under the natural identification  $H_c^2(M) = H_2(M, \mathbb{R})$  arising from Poincaré–Lefschetz duality, we then have  $\clubsuit(-c_1) = \sum_{i=1}^k [E_i]$ , as may be checked by integrating/intersecting both sides against each of the homology generators  $[E_j]$ . Thus, the mass formula (3.1) tells us in the AE case that

$$m(M, g) = \frac{1}{3\pi} \sum_{j=1}^k \int_{E_j} \omega + \frac{1}{12\pi^2} \int_M s_g d\mu_g.$$

We now show that each of the homology classes  $E_i$  can actually be represented by a finite sum of holomorphic curves  $D_j$  in  $(M, J)$  with positive integer coefficients. We do this by first carrying out our construction of the compact symplectic manifold  $(N, \hat{\omega})$  rather more carefully. First, notice that our metric fall-off condition (ii) guarantees that the vector field  $\eta := (\varrho \nabla \varrho) / |\nabla \varrho|^2$  defined in terms of the Euclidean radius  $\varrho$  and the metric  $g$ , satisfies  $\mathcal{L}_\eta g = 2g + O(\varrho^{-1-\varepsilon})$ ; moreover, the generalized Euler vector field  $\eta$  is normal to the spheres  $\varrho = \text{constant}$ , and its flow just rescales the radial function  $\varrho$  by positive constants. For  $c$  sufficiently large, we may therefore define a distance-non-increasing piecewise differentiable map  $\Psi : M \rightarrow M$  that sends the inner region  $\varrho \leq c$  to itself by the identity, and that sends the outer region  $\varrho \geq c$  to the boundary sphere  $\varrho = c$  by the backward flow of  $\eta$ . By choosing



$c$  to be sufficiently large, we can also arrange that the restriction of  $\Psi$  to the region  $\varrho \geq 3c$  actually contracts distances by a factor of at least 2.

We next apply the coordinate transformation  $\Phi$  given by Proposition 1.1 in order to identify the Kähler form  $\omega$  on the asymptotic region of  $(M, g, J)$  with the standard symplectic form  $\omega_0$  on  $\mathbb{C}^2$ . Because the derivative of  $\Phi$  satisfies  $\Phi_* = I + O(|x|^{-1-\varepsilon})$ , the image  $\Phi_*J_0$  is uniformly as close as we like to  $J_0$  in the image of the region  $\varrho \geq c$ , provided we again take  $c$  to be sufficiently large. Our fall-off condition  $J = J_0 + O(|x|^{-1-\varepsilon})$  now also guarantees that  $\tilde{J} = \Phi_*J$  is similarly uniformly close to  $J_0$ . In particular, we may arrange that  $T_j^{1,0} \cap T_{j_0}^{0,1} = 0$ , which then allows us to represent  $T_j^{1,0}$  by a tensor field  $\phi \in \Lambda_{J_0}^{0,1} \otimes T_{j_0}^{1,0}$ , and the fact that  $\tilde{J}$  and  $J_0$  are both  $\omega_0$  compatible is then encoded by the statement that  $\phi \lrcorner \omega_0 \in \Lambda_{J_0}^{0,1} \otimes \Lambda_{J_0}^{0,1}$  is symmetric. Since the latter condition is linear in  $\phi$ , the almost-complex structure  $\tilde{J}$  corresponding to  $f\phi$ , will also be  $\omega_0$  compatible, where we now take  $f = f(\varrho)$  to be a smooth, non-increasing cutoff function which is  $\equiv 1$  for  $\varrho \leq 4c$  and  $\equiv 0$  for  $\varrho \geq 5c$ . Because this almost-complex structure is still uniformly close to  $\tilde{J}$ , the corresponding Riemannian metric  $\hat{g} = \omega_0(\cdot, \tilde{J}\cdot)$  is uniformly close to  $g$  in the exterior region, and we can therefore arrange that  $\Phi^*\hat{g} \geq g/2$  in the region  $\varrho \geq 3c$ , while nonetheless keeping  $\Phi^*\hat{g} = g$  in the region  $\varrho \leq 3c$ . Thus, the constructed map  $\Psi : M \rightarrow M$  is distance non-increasing with respect to  $\hat{g}$  as well as with respect to  $g$ ; and it is moreover *strictly* distance decreasing outside of the region where  $\varrho \leq c$  in our original coordinates.

To cap off the end, we next choose a Kähler metric  $h$  on  $\mathbb{CP}_2$  that is identically Euclidean on the unit ball in  $\mathbb{C}^2 \subset \mathbb{CP}_2$ . By multiplying  $h$  by a large positive constant  $\lambda > 25c^2$ , we then obtain a Kähler metric  $\lambda h$  on  $\mathbb{CP}_2$  which contains an isometric copy of a Euclidean ball of radius  $> 5c$ . We then cut a Euclidean ball  $B$  of radius  $5c$  out of this larger ball, and glue in the region  $\mathcal{U} \subset M$  that is given by  $\varrho \leq 5c$  in our symplectic coordinates. The resulting symplectic 4-manifold  $(N, \hat{\omega})$  thus comes equipped with an almost-Kähler metric  $\hat{g}$  which is given by  $\lambda h$  on  $\mathbb{CP}_2 - B$ , by  $g$  on the region  $\mathcal{V} \subset \mathcal{U}$  corresponding  $\varrho \leq c$  in our initial coordinates, and by the constructed interpolation  $\hat{g}$  on the transition annulus  $\mathcal{U} - \mathcal{V}$ .

However, because  $(N, \hat{\omega})$  is a symplectic manifold with  $b_+ = 1$ , a result of Taubes [22] therefore tells us that the perturbed Seiberg–Witten invariant of  $N$  is nonzero for the the  $\text{spin}^c$  structure  $c$  determined by  $J$  and the chamber containing large multiples of  $-\hat{\omega}$ . However, because  $N$  also admits self-diffeomorphisms which act on  $H^2(N)$  by  $[E_i] \mapsto -[E_i]$  and by the identity on  $[E_i]^\perp$ , the analogous perturbed Seiberg–Witten invariant is also nonzero for the images of  $c$  under all these reflections. It therefore follows [13, 23] that each of the classes  $[E_i]$  is represented by a (possibly singular)  $\hat{J}$ -holomorphic curve  $\mathcal{E}_i$ . Moreover,  $\mathcal{E}_i$  is the zero locus of a section  $u$  of a line bundle  $\mathcal{L}_i \rightarrow M$  with Chern class  $c_1(\mathcal{L}_i) = [E_i]$  with the property that  $u$  is approximately holomorphic near  $\mathcal{E}_i$ .

Now, the truncated capsule region  $\mathbb{CP}_2 - B$  of  $N$  is a union of projective lines, and these, by construction, are all  $\hat{J}$ -holomorphic curves. Since  $u$  is approximately holomorphic near  $\mathcal{E}_i$ , the number of zeroes of the restriction of  $u$  to any such projective line  $P$ , counted with the obvious nonnegative multiplicities, is exactly  $\int_P c_1(\mathcal{L}_i)$ . However,  $\int_P c_1(\mathcal{L}_i)$  is also exactly the intersection pairing of  $[E_i]$  and  $[P]$ , which we have known from the outset to be zero. It follows that  $u$  is everywhere nonzero on every such projective line  $P$ , so that we always have  $\mathcal{E}_i \cap P = \emptyset$ . But since  $\mathbb{CP}_2 - B$  is a union of such projective lines  $P$ , this implies that  $\mathcal{E}_i \subset N - (\mathbb{CP}_2 - B) = \mathcal{U}$ .

This means that  $\mathcal{E}_i$  is a pseudo-holomorphic curve in  $(\mathcal{U}, \hat{J})$ , and thus in  $(M, \hat{J})$ , after we recall that our interpolated almost-complex structure  $\hat{J}$  was initially defined in symplectic coordinates on the entire end  $M_\infty$ . Here, it is worth pointing out that, while  $\mathcal{E}_i$  may very well be singular, the corresponding pseudo-holomorphic curves for generic perturbations



$J'$  of  $\hat{J}$  are embedded 2-spheres, because  $[E_i]^2 = -1$  and  $c_1 \cdot [E_i] = +1$ ; by Gromov compactness [6, 16],  $\mathcal{E}_i$  can therefore be, at worst, a finite tree of branched minimal 2-spheres. We now recall that, since these 2-spheres are all calibrated submanifolds of the almost-Kähler manifold  $(M, \hat{g}, \omega)$ , each one has least area among surfaces in its homology class. But we have carefully arranged for the piecewise smooth map  $\Psi : M \rightarrow M$  to be distance non-increasing with respect to  $\hat{g}$ , and to even be strictly distance decreasing on  $M - \mathcal{V}$ ; moreover,  $\Psi : M \rightarrow M$  was also constructed as a deformation retraction of  $M$  to  $\mathcal{V}$ . It therefore follows that none of the 2-spheres that make up  $\mathcal{E}_i$  can meet  $M - \mathcal{V}$ , because applying  $\Psi : M \rightarrow M$  to such a 2-sphere would otherwise produce a homotopic 2-sphere of strictly smaller area. It therefore follows that each spherical piece of  $\mathcal{E}_i$ , and hence the entire pseudo-holomorphic curve  $\mathcal{E}_i$  itself, must be contained in  $\mathcal{V}$ , where  $\hat{J}$  coincides with the original integrable complex structure  $J$  of  $(M, J)$ . In other words, each  $\mathcal{E}_i$  is actually a holomorphic curve in our original Kähler manifold  $(M, g, J)$ . This means that  $\int_{\mathcal{E}_i} \omega$  is in fact exactly the area of  $\mathcal{E}_i$ , counted with multiplicities, and our mass formula can therefore be rewritten as

$$m(M, g) = \frac{1}{3\pi} \sum_i \text{Vol}(\mathcal{E}_i) + \frac{1}{12\pi^2} \int_M s_g d\mu_g.$$

If the  $D_j$  are the various spherical components of the various  $\mathcal{E}_i$ , and if  $n_j$  is the multiplicity with which a given  $D_j$  occurs in this way, we can then rewrite this as

$$m(M, g) = \frac{1}{3\pi} \sum_j n_j \text{Vol}(D_j) + \frac{1}{12\pi^2} \int_M s_g d\mu_g.$$

If  $s_g \geq 0$ , this then gives us the Penrose-type inequality

$$m(M, g) \geq \frac{1}{3\pi} \sum_j n_j \text{Vol}(D_j),$$

where equality iff  $g$  is scalar-flat Kähler.  $\square$

There is one respect in which this result remains noticeably weaker than [7, Theorem E]. Indeed, the earlier argument shows that, assuming stronger fall-off conditions, the underlying complex surface of an AE  $(M, g, J)$  must be an iterated blow-up of  $\mathbb{C}^2$ . When combined with [6, Theorem 0.3C], our present result instead shows that the weaker fall-off conditions (i)–(ii) imply that  $(M^4, \omega)$  is an iterated symplectic blow-up of  $(\mathbb{R}^4, \omega_0)$ . Nonetheless, this is quite good enough for applications like the following:

**Corollary 3.6** (Positive mass theorem for Kähler manifolds) *Let  $(M^{2m}, g, J)$  be an AE Kähler manifold, where the metric merely satisfies the Chruściel fall-off hypotheses (i)–(ii) for some  $\varepsilon > 0$  in some real asymptotic coordinate system. If  $g$  has scalar curvature  $s_g \geq 0$  everywhere, then  $m(M, g) \geq 0$ , with equality iff  $(M, g)$  is Euclidean space.*

**Proof** By Theorem 3.5, we merely need to consider the case when  $M$  is diffeomorphic to  $\mathbb{R}^{2m}$  and the metric  $g$  is scalar-flat Kähler. However, this implies that the Ricci-form  $\rho$  of  $g$  is harmonic, and is an  $L^2$  harmonic form. Since de Rham classes on an ALE manifold have unique  $L^2$  harmonic representatives, this means that  $g$  must be Ricci-flat, because we have assumed that  $M$  is contractible. But since the asymptotic volume growth of an AE metric is exactly Euclidean, the Bishop–Gromov equality therefore implies that the exponential map gives an isometry between any tangent space and  $(M, g)$ .  $\square$

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# References

1. Barth, W., Peters, C., Van de Ven, A.: Compact Complex Surfaces, Vol. 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete* (3). Springer, Berlin (1984)
2. Bartnik, R.: The mass of an asymptotically flat manifold. *Commun. Pure Appl. Math.* **39**, 661–693 (1986)
3. Bray, H.L.: Proof of the Riemannian Penrose inequality using the positive mass theorem. *J. Differ. Geom.* **59**, 177–267 (2001)
4. Calderbank, D.M.J., Singer, M.A.: Einstein metrics and complex singularities. *Invent. Math.* **156**, 405–443 (2004)
5. Chruściel, P.: Boundary conditions at spatial infinity from a Hamiltonian point of view. In: *Topological Properties and Global Structure of Space–Time* (Erice, 1985), vol. 138 of NATO Adv. Sci. Inst. Ser. B Phys., Plenum, New York, 1986, pp. 49–59. <http://homepage.univie.ac.at/piotr.chrusciel/scans/index.html>
6. Gromov, M.: Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.* **82**, 307–347 (1985)
7. Hein, H.-J., LeBrun, C.: Mass in Kähler geometry. *Commun. Math. Phys.* **347**, 183–221 (2016)
8. Hirzebruch, F.: Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen. *Math. Ann.* **126**, 1–22 (1953)
9. Huiskens, G., Ilmanen, T.: The Riemannian Penrose inequality. *Int. Math. Res. Not.* **1997**, 1045–1058 (1997)
10. LeBrun, C.: Counter-examples to the generalized positive action conjecture. *Commun. Math. Phys.* **118**, 591–596 (1988)
11. Lee, J., Parker, T.: The Yamabe problem. *Bull. Am. Math. Soc.* **17**, 37–91 (1987)
12. Lerman, E.: Symplectic cuts. *Math. Res. Lett.* **2**, 247–258 (1995)
13. Li, T.-J., Liu, A.-K.: The equivalence between SW and Gr in the case where  $b^+ = 1$ . *Int. Math. Res. Not.* **1999**, 335–345 (1999)
14. McDuff, D.: The structure of rational and ruled symplectic 4-manifolds. *J. Am. Math. Soc.* **3**, 679–712 (1990)
15. McDuff, D.: Immersed spheres in symplectic 4-manifolds. *Ann. Inst. Fourier (Grenoble)* **42**, 369–392 (1992)
16. McDuff, D., Salamon, D.: *Introduction to Symplectic Topology*. Oxford University Press, New York (1995)
17. Moser, J.: On the volume elements on a manifold. *Trans. Am. Math. Soc.* **120**, 286–294 (1965)
18. Penrose, R.: Naked singularities. *Ann. N. Y. Acad. Sci.* **224**, 125–134 (1973). Sixth Texas Symposium on Relativistic Astrophysics
19. Schoen, R., Yau, S.T.: Incompressible minimal surfaces, three-dimensional manifolds with nonnegative scalar curvature, and the positive mass conjecture in general relativity. *Proc. Nat. Acad. Sci. USA* **75**, 2567 (1978)
20. Schoen, R.M., Yau, S.T.: Complete manifolds with nonnegative scalar curvature and the positive action conjecture in general relativity. *Proc. Nat. Acad. Sci. USA* **76**, 1024–1025 (1979)
21. Şuvaina, I.: ALE Ricci-flat Kähler metrics and deformations of quotient surface singularities. *Ann. Glob. Anal. Geom.* **41**, 109–123 (2012)
22. Taubes, C.H.: The Seiberg–Witten invariants and symplectic forms. *Math. Res. Lett.* **1**, 809–822 (1994)
23. Taubes, C.H.: The Seiberg–Witten and Gromov invariants. *Math. Res. Lett.* **2**, 221–238 (1995)
24. Thurston, W.P.: The geometry and topology of 3-manifolds. Unpublished Princeton Lecture Notes (1980). <http://library.msri.org/books/gt3m/>
25. Whitney, H.: Differentiable manifolds. *Ann. Math. (2)* **37**, 645–680 (1936)
26. Witten, E.: A new proof of the positive energy theorem. *Commun. Math. Phys.* **80**, 381–402 (1981)
27. Wright, E.P.: Quotients of gravitational instantons. *Ann. Glob. Anal. Geom.* **41**, 91–108 (2012)

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