Transformation Groups
Vol. 25, No. 3, 2020, pp. 843–858

VERSAL TORSORS AND RETRACTS

A. S. MERKURJEV*

Department of Mathematics
University of California
Los Angeles
CA, 90095-1555 USA

merkurev@math.ucla.edu

Abstract. Let G be an algebraic group over F and p a prime integer. We introduce the notion of a p-retract rational variety and prove that if $Y \to X$ is a p-versal G-torsor, then $\mathrm{B}G$ is a stable p-retract of X. It follows that the classifying space $\mathrm{B}G$ is p-retract rational if and only if there is a p-versal G-torsor $Y \to X$ with X a rational variety, that is, all G-torsors over infinite fields are rationally parameterized. In particular, for such groups G the unramified Galois cohomology group $H^n_{\mathrm{nr}}\big(F(\mathrm{B}G), \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$ coincides with $H^n\big(F, \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$.

Introduction

Let G be an algebraic group over a field F. In the present paper we study G-torsors $E \to \operatorname{Spec} K$ for field extensions K/F. In many cases G-torsors are related to classical algebraic objects. For example, if $G = \mathbf{PGL}_n$ such objects are central simple algebras A of degree n over K. Every $\mathbf{PGL}_n$-torsor over $\operatorname{Spec} K$ is the torsor of isomorphisms between A and the matrix algebra $M_n(K)$.

A G-torsor $f : Y \to X$ is called *versal* if every G-torsor $E \to \operatorname{Spec} K$ for an extension K/F with K an infinite field is isomorphic to the pull-back of f with respect to a morphism (a point) $\operatorname{Spec} K \to X$ and the set of images of such morphisms is dense in X. Thus, a versal G-torsor keeps information about all G-torsors over field extensions K/F.

Versal G-torsors exist. For example, let V be a generically free representation of G (that is the generic stabilizer of a vector in V is trivial). There is a nonempty G-invariant open subset $I \subset V$ and a G-torsor $I \to Z$ for some variety Z over F. (One can think of Z as the variety of orbits I/G.) It appears that $I \to Z$ is a versal G-torsor. We call such torsors *standard* versal G-torsors. We think of the variety Z as an "approximation" of the stack $\mathrm{B}G$ of all G-torsors, which we call the *classifying space* of G.

If $I \to Z$ and $I' \to Z'$ are two standard versal G-torsors, then the varieties Z and Z' are stably birationally isomorphic. In other words, the stable birational type of the classifying space $\mathrm{B}G$ is well defined.

DOI: 10.1007/s00031-019-09521-y
*Supported by NSF grant DMS #1801530.
Received May 22, 2018. Accepted October 13, 2018.
Published online April 6, 2019.
Corresponding Author: A. S. Merkurjev, e-mail: merkurev@math.ucla.edu

If $I \to Z$ is a standard versal G-torsor and $Y \to X$ is a versal G-torsor, then Z and X may not be stably birationally isomorphic. But we prove in Theorem 9 that Z is a stable retract of X, that is, there are rational morphisms $f : Z \dashrightarrow X \times \mathbb{A}^n_F$ and $g : X \times \mathbb{A}^n_F \dashrightarrow Z$ for some n such that the composition $g \circ f$ is defined and equals the identity of Z.

We say that all G-torsors over infinite fields for an algebraic group G are *rationally parameterized* if there is a versal G-torsor $Y \to X$ with X a rational variety. We prove (Theorem 10) that all G-torsors over infinite fields are rationally parameterized if and only if BG (that is, its approximation Z for a standard versal torsor $I \to Z$) is a retract of a rational variety.

We also consider the local setting. Namely, for a prime integer p we consider p-versal torsors and define p-retracts, roughly, by ignoring the effects given by dominant morphisms of finite degree prime to p. We prove local analogs of the theorems mentioned above.

In Section 6 we prove (Theorem 15) that if X and X' are smooth varieties over F such that X is a p-retract of X', then there is an injective homomorphism of the groups of unramified cohomology

$$H^n_{\mathrm{nr}}\big(F(X), \mathbb{Q}_p/\mathbb{Z}_p(j)\big) \to H^n_{\mathrm{nr}}\big(F(X'), \mathbb{Q}_p/\mathbb{Z}_p(j)\big).$$

In particular, if X is a p-retract rational smooth variety over F, then the natural homomorphism

$$H^n\big(F, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) \to H^n_{\mathrm{nr}}\big(F(X), \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$$

is an isomorphism.

We use the following notation. A *variety* over a field F is an integral separated scheme of finite type over F. We write $F(X)$ for the function field of X over F. An *algebraic group* over F is an affine group scheme of finite type over F (not necessarily smooth or connected). The *degree* of a dominant morphism $Y \to X$ of varieties is the integer $[F(X) : F(Y)]$. We write $X \approx Y$ if X and Y are birationally isomorphic, i.e., $F(X) \simeq F(Y)$ over F. If X is a scheme over F and L/F is a field extension, we write X_L for the scheme $X \times_F \operatorname{Spec} L$ over L. The *generic fiber* of a dominant rational morphism $f : Y \dashrightarrow X$ of varieties over F is the scheme $U \times_X \operatorname{Spec} K$ over $K = F(X)$, where $U \subset Y$ is the domain of definition of f. We write pt for $\operatorname{Spec} F$.

Let $Y \to X$ be a G-torsor with X a variety over F. The trivial vector bundle $Y \times V \to Y$ with the diagonal G-action on $Y \times V$ descends to a vector bundle $Y^V \to X$ (see [BK] and [V, Chap. 4]).

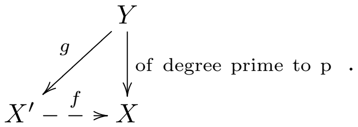
The letter p in the paper denotes either a prime integer or 0. An integer k is said to be *prime to p* when k is prime to p if $p > 0$ and $k = 1$ if $p = 0$.

We collect technical (mostly known) results in the Appendix.

Acknowledgment. The author is grateful to Jean-Louis Colliot-Thélène, David Harbater and Zinovy Reichstein for useful comments.

1. Split rational morphisms

A rational dominant morphism of varieties $f : X' \dashrightarrow X$ over a field F is called (*rationally*) *p-split* if for every nonempty open subset $U' \subset X'$ in the domain of definition of f, there is a morphism of varieties $g : Y \to X'$ such that $\operatorname{Im}(g) \subset U'$ and the composition $f \circ g : Y \to X$ is dominant of finite degree prime to p:



Clearly, a dominant morphism $f : X' \dashrightarrow X$ is p-split if and only if the set of closed points of degree prime to p in the generic fiber of f is everywhere dense.

Remark 1. By Lemma 21, the set of closed points of degree prime to $p > 0$ in a regular variety is either empty or everywhere dense. It follows that in the case the generic fiber of a dominant rational morphism $f : X' \dashrightarrow X$ is regular, the density condition in the definition of p-retract for f can be removed if $p > 0$.

Example 1. If $f : X' \dashrightarrow X$ is a dominant rational morphism of finite degree prime to p, then f is p-split. Indeed, we can take Y the domain of definition of f and g the inclusion of Y into X'.

We say that f is *split* if f is p-split for $p = 0$. By definition, f is split if and only if for every nonempty open subset U' in the domain of definition of f, there is a rational morphism $g : X \dashrightarrow X'$ such that $\operatorname{Im}(g) \cap U' \neq \varnothing$ and $f \circ g = 1_X$. Equivalently, f is split if and only if the rational points in the generic fiber of f are everywhere dense.

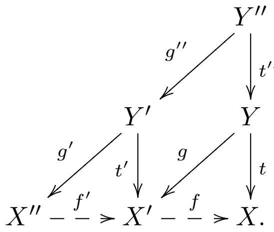
Lemma 1.

(1) *If $f : X' \dashrightarrow X$ and $f' : X'' \dashrightarrow X'$ are p-split morphisms of varieties over F, then so is $f \circ f'$.*
(2) *Every birational isomorphism is split.*
(3) *If X is a variety over F such that the field $F(X)$ is infinite, then the projection $X \times \mathbb{A}^n_F \to X$ is split for all n.*

Proof. (1): Let $U'' \subset X''$ be a nonempty open subset in the domains of definition of f' and $f \circ f'$. Choose a morphism $g' : Y' \to X''$ such that $\operatorname{Im}(g') \subset U''$ and $t' := f' \circ g'$ is dominant of finite degree prime to p.

By Lemma 19, there exists a nonempty open subset $U' \subset X'$ in the domain of definition of f such that for every point $u \in U'$ there is $y \in Y'$ with $t'(y) = u$ and finite $[F(y) : F(u)]$ prime to p. Since f is p-split, there is a morphism $g : Y \to X'$ with $\operatorname{Im}(g) \subset U'$ such that $t := f \circ g$ is dominant of finite degree prime to p.

By Lemma 20, there exists a variety Y'', a morphism $g'' : Y'' \to Y'$ and a dominant morphism $t'' : Y'' \to Y$ of finite degree prime to p such that $t' \circ g'' = g \circ t''$:



We have

$$(f \circ f') \circ (g' \circ g'') = f \circ t' \circ g'' = f \circ g \circ t'' = t \circ t''$$

and $\mathrm{Im}(g' \circ g'') \subset \mathrm{Im}(g') \subset U''$. Moreover, $\deg(t \circ t'') = \deg(t)\deg(t'')$ is prime to p. Therefore, $f \circ f'$ is p-split.

(2) follows immediately from the definition.

(3): Under the assumption, the $F(X)$-points are dense in the generic fiber $\mathbb{A}^n_{F(X)}$ of the projection $X \times \mathbb{A}^n_F \to X$. □

Lemma 2. *If $f : X' \dashrightarrow X$ is p-split, then so is $f \times 1 : X' \times \mathbb{A}^n_F \dashrightarrow X \times \mathbb{A}^n_F$ for every n.*

Proof. Let W be the domain of definition of f and $U \subset W \times \mathbb{A}^n_F$ a nonempty open subset. As the projection $p : W \times \mathbb{A}^n_F \to W$ is flat, it is an open morphism, hence the image $U' := p(U)$ is open in W. As f is p-split, there is a morphism of varieties $g : Y \to X'$ such that $\mathrm{Im}(g) \subset U'$ and the composition $f \circ g : Y \to X$ is dominant of finite degree prime to p. It follows that the image of $g \times 1 : Y \times \mathbb{A}^n_F \to X' \times \mathbb{A}^n_F$ intersects U. Therefore, the subset $T := (g \times 1)^{-1}(U) \subset Y \times \mathbb{A}^n_F$ is nonempty open. Then the restriction $h := (g \times 1)|_T : T \to X' \times \mathbb{A}^n_F$ satisfies $\mathrm{Im}(h) \subset U$ and the composition $(f \times 1) \circ h : T \to X \times \mathbb{A}^n_F$ is dominant of finite degree prime to p. □

2. Retracts

We say that a variety X *is a* (*rational*) *p-retract* of a variety X' if there is a p-split rational morphism $f : X' \dashrightarrow X$. We write $X <_p X'$ if X is a p-retract of X'.

If $p = 0$, we simply write $X < X'$ for $X <_p X'$ and call X a *retract of* X'. Clearly, $X < X'$ implies $X <_p X'$ for every p.

Our definition of retract coincides with the one in [R, Def. 1.1].

Example 2. If $f : X' \dashrightarrow X$ is a dominant rational morphism of finite degree prime p, then $X <_p X'$ (see Example 1).

In the case $p = 0$, the following lemma was proved in [R, Lems. 1.3, 1.4, Ex. 1.5a].

Lemma 3.

(1) *If* $X <_p X'$ *and* $X' <_p X''$*, then* $X <_p X''$*.*
(2) *If* $X \approx Y$*, then* $X <_p Y <_p X$ *for all* p*.*
(3) *If* $X <_p X'$*,* $X \approx Y$ *and* $X' \approx Y'$*, then* $Y <_p Y'$*.*
(4) *If* X *is a variety over* F *such that the function field* $F(X)$ *is infinite, then* $X < (X \times \mathbb{A}^n_F)$ *for all* n*.*

Proof. (3) follows from (1) and (2). The other statements are proved in Lemma 1. □

Lemma 3 shows that the relation $<_p$ can be defined on the set of birational isomorphism classes of varieties over F.

The following statement is proved in Lemma 2.

Lemma 4. *If* $X <_p X'$*, then* $(X \times \mathbb{A}^n_F) <_p (X' \times \mathbb{A}^n_F)$ *for every* n*.*

We say that X *is a stable* p*-retract of* X' and write $X \lhd_p X'$ if $X <_p (X' \times \mathbb{A}^n_F)$ for some $n \geq 0$ (cf. [R, Def. 4.1]). If X is a p-retract of X', i.e., $X <_p X'$, then $X \lhd_p X'$.

Corollary 5. *If* $X \lhd_p X'$ *and* $X' \lhd_p X''$*, then* $X \lhd_p X''$*.*

Proof. We have $X <_p X' \times \mathbb{A}^m_F$ and $X' <_p X'' \times \mathbb{A}^n_F$ for some m and n. By Lemma 4, $X' \times \mathbb{A}^m_F <_p X'' \times \mathbb{A}^{n+m}_F$, hence $X <_p X'' \times \mathbb{A}^{n+m}_F$ in view of Lemma 3. □

If X and Y are varieties over F, we write $X \overset{\text{s.b.}}{\approx} Y$ if X and Y are stably birational, i.e., $X \times \mathbb{A}^m_F \approx Y \times \mathbb{A}^n_F$ for some m and n.

Corollary 6. *If* $F(Y)$ *is infinite,* $X \lhd_p X'$*,* $X \overset{\text{s.b.}}{\approx} Y$ *and* $X' \overset{\text{s.b.}}{\approx} Y'$*, then* $Y \lhd_p Y'$*.*

Proof. We have birational isomorphisms $X \times \mathbb{A}^m_F \approx Y \times \mathbb{A}^n_F$, $X' \times \mathbb{A}^r_F \approx Y' \times \mathbb{A}^k_F$ and $X <_p X' \times \mathbb{A}^s_F$ for some m, n, r, k, s. By Lemmas 3 and 4,

$$Y <_p (Y \times \mathbb{A}^{n+r}_F) <_p (X \times \mathbb{A}^{m+r}_F) \lhd_p (X \times \mathbb{A}^r_F) <_p (X' \times \mathbb{A}^{r+s}_F) <_p (Y' \times \mathbb{A}^{k+s}_F) \lhd_p Y'.$$

By Corollary 5, $Y \lhd_p Y'$. □

A variety X is called *p-retract rational* if X is a p-retract of a rational variety. Equivalently, by Lemma 3, X is p-retract rational if and only if $X \lhd_p \mathrm{pt}$. A variety X is called *retract rational* if X is p-retract rational for $p = 0$.

3. Versal torsors

Let G be an algebraic group over F. We consider G-torsors $Y \to X$ over a variety X. Note that we do not assume that Y is a variety, i.e., Y is integral.

A G-torsor $Y \to X$ over a variety X is called *p-versal* if for every G-torsor $E \to \mathrm{Spec}\,(K)$ for a field extension K/F with K an infinite field and every nonempty open subset $U \subset X$, there is a finite field extension L/F of degree prime to p such that the G-torsor $E_L \to \mathrm{Spec}\, L$ is isomorphic to the pull-back of $Y \to X$ with respect to a point $x : \mathrm{Spec}\,(L) \to X$ with $\mathrm{Im}(x) \in U$ (see [DR]).

A G-torsor $Y \to X$ is called *versal* if it is p-versal for $p = 0$ (see [GMS]). Every versal torsor is p-versal for every p.

Proposition 7. *Let $f : X_1 \to X_2$ be a dominant morphism of varieties over F, $Y_2 \to X_2$ a G-torsor and $Y_1 \to X_1$ the pull-back of $Y_2 \to X_2$ with respect to f. Then*

(1) *If $Y_1 \to X_1$ is a p-versal G-torsor, then so is $Y_2 \to X_2$.*
(2) *If $Y_2 \to X_2$ is a p-versal G-torsor and f is p-split, then $Y_1 \to X_1$ is p-versal.*

Proof. (1) Let $E \to \operatorname{Spec} K$ be a G-torsor, where K is a field extension of F such that K is an infinite field, and $U_2 \subset X_2$ a nonempty open subset. As f is dominant, the open subset $U_1 := f^{-1}(U_2) \subset X_1$ is nonempty. Since $Y_1 \to X_1$ is a p-versal torsor, there is a field extension L/K of finite degree prime to p and a point $x_1 : \operatorname{Spec} L \to X_1$ with $\operatorname{Im}(x_1) \subset U_1$ such that the torsor $E_L \to \operatorname{Spec} L$ is isomorphic to the pull-back of $Y_1 \to X_1$ with respect to x_1. If $x_2 := f \circ x_1 : \operatorname{Spec} L \to X_2$, then $\operatorname{Im}(x_2) \subset U_2$ and $E_L \to \operatorname{Spec} L$ is isomorphic to the pull-back of $Y_2 \to X_2$ with respect to x_2.

(2) Let $E \to \operatorname{Spec} K$ be a G-torsor, where K is a field extension of F such that K is an infinite field, and $U_1 \subset X_1$ a nonempty open subset. Since f is p-split, there is a morphism of varieties $g : Y \to X_1$ such that $\operatorname{Im}(g) \subset U_1$ and the composition $f \circ g : Y \to X_2$ is finite of degree prime to p. In view of Lemma 19 applied to the morphism $f \circ g : Y \to X_2$ of finite degree prime to p, we find a nonempty open subset $U_2 \subset X_2$ such that for every point $x_2 \in U_2$ there is a point $y \in Y$ with the property that $f(g(y)) = x_2$ and the field extension $F(y)/F(x_2)$ is finite of degree prime to p.

As $Y_2 \to X_2$ is a p-versal G-torsor, there is a field extension L/K of finite degree prime to p and a morphism $h : \operatorname{Spec} L \to X_2$ such that $\{x_2\} := \operatorname{Im}(h) \subset U_2$ and $E_L \to \operatorname{Spec} L$ is isomorphic to the pull-back of the torsor $Y_2 \to X_2$ with respect to h. Choose a point $y \in Y$ such that $f(g(y)) = x_2$ and the field extension $F(y)/F(x_2)$ is finite of degree prime to p. By Corollary 18, applied to the morphism $f \circ g : Y \to X_2$, there is a field extension L'/L of finite degree prime to p and a morphism $k : \operatorname{Spec} L' \to Y$ such that $\operatorname{Im}(k) = \{y\}$ and the composition of $\operatorname{Spec} L' \to \operatorname{Spec} L$ with h coincides with $f \circ g \circ k$:

$$\begin{array}{ccccc} \operatorname{Spec} L' & \longrightarrow & \operatorname{Spec} L & & \\ {\scriptstyle k}\downarrow & & & \searrow^{h} & \\ Y & \xrightarrow{\ g\ } & X_1 & \xrightarrow{\ f\ } & X_2. \end{array}$$

It follows that $E_{L'} \to \operatorname{Spec} L'$ is isomorphic to the pull-back of the torsor $Y_1 \to X_1$ with respect to $g \circ k$ and $[L' : K] = [L' : L] \cdot [L : K]$ is prime to p. Finally, $\operatorname{Im}(g \circ k) = g(\operatorname{Im}(k)) = \{g(y)\} \subset U_1$. It follows that $Y_1 \to X_1$ is a p-versal torsor. □

4. Standard versal torsors

Let G be an algebraic group over F. Let V be a generically free G-representation and $I \subset V$ a nonempty G-invariant open subset together with a G-torsor $I \to Z$, where Z is a variety over F. We call $I \to Z$ a *standard* G-torsor. We always assume that I is chosen so that $\dim(Z) > 0$, hence the field $F(Z)$ is infinite.

Example 3. Embed G into $\mathbf{GL}_n$ as a closed subgroup. Then the natural morphism $\mathbf{GL}_n \to \mathbf{GL}_n/G$ is a standard G-torsor since $\mathbf{GL}_n$ is an open subset of the affine space of $M_n(F)$ and G acts on $M_n(F)$ by multiplication generically freely.

Let $Y \to X$ be a G-torsor with X a variety over F. The trivial vector bundle $Y \times V \to Y$ with the diagonal G-action on $Y \times V$ descends to a vector bundle $Y^V \to X$. The open nonempty G-invariant subset $Y \times I \subset Y \times V$ descends to an open subset $Y^I \subset Y^V$. In particular, Y^I is a variety over F birational to $X \times V$, therefore, $Y^I \overset{\text{s.b.}}{\approx} X$. The projection $Y \times I \to I$ yields a morphism $Y^I \to Z$.

Let $E \to \operatorname{Spec} K$, where $K = F(Z)$, be the generic fiber of $I \to Z$. Write $Y^E \to \operatorname{Spec} K$ for the generic fiber of $Y^I \to Z$. As Y^E is a localization of Y^I, Y^E is a variety over K.

If $I_1 \to Z_1$ and $I_2 \to Z_2$ are two standard G-torsors, then

$$Z_1 \overset{\text{s.b.}}{\approx} (I_1)^{I_2} \simeq (I_2)^{I_1} \overset{\text{s.b.}}{\approx} Z_2,$$

hence Z_1 and Z_2 are stably birationally isomorphic.

If Y is a variety, we write $\mathrm{B}G \lhd_p Y$ if $Z \lhd_p Y$ for a standard G-torsor $I \to Z$. By Corollary 6, this makes sense. We say that $\mathrm{B}G$ is stably rational (respectively, p-retract rational) if so is Z.

Example 4. If $\operatorname{char}(F) = p > 0$ and G is a finite p-group, then $\mathrm{B}G$ is stably rational (see [G] and [JLY, §5.6]).

Example 5. Let $H \subset G$ be a subgroup of finite index prime to p and $I \to Z$ a standard G-torsor. Then $I \to T := I/H$ is a standard H-torsor. Since the natural morphism $T \to Z$ is of degree $[G : H]$ prime to p, we have $Z <_p T$ by Example 2. In other words, $\mathrm{B}G <_p \mathrm{B}H$.

By [GMS, Part 1, §5.4], every standard G-torsor $I \to Z$ is versal.

Proposition 8. *Let $Y \to X$ be a G-torsor with X a variety and let $I \to Z$ be a standard G-torsor. Then $Y \to X$ is p-versal if and only if the morphism $Y^I \to Z$ is p-split.*

Proof. $\Rightarrow$: Let $K = F(Z)$ and $E \to \operatorname{Spec} K$ the generic fiber of $I \to Z$. It suffices to show that closed points of degree prime to p are dense in Y^E.

Let $U \subset Y^E$ be a nonempty open subset. We will show that U contains a closed point of degree prime to p.

Since Y^E is a localization of Y^I, there is an open subset $U' \subset Y^I$ such that U is the pull-back of U' under the natural morphism $Y^E \to Y^I$. As the morphism $Y^I \to X$ is flat, it is open and the image W of U' is an open subset of X.

As $Y \to X$ is a p-versal torsor, there exists a field extension L/K of finite degree prime to p and a point $x : \operatorname{Spec} L \to X$ such that $\operatorname{Im}(x) \subset W$ and the torsor $E_L \to \operatorname{Spec} L$ is isomorphic to the pull-back of $Y \to X$ with respect to x. We can find a variety Z' over F, a morphism $s : Z' \to Z$ of varieties over F such that the field extension $F(Z')/F(Z)$ given by s is isomorphic to L/K, a morphism $t : Z' \to X$ such that the composition $\operatorname{Spec} L \overset{\sim}{\to} Z' \overset{t}{\to} X$ coincides with x and

$\mathrm{Im}(t) \subset W$ such that there is a commutative diagram

$$\begin{array}{ccccc} I & \xleftarrow{a} & I' & \xrightarrow{b} & Y \\ \downarrow & & \downarrow & & \downarrow \\ Z & \xleftarrow{s} & Z' & \xrightarrow{t} & X \end{array}$$

with two fiber product squares.

The diagram

$$\begin{array}{ccccc} I \times I & \xleftarrow{a\times 1} & I' \times I & \xrightarrow{b\times 1} & Y \times I \\ \downarrow & & \downarrow & & \downarrow \\ I & \xleftarrow{a} & I' & \xrightarrow{b} & Y \end{array},$$

where the vertical maps are first projections, yields a fiber product diagram

$$\begin{array}{ccccc} I^I & \xleftarrow{g} & (I')^I & \xrightarrow{f} & Y^I \\ \downarrow & & \downarrow & & \downarrow \\ Z & \xleftarrow{s} & Z' & \xrightarrow{t} & X \end{array}.$$

Since $\mathrm{Im}(t) \subset W$, we have $\mathrm{Im}(f) \cap U' \neq \varnothing$. Therefore, the open subset $T' := f^{-1}(U') \subset (I')^I$ is nonempty. As $(I')^E$ is a localization of $(I')^I$, the inverse image T of T' under the natural morphism $(I')^E \to (I')^I$ is a nonempty open subset of $(I')^E$. The commutativity of the diagram

$$\begin{array}{ccc} (I')^E & \xrightarrow{h} & Y^E \\ \downarrow & & \downarrow \\ (I')^I & \xrightarrow{f} & Y^I \end{array}$$

implies that $h(T) \subset U$.

The natural morphism $g' : (I')^E \to I^E$ of varieties over K induced by g is dominant of finite degree prime to p. By Lemma 19 applied to the restriction $k : T \to I^E$ of g', there is a nonempty open subset $U'' \subset I^E$ such that for every point $x \in U''$ there is a point $t \in T$ with the property that $k(t) = x$ and the field extension $K(t)/K(x)$ is finite of degree prime to p.

The variety I is a nonempty G-invariant open subset of a vector space V. Therefore, I^E is open in the twist $(V \times E)/G$ of V by E. By a variant of the classical Hilbert Theorem 90, V^E is a vector space over K, hence $V^E \simeq V_K$. It follows that $I^E \approx I_K$ over K. Therefore, as K is an infinite field, the K-points of I^E are everywhere dense. Choose a K-point $x \in U'' \subset I^E$. There is a closed point $t \in T$ of degree prime to p such that $k(t) = x$. Then $h(t) \in U \subset Y^E$ is a closed point of degree prime to p.

$\Leftarrow$: Consider the following diagram with two fiber product squares

$$\begin{array}{ccccc} I & \xleftarrow{p_2} & Y \times I & \xrightarrow{p_1} & Y \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longleftarrow & Y^I & \longrightarrow & X \end{array}.$$

As $I \to Z$ is versal and $Y^I \to Z$ is p-split, by Proposition 7(2), the torsor $Y \times I \to Y^I$ is p-versal. It follows from Proposition 7(1) that $Y \to X$ is p-versal. $\square$

Remark 2. It was shown in [DR] that if $Y \to X$ is a versal torsor, the rational points are dense in Y. The p-local analog is false if $p > 0$.

Example 6. Let $p = 2$ and $G = \mu_3$ over a field F of characteristic not 3 such that $G(F) = 1$. If K/F is a field extension and $a \in K^\times$, write $K_a := K[x]/(x^3 - a)$ and set $Y_a = \operatorname{Spec} K_a$. Then $Y_a \to \operatorname{Spec} K$ is a G-torsor and every G-torsor over $\operatorname{Spec} K$ is of this form. If $a \in K^{\times 3}$, the torsor Y_a is trivial. Otherwise, K_a is a field, hence Y_a is a variety. Therefore, a nontrivial G-torsor Y_a is split over the cubic field extension K_a/K. It follows that the trivial G-torsor $G \to \operatorname{Spec} F$ is 2-versal. But since $G = \operatorname{Spec} F + \operatorname{Spec} L$, where L/F is a quadratic field extension, the closed points of G of odd degree are not dense in G.

Theorem 9. *Let $Y \to X$ be a p-versal G-torsor. Then* BG *is a stable p-retract of X.*

Proof. As $Y^I \overset{\text{s.b.}}{\approx} X$, we have $Y^I \lhd X$ by Corollary 6. In view of Proposition 8, the morphism $Y^I \to Z$ is p-split. Therefore, Z is a p-retract of Y^I, i.e., $Z <_p Y^I$. Finally, $Z \lhd_p X$ by Corollary 5. $\square$

Theorem 10. *Let G be an algebraic group over F. Then* BG *is p-retract rational if and only if there is a p-versal G-torsor $Y \to X$ with X a rational variety.*

Proof. $\Rightarrow$: Choose a standard G-torsor $I \to Z$ over F. By assumption, Z is a p-retract of a rational variety X, i.e., there is a p-split rational dominant morphism $f : X \dashrightarrow Z$. Shrinking X, we may assume that f is regular. Let $Y \to X$ be the pull-back of $I \to Z$ with respect to f. By Proposition 7(2), the torsor $Y \to X$ is p-versal.

$\Leftarrow$: Let $Y \to X$ be a p-versal G-torsor with X a rational variety. By Theorem 9, B$G \lhd_p X$. As X is rational, BG is p-retract rational. $\square$

Corollary 11. *Let G be an algebraic group over F. Then* BG *is retract rational if and only if all G-torsors over field extensions of F can be rationally parameterized, i.e., there is a versal G-torsor $Y \to X$ with X a rational variety.*

Note that in the case G is a finite group and F is infinite, the corollary was proved in [DM, Lem. 5].

5. An example

The classifying space of the alternating group A_n is stably rational if $n \leq 5$ (see [Ma] and [CS2, §4.7]). The case $n \geq 6$ remains open.

Theorem 12. *The classifying space* $\mathrm{B}A_n$ *of the alternating group* A_n *is* p*-retract rational for every prime integer* p.

Proof. Let p be a prime integer.

Case 1: $p = \mathrm{char}(F)$. Let P be a Sylow p-subgroup of A_n. The space $\mathrm{B}P$ is stably rational by Example 4. As $\mathrm{B}A_n <_p \mathrm{B}P$ in view of Example 5, the classifying space $\mathrm{B}A_n$ is p-retract rational.

Case 2: $p \neq \mathrm{char}(F)$ and p is odd. We prove that $\mathrm{B}A_n$ is p-retract rational by induction on n. Let $m := [n/p]$. Consider the subgroup $H := C^m \rtimes A_m$ of A_n, where $C := \mathbb{Z}/p\mathbb{Z}$. Let $F' := F(\xi_p)$, where ξ_p is a primitive root of unity of degree p. We consider C as the subgroup generated by ξ_p of the quasi-trivial torus $S := R_{F'/F}(\mathbb{G}_m)$ over F and set $T := S/C$.

The group C acts by multiplication by p-th roots of unity on the affine space $\mathbb{A}(F')$ of F' over F. Therefore, H acts faithfully naturally linearly on the affine space $\mathbb{A}(F'^m)$. As S^m is an open H-invariant subset of $\mathbb{A}(F'^m)$, we have

$$\mathrm{B}H \overset{\text{s.b.}}{\approx} S^m/H = T^m/A_m. \tag{1}$$

The torus T is split by the cyclic cyclotomic field extension F'/F. Choose a flasque resolution $1 \to S \to P \to T \to 1$ of T split by F'/F (see [CS1]). As every flasque module over a cyclic group is invertible (see [EM]), there is a torus S' such that the torus $S \times S'$ is quasi-split. By [CS1, §2], there is a torus T' over F split by F'/F such that the torus $T \times T'$ is rational. The group A_m acts by permutations on $T^m \times T'^m$, hence

$$\mathrm{B}A_m \overset{\text{s.b.}}{\approx} (T^m \times T'^m)/A_m. \tag{2}$$

The generic fiber of the projection $f : (T^m \times T'^m)/A_m \to T^m/A_m$ is equal to

$$(T'^m \times \operatorname{Spec} L)/A_m,$$

where $L := F(T'^m)$. This is a torus $\widetilde{T}$ over $K := F(T'^m/A_m) = L^{A_m}$ split by $F' \otimes_F L$. As K is infinite, the K-rational points are dense in the torus $\widetilde{T}$, i.e., f is split and hence

$$T^m/A_m < (T^m \times T'^m)/A_m. \tag{3}$$

It follows from (1), (2) and (3) that $\mathrm{B}H$ is a stable retract of $\mathrm{B}A_m$. By the induction hypothesis, $\mathrm{B}A_m$ is p-retract rational, then so is $\mathrm{B}H$. Since the index $[A_n : H]$ is prime to p, we have $\mathrm{B}A_n <_p \mathrm{B}H$ by Example 5. Therefore, $\mathrm{B}A_n$ is p-retract rational.

Case 3: $\mathrm{char}(F) \neq 2$ and $p = 2$. Let $m := [n/2]$ and let B be the kernel of the map $(\mathbb{Z}/2\mathbb{Z})^m \to \mathbb{Z}/2\mathbb{Z}$ taking (a_i) to $\sum a_i$. The symmetric group S_m acts by permutations on B. The group $D := B \rtimes S_m$ is a subgroup of A_n. The group $(\mathbb{Z}/2\mathbb{Z})^m$ acts on $\mathbb{A}^m_F = \operatorname{Spec} F[t_1, \dots, t_m]$ by $t_i \to \pm t_i$ and S_m acts by permutations of the t_i. Therefore, D acts faithfully and linearly on $\mathbb{A}^m_F$ with

$$\mathbb{A}^m_F/D = \operatorname{Spec} F[s_1, \dots s_{m-1}, t] \simeq \mathbb{A}^m_F,$$

where s_i is the i-th symmetric function on $t_1^2, \dots, t_m^2$ and $t = t_1 \cdots t_m$. Thus, $\mathrm{B}D$ is stably rational. As the index $[A_n : D]$ is odd, $\mathrm{B}A_n <_2 \mathrm{B}D$ by Example 5, and hence $\mathrm{B}A_n$ is 2-retract rational. □

6. Unramified cohomology

For every integer $j \geq 0$ and a prime integer p, let $\mathbb{Q}_p/\mathbb{Z}_p(j)$ denote an object in the derived category of sheaves of abelian groups on the big étale site of $\operatorname{Spec} F$, where

$$\mathbb{Q}_p/\mathbb{Z}_p(j) = \operatorname{colim}\ (\mu_{p^n})^{\otimes j},$$

if $p \neq \operatorname{char}F$, with μ_{p^n} the sheaf of p^n-th roots of unity. If $p = \operatorname{char}F > 0$, the complex $\mathbb{Q}_p/\mathbb{Z}_p(j)$ is defined via logarithmic de Rham–Witt differentials (see [I, I.5.7] or [K]). In particular, $\mathbb{Q}_p/\mathbb{Z}_p(0) = \mathbb{Q}_p/\mathbb{Z}_p$.

If X is a scheme over F, we write $H^n\big(X, \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$ for the degree n étale cohomology group of X with values in $\mathbb{Q}_p/\mathbb{Z}_p(j)$. If $X = \operatorname{Spec} R$ for a commutative ring R, we simply write $H^n\big(R, \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$ for $H^n\big(X, \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$. For example, if $\operatorname{char}(F) = p > 0$ (see [BM]),

$$H^n(F, \mathbb{Q}_p/\mathbb{Z}_p(j)) = \begin{cases} K_j^M(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p, & \text{if } n = j; \\ H^1(F, K_j^M(F_{\text{sep}}) \otimes \mathbb{Q}_p/\mathbb{Z}_p), & \text{if } n = j+1; \\ 0, & \text{otherwise,} \end{cases}$$

where K_j^M are Milnor K-groups.

If L/F is a field extension, there is a natural homomorphism

$$\beta_{L/F} : H^n(F, \mathbb{Q}_p/\mathbb{Z}_p(j)) \to H^n(L, \mathbb{Q}_p/\mathbb{Z}_p(j)).$$

If L/F is finite, the norm map for Milnor K-groups and the corestriction in cohomology yield the norm (corestriction) homomorphism

$$\gamma_{L/F} : H^n(L, \mathbb{Q}_p/\mathbb{Z}_p(j)) \to H^n(F, \mathbb{Q}_p/\mathbb{Z}_p(j)).$$

The composition $\gamma_{L/F} \circ \beta_{L/F}$ is multiplication by $[L : F]$.

We write $\mathcal{H}_X^n\big(\mathbb{Q}_p/\mathbb{Z}_p(j)\big)$ for the Zariski sheaf on X associated with the presheaf

$$U \mapsto H^n\big(U, \mathbb{Q}_p/\mathbb{Z}_p(j)\big).$$

Let $O_{X,x}$ denote the local ring of X at a point $x \in X$.

Proposition 13 (see [CHK, §2.1] and [GS, Thm. 1.4])**.** *Let X be a smooth variety over F. Then the pull-back to the generic point yields an injective homomorphism*

$$H^0_{\text{Zar}}\big(X, \mathcal{H}_X^n(\mathbb{Q}_p/\mathbb{Z}_p(j))\big) \to H^0_{\text{Zar}}\big(\operatorname{Spec} F(X), \mathcal{H}^n_{F(X)}(\mathbb{Q}_p/\mathbb{Z}_p(j))\big) = H^n\big(F(X), \mathbb{Q}_p/\mathbb{Z}_p(j)\big).$$

Its image coincides with the intersection of images of the natural homomorphisms

$$H^n\big(O_{X,x}, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) \to H^n\big(F(X), \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$$

for all points $x \in X$ of codimension 1.

Let K/F be a field extension and v a discrete valuation of K over F with valuation ring O_v. Following [C], [CO], we say that an element $a \in H^n\big(K, \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$ is *unramified with respect to* v if a belongs to the image of the map

$$H^n\big(O_v, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) \to H^n\big(K, \mathbb{Q}_p/\mathbb{Z}_p(j)\big).$$

We write $H^n_{\mathrm{nr}}\big(K, \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$ for the subgroup of all elements in $H^n\big(K, \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$ that are unramified with respect to all discrete valuations of K over F. We have the natural homomorphism

$$H^n\big(F, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) \to H^n_{\mathrm{nr}}\big(K, \mathbb{Q}_p/\mathbb{Z}_p(j)\big). \tag{4}$$

Proposition 14 ([LM, Prop. 3.1])**.** *Let K/F be a purely transcendental field extension. Then the map* (4) *is an isomorphism.*

Let X be a smooth variety over F. If $x \in X$ is a point of codimension 1, the local ring $O_{X,x}$ is a discrete valuation ring. It follows from Proposition 13 that the image of the injective homomorphism $H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n_X(\mathbb{Q}_p/\mathbb{Z}_p(j))\big) \to H^n\big(F(X), \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$ contains the subgroup $H^n_{\mathrm{nr}}\big(F(X), \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$.

Theorem 15. *Let X and X' be smooth varieties over F such that X is a p-retract of X'. Then there is a commutative diagram*

$$\begin{array}{ccc}
H^n\big(F, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) & = & H^n\big(F, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) \\
\downarrow & & \downarrow \\
H^n_{\mathrm{nr}}\big(F(X), \mathbb{Q}_p/\mathbb{Z}_p(j)\big) & \xrightarrow{\alpha} & H^n_{\mathrm{nr}}\big(F(X'), \mathbb{Q}_p/\mathbb{Z}_p(j)\big)
\end{array}$$

with α an injective homomorphism.

Proof. There is a rational dominant morphism $f : X' \dashrightarrow X$ and a morphism $g : Y \to X'$ with $\mathrm{Im}(g)$ in the domain of definition of f such that the composition $f \circ g$ is dominant of finite degree prime to p. Shrinking X' and Y, we may assume that f is regular. We have the following commutative diagram:

$$\begin{array}{ccccc}
H^n\big(F, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) & = & H^n\big(F, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) & & \\
\downarrow & & \downarrow & & \\
H^n_{\mathrm{nr}}\big(F(X), \mathbb{Q}_p/\mathbb{Z}_p(j)\big) & \xrightarrow{\alpha} & H^n_{\mathrm{nr}}\big(F(X'), \mathbb{Q}_p/\mathbb{Z}_p(j)\big) & & \\
\downarrow & & \downarrow & & \\
H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n_X(\mathbb{Q}_p/\mathbb{Z}_p(j))\big) & \to & H^0_{\mathrm{Zar}}\big(X', \mathcal{H}^n_{X'}(\mathbb{Q}_p/\mathbb{Z}_p(j))\big) & \to & H^0_{\mathrm{Zar}}\big(Y, \mathcal{H}^n_Y(\mathbb{Q}_p/\mathbb{Z}_p(j))\big) \\
\downarrow & & & & \downarrow \\
H^n\big(F(X), \mathbb{Q}_p/\mathbb{Z}_p(j)\big) & & \xrightarrow{\qquad\beta\qquad} & & H^n\big(F(Y), \mathbb{Q}_p/\mathbb{Z}_p(j)\big).
\end{array}$$

The maps α and β are the pull-back homomorphisms induced by the field extensions $F(X')/F(X)$ and $F(Y)/F(X)$, respectively. For every $a \in \mathrm{Ker}(\beta)$, we have

$$0 = \gamma(\beta(a)) = [F(Y) : F(X)] \cdot a,$$

where $\gamma : H^n\big(F(Y), \mathbb{Q}_p/\mathbb{Z}_p(j)\big) \to H^n\big(F(X), \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$ is the norm homomorphism. As $[F(Y) : F(X)]$ is prime to p, we have $a = 0$, i.e., β is injective. It follows that α is also injective. □

Corollary 16. *Let X be a p-retract rational smooth variety over F. Then the natural homomorphism*

$$H^n\big(F, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) \to H^n_{\mathrm{nr}}\big(F(X), \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$$

is an isomorphism.

Proof. Let X be a p-retract of a rational variety X'. As $F(X')$ is purely transcendental over F, the map

$$H^n\big(F, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) \longrightarrow H^n_{\mathrm{nr}}\big(F(X'), \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$$

is an isomorphism by Proposition 14. The statement now follows from Theorem 15. □

Remark 3. The same argument shows that Corollary 16 holds for the cohomology groups with coefficients in $\mathbb{Z}/p^m\mathbb{Z}(j)$ for all m in place of $\mathbb{Q}_p/\mathbb{Z}_p(j)$.

Example 7. Let p be a prime integer and F an algebraically closed field of characteristic not p. The classifying spaces $\mathrm{B}G$ for all p-groups of order dividing p^4 and 32 are stably rational by [CK] and [CHKP]. There are finite groups G such that $H^2_{\mathrm{nr}}\big(F(\mathrm{B}G), \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \neq 0$ (see [S]). In [HKK] such groups of order p^5 (if p odd) and 64 (if $p = 2$) are given. By Corollary 16, $\mathrm{B}G$ is not p-retract rational for finite groups G with $H^2_{\mathrm{nr}}\big(F(\mathrm{B}G), \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \neq 0$.

Example 8. Let G be a finite group and F a field of characteristic $p > 0$. Let V be a generically free representation of G and $I \subset V$ a nonempty G-invariant open subset together with a G-torsor $I \to Z$. If H is a Sylow p-subgroup of G, the H-torsor $I \to S := I/H$ is standard and the degree $[G : H]$ of the natural dominant morphism $S \to Z$ is prime to p. By Example 5, $Z <_p S$, hence $\mathrm{B}G <_p \mathrm{B}H$. In view of Example 4, $\mathrm{B}H$ is stably rational, therefore, the classifying space $\mathrm{B}G$ is p-retract rational over F. It follows from Corollary 16 that

$$H^n\big(F, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) \to H^n_{\mathrm{nr}}\big(F(\mathrm{B}G), \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$$

is an isomorphism.

Example 9. It follows from Theorem 12 and Corollary 16 that the natural homomorphism $H^n\big(F, \mathbb{Q}_p/\mathbb{Z}_p(j)\big) \to H^n_{\mathrm{nr}}\big(F(\mathrm{B}A_n), \mathbb{Q}_p/\mathbb{Z}_p(j)\big)$ is an isomorphism for all p. In the case F is algebraically closed this was proved in [BP].

7. Appendix

In the appendix we collect a few technical results used in the paper.

Lemma 17 ([KM, Lem. 3.3]). *Let K'/K be a field extension of finite degree prime to p, and $K \to L$ a field homomorphism. Then there exist a field extension L'/L of finite degree prime to p and a field homomorphism $K' \to L'$ extending $K \to L$.*

Corollary 18. *Let $f : X' \to X$ be a morphism of varieties over F, and let $x' \in X'$ and $x \in X$ be points such that $f(x') = x$ and the field extension $F(x')/F(x)$ is finite of degree prime to p. Let L/F be a field extension and $v : \operatorname{Spec} L \to X$ a morphism over F with image $\{x\}$. Then there exists a field extension L'/L of finite degree prime to p and a commutative diagram of morphisms over F*

$$\begin{array}{ccc} \operatorname{Spec} L' & \longrightarrow & \operatorname{Spec} L \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow{\ f\ } & X \end{array}$$

such that $\operatorname{Im}(v') = \{x'\}$.

Proof. Apply Lemma 17 to the field extension $F(x')/F(x)$ and the field homomorphism $F(x) \to L$. $\square$

Lemma 19 ([Me, Lem. 6.2]). *Let $f : X' \to X$ be a morphism of varieties over F of degree prime to p. Then there is a nonempty open subset $U \subset X$ such that the restriction $f^{-1}(U) \to U$ is finite flat and for every $x \in U$ there exists a point $x' \in X'$ with $f(x') = x$ and the degree $[F(x') : F(x)]$ is prime to p.*

Lemma 20 ([Me, Lem. 6.3]). *Let $g : X \to Y$ and $h : Y' \to Y$ be morphisms of varieties over F. Let $y \in Y$ be the image of the generic point of X. Suppose that there is a point $y' \in Y'$ such that $h(y') = y$ and $[F(y') : F(y)]$ is finite and prime to p. Then there exists a commutative square of morphisms of varieties*

$$\begin{array}{ccc} X' & \xrightarrow{\ m\ } & X \\ \downarrow & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{\ h\ } & Y \end{array}$$

with m dominant of finite degree prime to p.

Lemma 21 ([GLL, Prop. 6.8]). *Let X be a regular algebraic variety over a field F, p a prime integer and S the set of all closed points in X of degree prime to p. Then if S is nonempty, then S is dense in X.*

References

[BK] Ф. А. Богомолов, П. И. Кацыло, *Рациональность некоторых фактор-многообразий*, Матем. сб. **126(168)** (1985), ном. 4, 584–589. Engl. transl.: F. A. Bogomolov, P. I. Katsylo, *Rationality of some quotient varieties*, Mat. USSR-Sb. **54** (1986), no. 2, 571–576.

[BM] S. Blinstein, A. Merkurjev, *Cohomological invariants of algebraic tori*, Algebra Number Theory **7** (2013), no. 7, 1643–1684.

[BP] F. Bogomolov, T. Petrov, *Unramified cohomology of alternating groups*, Cent. Eur. J. Math. **9** (2011), no. 5, 936–948.

[CHKP] H. Chu, S.-J. Hu, M.-C. Kang, Y. G. Prokhorov, *Noether's problem for groups of order* 32, J. Algebra **320** (2008), no. 7, 3022–3035.

[CK] H. Chu, M.-C. Kang, *Rationality of p-group actions*, J. Algebra **237** (2001), no. 2, 673–690.

[C] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, in: *K-Theory and Algebraic Geometry: Connections with Quadratic Forms and Division Algebras* (Santa Barbara, CA, 1992), Amer. Math. Soc., Providence, RI, 1995, pp. 1–64.

[CHK] J.-L. Colliot-Thélène, R. T. Hoobler, B. Kahn, *The Bloch–Ogus-Gabber theorem*, in: *Algebraic K-Theory* (Toronto, ON, 1996), Amer. Math. Soc., Providence, RI, 1997, pp 31–94.

[CO] J.-L. Colliot-Thélène, M. Ojanguren, *Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford*, Invent. Math. **97** (1989), no. 1, 141–158.

[CS1] J.-L. Colliot-Thélène, J.-J. Sansuc, *La R-équivalence sur les tores*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 2, 175–229.

[CS2] J.-L. Colliot-Thélène, J.-J. Sansuc, *The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group)*, in: *Algebraic Groups and Homogeneous Spaces*, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007, pp. 113–186.

[DM] F. R. DeMeyer, *Generic polynomials*, J. Algebra **84** (1983), no. 2, 441–448.

[DR] A. Duncan, Z. Reichstein, *Versality of algebraic group actions and rational points on twisted varieties*, J. Algebraic Geom. **24** (2015), no. 3, 499–530, with an appendix containing a letter from J.-P. Serre.

[EM] S. Endô, T. Miyata, *On a classification of the function fields of algebraic tori*, Nagoya Math. J. **56** (1975), 85–104.

[G] W. Gaschütz, *Fixkörper von p-Automorphismengruppen rein-transzendener Körpererweiterungen von p-Charakteristik*, Math. Z. **71** (1959), 466–468.

[GLL] O. Gabber, Q. Liu, D. Lorenzini, *The index of an algebraic variety*, Invent. Math. **192** (2013), no. 3, 567–626.

[GMS] R. Garibaldi, A. Merkurjev, J.-P. Serre, *Cohomological Invariants in Galois Cohomology*, American Mathematical Society, Providence, RI, 2003.

[GS] M. Gros, N. Suwa, *La conjecture de Gersten pour les faisceaux de Hodge–Witt logarithmique*, Duke Math. J. **57** (1988), no. 2, 615–628.

[HKK] A. Hoshi, M-C. Kang, B. E. Kunyavskii, *Noether's problem and unramified Brauer groups*, Asian J. Math. **17** (2013), no. 4, 689–713.

[I] L. Illusie, *Complexe de de Rham–Witt et cohomologie cristalline*, Ann. Sci. École Norm. Sup. (4) **12** (1979), no. 4, 501–661.

[JLY] C. U. Jensen, A. Ledet, N. Yui, *Generic Polynomials. Constructive Aspects of the Inverse Galois Problem*, Mathematical Sciences Research Institute Publications, Vol. 45, Cambridge University Press, Cambridge, 2002.

[K] B. Kahn, *Applications of weight-two motivic cohomology*, Doc. Math. **1** (1996), no. 17, 395–416.

[KM] N. A. Karpenko, A. S. Merkurjev, *Essential dimension of finite p-groups*, Invent. Math. **172** (2008), no. 3, 491–508.

[LM] D. Laackman, A. Merkurjev, *Degree three cohomological invariants of reductive groups*, Comment. Math. Helv. **91** (2016), no. 3, 493–518.

[Ma] T. Maeda, *Noether's problem for A_5*, J. Algebra **125** (1989), no. 2, 418–430.

[Me] A. S. Merkurjev, *Essential dimension*, in: *Quadratic Forms — Algebra, Arithmetic, and Geometry*, Contemp. Math., Vol. 493, Amer. Math. Soc., Providence, RI, 2009, pp. 299–325.

[R] Z. Reichstein, *A note on retracts and lattices* (*after D. J. Saltman*), preprint, `https://www.math.ubc.ca/~reichst/retract6.pdf` (2006).

[S] D. J. Saltman, *Noether's problem over an algebraically closed field*, Invent. Math. **77** (1984), no. 1, 71–84.

[V] A. Vistoli, *Grothendieck topologies, fibered categories and descent theory*, in: *Fundamental Algebraic Geometry*, Math. Surveys Monogr., Vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.