A LARGE PLANE SET INTERSECTING LINES IN INFINITELY MANY DIRECTIONS IN AT MOST ONE POINT

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Abstract. We prove that for every at most countable family $\{f_k(x)\}$ of real functions on $[0, 1)$ there is a single-valued real function $F(x)$, $x \in [0, 1)$, such that the Hausdorff dimension of the graph Γ of $F(x)$ equals 2, and for every $C \in \mathbb{R}$ and every k, the intersection of Γ with the graph of the function $f_k(x) + C$ consists of at most one point. We also construct a family of functions of cardinality continuum and a function F with similar properties.

1. Introduction

The motivation of this note comes from the following question by Sergei Treil (August 2018, private communication). *Let E be a set in $\mathbb{R}^n$, and let K be an n-dimensional cone in $\mathbb{R}^n$. Suppose that for every line l in K and for every vector* $\mathbf{b}$*, the intersection $E \cap (l + \mathbf{b})$ is at most countable. Does it follow that the Hausdorff dimension of E is less than n?* Recall that for $s \geq 0$, the Hausdorff measure $H^s(E)$ of a set E is defined by

$$H^s(E) = \lim_{\delta \to 0} \inf_{r_i < \delta} \sum r_i^s,$$

where the $\inf_{r_i<\delta}$ is taken over all at most countable covers of E by disks with radii $r_i < \delta$. The Hausdorff dimension $\dim_H(E)$ is given by

$$\dim_H(E) = \sup\{s : H^s(E) = \infty\} = \inf\{s : H^s(E) = 0\}.$$

For $n = 2$, an affirmative answer to Treil's question was given by Marstrand [4] under the additional assumptions that E is measurable with respect to s-dimensional Hausdorff measure H^s, and $0 < H^s(E) < \infty$. Marstrand proved that if $1 < s \leq 2$, then at H^s-almost all points $x \in E$ the following is true: for almost all straight lines l passing through x, $H^{s-1}(E \cap l) < \infty$ and the Hausdorff dimension of $E \cap l$ is equal to $s - 1$. See [5, 8] and references therein for generalizations and related results.

We consider the case $n = 2$ and try to approach the question from the other end: for which sets K of directions (not necessarily n-dimensional) is the answer negative? Marstrand's theorem quoted above implies only that under additional assumptions, K has zero measure. P. Mattila [6, 7, 8] and T. Orponen [10] obtained estimates for exceptional sets of points x and for exceptional sets of directions in terms of Hausdorff dimension in a much more general setting; see also [9]. As a special case, Corollary 5.3 in [10] (also see [8, Theorem 5.2]) yields the following assertion: Let $s > 1$ and let E be a Borel set in $\mathbb{R}^2$ with non-zero finite Hausdorff measure H^s. Then there exists a set of Hausdorff dimension $2 - s$ consisting of line directions in the plane such that in any other direction there exists a line intersecting E in infinitely many points. It is known

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[3, Theorem 2] that if $H^s(E) = \infty$, then E has a closed subset of Hausdorff measure 1. If E has Hausdorff dimension 2, it follows that $H^s(E) = \infty$ for all $s = 2 - \varepsilon$, $\varepsilon > 0$. Therefore the Hausdorff dimension of the set of directions is $2 - s = \varepsilon$, and it follows that the set of exceptional directions has Hausdorff dimension 0.

Our goal is to construct examples which provide us with more detailed information about K and intersections $E \cap (l + \mathbf{b})$. The case when K consists of only one direction is known—see for example [1]. Namely, there exists a function $F(x)$ (which can even be continuous!) whose graph has Hausdorff dimension 2. So, the intersection of the graph of $F(x)$ with every vertical line consists of at most one point.

We show that the answer to Treil's question is negative in the case of any countable set of directions. In fact we prove a much more general assertion.

Theorem 1.1. *For every at most countable family $\mathcal{F}$ of real functions on $[0, 1)$ there is a (single-valued) function $F(x)$, $x \in [0, 1)$, such that*

(i) the Hausdorff dimension of the graph Γ of $F(x)$ equals 2;

(ii) the intersection of Γ with the graph of any function $f_k(x) + C$, where $f_k(x) \in \mathcal{F}$, $C \in \mathbb{R}$, consists of at most one point.

Note that in the special case that the f_k are all linear, this means that for every countable set of directions there is Γ such that every line intersects Γ in at most one point. It turns out that there are even uncountable families of directions with similar properties—see Section 3 for a result in this direction.

T. Keleti [2] constructed a compact subset of $\mathbb{R}$ with Hausdorff dimension 1 that intersects each of its non-identical translates in at most one point. This result and our theorem have a similar flavor, but their proofs are completely different.

Our main idea is to regard the function $F\colon [0, 1) \to [0, 1)$ as sending one infinite sequence of bits (that is digits in the binary representation of a number) to another. Most low order bits of the output sequence $F(x)$ are controlled by much higher order bits of the input sequence (that is, bits appearing later in the sequence x). Consequently, thinking of both input and output as real numbers, we find that F is highly oscillatory on every scale, with the result that it is surprisingly hard to cover its graph Γ with small disks; in order to cover a short interval of x-values, one needs a comparatively large disk to encompass the range of possible y-values. This enables us to prove that the Hausdorff dimension of Γ is 2.

In addition to the usual output bits, which are controlled by higher order input bits, there is a thin but infinite sequence of special bits which are controlled by much lower order input bits. These are assigned to particular slopes m, and they are reserved to encode information which enables one to reconstruct x from the y-intercept of the line through $(x, F(x))$ of slope m. This will show that there is a unique point on the intersection of the graph Γ with any line of that slope.

Conceptually, one bit, B, is reserved for each pair consisting of a specified slope m and a specified bit b of x. The simplest version of this idea would be to set the Bth bit of $F(x)$ so that the Bth bit of $F(x) - mx$ would equal the bth bit of x. Unfortunately, even knowing mx, we cannot determine which value of the Bth bit of $F(x)$ gives a desired value for the Bth bit of $F(x) - mx$ without using information about later bits of $F(x)$, which themselves may depend on still later bits. To avoid this regress, we use a three bit field in $F(x)$ to capture a single bit of x. We remark that the argument sketched above uses no special properties of the class of functions $\{mx\}$, so Theorem 1.1 is expressed in terms of arbitrary sequences of functions. When the functions are continuous, we can prove that F is measurable and Baire one.

Since there are only countably many output bits available, our method is apparently capable of handling only countably many directions. However, if the slopes are chosen to form a suitable Cantor set $\mathcal{K}$, it is possible for each bit field to cover a whole interval in $\mathcal{K}$. This is the additional idea behind the proof of Theorem 3.1 below. In the construction given, $\mathcal{K}$ is of Hausdorff dimension zero; in a certain sense this is a counterpart of the result mentioned above.

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2. Proof of Theorem 1.1

Every $x \in \mathbb{R}$ can be written in the form

$$x = \lfloor x \rfloor + \{x\} = \lfloor x \rfloor + \sum_{i=1}^{\infty} x_i 2^{-i},$$

where $\lfloor x \rfloor$, $\{x\}$ are the integer and the fractional parts of x correspondingly, and each x_i is either 0 or 1. We write $(0100 \cdots 0 \cdots)$ instead of $(0011 \cdots 1 \cdots)$. In other words, the binary expansion of every number $\{x\}$ in $[0,1)$ contains infinitely many zeros. Such a representation is unique.

We partition the set $\mathbb{N}$ of positive integers into a set T and a collection of 3-element sets S_{ij} indexed by ordered pairs (i,j) of positive integers in such a way that the following statements hold:

(1) The density of T in the positive integers is 1.
(2) Each S_{ij} is of the form $\{s_{ij}, s_{ij}+1, s_{ij}+2\}$ for some positive integer s_{ij}.
(3) All sets S_{ij} and T are mutually disjoint.

If $x = \sum_i x_i 2^{-i} \in [0,1)$ and s is a positive integer, we define

$$g_s(x) := x_s 2^{-s} + x_{s+1} 2^{-s-1}. \tag{2.1}$$

We extend $g_s(x)$ to a function on $\mathbb{R}$ by imposing periodicity: $g_s(x+1) = g_s(x)$. In other words, we set $g(x) := g(\{x\})$.

Lemma 2.1. *Let s be a positive integer, U a subset of the positive integers which is disjoint from $\{s, s+1, s+2\}$, and $a \in \mathbb{R}$. Let*

$$A := g_s(a) + \sum_{i \in U} 2^{-i} - a; \quad B := g_s(2^{-s} + a) + \sum_{i \in U} 2^{-i} - a.$$

Then

$$\{2^{s-1}A\} \in [0, 1/8] \cup [3/4, 1); \quad \{2^{s-1}B\} \in [1/4, 5/8].$$

Proof. We partition U into $U^+ := U \cap [1, s-1]$ and $U^- := U \cap [s+3, \infty)$. Thus

$$\sum_{i \in U} 2^{-i} = \sum_{i \in U^+} 2^{-i} + \sum_{i \in U^-} 2^{-i} = \frac{m}{2^{s-1}} + \delta$$

for some $m \in \mathbb{Z}$, $\delta \in [0, 2^{-s-2}]$. As $\{2^{s-1}A\}$ and $\{2^{s-1}B\}$ only depend on $\{a\}$ and U^-, we may assume $a \in [0,1)$. We can therefore write

$$a - g_s(a) = \sum_{i\in A^+} 2^{-i} + \sum_{i\in A^-} 2^{-i},$$

where $A^+ \subset [1, s-1]$ and $A^- \subset [s+2, \infty)$ are sets of integers. Thus,

$$a - g_s(a) = \frac{n}{2^{s-1}} + \varepsilon$$

for some $n \in \mathbb{Z}$, $\varepsilon \in [0, 2^{-s-1}]$. It follows that

$$2^{s-1}A = m - n + 2^{s-1}(\delta - \varepsilon) \in [m - n - 1/4, m - n + 1/8].$$

Likewise,

$$a - g_s(2^{-s} + a) = (2^{-s} + a) - g_s(2^{-s} + a) - 2^{-s} = \frac{n - 1/2}{2^{s-1}} + \varepsilon$$

for some integer n and $\varepsilon \in [0, 2^{-s-1}]$, so

$$2^{s-1}B = m - n + 1/2 + 2^{s-1}(\delta - \varepsilon) \in [m - n + 1/4, m - n + 5/8].$$

Lemma 2.1 is proved. □

For positive integers i and j, we define

$$h_{ij}(x) = g_{s_{ij}}(f_i(x) + x_j 2^{-s_{ij}}), \quad f_i \in \mathcal{F}.$$

Define $F(x)$, $x \in [0, 1)$, by the equality

$$F(x) = F\left(\sum_{i=1}^{\infty} x_i 2^{-i}\right) = \sum_{i\in T} x_{i^2} 2^{-i} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} h_{ij}(x). \tag{2.2}$$

Lemma 2.2. *The function $F(x) - f_i(x)$ is one-to-one on $[0, 1)$ for every $f_i \in \mathcal{F}$.*

Proof. Fix i, j, and $x \in [0, 1)$, and observe that the $h_{ij}(x)$ can have a non-zero bit only in position s_{ij} or $s_{ij} + 1$, while $\sum_{k\in T} x_{k^2} 2^{-k}$ and $\sum_{(k,l)\neq(i,j)} h_{kl}(x)$ are sums of 2^{-k} over some subsets of T and of $\mathbb{N} \setminus T$ which are both disjoint from S_{ij}. Thus,

$$F(x) - f_i(x) = \begin{cases} g_{s_{ij}}(f_i(x)) + \sum_{i\in U} 2^{-i} - f_i(x), & x_j = 0, \\ g_{s_{ij}}(f_i(x) + 2^{-s_{ij}}) + \sum_{i\in U} 2^{-i} - f_i(x), & x_j = 1, \end{cases}$$

where U is a set of positive integers which is disjoint from S_{ij}.

Choose $x \neq y$. There exists j such that $x_j \neq y_j$. According to Lemma 2.1,

$$\{2^{s_{ij}-1}(F(x) - f_i(x))\} \neq \{2^{s_{ij}-1}(F(y) - f_i(y))\}$$

for every positive integer i. Hence, $F(x) - f_i(x) \neq F(y) - f_i(y)$. □

Note that Lemma 2.2 implies (ii) of the main theorem.

The following lemma establishes the validity of (i).

Lemma 2.3. *Let T and S_{ij} be defined as above, and for each pair of positive integers (i, j), let $h_{ij}\colon \mathbb{R} \to \mathbb{R}$ denote any function whose range is contained in*

$$\{k 2^{-s_{ij}-1} \mid k \in [0, 3] \cap \mathbb{Z}\}.$$

Let Γ denote the graph of $F(x)$ defined as in (2.2). Then

$$\dim_H(\Gamma) = 2.$$

The proof of Lemma 2.3 is based on the following assertion.

Lemma 2.4. *Let functions $h_{ij}(x)$ be chosen as above. For every $\alpha < 2$ and $\varepsilon > 0$ there exists $\delta = \delta(\alpha, \varepsilon) > 0$ with the following property. For every disk $D(r)$ with radius $r < \delta$, the length of the projection of $\Gamma \cap D(r)$ onto the x-axis is less than εr^α.*

Let us show that Lemma 2.4 implies Lemma 2.3.

Proof of Lemma 2.3. Suppose that $\dim_H(\Gamma) < 2$. Choose β so that $\dim_H(\Gamma) < \beta < 2$. Let $\delta := \delta(\beta, 1)$ be the number in Lemma 2.4. There is an at most countable family of disks $D_i(r_i)$ such that $r_i < \delta$, $\Gamma \subset \bigcup_i D_i(r_i)$, and $\sum_i r_i^\beta < 1$. We have

$$|\mathrm{Pr}(\Gamma)| \le \sum_i |\mathrm{Pr}(\Gamma \cap D_i(r_i))| \le \sum_i r_i^\beta < 1,$$

where $|\mathrm{Pr}(A)|$ denotes the length of the projection of a set A onto the x-axis. Since Γ projects onto the whole of $[0, 1)$, this contradiction proves that $\dim_H(\Gamma) = 2$. □

Proof of Lemma 2.4. By (2.2), for every $x \in [0, 1)$, the value $y = F(x)$ can be written as $\sum_{i \in U} 2^{-i}$ for some set U of positive integers which does not contain any integer of the form $s_{ij} + 2$ but does contain i whenever $i \in T$ and $x_{i^2} = 1$. Since there are infinitely many integers of the form $s_{ij} + 2$, we have $y_i = 1$ if and only if $i \in U$. Hence, $y_i = x_{i^2}$ for $i \in T$.

We may assume that $\delta < 1/2$. Let N be such that $2^{-N-1} \le r < 2^{-N}$. Then a disk $D(r)$ intersects at most nine dyadic squares with side length 2^{-N}. Hence, it suffices to prove the existence of N_0 such that for every open dyadic square Q with side length less than 2^{-N}, where $N > N_0$,

$$|\mathrm{Pr}(Q \cap \Gamma)| < \varepsilon 2^{-N\alpha}, \quad N > N_0.$$

Fix Q. For all points $(x, y) \in Q$, the first N digits $x_1, \dots, x_N$ in the binary representations of x are determined by Q, and likewise for $y_1, \dots, y_N$. Let $M = M(N)$ be the number of positive integers in the set $[1, N] \cap T$. By (2.2), for all $(x, y) \in \Gamma$ and $n \in T$, we have $x_{n^2} = y_n$. Therefore, for all $(x, y) \in \Gamma \cap Q$, x_m is contant for $m \in [1, N]$ and also for $m \in \{n^2 \mid n \in T \cap [1, N]\}$. The union of these two sets has at least $M + N - \sqrt{N}$ elements. Since $\lim_{N\to\infty} M/N = 1$ and $\alpha < 2$, we have

$$|\mathrm{Pr}(Q \cap \Gamma)| \le 2^{-(N+M-\sqrt{N})} = 2^{-N(1+M/N-\alpha-\sqrt{N}/N)} 2^{-N\alpha} < \varepsilon 2^{-N\alpha},$$

if N is sufficiently large. Lemma 2.4 is proved □

Proposition 2.5. *If the f_i are continuous, the function F can be taken to be Borel and even of Baire class one. In particular, the graph Γ is Borel.*

Proof. Indeed, (2.2) expresses F as a uniformly convergent sum of two types of functions: $x \mapsto x_{i^2} 2^{-i}$ and $h_{ij}(x)$. Each function of the first type can obviously be written as a finite linear combination of characteristic functions of half-open intervals and therefore is a Baire one function. Any pointwise limit of Borel functions is Borel, and any uniform limit of Baire one functions is Baire one. Finally, the graph of any Borel function is a Borel set. Thus, it suffices to prove that each $h_{ij}(x)$ is a Baire one function. It suffices to check this on each part of a finite partition of the domain into half-open intervals, so we may assume that x_j is constant on the domain.

Thus, we need only show that $g(f(x))$ is Baire one when g is a right-continuous piecewise constant function and f is continuous and bounded on an interval $[a,b)$. On the range of f, we can express g as a finite linear combination of characteristic functions of intervals $[c,\infty)$, so we may assume g is of this form. Let $U = f^{-1}((-\infty,c))$. This is a countable union of disjoint open intervals and can therefore be expressed as an increasing union $U = \bigcup_{n=1}^{\infty} I_n$, where each I_n is a finite union of disjoint closed intervals. For each I_n there exists an open set J_n containing I_n and disjoint from the complement of U. By the Tietze extension theorem, there exists a continuous real-valued function ϕ_n on $[a,b)$ which is 0 on I_n and 1 on the complement of J_n (and therefore on the complement of U). Then $g \circ f$ is the pointwise limit of the functions ϕ_n.

□

3. Families $\mathcal{F}$ of cardinality continuum

Theorem 3.1. *For every real function $f(x)$ on [0,1) there exist a set $\mathcal{K}$ of real numbers and a single-valued function $F : [0,1) \to [0,1)$ such that*

(i) $\mathcal{K}$ has cardinality continuum;

(ii) the Hausdorff dimension of the graph Γ of $F(x)$ equals 2;

(iii) the intersection of Γ with the graph of any function $kf(x)+C$, where $k \in \mathcal{K}$, $C \in \mathbb{R}$, consists of at most one point.

Proof. We may assume that $0 \notin \mathcal{K}$. Let

$$\Lambda := \{\lambda : \lambda = 1/k,\ k \in \mathcal{K}\}.$$

Then (iii) in Theorem 3.1 is equivalent to the following statement: the function $\lambda F(x) - f(x)$, $x \in [0,1)$, is one-to-one for every $\lambda \in \Lambda$.

Let

$$s_0 = 5, \quad s_j = (j+1)s_{j-1},\ j \geq 1, \quad V := [0,5] \cup \left(\bigcup_{j=1}^{\infty} [s_j - s_{j-1}, s_j + s_{j-1}] \right), \quad T := \mathbb{N} \backslash V.$$

Note that the density of T in the positive integers is 1, and $i \geq 16$ for $i \in T$.

Define $F(x),\ x \in [0,1)$, by the equality

$$F(x) = F\left(\sum_{i=1}^{\infty} x_i 2^{-i} \right) = \sum_{i \in T} x_{i^2} 2^{-i} + \sum_{j=1}^{\infty} h_j(x), \tag{3.1}$$

where $h_j(x) = g_{s_j}(f(x) + x_j 2^{-s_j})$, and g_s is defined by (2.1). Essentially the same arguments as in the proof of Lemmas 2.3 and 2.4 yield (ii). Let

$$\Lambda := \{\lambda \in \mathbb{R} : \lambda = 1 + \sum_{i=1}^{\infty} \lambda_i 2^{-s_i}\},$$

where each λ_i is either 0 or 1. Obviously, Λ (and hence $\mathcal{K}$) has cardinality continuum.

To establish (iii), it suffices to prove the following claim: for every $x \in [0,1)$, $j \in \mathbb{N}$, and $\lambda \in \Lambda$, there exists a set $U = U(x,j,\lambda)$ of nonnegative integers which is disjoint from $\{s_j, s_j+1, s_j+2\}$ and such that

$$\lambda F(x) = h_j(x) + \sum_{i \in U} 2^{-i}. \tag{3.2}$$

Indeed, the claim implies that $\lambda F(x) - f(x)$ is one-to-one for every $\lambda \in \Lambda$ exactly by the same arguments as in the proof of Lemma 2.2.

We fix $x \in [0,1)$, $j \in \mathbb{N}$, and $\lambda \in \Lambda$, and split T into $T^- = T \cap [0, s_j - s_{j-1} - 1]$ and $T^+ = T \cap [s_j + s_{j-1} + 1, \infty)$ (for $j = 1$, the set T^- is empty). Thus,

$$\begin{aligned}\lambda F(x) &= F(x) + \left(\sum_{k=1}^{\infty} \lambda_k 2^{-s_k}\right) \cdot \left(\sum_{i \in T} x_{i^2} 2^{-i} + \sum_{i=1}^{\infty} h_i(x)\right) \\ &= F(x) + \sum_{k=1}^{j-1} \sum_{i \in T^-} \lambda_k x_{i^2} 2^{-(s_k+i)} + \sum_{k=1}^{j-1} \sum_{i \in T^+} \lambda_k x_{i^2} 2^{-(s_k+i)} \\ &+ \sum_{k=j}^{\infty} \sum_{i \in T} \lambda_k x_{i^2} 2^{-(s_k+i)} + \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \lambda_k 2^{-s_k} h_i(x),\end{aligned}$$

which we write $F(x) + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$. Note that for $j = 1$, $\Sigma_1 = \Sigma_2 = 0$.

By (3.1), we may write $F(x)$ in the form

$$F(x) = \sum_{i \in I_F^-} 2^{-i} + h_j(x) + \sum_{i \in I_F^+} 2^{-i},$$

where $I_F^- \subset [0, s_j - s_{j-1} - 1]$, $I_F^+ \subset [s_j + s_{j-1} + 1, \infty)$; for $j = 1$, the set I_F^- is empty.

For the sum Σ_1 we have $s_k + i \leq s_{j-1} + s_j - s_{j-1} - 1 = s_j - 1$. Hence,

$$\Sigma_1 = \sum_{i \in I_1} 2^{-i}, \quad I_1 \subset [0, s_{j-1}].$$

As $T^+ \subset [s_j + s_{j-1} - 1, \infty)$,

$$\Sigma_2 < \sum_{k=1}^{j-1} 2^{-(s_k + s_j + s_{j-1})} < 2^{-(s_j + s_{j-1})}.$$

Since $i \geq 16$ for $i \in T$,

$$\Sigma_3 < \sum_{k=j}^{\infty} 2^{-(s_k + 15)} < 2^{-(s_j + 14)}.$$

Finally, Σ_4 may be decomposed into two sums. The first sum

$$\Sigma_{4,1} = \sum_{k=1}^{j-1} \sum_{i=1}^{j-1} \lambda_k 2^{-s_k} h_i(x) = \sum_{i \in I_4^-} 2^{-i}, \quad I_4^- \subset [0, s_j),$$

since $s_k + s_i + 1 < s_j$. In the second sum at least one index k or i is greater than or equal to j. If $i \geq j$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \sum_{i=j}^{\infty} \lambda_k 2^{-s_k} h_i(x) &\leq \sum_{k=1}^{\infty} \sum_{i=j}^{\infty} 2^{-s_k} (2^{-s_i} + 2^{-s_i - 1}) \\ &< \sum_{k=1}^{\infty} 2^{-s_k} \cdot 2^{-s_j + 2} \leq 2^{-s_j - 7},\end{aligned}$$

since $s_k \leq 10$ as $k \geq 1$. The case $k \geq j$ is analogous to the previous one.

Combining these estimates, we obtain (3.2), which implies the theorem. □

References

[1] F. Bayart and Y. Heurteaux, *On the Hausdorff Dimension of Graphs of Prevalent Continuous Functions on Compact Sets*, J. Barral and S. Seuret (eds.), Further Developments in Fractals and Related Fields, Trends in Mathematics, DOI 10.1007/978-0-8176-8400-6-2, Springer Science+Business Media New York 2013, 25–34.

[2] T. Keleti, A 1-dimensional subset of the reals that intersects each of its translates in at most a single point, *Real Anal. Exchange* **24** (1998/99), no. 2, 843–844.

[3] D. G. Larman, On Hausdorff measure in finite-dimensional compact metric spaces. *Proc. London Math. Soc. (3)* **17** (1967), 193–206.

[4] J. M. Marstrand, Some fundamental geometrical properties of plane sets of fractional dimensions, *Proc. London Math. Soc. (3)* **4** (1954), 257–302.

[5] P. Mattila, Hausdorff dimension, orthogonal projections and intersections with planes, *Ann. Acad. Sci. Fenn. A Math.* **1** (1975), 227–244.

[6] P. Mattila, *Fourier Analysis and Hausdorff Dimension*, Cambridge University Press, Cambridge, 2015.

[7] P. Mattila, Exceptional set estimates for the Hausdorff dimension of intersections, *Ann. Acad. Sci. Fenn. A Math.* **42** (2017), 611–620.

[8] P. Mattila, Hausdorff dimension, projections, intersections, and Besicovitch sets, arXiv: 1712.09199.

[9] P. Mattila and T. Orponen, Hausdorff dimension, intersections of projections and exceptional plane sections, *Proc. Amer. Math. Soc.* **144** (2016), 3419–3430.

[10] T. Orponen, Slicing sets and measures, and the dimension of exceptional parameters, *J. Geom. Anal.* **24** (2014), 47–80.

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