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Discrete Convex Analysis and Its Applications in Operations: A Survey

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Discrete convexity, in particular, $L^\natural$-convexity and $M^\natural$-convexity, provides a critical opening to attack several classical problems in inventory theory, as well as many other operations problems that arise from more recent practices, for instance, appointment scheduling and bike sharing. As a powerful framework, discrete convex analysis is becoming increasingly popular in the literature. This review will survey the landscape of the approach. We start by introducing several key concepts, namely, $L^\natural$-convexity and $M^\natural$-convexity and their variants, followed by a discussion of some fundamental properties that are most useful for studying operations models. We then illustrate various applications of these concepts and properties. Examples include network flow problem, stochastic inventory control, appointment scheduling, game theory, portfolio contract, discrete choice model, and bike sharing. We focus our discussion on demonstrating how discrete convex analysis can shed new insights on existing problems, and/or bring about much more simpler analyses and algorithm developments than previous methods in the literature. We also present several results and analyses that are new to the literature.

Key words: discrete convexity; inventory management; appointment scheduling; bike sharing
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1. Introduction

Convexity plays a fundamental role in continuous optimization. It is often regarded as the synonym for computational tractability, and widely used to characterize structures of optimal policies in dynamic operations problems. However, discreteness or combinatorial structure often appears in many practical problems which the traditional convex analysis is not applicable or inadequate. For example, products may be produced only in batches, and the number of customers or patients waiting in a queue must be integral. For many operations models with multi-dimensional decisions and/or parameters, meaningful results can be derived only under certain combinatorial structures characterized by properties like submodularity/supermodularity in addition to convexity. In view of these, discrete convexity, which extends important concepts and properties in convex analysis to discrete spaces with salient combinatorial structures, is becoming a prominent analytical framework in the operations literature. It provides powerful tools for characterizing the structures of optimal policies, establishing the existence of equilibrium, and/or facilitating the design of efficient algorithms in many operations models.

The study of discrete convexity in operations dates back to Miller (1971), who introduces the concept of "discretely convex" to analyze an aircraft maintenance problem. Over the years, various concepts about different types of discrete convexity have been introduced (see Murota 2003). Our review is not intended to provide a comprehensive coverage of the aforementioned developments of discrete convex analysis. Instead, we give a concise introduction to major concepts and key technical results in a self-contained manner. Moreover, our discussion will be oriented toward operations models. Specifically, we draw applications from diverse areas ranging from network flow optimization, inventory management, revenue management, healthcare appointment scheduling, portfolio contract selection, choice modeling, to bike sharing, and game theory models. We show that with the introduction of concepts and properties of discrete convex analysis, the analysis of many operations models, especially several classical ones, can be significantly simplified or unified and their results can often be established under much more general conditions. In these cases, concise concepts and general properties capture essential mathematical structures of a variety of operations models that differ in contexts and details, which in turn, facilitates technical analysis, and crystalize its presentation here.

A closely related survey paper is Chen (2017a), which however only covers $L^\natural$-convexity and

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applications associate with it. In comparison, this review covers a broader scope that involves both $L^\natural$-convexity and $M^\natural$-convexity and their variants, and many more applications in diverse areas. We have also included several new results and analyses that are new to the literature. For example, for a continuous review assemble-to-order inventory model, we use the convex conjugate relationship between $L^\natural$-convexity and $M^\natural$-convexity to provide a simple proof for a key result in Doğru et al. (2017). Our analysis for some production control problems with two products (facilities) shows that $M^\natural$-convexity provides the needed concept to unify different approaches in several different models. We provide a sufficient and necessary condition for the substitutability of a discrete choice model. For a dock reallocation problem in bike sharing, we present an alternative and intuitive proof of the multimodularity of the objective function by the use of preservation properties of elementary operations of $L^\natural$-convexity.

The organization of this study is as follows. In section 2, we first provide a brief review of several key concepts in discrete convex analysis, $L^\natural$/$M^\natural$-convexity and their variants, followed by some examples. We then present in section 3 some fundamental properties that are important to operations models including monotonicity of optimal solutions and preservation properties of $L^\natural$/$M^\natural$-convexity under various operations. In section 4, we discuss many different applications of discrete convex analysis in studies of the aforementioned operations models. Our concluding remarks on future research is given in section 5.

1.1. Notations and Terminologies

For convenience, we list the notations and terminologies used throughout this study. The real space and the integer space are denoted by $\mathbb{R}$ and $\mathbb{Z}$, respectively. We use $\mathcal{F}$ to represent $\mathbb{R}$ or $\mathbb{Z}$, $\overline{\mathcal{F}}$ to denote $\mathcal{F} \cup \{-\infty, +\infty\}$ and $\mathcal{F}_+$ to denote the set of all nonnegative numbers in $\mathcal{F}$. For any subset $S \subseteq \mathbb{R}^n$, we denote δ_S its indicator function, that is, $\delta_S(x) = 0$ if $x \in S$, and $\delta_S(x) = +\infty$ otherwise. This definition of indicator function is commonly used in the convex analysis literature (see Rockafellar 1970). We caution the readers that this definition is different from another commonly used one under the same name which specifies $\delta_S(x) = 1$ if $x \in S$ and $\delta_S(x) = 0$ otherwise. For a positive integer n, we denote $[n]$ the set $\{1,2,\ldots,n\}$. The power set and cardinality of a set X are denoted by 2^X and $|X|$, respectively. Denote $\mathbf{e}_i$ the vector with 1 in its i-th coordinate and 0 otherwise, $\mathbf{e}_0$ the zero vector, $\mathbf{e}$ the vector with 1 in each component, and $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$ for any subset S of the index set. The dimensions of $\mathbf{e}_i, \mathbf{e}_0, \mathbf{e}, \mathbf{e}_S$ should be clear from the context. The component minimum and maximum of two vectors $x, y \in \mathbb{R}^n$ are denoted by $x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$ and $x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$, respectively. The positive and negative index set of a vector x are denoted by $\text{supp}^+(x) = \{i | x_i > 0\}$ and $\text{supp}^-(x) = \{i | x_i < 0\}$, respectively. The max norm and L_1 norm of a vector $x \in \mathbb{R}^n$ are $\|x\|_\infty = \max_{1 \le i \le n}\{|x_i|\}$ and $\|x\|_1 = \sum_{i=1}^n |x_i|$, respectively. The feasible domain of a function $f : \mathcal{F}^n \to \overline{\mathbb{R}}$ is denoted by $\text{dom}(f) = \{x \in \mathcal{F}^n | -\infty < f(x) < \infty\}$. For any $p \in \mathbb{R}^n$, the function $f[p] : \mathcal{F}^n \to \overline{\mathbb{R}}$ is defined as $f[p](x) = f(x) + p^T x$. A *linearity domain* of f is defined by $\arg\min_{x \in \mathcal{F}^n} f[p](x)$ for some $p \in \mathbb{R}^n$. The name is from the fact that f is a linear function on any linearity domain. The gradient and Hessian of a function f are denoted by ∇f and $\nabla^2 f$, respectively. For a random variable ξ, we denote $\text{supp}(\xi)$ its support. A set X and a binary operation $\preceq$ on X is called a partially ordered set if $(X, \preceq)$ satisfies (i) reflexivity: $a \preceq a$ for any $a \in X$ (ii) antisymmetry: if $a \preceq b$ and $b \preceq a$, then $a = b$ (iii) transitivity: if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

2. Key Concepts and Examples

In this section, we review definitions and examples of $L/L^\natural$-convexity, $M/M^\natural$-convexity and some of their variants. All the materials for which no reference is provided can be found in Murota (2003) and Simchi-Levi et al. (2014). Although we focus on discrete convexity concepts, we will freely refer to their concave counterparts, whose definitions are straightforward, when needed.

2.1. Definitions

We first recall that a function $f : \mathbb{R}^n \to (-\infty, \infty]$ is *convex* if it satisfies the following inequality

$$(1-\lambda)f(x) + \lambda f(y) \ge f((1-\lambda)x + \lambda y), \ \forall x, y \in \mathbb{R}^n, \ \lambda \in [0,1]. \tag{1}$$

Convex functions play an important role in continuous optimization. The class of convex programming problems, which minimize convex functions subject to convex feasible set constraints, is usually regarded as tractable since local optimality implies global optimality among some other properties.

In an attempt to extend the tractability of convex programming to integer space, one may define "discrete convex" functions and impose at least the following properties:

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(a) Convex extensible to real spaces, that is, there exists a convex function $\tilde{f}$ such that $\tilde{f}(x) = f(x)$ for all $x \in \mathbb{Z}^n$,
(b) Local optimality (in some sense) implies global optimality.

One possible approach to define such a discrete convex function $f : \mathbb{Z}^n \to (-\infty, \infty]$ is to consider its local convex extension

$$\tilde{f}(x) = \sup_{p \in \mathbb{R}^n, \alpha \in \mathbb{R}} \{p^T x + \alpha | p^T y + \alpha \le f(y), \forall y \text{ with } \|y - x\|_\infty < 1\}$$

defined on $\mathbb{R}^n$ (see Murota 2003, p. 93). If $\tilde{f}$ is convex, then f is called *integrally convex*. In one-dimensional spaces, $\tilde{f}$ is constructed by the linear interpolation between $(x, f(x))$ and $(x + 1, f(x + 1))$ for any $x \in \mathbb{Z}$. In this case, a discrete function f is integrally convex if and only if $f(x - 1) + f(x + 1) \ge 2f(x)$ for any $x \in \mathbb{Z}$ and dom(f) is a set of consecutive integers. Such one-dimensional functions are also called *univariate discrete convex functions*[1].

Integrally convex functions clearly satisfy property (a), but a convex-extensible function is not necessarily integrally convex (see Example 3.20 in Murota 2003). It is shown that an integrally convex function f satisfies property (b) in the following sense: x is a global minimizer if and only if

$$f(x) \le f(x + \mathbf{e}_S - \mathbf{e}_T) \text{ for all } S, T \subseteq [n].$$

Although integrally convex functions satisfy property (b), the computational complexity of checking the local optimality is exponential in n in general (see Note 3.23 in Murota 2003). In view of this, it is desirable to restrict to tractable subclasses of integrally convex functions. Among various function classes proposed in the literature, L$^\natural$-convex functions and M$^\natural$-convex functions are arguably the most important ones.

L$^\natural$-convexity can be motivated from the concept of midpoint convexity, which requires a function f to satisfy inequality (1) for $\lambda = \frac{1}{2}$, that is,

$$f(x) + f(y) \ge 2f\left(\frac{x+y}{2}\right), \ \forall x, y \in \mathbb{R}^n. \tag{2}$$

It is well known that for lower semi-continuous functions, convexity is equivalent to midpoint convexity (see, e.g., Donoghue 1969, p. 12). To accommodate the integrality requirement, one can impose certain integral conditions on the points involved in the midpoint convexity definition. Specifically, we can replace the midpoint in Equation (2) by rounding it down and up to the nearest integral points $\lfloor \frac{x+y}{2} \rfloor$ and $\lceil \frac{x+y}{2} \rceil$ (see Figure 1) and impose the *discrete midpoint convexity*, that is,

$$f(x) + f(y) \ge f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) + f\left(\left\lceil \frac{x+y}{2} \right\rceil\right), \ \forall x, y \in \mathbb{Z}^n.$$

A function defined on an integer space is called L$^\natural$-convexity if it satisfies the discrete midpoint convexity. Here L$^\natural$ stands for "lattice" in view of the lattice structure of the integral points.

It is not hard to show that a function $f : \mathbb{Z}^n \to (-\infty, \infty]$ is L$^\natural$-convex if and only if $f(x - \xi \mathbf{e})$ is submodular in $(x, \xi) \in \mathbb{Z}^n \times \mathbb{Z}$. Here, a function $g : \mathcal{F}^n \to (-\infty, \infty]$ is *submodular* if

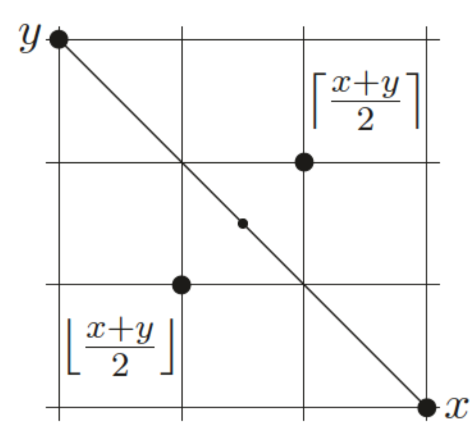
$$g(x) + g(y) \ge g(x \wedge y) + g(x \vee y), \ \forall x, y \in \mathcal{F}^n.$$

This equivalent definition makes it convenient to verify discrete L$^\natural$-convexity via submodularity. Interestingly, its extension to the continuous space is also straightforward. In the following, we present a unified definition of L$^\natural$-convexity for both integer and continuous spaces, which we feel is more convenient for operations applications.

Definition 2.1. A lower semi-continuous[2] function $f : \mathcal{F}^n \to (-\infty, \infty]$ is called *L$^\natural$-convex* on $\mathcal{F}^n$ if $f(x - \xi \mathbf{e})$ is submodular in $(x, \xi) \in \mathcal{F}^n \times \mathcal{F}$.

A closely related sibling of L$^\natural$-convexity is L-convexity. A function $f : \mathcal{F}^n \to (-\infty, \infty]$ is called *L-convex* on $\mathcal{F}^n$ if $f : \mathcal{F}^n \to (-\infty, \infty]$ is L$^\natural$-convex on $\mathcal{F}^n$ and f is linear in the direction of $\mathbf{e}$, that is, $f(x + \alpha \mathbf{e}) = f(x) + \alpha r$ for some $r \in \mathbb{R}$ and any $\alpha \in \mathcal{F}$. Clearly, any L$^\natural$ (or L)-convex function is submodular. In addition, a lower semi-continuous L$^\natural$ (or L)-convex function defined in a continuous space is both convex and submodular.

Figure 1 Discrete Midpoint Convexity



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L$^\natural$-convexity is also closely related to the concept of *multimodularity* introduced by Hajek (1985) to study an optimal admission control problem in a single queue model under no queue information and independently by Weber and Stidham (1987) under a different name to study a dynamic control of queues problem with full state information. A lower semi-continuous function $f : \mathcal{F}^n \to (-\infty, \infty]$ is called *multimodular* if

$$\tilde{f}(x_0, x) = f(x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1})$$

is submodular in $(x_0, x) \in \mathcal{F}^{n+1}$. By definition of L$^\natural$-convexity, f is multimodular if and only if $\tilde{f}(0, x)$ is L$^\natural$-convex (see Murota 2005). We refer to Glasserman and Yao (1994) and Altman et al. (2003) for more applications of multimodularity to static and dynamic control of queues problems and Moriguchi and Murota (2018) for fundamental operations related to multimodularity.

M$^\natural$-convexity is motivated by another equivalent definition of convexity in a continuous space, equal-distance convexity, which imposes the following inequality:

$$f(x) + f(y) \geq f(x - \alpha(x - y)) + f(y + \alpha(x - y)),$$
$$\forall x, y \in \mathbb{R}^n, \alpha \in [0, 1].$$

For lower semi-continuous functions, convexity is equivalent to the equal-distance convexity as it implies midpoint convexity. In the definition, the two points $x(\alpha) := x - \alpha(x - y)$ and $y(\alpha) := y + \alpha(x - y)$ maintain equal distance to x and y, respectively, or equivalently they are symmetric with respect to the midpoint $\frac{x+y}{2}$. In addition, one can easily show that the sum of the function values over the two points $x(\alpha)$ and $y(\alpha)$ is non-increasing when they move toward each other through the line segment connecting them (see Figure 2a).

It is tempting to mimic this intuition when we switch to integer spaces. Ideally, we would like to construct the trajectories of two integral points $x(\alpha)$ and $y(\alpha)$, starting from x and y, respectively, and symmetrical with respect to the midpoint $\frac{x+y}{2}$, such that the sum of their function values is non-increasing as the two points move closer. To keep integral, we restrict the movement of the two points only along the axes (see Figure 2b). However, unlike the continuous space case in which $x(\alpha)$ and $y(\alpha)$ move along a one-dimensional line segment, there are different directions the two points can move along (this becomes more prominent when the space dimension is larger than two), and M$^\natural$-convexity only imposes the existence of such a direction. Similar to L$^\natural$-convexity, the same definition of M$^\natural$-convexity for discrete functions applies to continuous spaces with minor modification.

DEFINITION 2.2. A lower semi-continuous[3] function $f : \mathcal{F}^n \to (-\infty, \infty]$ is called *M$^\natural$-convex* on $\mathcal{F}^n$ if it satisfies the following exchange condition: for any x, $y \in \mathrm{dom}(f)$ and $i \in \mathrm{supp}^+(x - y)$, there exist $j \in \mathrm{supp}^-(x - y) \cup \{0\}$ and a positive number $\alpha_0 \in \mathcal{F}_+$ such that

$$f(x) + f(y) \geq f(x - \alpha(\mathbf{e}_i - \mathbf{e}_j)) + f(y + \alpha(\mathbf{e}_i - \mathbf{e}_j))$$

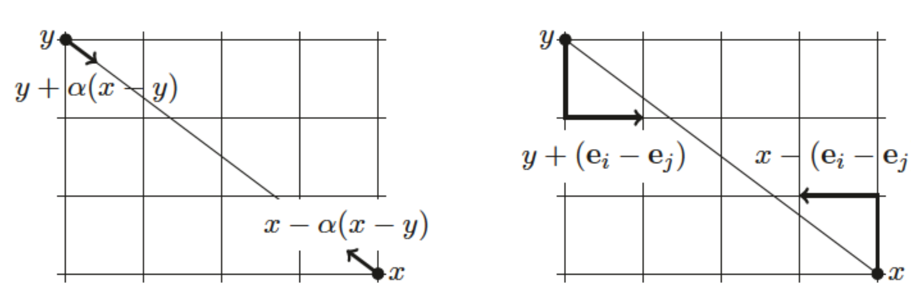
for every $\alpha \in [0, \alpha_0] \cap \mathcal{F}$.

Here "M" stands for matroid in view of the connection of the exchange condition to the definition of matroid.

We can also define the sibling of M$^\natural$-convexity, M-convexity. A function $f : \mathcal{F}^n \to (-\infty, \infty]$ is called *M-convex* on $\mathcal{F}^n$ if f is M$^\natural$-convex on $\mathcal{F}^n$ and its domain is contained in a hyperplane $\{x \in \mathcal{F}^n | \sum_{i=1}^n x_i = r\}$ for some $r \in \mathcal{F}$.

Murota (2003) provides a host of weak variants of L$^\natural$-convexity and M$^\natural$-convexity, most of which have not been used in the operations literature. Here we only introduce *semistrictly quasi* M$^\natural$- (SSQM$^\natural$-) convexity, which is useful in the analysis of monotone comparative statics in parametric optimization problems

Figure 2 Convexity and M$^\natural$-Convexity



(a) Equal-distance convexity **(b)** Exchange condition

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(see section 3). The concept is motivated by quasi-convexity in continuous spaces, a relaxed version of convexity, which requires

$$f((1-\lambda)x+\lambda y)\leq \max\{f(x),f(y)\},\ \text{for any } x,y\in\mathbb{R}^n,$$

or equivalently

$$f((1-\lambda)x+\lambda y)\leq f(x) \text{ or } f((1-\lambda)x+\lambda y)\leq f(y).$$

Specifically, a lower semi-continuous function $f:\mathcal{F}^n\to(-\infty,\infty]$ is SSQM$^\natural$-convex if for any x, $y\in \text{dom}(f)$ and $i\in\text{supp}^+(x-y)$, there exist $j\in\text{supp}^-(x-y)\cup\{0\}$ and a positive number $\alpha_0\in\mathcal{F}_+$ such that for any $\alpha\in[0,\alpha_0]\cap\mathcal{F}$,

$$\begin{aligned} &(a) && f(x-\alpha(\mathbf{e}_i-\mathbf{e}_j))<f(x)\\ \text{or}\quad &(b) && f(y+\alpha(\mathbf{e}_i-\mathbf{e}_j))<f(y)\\ \text{or}\quad &(c) && f(x-\alpha(\mathbf{e}_i-\mathbf{e}_j))-f(x)\\ & &&=f(y+\alpha(\mathbf{e}_i-\mathbf{e}_j))-f(y)=0. \end{aligned}$$

SSQM$^\natural$-convexity is defined by relaxing the exchange condition of M$^\natural$-convexity. Therefore, M$^\natural$-convex functions are SSQM$^\natural$-convex. The name "semistrictly" comes from the strict inequality requirement in (a) and (b).

Our discrete convexity concepts are defined for extended functions, that is, functions can take the value +∞. This naturally imposes conditions on their domains. A closed set $S\subseteq\mathcal{F}^n$ is called L$^\natural$-convex, M$^\natural$-convex, SSQM$^\natural$-convex or a sublattice (of $\mathcal{F}^n$) if its indicator function δ_S is L$^\natural$-convex, M$^\natural$-convex, SSQM$^\natural$-convex or submodular, respectively (recall $\delta_S(x)=0$ if $x\in S$ and $\delta_S(x)=+\infty$ otherwise). Note that any L$^\natural$-convex set S is a sublattice since δ_S is L$^\natural$-convex, and thus submodular. See Propositions 3.1 and 3.9 for full characterizations of L$^\natural$/M$^\natural$-convex sets.

2.2. Examples

In the following, we present several examples of L$^\natural$/M$^\natural$-convex sets and functions.

Examples of L$^\natural$-convex sets and functions

- Let $U,V\subseteq[n]$, $A\subseteq[n]\times[n]$, $l_i,u_j,a_{kl}\in\mathbb{R}$ for $i\in U$, $j\in V$, $(k,l)\in A$. The polyhedron $\{x\in\mathbb{R}^n\ |x_i\geq l_i,\ x_j\leq u_j,\ x_k-x_l\leq a_{kl}$, for all $i\in U$, $j\in V$, $(k,l)\in A\}$ is an L$^\natural$-convex set. Its restriction on $\mathbb{Z}^n$ is a discrete L$^\natural$-convex set.
- Let $f:\mathcal{F}\to(-\infty,\infty]$ be a univariate convex function. Then f is L$^\natural$-convex and $h(x,y)=f(x-y)$ is L$^\natural$-convex in (x,y).
- For any nondecreasing univariate convex function h and $b_0,\ldots,b_n\in\mathbb{R}$, $f(x)=h(\max\ b_0, x_1+b_1,\ldots,x_n+b_n\})$ is L$^\natural$-convex on $\mathcal{F}^n$. If $f(x,t)$ is L$^\natural$-convex on $\mathcal{F}^m\times\mathcal{F}$ and nondecreasing in t, then $g(x,y)=f(x,\max\{y_1,\ldots,y_n\})$ is L$^\natural$-convex on $\mathcal{F}^m\times\mathcal{F}^n$.
- A separable convex function on $\mathcal{F}^n$ is L$^\natural$-convex. Here a function $f:\mathcal{F}^n\to(-\infty,\infty]$ is separable convex on $\mathcal{F}^n$ if $f(x)=\sum_{i=1}^n f_i(x_i)$, where $f_i:\mathcal{F}\to(-\infty,\infty]$ $(i=1,..,m)$ are univariate convex functions.
- A quadratic function $f(x)=x^TAx$ with the matrix A being symmetric is L$^\natural$-convex on $\mathcal{F}^n$ if and only if A is a diagonally dominant M-matrix, that is,

$$a_{ij}\leq 0\ \forall\ i\neq j,\ \text{and}\ \sum_{j=1}^n a_{ij}\geq 0\ \forall\ i,$$

  where a_{ij} refers to the ij-th component of matrix A.
- Any submodular function on $\{0,1\}^n$ is L$^\natural$-convex.

Examples of M$^\natural$-convex sets and functions

- Let f be a submodular set function with a ground set V and $n=|V|$. Its base polyhedron $B(f)=\{x\in\mathbb{R}^n|\sum_{i\in U}x_i\leq f(U)$ for all $U\subset V$ and $\sum_{i\in V}x_i=f(V)\}$ is an M$^\natural$-convex set, and so is the projection $P=\{y\in\mathbb{R}^{n-1}|(x_1,y)\in B(f)$ for some $x_1\in\mathbb{R}\}$. Both $B(f)\cap\mathbb{Z}^n$ and $P\cap\mathbb{Z}^n$ are discrete M$^\natural$-convex sets.
- A separable convex function $f:\mathcal{F}\to(-\infty,\infty]$ is M$^\natural$-convex.
- *Laminar convex functions.* A nonempty set $\mathcal{L}\subseteq 2^{[n]}$ is called a laminar family if for any $A,B\in\mathcal{L}$, $A\cap B=\emptyset$ or $A\subseteq B$ or $B\subseteq A$. A function $f:\mathcal{F}^n\to(-\infty,\infty]$ is called a laminar convex function if it can be represented as

$$f(x)=\sum_{S\in\mathcal{L}} f_S\left(\sum_{i\in S}x_i\right),$$

  where f_S $(S\in\mathcal{L})$ are univariate convex functions and $\mathcal{L}$ is a laminar family.
- A quadratic function $f(x)=x^TAx$ with a symmetric matrix A is M$^\natural$-convex on $\mathbb{Z}^n$ if and only if the following conditions hold:

$$A_{ij}\geq 0\ \forall\ i,j\in[n],\ \text{and}\ A_{ij}\geq\min\{A_{ik},A_{jk}\}\ \forall\ k\neq i,j,$$

  which is also equivalent to the laminar convexity of f.[4]
- A quadratic function $f(x)=x^TAx$ with a symmetric matrix A is M$^\natural$-convex on $\mathbb{R}^n$ if and only if for any $\lambda>0$, $A+\lambda I$ is nonsingular and $(A+\lambda I)^{-1}$ is a diagonally dominant M-matrix.

- (Moriguchi and Murota 2018) A function $f : \mathbb{Z}^2 \to (-\infty, \infty]$ is multimodular if and only if it is M$^\natural$-convex.[5]
- Let (V,I) be a matroid with the ground set V and the independent sets $I \subseteq \{0,1\}^n$, where $n = |V|$. Then $f(x) = \max\{a^T y | y \in I,\ y \leq x\}$ is M$^\natural$-concave on $\{0,1\}^n$ for any $a \in \mathbb{R}^n_+$. In particular, the rank function is M$^\natural$-concave.

3. Characterizations and Properties

Similar to convexity, L$^\natural$-convexity and M$^\natural$-convexity admit many salient characterizations and properties that render broad applications. We focus on L$^\natural$-convexity first and then move on to M$^\natural$-convexity.

We first provide some characterizations of L$^\natural$-convexity below. For L$^\natural$-convex sets, the following proposition shows that a closed set is L$^\natural$-convex if and only if it is the intersection of $\mathcal{F}^n$ with a polyhedron whose facets are perpendicular to $\mathbf{e}_i - \mathbf{e}_j, i,j \in \{0,1,\ldots,n\}, i \neq j$.

Proposition 3.1. (Murota 2003, section 5). *A closed subset of $\mathcal{F}^n$ is L$^\natural$-convex if and only if it can be expressed as $\{x \in \mathcal{F}^n | l \leq x \leq u,\ x_i - x_j \leq a_{ij},\ i \neq j\}$ with $l, u \in \overline{\mathcal{F}}^n$ and $a_{ij} \in \overline{\mathcal{F}}$.*[6]

Any discrete L$^\natural$-convex function f is convex extensible as it is integrally convex. Furthermore, one of its convex extensions can be constructed efficiently by first extending on each unit hypercube using the so-called Lovász extension (e.g., see Murota 2003, p. 16) and then splicing them together. Hence, a discrete L$^\natural$-convex functions can be regarded as the restriction of certain (not any arbitrary) convex function on an integer space. In contrast, the following result shows that a discrete L$^\natural$-convex function can also be regarded as a function obtained by consistently splicing linear functions defined on L$^\natural$-convex sets. It says that for a discrete L$^\natural$-convex function f, if we move any hyperplane to touch its epigraph $\{(x,t) | t \geq f(x), x \in \mathrm{dom}(f)\}$, the touch points form an L$^\natural$-convex set and vice versa.

Proposition 3.2. (Murota 2003, Theorem 7.17). *Assume that $f : \mathbb{Z}^n \to (-\infty, \infty]$ has a nonempty bounded dom(f). f is L$^\natural$-convex if and only if its linearity domain arg min f[p] is an L$^\natural$-convex set for all $p \in \mathbb{R}^n$.*

For twice continuously differentiable functions, a characterization in terms of the Hessian matrices is provided below.

Proposition 3.3. (Murota and Shioura 2004). *A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is L$^\natural$-convex if and only if its Hessian matrix $\nabla^2 f(x)$ is a diagonally dominant M-matrix for all $x \in \mathbb{R}^n$.*

As an integrally convex function, local optimality of discrete L$^\natural$-convex functions implies global optimality in the following sense.

Proposition 3.4. (Murota 2003, Theorem 7.14). *Let $f : \mathbb{Z}^n \to (-\infty, \infty]$ be a discrete L$^\natural$-convex function. If $f(x) \leq \min\{f(x + \mathbf{e}_S), f(x - \mathbf{e}_S)\}$ for all $S \subseteq [n]$, then x is a global optimum.*

It seems that one needs to check 2^{n+1} points to verify the local optimality. Interestingly, as f is submodular on both $\{x + \mathbf{e}_S | S \subseteq [n]\}$ and $\{x - \mathbf{e}_S | S \subseteq [n]\}$, the local optimality of a given point x can be verified by conducting polynomial number of function evaluations (see Orlin 2009). Based on this, Proposition 3.4 suggests an efficient steepest descent algorithm of finding a global minimizer of an L$^\natural$-convex function, which can be further accelerated with a scaling technique. The computational complexity of this steepest descent scaling algorithm is $O(\sigma(n) v_f n^2 \log_2(\frac{K}{2n}))$, where $\sigma(n) = O(n^5)$ is the number of function evaluations of minimizing a submodular function over a set with n elements, v_f is an upper bound of the time of evaluating function f, and $K = \max\{\|x - y\|_\infty | x, y \in \mathrm{dom}(f),\ x_i = y_i \text{ for some } i \in [n]\}$ is (roughly) the diameter of the domain of f (see Murota 2003, p. 305–308).

The intimate connection with submodularity allows one to develop powerful monotone comparative statics results using L$^\natural$-convexity. For a given function $f(\cdot,\cdot), \mathcal{F}^m \times \mathcal{F}^n \to (-\infty, \infty]$, consider the parametric optimization problem

$$g(x) = \inf_{y:(x,y) \in S} f(x,y), \tag{3}$$

where S refers to the graph of the feasible set. Let $\mathcal{Y}_x = \arg\min_{y:(x,y)\in S} f(x,y)$ be the optimal solution set given the parameter x.

Before we proceed to present the technical results, we review the concept of induced set ordering in lattice programming. Let $\mathbb{Y} \subseteq \mathcal{F}^n$ be a set. The *induced set ordering* $\sqsubseteq$ over $2^{\mathbb{Y}} \setminus \{\emptyset\}$ is defined as: for $\mathcal{Y}, \mathcal{Y}' \in 2^{\mathbb{Y}} \setminus \{\emptyset\}$, $\mathcal{Y} \sqsubseteq \mathcal{Y}'$ if for any $y \in \mathcal{Y}$ and $y' \in \mathcal{Y}'$, $y \wedge y' \in \mathcal{Y}$ and $y \vee y' \in \mathcal{Y}'$ (see Topkis 1998, p. 32). Let X be a partially ordered set with partial order $\preceq$ and $\mathcal{Y}_x$ be a nonempty set parameterized by $x \in X$. $\mathcal{Y}_x$ is said to be nondecreasing (nonincreasing) in $x \in X$ with respect to the induced set ordering if for any $x \preceq x'$, $\mathcal{Y}_x \sqsubseteq \mathcal{Y}_{x'}$ ($\mathcal{Y}_{x'} \sqsubseteq \mathcal{Y}_x$). Throughout the paper, the partial order on a subset of $\mathcal{F}^n$ refers to the component-wise order $\leq$.

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The following proposition is a classical result in lattice programming, which is widely used for monotone comparative statics analyses in the operations research and economics literature. Part (a) says the optimal solution set is monotone and part (b) implies submodularity is preserved under the parametric optimization operation.

PROPOSITION 3.5. (TOPKIS 1998, SECTION 2.7). *Consider the parametric optimization problem (3). If $f(\cdot,\cdot), \mathcal{F}^m \times \mathcal{F}^n \to (-\infty, \infty]$ is submodular on $\mathcal{F}^m \times \mathcal{F}^n$ and the graph of the feasible set S is a sublattice (i.e., $a,b \in S$ implies $a \wedge b, a \vee b \in S$), then the following two statements hold.*

(a) *The optimal solution set $\mathcal{Y}_x$ is nondecreasing in $x \in \{z : \mathcal{Y}_z \neq \emptyset\}$ with respect to the induced set ordering.*
(b) *The optimal value function $g(x)$ is submodular on $\mathcal{F}^m$ provided $g(x) > -\infty$ for any $x \in \mathcal{F}^m$.*

The following monotonicity result of L$^\natural$-convexity can be directly obtained from Proposition 3.5(a).

PROPOSITION 3.6. (CHEN ET AL. 2018). *Consider the parametric optimization problem (3). If $f(\cdot,\cdot), \mathcal{F}^m \times \mathcal{F}^n \to (-\infty, \infty]$ is an L$^\natural$-convex function in $\mathcal{F}^m \times \mathcal{F}^n$ and $S \subseteq \mathcal{F}^m \times \mathcal{F}^n$ is an L$^\natural$-convex set in $\mathcal{F}^m \times \mathcal{F}^n$, then $\mathcal{Y}_x$ is nondecreasing in $x \in \{z : \mathcal{Y}_z \neq \emptyset\}$ with respect to the induced set ordering, and in addition $\mathcal{Y}_{x+\alpha e} \sqsubseteq \mathcal{Y}_x + \alpha e$ for any $\alpha \in \mathcal{F}_+$.*

Proposition 3.6 implies that the optimal solution set is nondecreasing in the parameter and has bounded sensitivity along the direction of the all-ones vector e with respect to the induced set ordering. If $\mathcal{Y}_x$ is compact for each x, one can show that the largest element in $\mathcal{Y}_x$ for each x (or the smallest elements in $\mathcal{Y}_x$ for each x), guaranteed to exist under the assumption in Proposition 3.6 and the nonemptyness and compactness of $\mathcal{Y}_x$, is nondecreasing in x and has bounded sensitivity.

Similar to convexity, L$^\natural$-convexity is preserved under basic operations of scaling, translation and addition. That is, if $f_1(x), f_2(x)$ are L$^\natural$-convex, then $\lambda f_1(x)$, $f_1(a + bx)$ and $f_1(x) + f_2(x)$ are L$^\natural$-convex in x for all $\lambda \in \mathbb{R}_+$, $a \in \mathcal{F}^n$ and $b \in \mathcal{F}$. L$^\natural$-convexity is also preserved under point-wise limit as submodularity is preserved under point-wise limit. Moreover, given a function $f : \mathcal{F}^n \times \mathcal{F}^m \to (-\infty, \infty]$ which is L$^\natural$-convex in the first component, $\mathbb{E}_\xi[f(\cdot, \xi)]$ is L$^\natural$-convex for any random vector ξ on $\mathcal{F}^m$ provided the expectation is well defined. Finally, the following result, a direct corollary of Proposition 3.5(b), illustrates that L$^\natural$-convexity is preserved under parametric optimization operations, which has been proven quite powerful to deal with dynamic operations models.

PROPOSITION 3.7. (CHEN ET AL. 2018). *Consider the parametric optimization problem (3). If $f(\cdot,\cdot), \mathcal{F}^m \times \mathcal{F}^n \to (-\infty, \infty]$ is an L$^\natural$-convex function on $\mathcal{F}^m \times \mathcal{F}^n$ and $S \subseteq \mathcal{F}^m \times \mathcal{F}^n$ is an L$^\natural$-convex set in $\mathcal{F}^m \times \mathcal{F}^n$, then the optimal objective value function $g(x)$ is L$^\natural$-convex on $\mathcal{F}^m$ provided $g(x) > -\infty$ for any $x \in \mathcal{F}^m$.*

This preservation property, together with the monotonicity property Proposition 3.6, is useful to analyze the monotone comparative statics of optimal decisions in the literature of inventory control and revenue management (to be discussed in section 4.2).

In operations models, we often end up with parametric optimization problems in which decisions are truncated by random variables of the following form:

$$g(x,z) = \inf_{u:(x,z,u)\in\mathcal{A}} \mathbb{E}[f(x, u \wedge (z + \xi))], \qquad (4)$$

where x and z are some state variables, u is the decision vector, and $\mathbb{E}$ is the expectation taking over the random vector ξ . For instance, in inventory models, one may face uncertain supply/production capacity due to random factors such as imperfect quality and unreliable production process. In this case, z and u may represent the initial inventory level and the target order-up-to level, respectively, ξ is the uncertain capacity level, $u \wedge (z + \xi)$ is the realized order-up-to level, and f is the cost incurred (x is dummy here). In capacity control of revenue management, u may represent the booking limits of different demand classes, $z + \xi$ is the realized demand with z being mean demand forecast, x is the available capacity, and f is the cost (or equivalently negative reward) of serving accepted demand $u \wedge (z + \xi)$. A technical challenge of problem (4) is that it is not a convex programming problem in general even if f is a convex function and $\mathcal{A}$ is a convex set. Interestingly, under certain conditions, Properties 3.6 and 3.7 can be extended to address such models (see Chen et al. 2018 and Chen and Gao 2019). For simplicity, we impose the following structure on $\mathcal{A}$:

$$\mathcal{A} = \{(x,z,u) \in \mathcal{F}^m \times \mathcal{F}^n \times \mathcal{F}^n | Au \leq b(x,z), u \geq \underline{u}(x,z)\}$$

where A is a matrix with an appropriate dimension, and b and $\underline{u}$ are vectors which may depend on x and z. Define

$$\mathcal{A}^{\Xi} = \{(x,z,w) | w = u \wedge (z + \xi),\ (x,z,u) \in \mathcal{A},\ \xi \in \mathrm{supp}(\xi)\}$$

and denote $\mathcal{U}^*(x,z)$ the optimal solution set of problem (4).

PROPOSITION 3.8. (CHEN ET AL. 2018). *Consider problem (4). Assume that* $f : \mathcal{F}^m \times \mathcal{F}^n \to (-\infty, \infty]$ *is an* L$^\natural$-*convex function in* $\mathcal{F}^m \times \mathcal{F}^n$, ξ *is a random vector in* $\mathcal{F}^n$ *with independent components and* $\mathcal{A} \subseteq \mathcal{F}^m \times \mathcal{F}^n \times \mathcal{F}^n$ *is a nonempty set. If the matrix A is nonnegative and* $z + \xi \geq \underline{u}(x,z)$ *with probability one, then the following statements hold.*

(a) *If* $\mathcal{A}^{\Xi}$ *is* L$^\natural$-*convex, then g is* L$^\natural$-*convex.*
(b) *If in addition* $\mathcal{A}$ *is* L$^\natural$-*convex, then* $\mathcal{U}^*(x,z)$ *is nondecreasing in (x,z) and* $\mathcal{U}^*((x,z) + \alpha\mathbf{e}) \sqsubseteq \mathcal{U}^*(x,z) + \alpha\mathbf{e}$ *for any* $\alpha \in \mathcal{F}_+$.

We now focus on M$^\natural$-convexity.

As integrally convex functions, discrete M$^\natural$-convex functions are convex extensible. In addition, the following characterizations of M$^\natural$-convexity are parallel to Propositions 3.1–3.4 for L$^\natural$-convexity.

PROPOSITION 3.9. (MUROTA 2003, SECTION 4). *A nonempty polyhedron P in* $\mathbb{R}^n$ *is* M$^\natural$-*convex if and only if the tangent cone* $T_P(x)$ *of P at x (i.e., the cone generated by vectors* $y-x$ *for all* $y \in P$*) is generated by vectors chosen from* $\{\mathbf{e}_i - \mathbf{e}_j | i,j = 0,1,\ldots,n\}$ *for all* $x \in P$.

PROPOSITION 3.10. (MUROTA 2003, THEOREM 6.30). *Assume that* $f : \mathbb{Z}^n \to (-\infty, \infty]$ *has a nonempty bounded dom(f). Then f is* M$^\natural$-*convex if and only if its linearity domain* $\arg\min f[p]$ *is an* M$^\natural$-*convex set for all* $p \in \mathbb{R}$.

PROPOSITION 3.11. (CHEN AND LI 2019). *Let* $a, b \in \overline{\mathbb{R}}^n$. *A twice continuously differentiable convex function* $f : (a,b) \to \mathbb{R}$ *is* M$^\natural$-*convex on (a,b) if and only if* $\nabla^2 f(x) + \lambda I$ *is nonsingular and* $(\nabla^2 f(x) + \lambda I)^{-1}$ *is a diagonally dominant M-matrix for all* $x \in (a,b)$ *and all positive number* λ.

PROPOSITION 3.12. (MUROTA 2003, THEOREM 6.26). $f : \mathbb{Z}^n \to (-\infty, \infty]$ *is* M$^\natural$-*convex, then* $x \in dom(f)$ *is a global minimizer of f if and only if x is a local minimizer, that is,* $f(x) \leq f(x - \mathbf{e}_i + \mathbf{e}_j)$ *for any i,* $j \in \{0,1,\ldots,n\}$.

Proposition 3.12 suggests a steepest descent algorithm of finding a global minimum of a discrete M$^\natural$-convex function *f*, and this algorithm has a scaling version as well (see Murota 2003, p. 281- 283). The computational complexity of the steepest descent scaling algorithm is $O(v_f n^2 K_1)$, where v_f is an upper bound of the time of evaluating function *f* and $K_1 = \max\{\|x - y\|_1 | x, y \in \mathrm{dom}(f)\}$, the diameter of the domain of *f* in ℓ_1-norm.

For two-dimensional functions, M$^\natural$-convexity, and thus equivalently multimodularity as we pointed out earlier, are characterized by the property of supermodularity and diagonal dominance which finds applications in several inventory and production models (see section 4.3).

PROPOSITION 3.13. *A function* $f : \mathcal{F}^2 \to (-\infty, \infty]$ *is* M$^\natural$-*convex if and only if f is supermodular, and satisfies the diagonal dominance property: for any* $x \in \mathcal{F}^2$, $\delta, \delta_1 \in \mathcal{F}_+$,

$$f(x_1, x_2) + f(x_1 + \delta + \delta_1, x_2 - \delta) \geq f(x_1 + \delta, x_2 - \delta) + f(x_1 + \delta_1, x_2).^7$$

For discrete functions, the diagonal dominance property means $\Delta_{ii} f(x) \geq \Delta_{ij} f(x), i \neq j, i,j \in \{1,2\}$, where $\Delta_{ij} f(x) = f(x + \mathbf{e}_i + \mathbf{e}_j) - f(x + \mathbf{e}_i) - f(x + \mathbf{e}_j) + f(x)$ is the twice difference of *f*. For twice continuously differentiable functions, the diagonal dominance property means $\frac{\partial^2 f}{\partial x_i \partial x_i}(x) \geq \frac{\partial^2 f}{\partial x_i \partial x_j}(x), i \neq j, i,j \in \{1,2\}$.

One may wonder whether M$^\natural$-convexity would allow us to derive some monotone comparative statics in the parametric optimization problem (3). The issue is more challenging than L$^\natural$-convexity because the traditional lattice programming typically considers parametric minimization (or maximization) problems with a submodular (or supermodular) objective function over a constraint set with a lattice structure while an M$^\natural$-convex function is supermodular as stated in the following proposition.

PROPOSITION 3.14. (MUROTA 2003, THEOREM 6.19; MUROTA AND SHIOURA 2004). M$^\natural$-*convex functions are supermodular.*

In general, there is not much one can say about the monotonicity of the optimal solution of problem (3) if the objective function is merely supermodular. However, since M$^\natural$-convexity is much stronger than supermodularity, we can indeed derive some monotone comparative statics using the concept of weak induced set ordering. Let $\mathbb{Y} \subseteq \mathcal{F}^n$ be a set. The *weak induced set ordering* $\sqsubseteq_w$ over $2^{\mathbb{Y}} \setminus \{\emptyset\}$ is defined as: for $\mathcal{Y}, \mathcal{Y}' \in 2^{\mathbb{Y}} \setminus \{\emptyset\}$, $\mathcal{Y} \sqsubseteq_w \mathcal{Y}'$ if for any $y \in \mathcal{Y}$, there exists $y' \in \mathcal{Y}'$ with $y \leq y'$, and for any $y' \in \mathcal{Y}'$, there exists $y \in \mathcal{Y}$ with $y \leq y'$ (see Topkis 1998, p. 38). Let *X* be a partially ordered set with partial order $\preceq$ and $\mathcal{Y}_x$ be a nonempty set parameterized by $x \in X$. $\mathcal{Y}_x$ is said to be nondecreasing (nonincreasing) in $x \in X$ with respect to the weak induced set ordering if for any $x \preceq x'$, $\mathcal{Y}_x \sqsubseteq_w \mathcal{Y}_{x'}$ ($\mathcal{Y}_{x'} \sqsubseteq_w \mathcal{Y}_x$).

PROPOSITION 3.15. (CHEN AND LI 2019). *Consider problem (3). If $f : \mathcal{F}^m \times \mathcal{F}^n \rightarrow (-\infty, \infty]$ is an* M$^\natural$*-convex function and $S = \mathcal{F}^m \times \mathcal{B}$ with $\mathcal{B}$ being a box in $\mathcal{F}^n$, that is, $\mathcal{B} = [l, u] \cap \mathcal{F}^n$ and $l, u \in \overline{\mathcal{F}}^n$, then the optimal solution set $\mathcal{Y}_x$ is nonincreasing in x and $\mathbf{e}^T\mathcal{Y}_x + \mathbf{e}^T x$ is nondecreasing in x with respect to the weak induced set ordering for $x \in \{z : \mathcal{Y}_z \neq \emptyset\}$.*

The first part of Proposition 3.15 provides a condition under which the optimal solution set is *nonincreasing* in the parameters. This is totally different from Propositions 3.5 and 3.6, where the optimal solution is nondecreasing (instead of nonincreasing) in the parameters. Interestingly, this result can be extended to SSQM$^\natural$-convexity: if f is SSQM$^\natural$-convex, then $\mathcal{Y}_x$ is nonincreasing in x with respect to the weak induced set ordering. The second part of Proposition 3.15 says $\mathbf{e}^T\mathcal{Y}_x + \mathbf{e}^T x$ is nondecreasing in x. In inventory applications, if y represents the ordering quantity and x the initial inventory, $\mathbf{e}^T\mathcal{Y}_x + \mathbf{e}^T x$ being nonincreasing means that the total optimal order-up-to level is nonincreasing in the initial inventory. See section 4.3 for an application of Proposition 3.15 in a multi-product inventory model. We would like to point out that Proposition 3.15 cannot be derived from classical results in lattice programming by making simple variable transformation from (x,y) to $(x,-y)$ or $(-x,y)$ as neither $f(x,-y)$ nor $f(-x,y)$ is submodular in general.

M$^\natural$-convexity is preserved under several commonly used operations. If $f(x)$ is M$^\natural$-convex, then $\lambda f(x)$, $f(a \pm x)$ and $f[p](x)$ are M$^\natural$-convex for all $\lambda \in \mathbb{R}_+$, $a \in \mathcal{F}^n$ and $p \in \mathbb{R}^n$. If $f(x)$ is M$^\natural$-convex (SSQM$^\natural$-convex, M-convex), then $f(x+y)$ is M$^\natural$-convex (SSQM$^\natural$-convex, M-convex) in (x,y). M$^\natural$-convexity is preserved under point-wise limit. The following result illustrates that M$^\natural$-convexity is preserved under infimal convolution.

PROPOSITION 3.16. (MUROTA 2003, THEOREMS 6.15 AND 6.50). *The infimal convolution of two* M$^\natural$*-convex functions $f_1, f_2 : \mathcal{F}^n \rightarrow (-\infty, \infty]$, defined as $(f_1 \Box f_2)(x) = \min_{u+v=x}(f_1(u) + f_2(v))$, remains* M$^\natural$*-convex in $\mathcal{F}^n$ if $(f_1 \Box f_2)(x) > -\infty$ for any $x \in \mathcal{F}^n$.*

Finally, M$^\natural$-convexity is preserved under parametric optimization operations in the setting of Proposition 3.15.

PROPOSITION 3.17. (MUROTA 2003, THEOREMS 6.15 AND 6.50). *Consider problem (3). If $f : \mathcal{F}^m \times \mathcal{F}^n \rightarrow (-\infty, \infty]$ is an* M$^\natural$*-convex function and $S = \mathcal{F}^m \times \mathcal{B}$ with $\mathcal{B}$ being a box in $\mathcal{F}^n$, that is, $\mathcal{B} = [l, u] \cap \mathcal{F}^n$ and $l, u \in \overline{\mathcal{F}}^n$, then $g(x)$ is* M$^\natural$*-convex on $\mathcal{F}^m$ provided $g(x) > -\infty$ for any $x \in \mathcal{F}^m$.*

Compared with L$^\natural$-convexity, M$^\natural$-convexity is much more challenging to deal with because M$^\natural$-convexity is not closed under some commonly used operations. For example, $f(bx)$ may not preserve M$^\natural$-convexity for $b \in \mathbb{R}$; the restriction of an M$^\natural$-convex function on an M$^\natural$-convex set is not necessarily an M$^\natural$-convex function; the summation of two M$^\natural$-convex functions may not be an M$^\natural$-convex function.

The result below links L/L$^\natural$-convexity with M/M$^\natural$-convexity through the Legendre transformation, which we will exploit to analyze an assemble-to-order inventory model in section 4.3. The Legendre transformation (or conjugate function) of a function $f : \mathcal{F}^n \rightarrow (-\infty, \infty]$ is a function $f^* : \mathcal{F}^n \rightarrow (-\infty, \infty]$ defined by

$$f^*(p) = \sup\{p^T x - f(x) | x \in \mathcal{F}^n\},\ p \in \mathcal{F}^n.$$

PROPOSITION 3.18. (MUROTA 1998, MUROTA AND SHIOURA 2004). *A function f is* M$^\natural$*-convex (*L$^\natural$*-convex) if and only if f^* is* L$^\natural$*-convex (*M$^\natural$*-convex). The statement remains true when we replace* L$^\natural$*-convex by L-convex and* M$^\natural$*-convex by M-convex, respectively.*

In the two-dimensional space, L$^\natural$-convexity and M$^\natural$-convexity are essentially equivalent subject to a simple variable transformation.

PROPOSITION 3.19. (FUJISHIGE 2005, LEMMA 17.4). *A two-dimensional function[8] $f : \mathcal{F}^2 \rightarrow (-\infty, \infty]$ is* M$^\natural$*-convex if and only if $f(-x_1, x_2)$ is* L$^\natural$*-convex in (x_1, x_2).*

As we mentioned earlier, M$^\natural$-convexity is not preserved under addition operation in general. Proposition 3.19 together with the preservation of L$^\natural$-convexity under the addition operation implies that the summation of two two-dimensional M$^\natural$-convex functions are M$^\natural$-convex.

PROPOSITION 3.20. *If two functions $f, h : \mathcal{F}^2 \rightarrow (-\infty, \infty]$ are* M$^\natural$*-convex, then $f + h$ is* M$^\natural$*-convex.*

This property allows us to show preservation of M$^\natural$-convexity in models involving expectation operations. See section 4.3 for a case study of a two-location joint inventory and transshipment model.

Proposition 3.19 together with the preservation of M$^\natural$-convexity under infimal convolution implies that L$^\natural$-convexity is preserved under infimal convolution if we restrict to a two-dimensional space.

PROPOSITION 3.21. (CHEN ET AL. 2013). *If $f_i : \mathcal{F}^2 \rightarrow (-\infty, \infty]$ $(i = 1, \ldots, n)$ are* L$^\natural$*-convex, then the function*

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$$f(x) = \min\left\{\sum_{i=1}^{n} f_i(y^i) | \sum_{i=1}^{n} y^i = x, y^i \in \mathcal{F}^2, i \in [n]\right\}$$

is $L^\natural$*-convex.*

4. Applications

In this section, we first review network flow problems in which $L^\natural$-convexity and $M^\natural$-convexity naturally arise. We then review some applications of $L^\natural$-convexity and finally those for $M^\natural$-convexity.

4.1. Network Flow Problem

$M^\natural$-convexity and $L^\natural$-convexity naturally arise in network flow problems. Let (V,A) be a directed graph with vertex set V and arc set A. For each arc a, denote $\xi(a)$ the flow on it and f_a the cost function of its flow which is assumed to be univariate convex. Denote $\xi \in \mathcal{F}^{|A|}$ the vector of flow on each arc. The boundary $\partial\xi \in \mathcal{F}^{|V|}$ of a flow vector ξ at a node $v \in V$ is defined as the total flows leaving v minus the total flows entering v. Let $T \subseteq V$ be the set of supply and demand nodes. The minimum cost of the network flows with a given supply and demand vector $x \in \mathcal{F}^{|T|}$[9] can then be written as

$$f(x) = \inf_{\xi}\left\{\sum_{a\in A} f_a(\xi_a) | (\partial\xi)_v = -x_v \text{ if } v \in T \text{ and } 0 \text{ otherwise}\right\}. \quad (5)$$

It is shown that function $f : \mathcal{F}^{|T|} \to (-\infty, \infty]$ is M-convex (see Murota 2003).[10] This result plays a critical role in Ma et al. (2018) to prove that a spatio-temporal pricing mechanism is subgame-perfect incentive compatible in a ride-sharing platform with strategic drivers.

$L^\natural$-convexity arises if we consider a network (V,A) of electric potentials. Let p_v be the electric potential on node $v \in V$ and $p \in \mathcal{F}^{|V|}$ be the electric potential vector. The coboundary of an electric potential vector p (viewed as voltages on the arcs) is a vector $\Delta p \in \mathcal{F}^{|A|}$ defined by $(\Delta p)_a = p_v - p_{v'}$ for $a = (v', v)$. Let g_a be the cost function associated with arc a, which is assumed to be univariate convex. Given the potential $q \in \mathcal{F}^{|T|}$ on a node subset $T \subseteq V$, the minimum cost of the network voltages can be written as

$$g(q) = \inf_{p}\left\{\sum_{a\in A} g_a((\Delta p)_a) | p_v = q_v,\ v \in T\right\}.$$

It is shown that function $g : \mathcal{F}^{|T|} \to (-\infty, \infty]$ is L-convex. Moreover, under the same network, if f_a and g_a are conjugate to each other for all $a \in A$, then f and g are conjugate to each other (see Murota 2003, Murota and Shioura 2004).

$M^\natural$-convexity and $L^\natural$-convexity also arise in a network with capacity constraints. In the above network (V,A), assume $f_a(\xi_a) = -w_a\xi_a$ with weight $w_a \in \mathcal{F}$ for every $a \in A$ and $T = \emptyset$. Let $l, u \in \mathcal{F}^{|A|}$ be vectors of the lower bound and the upper bound of the capacity on each arc, respectively. Then problem (5) becomes a max-weight circulation problem

$$F(w,l,u) = \sup_{\xi\in\mathcal{F}^{|A|}}\left\{\sum_{a\in A} \xi_a w_a | \partial\xi = 0,\ l \le \xi \le u\right\}.$$

The function F can be $M^\natural$-convex (concave) or $L^\natural$-convex (concave) in some of its variables depending on whether their associated arcs are parallel or series. Two arcs are said to be parallel (series) if they are in the opposite (same) direction for every simple cycle containing them. A set of arcs is said to be parallel (series) if it consists of pairwise parallel (series) arcs. For a series set S and a parallel set P, it is shown in Murota and Shioura (2005) that $F(w,l,u)$ is $M^\natural$-convex in $(w_a)_{a\in S}$ and separately $M^\natural$-concave in $(l_a)_{a\in P}$ and $(u_a)_{a\in P}$ ($F(w,l,u)$ is $L^\natural$-convex in $(w_a)_{a\in P}$ and separately $L^\natural$-concave in $(l_a)_{a\in S}$ and $(u_a)_{a\in S}$). This result implies that $F(w,l,u)$ is separately supermodular in $(w_a)_{a\in S}, (l_a)_{a\in S}, (u_a)_{a\in S}$ (separately submodular in $(w_a)_{a\in P}, (l_a)_{a\in P}, (u_a)_{a\in P}$), a crucial result in Gale and Politof (1981) which plays an important role in the analysis of the process flexibility in manufacturing (see Simchi-Levi et al. 2014, Chap. 13).

Many important properties and algorithms of classical min-cost flow problems can be extended to settings with $M^\natural$-convex objectives. Consider the fundamental form of the min-cost flow problem:

$$\min_{\xi\in\mathbb{R}^{|A|}}\left\{\sum_{a\in A} \xi_a w_a | \partial\xi = -x,\ l \le \xi \le u\right\}, \quad (6)$$

that is, problem (5) with $\mathcal{F} = \mathbb{R}$, $T = V$, $f_a(\xi_a) = w_a\xi_a$ for every $a \in A$ and a capacity constraint $[l,u]$. Numerous properties for problem (6) have been developed, for example, optimal criteria in terms of potentials (i.e., dual variables) and negative cycles as well as the integrality conditions of optimal flows. There are also efficient algorithms of finding an optimal solution such as the cycle canceling algorithm. We refer to Klein (1967) and Ahuja et al. (1993) for more results on min-cost flow problems. Equipped with $M^\natural$-convexity, these results can be extended to a more general submodular flow problem with a nonseparable objective function. The submodular flow problem solves

$$\begin{aligned} \min_{\xi} & \sum_{a \in A} \xi_a w_a + f(\partial \xi) \\ \text{s.t. } & l \le \xi \le u \\ & \partial \xi \in \text{dom}(f) \\ & \xi \in \mathbb{R}^{|A|}, \end{aligned} \tag{7}$$

where $f: \mathbb{R}^{|V|} \to (-\infty, \infty]$ is a function with dom(f) being a base polyhedron of some submodular function. It is shown in Murota (1998) and Murota (1999) that if f is M-convex (the corresponding problem is referred to as M-convex submodular flow problem or MSFP for short), the results on optimal criteria in terms of the potentials and negative cycles and integrality conditions of optimal flows can be extended to problem (7), and the optimal solution can be obtained by the cycle cancelling algorithm. These results of MSFP are used by Candogan et al. (2016) to analyze the competitive equilibrium in a trading network where agents have M$^\natural$-concave valuations. Specifically, they transform the problem of finding a competitive equilibrium as a MSFP and show that a competitive equilibrium exists and can be computed efficiently.

4.2. Applications of L$^\natural$-Convexity

4.2.1. Inventory Model with a Positive Lead Time. L$^\natural$-convexity is introduced by Zipkin (2008) to analyze the structure of the optimal policy of the classical single-product stochastic inventory model with lost sales and a positive lead time. In the model, at the beginning of each period, we observe the state represented by $s = (s_0, s_1, \ldots, s_{l-1})$, where l is the fixed lead time, and s_i is the on-hand inventory plus the orders to be arrived within i periods. Specifically, s_0 is the on-hand inventory level and s_{l-1} is the inventory position. An order is then placed which incurs a cost of c per unit and will be received l periods later. Any leftover inventory is carried over to the next period incurring a holding cost of h^+ per unit, and unsatisfied demand is lost incurring a lost-sales cost of h^- per unit. The cost-to-go function $f_t(s)$ at period t with a state s satisfies the following Bellman equation.

$$f_t(s) = \min_{y \ge s_{l-1}} c(y - s_{l-1}) + \mathbb{E}[g_t(s, y, D)],$$

and

$$g_t(s, y, D) = h^+(s_0 - D)^+ + h^-(D - s_0)^+ + f_{t+1}(s').$$

where y is the inventory position after ordering, D is the random demand, and $s' = (s_1 - s_0 \wedge D, s_2 - s_0 \wedge D, \ldots, y - s_0 \wedge D)$ is the state of the next period. At the end of the planning horizon, any remaining on-hand inventory is assumed to have a salvage value of c per unit.

One can show that $g_t(s, y, D)$ equals the optimal objective value of the following problem.

$$\begin{aligned} \min\, & h^+(s_0 - u) + h^-(u - s_0) + f_{t+1}(s_1 - u, s_2 - u, \ldots, y - u) \\ \text{s.t. } & 0 \le u \le D,\ u \le s_0, \end{aligned}$$

which basically says that even with the flexibility of satisfying only a portion of the demand, it is always optimal to satisfy the demand as much as possible. Based on this fact and Proposition 3.7, Zipkin (2008) shows by induction that $f_t(s)$ and $g_t(s, y, D)$ are L$^\natural$-convex, which together with Proposition 3.6 proves that the optimal order-up-to level $y^*(s)$ is nondecreasing in s but the increased amount is bounded by that of the on-hand inventory level (everything else unchanged). The monotonicity result implies that the optimal ordering quantity $y^*(s) - s_{l-1}$ is nonincreasing in the on-hand inventory s_0 and the outstanding orders $s_1 - s_0, \ldots, s_{l-1} - s_{l-2}$. In addition, it is more sensitive to more recent orders with bounded sensitivities (i.e., the decreased amount of the optimal order quantity is no more than the increased amount in the on-hand inventory plus the outstanding orders).

The analysis based on L$^\natural$-convexity greatly simplifies that for the lost-sales inventory model with positive lead time in the literature, and has been explored for several other fundamental models in inventory control and revenue management. Pang et al. (2012) consider a single product joint inventory-pricing control problem with backlogging and a positive lead time. Focusing on the case in which demand is a deterministic function of the selling price with an additive random noise independent of price, they show that the profit-to-go function is L$^\natural$-concave. Based on this, they establish a similar sensitivity result for the optimal ordering quantity, and show that the optimal demand is nondecreasing in the on-hand inventory and the inventory in transit with bounded sensitivities.

Chen et al. (2014b) investigate a joint inventory-pricing-disposal control problem for a perishable product with a fixed life time and a positive lead time. They provide monotonicity results of the optimal order-up-to level, the optimal ordering quantity and the optimal demand level for both backlogging and lost-sales cases. A similar monotonicity result for the optimal ordering policy has been established by Fries (1975) and Nahmias (1975) under a special case with fixed prices, zero lead time and continuous demand distributions using significantly more complicated analysis. However, as pointed out by Nahmias (2011) (page 10), "The main theorem requires 17 steps and is proven via a complex induction argument." In addition, for models with discrete demand, Nahmias and

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Schmidt (1986) developed a separate argument using a sequence of continuous demand distributions to approximate the discrete demand distribution, while the proof of Chen et al. (2014b) is applicable for both discrete demand and continuous demand.

Feng et al. (2019) consider a dynamic inventory system of two substitutable products with positive replenishment lead times. With a mild assumption on the parameters, they show that the cost-to-go function is L$^\natural$-convex and provide sensitivity analyses for the optimal order-up-to levels, optimal ordering quantities and optimal substitution quantities. By establishing Proposition 3.8, Chen et al. (2018) develop methodologies to handle a class of optimization models in which decision variables are truncated by random variables and illustrate their applications on several models including dual sourcing systems with random supply capacity (see also Chen 2017b), assemble-to-order systems with random supply capacity, and capacity allocation in network revenue management based on booking limit controls. Applications of L$^\natural$-convexity to optimal control in several other dynamic inventory models refer to Huh and Janakiraman (2010), Gong and Chao (2013) and Chen (2017a). We also refer to Li and Yu (2014) for several dynamic inventory control applications in terms of multimodularity.

So far L$^\natural$-convexity is only used to derive structures of optimal policies. It is useful to facilitate the computation of a class of dynamic program problems including the above inventory models as well. Consider a dynamic program with a planning horizon of T periods. Assume that the cost-to-go function f_t of period t satisfies the Bellman equation

$$f_t(x_t) = \min_{a_t \in A_t(x_t)} H_t(x_t, a_t) + \mathbb{E}_{w_t}[f_{t+1}(g_t(x_t, a_t, w_t))], \quad (8)$$

where $x_t \in \mathbb{Z}^{k_1}$ is the state, $a_t \in \mathbb{Z}^{k_2}$ is the control, $A_t(x_t)$ is the feasible action space, $w_t \in \mathbb{Z}^{k_3}$ is a random vector, H_t is an integer-valued single-period cost function and g_t is the state transition function. Halman et al. (2009) show that solving this dynamic program is NP-hard even for the one-dimensional case, that is, $k_1 = k_2 = k_3 = 1$. Chen et al. (2014a) further show that no algorithm can provide an ε-approximation of f_1 in a polynomial time in the input size of the dynamic program and $\frac{1}{\epsilon}$ for any $\varepsilon > 0$. Here a function $\tilde{f}_1(x)$ is called an ε-approximation of $f_1(x)$ if $|f_1(x) - \tilde{f}_1(x)| < \epsilon$. Interestingly, by assuming that $f_t(\cdot)$, $H_t(\cdot,\cdot)$, $f_t(g(\cdot,\cdot,\cdot))$ are L$^\natural$-convex and the supports of x_t, a_t and w_t are boxes, Chen et al. (2014a) provide an approximation algorithm which outputs an ε-approximation of $f_1(x)$ with a running time that is pseudo-polynomial in the input size and polynomial in $\frac{1}{\epsilon}$.

4.2.2. Assemble-to-Order Inventory Model. L$^\natural$-convexity is useful in several assemble-to-order (ATO) inventory models. For example, as we mentioned in the above subsection, Chen et al. (2018) consider the replenishment of different components which are allocated and assembled to satisfy demands in a periodic review stochastic model. For a special class of ATO systems, a generalized M-system in which a product uses all components while any other product is associated with only one component, they show that the optimal cost-to-go functions are L$^\natural$-convex and provide a characterization of the optimal ordering policy.

Continuous-review ATO models are arguably the first application of L$^\natural$-convexity in inventory literature (see Lu and Song 2005 and Bolandnazar et al. 2019). Here we focus on the models analyzed in Reiman and Wang (2015) and Doğru et al. (2017). Reiman and Wang (2015) consider a continuous-review ATO system with m products and n components. At each moment, the manager observes the arrived random demand and receives the components ordered before. Assume that all the components have the same replenishment lead time L. Then the manager determines how to allocate current available components to fulfill the demand and how many components to order. Assume that unsatisfied demand is backlogged and unused components remain in the inventory. The objective is to find a replenishment policy Γ and an allocation policy Π to minimize the long-run average expected cost

$$C^{\Gamma,\Pi} = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}[b^T B(t) + h^T I(t)]dt. \quad (9)$$

Here, b is the vector of each product's backlogging cost per unit of time, h is the vector of each component's inventory holding cost per unit of time, $B(t)$ is the vector of backlog of each product at time t, and $I(t)$ is the vector of inventory level of each component at time t.

Reiman and Wang (2015) show that for a general bill of materials (BOM), it is asymptotically optimal to follow a base stock replenishment policy for each component and certain component allocation principle when the lead time L tends to infinity. The base stock level can be set as arbitrary convex combination of optimal solutions from the following optimization problem (10) and its relaxation (11):

$$\inf_{y \geq 0} C(y), \quad (10)$$

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where $C(y) = b^T\mathbb{E}[D] + h^T y - \mathbb{E}[\phi(y;D)]$, $\phi(y;d) = \max\{c^T z | 0 \le z \le d,\ Az \le y\}$, D is the lead time demand vector, the matrix A is the BOM matrix with a_{ji} being the amount of component j needed to assemble one unit of product i, and $c = b + A^T h$;

$$\inf_y \hat{C}(y), \tag{11}$$

where $\hat{C}(y) = b^T\mathbb{E}[D] + h^T y - \mathbb{E}[\varphi(y;D)]$ and $\varphi(y;d) = \max\{c^T z | z \le d, Az \le y\}$.

Motivated by a thought-provoking discussion in Zipkin (2016) on combining polymatroids and discrete convexity to analyze specially structured ATO models, Doğru et al. (2017) consider a class of ATO systems with a chained BOM in which every product needs at most one unit of each component for production (i.e., it is a binary BOM), and for any two products which share some common component, the set of components used for one product contains the set of components used for the other one. Denote S_i the set of components needed for product i. It is clear that a chained BOM is exactly a binary BOM with $\{S_i\}_{1 \le i \le m}$ forming a laminar family.

For an ATO system with a chained BOM, Doğru et al. (2017) show that the objective functions $C(y)$ and $\hat{C}(y)$ are L$^\natural$-convex, which allows them to greatly simplify the solution procedure by taking advantage of the existing optimization algorithms for L$^\natural$-convex functions. Their proof is lengthy. Here we give a significantly simpler proof of the L$^\natural$-convexity of $C(y)$ (the proof for $\hat{C}(y)$ follows the same argument) by exploiting the conjugate relationship between L$^\natural$-convexity and M$^\natural$-convexity.

For this purpose, we only need to show that $\phi(y;d)$ is L$^\natural$-concave. By Proposition 3.18, it suffices to prove that the conjugate function of $-\phi(y;d)$, denoted by $g(w)$, is M$^\natural$-convex. By definition,

$$\begin{aligned} g(w) &= \sup_y w^T y + \phi(y;d) \\ &= \sup w^T y + c^T z \\ &\text{s.t. } Az \le y \\ &\quad 0 \le z \le d. \end{aligned} \tag{12}$$

By linear programming strong duality, the function value $g(w)$ equals the optimal objective value of the dual of problem (12), which can be written in a closed form

$$d^T(c + A^T w)^+ + \delta_{w \le 0}(w) = \sum_{i=1}^m d_i (c_i + \sum_{j \in S_i} w_j)^+ + \sum_{i=1}^m \delta_{w_i \le 0}(w_i)$$

Since A is a chained BOM, $g(w)$ is laminar convex, and thus M$^\natural$-convex.

4.2.3. Appointment Scheduling Problems.

L$^\natural$-convexity and the closely related concept multimodularity find applications in appointment scheduling problems when appointment times (Begen and Queyranne 2011, Ge et al. 2013) or the numbers of appointments at given times (Kaandorp and Koole 2007, Wang et al. 2020, Zacharias and Pinedo 2017, Zacharias and Yunes 2020, Zeng et al. 2010) need to be determined.

Consider an appointment scheduling problem where a decision maker needs to determine the appointment times of n jobs with random processing times. Denote $A = \{A_1, A_2, \ldots, A_{n+1}\}$ with A_i being the appointment time of job i (A_{n+1} refers to the ending time), and $p_i(w)$ the random processing time of job i, where w represents one scenario from a sample space Ω. Let $S_i(w)$ and $C_i(w)$ be the start time and completion time of job i, respectively. Assuming all jobs arrive on time, the waiting of job i+1 incurs an overage cost $o_i((C_i(w) - A_{i+1})^+)$, while the idle time due to the earlier completion of job i incurs an underage cost $u_i((A_{i+1} - C_i(w))^+)$. The objective of the decision maker is to minimize the expected total underage cost and overage cost which can be formulated as the following optimization problem.

$$\begin{aligned} \min_{A,C}\ & \mathbb{E}_w\left[\sum_{i=1}^n u_i((A_{i+1} - C_i(w))^+) + o_i((C_i(w) - A_{i+1})^+)\right] \\ \text{s.t. } & C_1(w) = p_1(w) \\ & C_i(w) = S_i(w) + p_i(w)\ \forall\, i = 2, \ldots, n, w \in \Omega \\ & S_i(w) = A_i \vee C_{i-1}(w)\ \forall\, i = 2, \ldots, n, w \in \Omega, \end{aligned} \tag{13}$$

where $C = \{C_i(w) | i \in [n], w \in \Omega\}$. Ge et al. (2013) show that problem (13) admits an integral optimal solution if u_i, o_i are continuous nondecreasing piecewise linear functions with integral breakpoints, $u_i(0) = o_i(0) = 0$, and the random processing times are integral and bounded. Moreover, with an additional assumption that o_j, u_j are convex and the summation of the smallest slopes of u_i and o_i is no less than the largest slope of u_{i+1}, they show that the integral optimal solution can be computed efficiently. More specifically, they prove that problem (13) is equivalent to the two-stage problem

$$\min_A G(A), \tag{14}$$

where $G(A) = \mathbb{E}_w[F_w(A)]$ and

$$\begin{aligned} F_w(A) = \min_C \ & \sum_{i=1}^{n} u_i(C_{i+1}(w) - C_i(w) - p_{i+1}(w)) + o_i \\ & (C_{i+1}(w) - A_{i+1} - p_{i+1}(w)) \\ \text{s.t. } & C_1(w) = p_1(w) \\ & C_i(w) \geq A_i + p_i(w) \ \forall\ i = 2, \ldots, n+1 \\ & C_i(w) \geq C_{i-1}(w) + p_i(w) \ \forall\ i = 2, \ldots, n+1. \end{aligned} \tag{15}$$

Note that the objective function in problem (15) is L$^\natural$-convex in $(C_1(w), \ldots, C_{n+1}(w), A)$ since u_i, o_i are univariate convex functions, and the graph of the constraint set of problem (15) is an L$^\natural$-convex set by Proposition 3.1. It follows that $F_w(A)$, and thus, $G(A)$ are L$^\natural$-convex by Proposition 3.7. The integral optimal solution can then be computed using a steepest descent scaling algorithm with a running time bounded by $O(n^7 v_G \log \hat{p})$. Here, $\hat{p}$ is the maximal processing time over all jobs and scenarios, v_G is an upper bound of the time to evaluate function $G(A)$. If the processing times of the n jobs are independent, the function value $G(A)$ can be computed in $O(n^2\hat{p}^2)$ time (see Begen and Queyranne 2011), which implies that the integral optimal solution can be computed in $O(n^9\hat{p}^2 \log \hat{p})$ time.

A different appointment scheduling problem is to decide the number of scheduled patients for each predetermined time slot. By assuming patient-homogeneous time-independent no-show probabilities and punctuality of patients, the objective function is shown to be multimodular in the following models: a single-server system with exponential service times (Kaandorp and Koole 2007, Zeng et al. 2010), a multi-server system with deterministic service times (Zacharias and Pinedo 2017), a single-server system with deterministic service times and a general walk-in process in which scheduled patients have priority over unscheduled patients (Wang et al. 2020), and a single-server system with general random service times and a general walk-in process in which the service discipline follows first-in-first-out and the scheduled patients have priority over unscheduled patients (Zacharias and Yunes 2020). The relationship between multimodularity and L$^\natural$-convexity implies that the objective functions are L$^\natural$-convex in terms of a new decision vector, whose ith component is the number of cumulative scheduled patients up to the ith time slot. Thus, the corresponding optimization problems can be solved using existing algorithms developed for L$^\natural$-convex function minimization problems.

4.3. Applications of M$^\natural$-Convexity

4.3.1. Exchange Economy. M$^\natural$-convexity plays an important role in establishing the existence of competitive equilibrium in an exchange economy with indivisible commodities. Here, the exchange economy is an economic model in which consumers exchange commodities through money so as to maximize their surplus (utility of acquired commodities minus the payment). For an exchange economy where commodities are indivisible and each has exactly one unit, Kelso and Crawford (1982) propose a concept called *gross substitutability* (GS) to show that a competitive equilibrium (i.e., a price vector and an allocation of all commodities such that the surplus of each agent is maximized under this allocation) exists if consumers' utility set functions[11] are monotone and satisfy (GS). Here, a function $f : \mathbb{Z}^n \to [-\infty, \infty)$ satisfies (GS) if for any two price vectors $p, q \in \mathbb{R}^n$ with $p \leq q$ and any $x \in \arg\max f[-p]$, there exists $y \in \arg\max f[-q]$ such that $y_i \geq x_i$ when $p_i = q_i$. A consumer's utility function satisfies (GS) means that if the prices of some commodities increase while the prices of the others keep unchanged, then the demands of the commodities with unchanged prices increase. For an exchange economy where more than one unit of each indivisible commodity can be consumed, Danilov et al. (2001) show that there exists a competitive equilibrium if utility functions defined on $\mathbb{Z}^n$ (n is the number of different commodities) are M$^\natural$-concave. Later, Murota and Tamura (2003) propose an efficient algorithm to compute the competitive equilibrium.

These results on exchange economy with indivisible commodities lead to a series of investigations of the relationship between (GS) and M$^\natural$-concavity. Fujishige and Yang (2003) prove that a set function is M$^\natural$-concave if and only if it satisfies (GS). Murota et al. (2013) extend this result by showing that a concave-extensible function with a bounded effective domain is M$^\natural$-concave if and only if it satisfies (GS) and the *law of the aggregate demand* (LAD). Here, a function $f : \mathbb{Z}^n \to [-\infty, \infty)$ satisfies (GS) and (LAD) if for any $p, q \in \mathbb{R}^n$ with $p \leq q$ and any $x \in \arg\max f[-p]$, there exists $y \in \arg\max f[-q]$ such that $y_i \geq x_i$ when $p_i = q_i$ and $\sum_{i=1}^n y_i \leq \sum_{i=1}^n x_i$. That is, when the prices of some commodities increase, the total amount of demands decreases while the demands of the commodities with unchanged prices increase.

4.3.2. Production Control Problems with Two Products (facilities). M$^\natural$-convexity is useful in the analysis of optimal policies of dynamic models in operations. In this subsection, we focus on a class of dynamic production models with two products (facilities). A common theme of these models is that their underlying states lie in two-dimensional spaces and their cost-to-go functions are supermodular and satisfy the diagonal dominance property mentioned in Proposition 3.13 (see, e.g., Ha 1997, Hu et al. 2008, Yang 2004, Yang and Qin 2007). Yet, the literature establishes these properties on a case by case basis and relies on a very careful analysis on the first-order and second-order derivatives of the cost-to-go functions in settings with real state spaces. Yang and Qin (2007) believe that "there must be some deeper theory that can unify all these different approaches. However, such a theory still seems elusive for the time being." We show in the following that $M^\natural$-convexity is exactly what one can use to unify the different approaches, and using $M^\natural$-convexity, the proof of showing the properties of supermodularity and diagonal dominance can be significantly simplified.

Ha (1997) studies a continuous-review dynamic production scheduling problem with a single server and two products. At each time $t \in [0,\infty)$, the product manager can take three actions: produce product 1, produce product 2 or do not produce. Assume that demands for product i are independent Poisson processes with rate λ_i ($i = 1,2$), and the production time for product i is exponential distributed with rate μ. The server follows a preemptive discipline and can only produce one product at each time. The inventory cost (rate), which is charged continuously over time, is $h(X_1(t), X_2(t)) = \sum_{i=1}^{2} h_i^+ X_i(t)^+ + h_i^- X_i(x)^-$, where $X_i(t)$ is the inventory level at time t for product i and h_i^+, h_i^- are its unit holding and backlogging cost rate, respectively. The objective is to dynamically determine the action for each time to minimize the expected discounted inventory cost over an infinite horizon. Denote $f(x_1, x_2)$ the infimum of the expected discounted cost over the infinite horizon with initial inventory $(x_1, x_2) \in \mathbb{Z}^2$ at time 0. One can show that function f satisfies a dynamic programming recursion $f = \mathcal{T}f$. Here $\mathcal{T}$ is an operator defined as

$$\begin{aligned}\mathcal{T}f(x_1,x_2) = h(x_1,x_2) + \lambda_1 f(x_1-1,x_2) + \lambda_2 f(x_1,x_2-1) \\ + \mu \min\{f(x_1+1,x_2), f(x_1,x_2 \\ +1), f(x_1,x_2)\},\end{aligned}$$

where the first term is the inventory cost rate, the second term and the third term correspond to the events of a unit demand of product 1 and product 2, respectively, and the last term refers to the decision to be made upon the completion of a production. A key result in Ha (1997), which is used to characterize the optimal policy, shows that the properties of supermodularity and diagonal dominance are preserved under operator $\mathcal{T}$. Interestingly, according to Proposition 3.13, this result is equivalent to the preservation of $M^\natural$-convexity under operator $\mathcal{T}$, which can be easily obtained by Proposition 3.16. To see this, define $g(x_1,x_2) = \min\{f(x_1+1,x_2), f(x_1,x_2+1), f(x_1,x_2)\}$. One can rewrite it as

$$g(x_1,x_2) = \min_{u+v=x}\{f(u_1,u_2) + \delta_B(v_1,v_2)\},$$

where $B=\{(0,-1),(-1,0),(0,0)\}$. It is clear that B is a discrete $M^\natural$-convex set, and thus $g(x)$ is the infimal convolution of two $M^\natural$-convex functions, which implies that $g(x)$ is $M^\natural$-convex by Proposition 3.16.

Yang and Qin (2007) investigate a periodic-review joint production and transshipment control problem in which a company manages two manufacturing facilities. Each facility has its own market and its random demand is assumed to be independent over time but can be dependent on the demand of the other facility. In each period k, the company divides its decisions into two stages. In the production stage, after observing the initial inventory (x_1, x_2), it decides the production quantity $z_i \in [0, T_i]$ of each facility $i = 1,2$, and pays a linear production cost, where T_i is a deterministic capacity. The production lead time is assumed to be 0. In the transshipment stage, after the random demand D_i is realized, the company decides the quantity of products transferred from one facility to the other one and pays a linear transshipment cost. At the end of period k, the demand of each facility is satisfied by its on-hand inventory and unsatisfied demand is backlogged. Let $H_i(\cdot)$ be the inventory carryover and backorder cost function of product i. The objective of the company is to decide the production and transshipment quantities so as to minimize the total discounted expected cost. Let $f_k(x_1, x_2)$ be the cost-to-go function when there are k periods left in the planning horizon and the initial inventory is (x_1, x_2).

We have the following dynamic program:

$$\begin{aligned}f_k(x_1,x_2) = &\min_{y_1,y_2} \mathbb{E}_{D_1,D_2}[c_1(y_1-x_1) + c_2(y_2-x_2) \\ &+ g_k(y_1-D_1, y_2-D_2|D_1,D_2)], \\ &\text{s.t. } x_1 \le y_1 \le x_1+T_1,\ x_2 \le y_2 \le x_2+T_2,\end{aligned} \tag{16}$$

and

$$\begin{aligned}g_k(u_1,u_2|D_1,D_2) = &\min_{w_1,w_2,v_1,v_2} g(w_1,w_2,v_1,v_2) \\ &\text{s.t. } w_1+v_1=u_1, w_2+v_2=u_2, v_1 \\ &+v_2=0, v_1 \le D_1, v_2 \le D_2,\end{aligned} \tag{17}$$

where $g(w_1,w_2,v_1,v_2) = H_1(w_1) + H_2(w_2) + s_1 v_1^+ + s_2 v_2^+ + \alpha f_{k-1}(w_1,w_2)$, s_i is the unit transshipment cost and α is the discount factor. In formulation (16), y_i is the inventory level after production. In formulation (17), w_i is the inventory level after transshipment and fulfillment of demand, v_i is the amount of demand of facility i assigned to the other facility, the constraint $v_1 + v_2 = 0$ comes from the fact that one only needs to consider unilateral transshipment, and the constraint $v_i \le D_i$ means that the amount of demand assigned to the other facility should not exceed D_i. Note that in the formulation v_i can be nonzero even if the net on-hand inventory level u_i is negative, which is appropriate when the transshipment simply models the assignment of a portion of demand generated in one facility to the other one (though in the formulation we allow w_i to be positive even when $v_i < 0$, at optimality this won't happen). Assume that $f^0(x_1, x_2) = -c_1x_1 - c_2x_2$, that is, the inventory leftover/backorders in facility i can be salvaged/fulfilled with a unit cost c_i.

Yang and Qin (2007) show that the properties of supermodularity and diagonal dominance of the cost-to-go functions are preserved through the dynamic programming induction, which is then used to characterize the optimal production and transshipment policies. As we mentioned earlier, their proof of the preservation result is based on a careful analysis on the derivatives of the cost-to-go functions and relies heavily on the characterization of optimal policies. Their derivation can be significantly simplified with the help of M$^\natural$-convexity by observing that optimization problems (16) and (17) can be written as infimal convolutions (one can put the constraints $v_1 + v_2 = 0$ and $v_i \leq D_i$ into the objective function by adding their corresponding indicator functions) and M$^\natural$-convexity is preserved under expectation operations for two-dimensional functions. A similar argument using M$^\natural$-convexity works for a production control problem with a single product and a single raw material studied in Yang (2004).

Hu et al. (2008) consider a joint production and transshipment model with random capacities and lost sales. Their model differ significantly from Yang and Qin (2007)'s model in that the capacity T_i is random and is realized after making the production decision y_i, and thus the effective inventory level is $y_i \wedge (x_i + T_i)$. Again, Hu et al. (2008) use a complicated induction argument to prove the preservation property of supermodularity and diagonal dominance of the cost-to-go function $f_k(x_1, x_2)$ (Hu et al. 2008 actually consider the profit-to-go function, which can be regarded as the negative of f_k) and obtain the optimal polices. Later, Chen et al. (2015) provide a new and simpler proof of this preservation result by showing L$^\natural$-convexity of a modified cost-to-go function $f_k(x_1, -x_2)$ (i.e., changing one variable to its negative). Interestingly, since by Proposition 3.19 $f_k(x_1, -x_2)$ is L$^\natural$-convex in (x_1, x_2) if and only if $f_k(x_1, x_2)$ is M$^\natural$-convex, we can directly employ M$^\natural$-convexity to prove the preservation property of supermodularity and diagonal dominance of the cost-to-go function $f_k(x_1, x_2)$ without the unnatural variable transformation by slightly modifying the argument in Chen et al. (2015).

4.3.3. Multi-Product Stochastic Inventory Model.

In this subsection, we illustrate the power of M$^\natural$-convexity on stochastic inventory models with more than two products. Consider the classical multi-product inventory model analyzed in Ignall and Veinott (1969) in which a company selling n products to m random demand classes over N periods. At the beginning of period t, after observing the initial inventory x_t, the company decides to raise the inventory to $y_t \geq x_t$ (assuming zero lead time) with a linear ordering cost $c^T(y_t - x_t)$, where $c = (c_1, \ldots, c_n)^T$ and c_i is the unit ordering cost of product i. Demand $D_t = (D_{t1}, \ldots, D_{tm})$ is then realized and fulfilled by on-hand inventory. Assume that D_t ($t = 1,\ldots,N$) are i.i.d. random vectors. At the end of this period, the company incurs a cost $g(y_t, D_t)$ depending on y_t and the realized demand D_t (e.g., holding/backlogging/lost-sales cost). The initial inventory of the next period is assumed to be a function $s(y_t, D_t)$ (e.g., $s(y_t, D_t) = y_t - D_t$ (or $(y_t - D_t)^+$) for the backlogging model (or the lost-sales model) when each demand class is associated with a unique product). Inventory of product i carried over to period $N+1$ will be returned with a unit price c_i. Let $\alpha \in (0,1]$ be a discount factor. The objective of the company is to determine the order-up-to levels $y_1, \ldots, y_N$ to minimize its total discounted expected cost, that is,

$$\begin{aligned} \min \ & \sum_{t=1}^{N} \alpha^{t-1} \mathbb{E}[c^T(y_t - x_t) + g(y_t, D_t)] - \alpha^N c^T x_{N+1} \\ \text{s.t. } & y_t \geq x_t, \\ & x_{t+1} = s(y_t, D_t), \\ & y_t \in Y, \ t = 1, \ldots, N, \end{aligned} \tag{18}$$

where Y denotes the feasible set of y_t (e.g., in the case with a storage capacity C, $Y = \{y | \mathbf{e}^T y \leq C\}$). A special case of the above model is the multi-product stochastic inventory model with lost-sales and a joint capacity constraint. In this case, $m = n$, $s(y_t, D_t) = (y_t - D_t)^+$, $Y = \{y | y \geq 0, \mathbf{e}^T y \leq C\}$ and $g(y_t, D_t) = \sum_{i=1}^n (h_i^+ (y_{ti} - D_{ti})^+ + h_i^- (D_{ti} - y_{ti})^+)$, where h_i^+ is the per unit holding cost and h_i^- is the per unit lost-sales cost of product i. If we modify the definition of $s(y_t, D_t)$ and Y to $s(y_t, D_t) = y_t - D_t$ and $Y = \{y | \mathbf{e}^T y^+ \leq C\}$, it corresponds to the backlogging model.

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Define a modified one-period cost function by

$$G(y) = \begin{cases} c^T y + \mathbb{E}_{D_t}[g(y, D_t) - \alpha s(y, D_t)] & \text{for } y \in Y \\ +\infty & \text{otherwise} \end{cases}$$

and denote a myopic policy by

$$\overline{y}(x) = \arg\min\{G(y) | y \geq x,\ y \in Y\}.$$

An important concept called substitute property is proposed by Ignall and Veinott (1969) to study the optimal policy. They show that the myopic policy $\overline{y}(x)$ is optimal if it (viewed as a function of x) satisfies the substitute property and certain regularity conditions. Here, a function f satisfies the substitute property if $f(x){-}x$ is nonincreasing in x.

Ignall and Veinott (1969) prove that if G is defined on a box and twice continuously differentiable with its Hessian matrix being a so-called *substitute matrix* at every point in this box, then $\overline{y}(x)$ satisfies the substitute property. Chen and Li (2019) show that a substitute matrix is exactly a symmetric inverse M-matrix. This observation together with Proposition 3.11 implies that $\overline{y}(x)$ has the substitute property if $G(y)$ is a twice continuously differentiable M$^\natural$-convex function defined on a box. One may also note that the substitute property of $\overline{y}(x)$ follows directly from Proposition 3.15 and the preservation property of M$^\natural$-convexity under variable summation operations (i.e., $G(x + z)$ is M$^\natural$-convex in (x,z) if $G(x)$ is M$^\natural$-convex in x).

For a special case in which (G,Y) has a so-called nested structure, Ignall and Veinott (1969) prove that the Hessian of the objective function at any feasible point is a substitute matrix and $\overline{y}(x)$ satisfies the substitute property. However, their proof relies on a complicated network analysis, and as pointed out in their paper, "Unfortunately, we have not been able to construct a simple proof that $\nabla^2 G(y)$ is a substitute matrix in the general nested case…." Interestingly, these results can be obtained readily by recognizing that the nested structure of (G,Y) essentially requires that G is laminar convex with dom$(G) = Y$, a special case of M$^\natural$-convexity (see Chen and Li 2019). We would like to mention that for the inventory model with lost-sales and a joint capacity constraint mentioned above, (G,Y) has a nested structure, which means that the myopic policy is optimal. For the corresponding backlogging model with a constraint $Y = \{y | \mathbf{e}^T y^+ \leq C\}$, though (G,Y) does not have a nested structure, Chen and Li (2019) show that the myopic policy still satisfies the substitute property by a simple variable transformation.

4.3.4. Portfolio Contract Model.

Consider a two-stage problem where a retailer reserves capacities in blocks from n competing suppliers to fulfill its random demand D. At the first stage, each supplier offers a block of capacities, its unit reservation fee and unit execution fee. The retailer then selects a subset of suppliers to reserve their capacities and pays a linear reservation fee for each selected supplier. Note that the retailer can reserve at most one block from each supplier. At the second stage, the random demand D is realized, the retailer allocates the reserved capacities to fulfill the demand and pays a linear execution fee for used capacities. The retailer gains a revenue ρ by fulfilling one unit of demand. The objective of the retailer is to determine the portfolio of suppliers and the allocation of capacities to maximize its expected profit. Let $x \in \{0,1\}^n$ represent the portfolio of suppliers, where $x_i = 1$ means that supplier i is selected and otherwise $x_i = 0$. Denote $\pi(x|d)$ the maximal profit by allocating capacities to fulfill the realized demand d. The optimization problem can then be written as

$$\max_{x \in \{0,1\}^n} \Pi(x), \tag{19}$$

where $\Pi(x) = \mathbb{E}_D[\pi(x|D)] - \sum_{l=1}^n x_l r_l K_l$. Here, r_l and K_l are the unit reservation fee and the block size of capacities from supplier l, respectively. Anderson et al. (2017) show that Problem (19), though with an objective function submodular on $\{0,1\}^n$, is NP-hard in general. For the special case of equal block sizes, they propose a dynamic programming approach and show that it solves problem (19) in a polynomial time. A key structual result in Anderson et al. (2017), used to analyze the equilibrium behavior of the suppliers, is that the optimal objective value function

$$\Pi^*(y) = \max\{\Pi(x) | x \leq y,\ x \in \{0,1\}^n\}$$

preserves submodularity when all block sizes are equal. It is clear that this preservation of submodularity does not follow from classical results in lattice programming. In fact, Anderson et al. (2017) give a rather lengthy proof which spans six pages. Interestingly, Chen and Li (2019) show that their proof can be significantly simplified using M$^\natural$-concavity. Note that $\pi(x|d)$ can be reformulated as

$$\pi(x|d) = \sum_{l=1}^n \rho_j \min\{(d - \sum_{l=1}^{j-1} x_l K_l)^+, x_j\} \tag{20}$$

$$= \rho_1 d - \sum_{j=2}^{n+1} (\rho_{j-1} - \rho_j)(d - \sum_{l=1}^{j-1} x_l K_l)^+, \tag{21}$$

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where ρ_j is the profit margin using one unit of capacity of supplier j, $\rho_{n+1} = 0$, and without loss of generality, the profit margin are ordered as $\rho_1 > \cdots > \rho_n$. Equation (20) follows from the fact that it is optimal to first use capacities from suppliers with higher profit margins, and Equation (21) is from $\min\{a, b\} = a - (a - b)^+$ and $(a^+ - b)^+ = (a - b)^+$ for $b \geq 0$. By formulation (21), it follows that $\pi(x|d)$ is laminar concave if K_l $(l = 1,\ldots,n)$ are equal, which implies that $\Pi(x)$ is laminar concave, and thus M$^\natural$-concave. By reformulating $\Pi^*(x)$ as a supremal convolution of M$^\natural$-concave functions, one can show that $\Pi^*(x)$ is M$^\natural$-concave and thus submodular by Propositions 3.14 and 3.16. This analysis can be extended to a more general setting with a risk-averse retailer whose risk preference is represented by a spectral risk measure. Moreover, with M$^\natural$-concavity of $\Pi(x)$ established, Chen and Li (2019) point out that the dynamic programming approach turns out to be the steepest ascent algorithm for M$^\natural$-concave function maximization problems.

4.3.5. Discrete Choice Model. In this subsection, we show that M-convexity is closely related to a concept of substitutability in discrete choice models proposed by Feng et al. (2018). Given n alternatives, let $\pi = (\pi_1, \ldots, \pi_n)$ be the deterministic utility of each alternative. A choice model can be represented by a choice probability function $q : \mathbb{R}^n \to \Delta_{n-1}$ which maps the utility vector π to a vector x in a $(n-1)$-dimensional simplex $\Delta_{n-1} = \{x|\mathbf{e}^T x = 1,\ x \geq 0\}$, where the ith component of x is the probability of choosing the ith alternative or the fraction of the entire population that chooses the ith alternative. We now introduce two discrete choice models: the representative agent model and the welfare-based choice model (See Feng et al. (2017) and Feng et al. (2018) for more details on these models).

In a representative agent model, an agent representing the entire population makes a choice x among n alternatives. Here x is a n-dimensional vector with its ith component representing the fraction of the population that chooses the ith alternative. The agent's objective is to maximize the average utility over all alternatives while taking into account some degree of diversification. More specifically, upon denoting $V(x)$, a convex and lower semi-continuous function, which models the penalty for centralization of choice x, the agent solves the optimization problem:

$$\max_{x \in \Delta_{n-1}} \pi^T x - V(x). \tag{22}$$

Since in (22) only values of $V(x)$ on Δ_{n-1} are relevant, we assume that $V(x)$ takes value $+\infty$ if $x \notin \Delta_{n-1}$. We also assume that for any π, (22) has a unique optimal solution $q^r(\pi)$, which is the choice probability function under the representative agent model.

We now introduce the welfare-based choice model. A function $w(\pi) : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called a *choice welfare function* if it satisfies the following properties:

- (Monotonicity) For any $\pi_1, \pi_2 \in \mathbb{R}^n$, $\pi_1 \geq \pi_2$ implies $w(\pi_1) \geq w(\pi_2)$;
- (Translation Invariance) For any $\pi \in \mathbb{R}^n$, $t \in \mathbb{R}$, $w(\pi + t\mathbf{e}) = w(\pi) + t$;
- (Convexity): For any $\pi_1, \pi_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$, $w(\lambda\pi_1 + (1-\lambda)\pi_2) \leq \lambda w(\pi_1) + (1-\lambda)w(\pi_2)$.

For a differentiable choice welfare function $w(\pi)$, the properties of monotonicity and translation invariance imply $q^w(\pi) = \nabla w(\pi) \in \Delta_{n-1}$, which is defined as the choice probability function in the welfare-based choice model.

A function $f : \mathbb{R} \to \mathbb{R}$ is *locally decreasing* at x if there exists $\delta > 0$ such that $f(x-h) \geq f(x) \geq f(x+h)$, $\forall h \in (0, \delta)$. A choice probability function $q : \mathbb{R}^n \to \Delta_{n-1}$ is said to have the *substitutability* (or its corresponding choice model is substitutable) if $q_i(\pi)$ is locally decreasing in π_j for any $i,j \in [n]$, $i \neq j$. Feng et al. (2018) show that a welfare-based choice model with a differentiable choice welfare function $w(\pi)$ is substitutable if and only if $w(\pi)$ is submodular. For the representative agent model, though it is equivalent to the welfare-based choice model as shown by Feng et al. (2017), only a necessary condition for the substitutability of q^r is provided in Feng et al. (2018). By establishing the relationship between M-convexity and choice welfare functions, we can give a sufficient and necessary condition as follows.

Theorem 4.1. *A representative agent model with an essentially strictly convex*[12] *$V(x)$ is substitutable if and only if $V(x)$ is M-convex.*

To prove Theorem 4.1, first note that for a given essentially strictly convex function $V(x)$ whose domain is contained in Δ_{n-1}, its conjugate function $V^*(\pi)$, which is exactly the optimal objective value function of the representative agent model (22), is a choice welfare function. Feng et al. (2018) observe that the choice probability function q^r in the representative agent model (22) is the same as the choice probability function q^w in the welfare-based choice model with the choice welfare function $w(\pi)$ given by $V^*(\pi)$. Based on this observation, the representative agent model with the penalty function $V(x)$ is substitutable if and only if the welfare-based choice model with the choice welfare function $w(\pi) = V^*(\pi)$ is substitutable. Observe that a choice welfare function is submodular if and only if it is L-convex. This, together with the characterization of the welfare-based choice model, implies that the welfare-based choice model with the choice

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welfare function $w(\pi) = V^*(\pi)$ is substitutable if and only if $w(\pi)$ is L-convex, which in turn is equivalent to the claim that $V(x)$ is M-convex by Proposition 3.18.

Theorem 3.1 allows us to directly derive two results on special representative agent models in Feng et al. (2018). For $V(x) = x^T Ax$ with A being a positive definite matrix, they show that the corresponding representative agent model is substitutable only if $A_{jk} - A_{ik} - A_{ij} + A_{ii} \geq 0$ for all distinct $i,j,k \in [n]$. By Theorem 4.1, this representative agent model is substitutable if and only if $V(x) = x^T Ax$ is M-convex. Their result then follows from the characterization of quadratic M-convex functions. For $V(x) = \sum_{i=1}^n V_i(x_i)$ on Δ_{n-1}, where $V_i(x_i), [0,1] \to \mathbb{R}$ is a strictly convex function for $i \in [n]$, they show that the corresponding representative agent model is substitutable. This result follows directly from our result by observing that a separable convex function with an effective domain Δ_{n-1} is M-convex.

4.3.6. Dock Reallocation Problem. Consider a bike-sharing system with n bike stations. Each bike station has some docks that can be used to store bikes. Suppose at the beginning of a day, station i has d_i open docks and b_i bikes (hence the station has $u_i = d_i + b_i$ docks in total). Denote by $X = (X_1, \ldots, X_s)$ a random sequence of customers arriving at station i during a day (X depends on i but for simplicity we omit the index i), where s is a random number denoting the total arrivals and the random variable X_t takes value 1 if the ith customer wants to rent a bike and -1 if the ith customer wants to return a bike. If there is no bike available when a customer wants to rent a bike, or there is no open dock when a customer wants to return a bike, an out-of-stock event occurs. Denote by $c_i^X(d_i, b_i)$ the number of out-of-stock events at station i, and $c_i(d_i, b_i) = \mathbb{E}_X[c_i^X(d_i, b_i)]$ the expected number of out-of-stock events, where the expectation is taken over all possible customer sequences. The objective is to determine the starting allocation $(b,d) = (b_1, \ldots, b_n, d_1, \ldots, d_n)$ to minimize the expected number of out-of-stock events under some budget constraints and an operational constraint. The optimization problem can be written as

$$
\begin{aligned}
\min_{d,b} \; & \sum_{i=1}^{n} c_i(d_i, b_i) \\
\text{s.t. } & \mathbf{e}^T(d+b) = D + B \\
& \mathbf{e}^T b \leq B \\
& \|d + b - (\overline{d} + \overline{b})\|_1 \leq 2\gamma \\
& l \leq d + b \leq u, \\
& d, b \in \mathbb{Z}_+^n.
\end{aligned} \tag{23}
$$

Here, the first constraint means that the total number of docks are fixed, the second constraint means that the total number of bikes does not exceed B, and the third operational constraint means that the reallocation (d,b) should not be too far from a fixed allocation $(\overline{d},\overline{b})$. Freund et al. (2017) show by definition that for each random sequence X, $c_i^X(d_i, b_i)$ is multimodular in (d_i, b_i), and thus so is $c_i(d_i, b_i)$, or equivalently $c_i(d_i, b_i)$ is M$^\natural$-convex by Example 2.2. Based on this observation, Freund et al. (2017) provide a steepest descent algorithm which outputs an optimal reallocation within γ iterations and a polynomial scaling algorithm.

Their problem is a special case of a more general problem of minimizing an M-convex function under an L$_1$-distance constraint (MML1), which solves

$$
\begin{aligned}
& \min f(x) \\
& \text{s.t. } \|x - x_c\|_1 \leq 2\gamma,
\end{aligned}
$$

where $f(x)$ is an M-convex function with $\text{dom}(f) \subseteq \{x | \mathbf{e}^T x = \theta\}$ and x_c is a point in dom(f). In the dock reallocation problem, define $f(x) = \min\{\sum_{i=1}^n c_i(d_i, b_i) | d + b = x, \; \mathbf{e}^T b \leq B, \; d, b \in \mathbb{Z}_+^n\}$ with $\text{dom}(f) = \{x \in \mathbb{Z}_+^n | \mathbf{e}^T x = D + B, \; l \leq x \leq u\}$. Shioura (2018) shows that $f(x)$ is M-convex, which means that problem (23) can be reformulated as a special case of problem (MML1). For the general problem (MML1), Shioura (2018) provides a variant of the steepest decent algorithm, and shows that a minor modification of his algorithm turns out to be the steepest descent algorithm provided in Freund et al. (2017). Moreover, Shioura (2018) shows that problem (MML1) can be reduced to an unconstrained M-convex function minimization problem which can be solved by a fast proximity scaling algorithm. Based on this, he proves that problem (MML1) can be solved in $O(n^6 \log^2(\gamma/n))$ time.

We now provide an alternative proof of the multimodularity of the number of out-of-stock events under sequence X, $c_i^X(d_i, b_i)$. For this purpose, let $f_t(d_{it}, u_i)$ be the number of out-of-stock events to occur starting from the time immediately before the tth customer's arrival with d_{it} open docks available until the end of the day. Recall u_i is the number of total docks. Upon the arrival of event X_t, an out-of-sock occurs if $d_{it} = 0$ and $X_t = -1$, or $d_{it} = u_i$ and $X_t = 1$. Thus,

$$
f_t(d_{it}, u_i) = |(d_{it} + X_t) - (d_{it} + X_t)^+ \wedge u_i| + f_{t+1}((d_{it} + X_t)^+ \wedge u_i, u_i).
$$

We claim that $f_t(d_{it}, u_i)$ can be derived by the following dynamic programming recursion:

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$$f_t^r(d_{it}, u_i) = \begin{array}{ll} \min & |(d_{it} + X_t) - z| + f_{t+1}^r(z, u_i) \\ s.t. & z \in [0, u] \cap \mathbb{Z}. \end{array}$$

and $f_{s+1}^r(d_{i,s+1}, u_i) = 0$. That is, $f_t(d_{it}, u_i) = f_t^r(d_{it}, u_i)$. To see this, we observe that $f_t^r(d_{it}, u_i)$ is a relaxation of $f_t(d_{it}, u_i)$ in the sense that we now have the flexibility of changing the number of open docks to keep upon the arrival of an event while incurring a unit cost for each change unnecessary from a myopic perspective. For example, if $d_{it} = 0$ and $X_t = -1$, an out-of-stock event occurs. The relaxed problem allows to set $z = 0$ which captures exactly the out-of-stock event and incurs a unit cost or set $z > 0$ which incurs a unit cost for out-of-stock plus a unit cost for each bike removed from an occupied dock to open up docks for future usage. It is not hard to show that the optimal solution of the dynamic programming is to set $z = (d_{it} + X_t)^+ \wedge u_i$, that is, if a change is unnecessary myopically, then it is unnecessary for the long run (one can simply use the fact that $|f_t^r(d_{it}, u_i) - f_t^r(d_{it} \pm 1, u_i)| \le 1$). One can show by induction that $f_t^r(d_{it}, u_i)$ is L$^\natural$-convex in (d_{it}, u_i) using Proposition 3.7, and thus $c_i^X(d_i, b_i) = f_0^r(d_i, d_i + b_i)$ is multimodular in (d_i, b_i) by the relationship between L$^\natural$-convexity and multimodularity. Our proof builds upon the basic preservation properties presented in the previous section and seems to be more intuitive than that directly using the definition of multimodularity.

4.3.7. Congestion Games on an Extension-Parallel Network. Let $G = (V,A)$ be a directed graph with a sink and a source, where V is the set of nodes and A is the set of arcs. Graph G is called an extension-parallel network if it is constructed by starting from finitely many copies of single-edge graph with a source and a sink, and repeatedly performing operations of source/sink extension and parallel join defined below. Given a network G, a source extension of G is a new network G' constructed by adding an arc a to G in a way that the head of a merges with the source of G and the tail of a becomes the source of G'. The sink extension is defined accordingly. Given two networks G_1, G_2, a parallel join of G_1, G_2 is a new network G' constructed by merging the sources of G_1, G_2 to form a source of G' and merging the sinks of G_1, G_2 to form a sink of G'.

Given an extension-parallel network $G = (V,A)$ with a source s and sink t, consider a congestion game with n players and A being the set of resources. The strategy set of player i, denoted by $\mathcal{P}_i$, is a set of elementary directed paths from source s to sink t (st-path) in graph G. For every resource $a \in A$, the congestion cost function on a, denoted by $c_a : \mathbb{Z}_+ \to \mathbb{R}_+$, is a nondecreasing function with $c_a(0) = 0$. Given a strategy profile $\mathbf{P} = (P_i : i \in [n])$, the congestion on resource a is defined as $v_{\mathbf{P}}(a) = |\{i \in [n] | a \in P_i\}|$, that is, the number of players using resource a, and the cost of player i is given by $\pi_i(\mathbf{P}) = \sum_{a \in P_i} c_a(v_{\mathbf{P}}(a))$, that is, the total congestion costs of the resources that player i uses. Note that any congestion game is a potential game with potential function $\Phi(\mathbf{P}) = \sum_{a \in A} \hat{c}_a(v_{\mathbf{P}}(a))$, where $\hat{c}_a(k) = \sum_{i=1}^k c_a(i)$ (see Monderer and Shapley 1996). That is, for any player i, $\pi_i(P_i, \mathbf{P}_{-i}) - \pi_i(P_i', \mathbf{P}_{-i}) = \Phi(P_i, \mathbf{P}_{-i}) - \Phi\ (P_i', \mathbf{P}_{-i})$ for any $P_i, P_i' \in \mathcal{P}_i$, where $\mathbf{P}_{-i} = (P_j : j \neq i)$, that is, given any two action profiles which differ only at the action of player i, the difference of player i's utility between these two profiles is given by the difference of the potential function values between these profiles. Fotakis (2010) shows that for any symmetric congestion game on an extension-parallel network with $\mathcal{P}_i$ being the set of all st-paths, the following best-response algorithm generates a pure Nash equilibrium in n steps.

Best-response algorithm:

Step 1. Starting from any strategy profile $\mathbf{P}$. Let $(i_1, \ldots, i_n)$ be any permutation of $\{1,\ldots,n\}$
Step 2. For $i = i_1, \ldots, i_n$, let $\mathbf{P} \leftarrow (\mathbf{P}_{-i}, \hat{P}_i)$, where $\hat{P}_i$ is a minimizer of $\Phi(\mathbf{P}_{-i}, Q)$ over $Q \in \mathcal{P}_i$.
Step 3. Output $\mathbf{P}$.

Note that in Step 2, the minimizer $\hat{P}_i$ is the best response of player i given the strategies $\mathbf{P}_{-i}$ of other players by the definition of potential functions. Interestingly, by observing the close relationship between this potential function $\Phi(\mathbf{P})$ and M-convexity, Fujishige et al. (2015) point out that the above best-response algorithm can be derived from the steepest descent algorithm for M-convex function minimization problems mentioned in section 3. To see the relationship between M-convexity and Φ, denote by $\mathcal{Q}_a$ the set of st-paths containing arc a. One can show that for an extension-parallel network, $\{\mathcal{Q}_a : a \in A\}$ forms a laminar family. Denote by $\mathcal{P}_{all}$ the set of all st-paths and treat each strategy profile $\mathbf{P}$ as a vector $x_{\mathbf{P}} \in \mathbb{Z}^{\mathcal{P}_{all}}$, where the value of $x_{\mathbf{P}}$ at coordinate $Q \in \mathcal{P}_{all}$ is the number of st-path Q in $\mathbf{P}$. With these notations, $\Phi(\mathbf{P}) = \sum_{a \in A} \hat{c}_a(v_{\mathbf{P}}(a)) = \sum_{a \in A} \hat{c}_a(\sum_{Q \in \mathcal{Q}_a} x_{\mathbf{P}}(Q))$, which can be regarded as a function $\tilde{\Phi}(x_{\mathbf{P}})$ over $\mathbb{Z}^{\mathcal{P}_{all}}$. Note that the effective domain of $\tilde{\Phi}(x_{\mathbf{P}})$ is contained in $\{x | \mathrm{e}^T x = n\}$ as each of the n players selects one st-path. Since $\hat{c}_a$ is univariate convex and $\{\mathcal{Q}_a : a \in A\}$ is a laminar family, $\tilde{\Phi}(x_{\mathbf{P}})$ is laminar convex and thus M-convex.

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5. Conclusion

In this survey, we have reviewed key concepts and fundamental properties of discrete convex analysis, with a focus on L$^\natural$-convexity, M$^\natural$-convexity and their variants. We show that these notions are natural abstractions of essential structures of many operations models. We show that as a consequence, structural analysis and algorithm design for many operations models, ranging from inventory management and revenue management to healthcare scheduling and bike sharing, can be simplified, enhanced, and generalized by focusing on discrete convexity, especially L$^\natural$-convexity and M$^\natural$-convexity.

To conclude this review, we point out that discrete convex analysis is a growing area of operations management, with new concepts and analysis continuing to be introduced. The new developments in turn calls for further efforts to extend the problem scope and cultivate deeper understanding of the approach. For example, Zipkin (2016) introduces the concept of cover-L$^\natural$-convexity, illustrates its use to analyze some ATO systems with special structures, and also follows up with a list of intriguing questions for future research. Chen and Li (2020) propose a new concept, which they refer to as substitute concavity, to capture the gross substitutability property in continuous spaces and develop monotone comparative statics results under conditions weaker than M$^\natural$-concavity. A weaker version of SSQM$^\natural$-concavity, referred to as SSQM$^\natural_w$-concavity, applies to a portfolio selection model developed by Chade and Smith (2006). An even weaker condition, SSQM$^{\natural\neq}_w$-concavity, features in a constrained assortment problem of Chen and Simchi-Levi (2017). It would be interesting to see how these concepts can be used to facilitate the design of efficient algorithms or the development of structural properties.

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Notes

[1]Note that in Murota (2003): f is a univariate discrete convex function if $f(x-1)+f(x+1) \geq 2f(x)$ for all $x \in \mathbb{Z}$. Without the assumption that the domain dom(f) is a set of consecutive integers, it includes functions that are not integrally convex. For example, $f(-2)=f(2)=0, f(x)=+\infty$ on other points.
[2]Note that in a continuous space, lower semi-continuity of f and submodularity of $f(x-\xi e)$ imply convexity of f, a condition explicitly imposed in Murota (2003)'s definition. Since lower semi-continuous convex functions are equivalent to closed convex functions, the L$^\natural$-convex functions defined in Definition 2.1 are closed L$^\natural$-convex functions. In this study, all L$^\natural$-convex functions refer to closed L$^\natural$-convex functions unless otherwise specified. Also note that a function defined in an integer space is automatically lower semi-continuous.
[3]Again a discrete function is automatically lower semi-continuous, and in a continuous space the lower semi-continuity combined with the exchange condition implies convexity, a condition explicitly imposed in Murota (2003)'s definition. The functions in continuous variables defined in Definition 2.2 are closed M$^\natural$-convex functions. In this study, all M$^\natural$-convex functions refer to closed M$^\natural$-convex functions unless otherwise specified.
[4]See Hochbaum et al. (1992) for a tardiness scheduling problem with a quadratic M$^\natural$-convex objective function.
[5]The equivalence is pointed out by Moriguchi and Murota (2018) only for discrete functions. Its extension to real spaces is straightforward.
[6]It means that the example of L$^\natural$-convex sets given in section 2.2 is exhaustive.
[7]The proof of Proposition 3.13 follows directly from the definition of multimodularity.
[8]If f is not lower semi-continuous, then $f(x_1, x_2)$ being M$^\natural$-convex does not imply $f(-x_1, x_2)$ being L$^\natural$-convex. For example, $f(x_1, x_2) = \delta_B(x_1, x_2)$ with $B = \{(x_1, x_2) | x_1 + x_2 = 0, x_1 \neq 0\}$ is M$^\natural$-convex but $f(-x_1, x_2) = \delta_{\tilde{B}}(x_1, x_2)$ is not L$^\natural$-convex, where $\tilde{B} = \{(x_1, x_2) | x_1 = x_2, x_1 \neq 0\}$.
[9]In the operations management literature, $v \in T$ is typically referred to as a supply node if $x_v > 0$ and a demand node if $x_v < 0$.
[10]This result still holds if x satisfies a box constraint $x \in [l, u]$.
[11]Since each commodity has only one unit, a consumer's utility is a function of the set of acquired commodities.
[12]Any strictly convex function is essentially strictly convex. See section 26 of Rockafellar (1970) for more on essentially strictly convexity.

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