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High School Students' Use of Technology to Make Sense of Functions Within the Context of Geometric Transformations

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Abstract
Although geometric transformations are functions, few studies have examined students' reasoning about these two important concepts. The purpose of this study was to examine the various ways students reasoned about functions in the context of pre-constructed, dynamic sketches of geometric transformations. We found that, regardless of prior experience, all students were able to reason about important aspects of the notion of function through dragging. Specifically, by using the idea of the semiotic potential of the artifact (the dragging tool), we were able to examine ways in which students with different backgrounds reasoned about functions and how the use of the dragging tool and semiotic mediation contributed to their descriptions of geometric transformations and functions.

Keywords Transformations · Function · Covariation · Dynamic geometry environment

Functions have long been viewed as an essential and important part of mathematics curricula. They can be used to describe and reason about relationships between quantities that we find all around us. For example, functions can be used to determine the cost of fruit as a function of its weight in pounds or money earned in an interest-generating account as a function of time (Kalchman & Koedinger, 2005). Several researchers have advocated for focusing on the ways quantities change to assist children in reasoning about linear and non-linear functions later on (Blanton & Kaput, 2011; Confrey, 1991; Confrey& Smith, 1995; Kaput, 1994). In articulating

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the essential understandings of the function concept for high school students, Cooney, Beckmann, and Lloyd (2010, p. 8) noted that it is essential that students understand:

a) Functions are single-valued mappings from one set—the domain of the function—to another, its range.
b) Functions apply to a wide range of situations. They do not have to be described by any specific expression.
c) The domain and range of functions do not have to be numbers.

For the past 40 years, the National Council of Teachers of Mathematics (NCTM) in the USA has encouraged the study of functions in school mathematics (NCTM, 1980, 1989, 2000). Research, however, suggests that students struggle to understand this important mathematical concept (e.g., Arnon et al., 2014; Breidenbach, Dubinsky, Hawks & Nichols, 1992; Carlson, 1998; Kjeldsen & Petersen, 2014; Sfard, 1992; Thompson & Carlson, 2017). "Two perspectives are generally taken in a functions approach to school algebra – a correspondence perspective and a coordination/covariation perspective" (Stephens, Ellis, Blanton & Brizuela, 2017, p. 398). Both of these perspectives tend to focus on quantities and numbers.

Although students often have early experiences with geometric transformations (e.g., reflections, rotations, translations), these experiences are rarely used to develop students' understanding of functions, even though transformations are functions (Martin, 1982). Geometric transformations provide opportunities for teachers to have explicit conversations with students about important function ideas, such as independent/dependent variables, mapping and variation without the use of numbers, and algebraic symbols or graphs. In fact, in the USA, standard G-CO2 of the *Common Core State Standards for Mathematics* (CCSS-M) states that students should "describe transformations as functions that take points in the plane as inputs and give other points as outputs" (CCSS-M, 2010).

Many have argued for the importance of providing students with concrete experience of objects that they can later describe using more abstract mathematical symbols, language, and models (e.g., Freudenthal, 1973; Piaget, 1952). However, interactions with technology are different from how one interacts with physical objects (Papert, 1993). In the case of dynamic geometry programs, the software is designed with mathematical rules so that its behavior is constrained by the underlying mathematics. Hence, as students work to control (i.e., drag) and understand a construction that embodies a function, they are discovering important mathematical relationships. Interactions with these technology tools may create new images and semiotic tools to reason about mathematical concepts (Abrahamson, 2009). The purpose of this study was to understand the various ways in which students' interactions with pre-constructed dynamic sketches related to the ways in which they reasoned about the notion of function.

Theoretical Framework: the Theory of Semiotic Mediation

Tools have been described in different ways. In particular, Vérillion and Rabardel (1995) make a distinction between artifact and instrument and refer to "the artifact, as a man-made material object, and the instrument, as a psychological construct" (p. 84). An instrument includes the tool and all of the ways a person thinks about how to use it. These are referred to as utilization schemes. While these are mental constructs that guide how a user approaches and employs an artifact, they are not directly observable: techniques are the observable interactions between the user and the artifact (Drijvers, Doorman, Boon, Reed & Gravemeijer, 2010). These interactions influence how one thinks.

According to Vygotsky (1978), signs and tools have mediating functions in orienting human behavior. Mediation is used for "referring to the potentiality of fostering the relation between pupils and mathematical knowledge, and mostly related to the accomplishment of a task" (Bartolini Bussi & Mariotti, 2008, p. 752). On the one hand, tools are externally oriented as they are auxiliary means for a mediated activity, in particular, "the tool's function is to serve as the conductor of human influence on the object of activity" (Vygotsky, 1978, p. 55). On the other hand, a sign is, for him, "a means of internal activity aimed at mastering oneself; the sign is internally oriented" (p.55). Through social and cultural interactions, a subject's use of a tool often changes drastically. For example, interacting with individuals in a cultural and social environment on a mediated activity (e.g., choosing from a range of options), the external tool-mediated actions become internalized. Vygotsky refers to this operation as an internalization and defines it as, "the internal reconstruction of an external operation" (p. 56).

It is important to note that "the process of internalization occurs through semiotic processes, in particular by the use of a semiotic system in social interaction" (Falcade, Laborde & Mariotti, 2007, p. 321). Signs generated by using a specific tool utilized to accomplish a task with a more capable peer leads to producing the externally oriented tool as a new psychological tool. When this occurs, the externally oriented tool is used as an instrument of semiotic mediation, which "sees knowledge-construction as a consequence of instrumented activity where signs emerge and evolve within social interaction" (Mariotti, 2009, p. 428).

Drawing upon Vygotsky's semiotic lens on signs and tools, Bartolini Bussi and Mariotti (2008) emphasize the process of producing signs with the use of an artifact to portray the emergence of new meanings in a mathematics community under the supervision of an expert (e.g., a teacher) or more capable individual (e.g., a more advanced peer). A semiotic relationship emerges through social and cultural interactions among individuals who have a common goal of accomplishing a task with the use of an artifact.

As Bartolini Bussi and Mariotti underscore, "personal meanings are related to the use of the artifact, in particular in relation to the aim of accomplishing the task; on the other hand, mathematical meanings may be related to the artifact and its use" (p. 754). Those meanings that evoke a double semiotic relationship with the artifact refer to the *semiotic potential of an artifact* in the theory of semiotic mediation. Designing and executing a didactical intervention, the teacher as an expert has an important role in identifying the semiotic potential of the artifact at stake to exploit it as a tool of semiotic mediation to guide students to produce *mathematical signs* (definition, proof, mathematical conclusion, generalization, etc.) to accomplish the didactical goals of the task.

In a didactical design, when students are given a task that requires use of an artifact, they most often produce signs foregrounding the use of that artifact or making an action that is accomplished with it. These researchers refer to those signs as *artifact signs*.

Under the teacher's supervision, the didactical goals of a lesson involve abandoning those artifact signs that can progress into producing mathematical signs in a mediated activity. This progression takes place in a didactical cycle containing a semiotic activity, individual, and collective production of signs.

For the existing study, we examined the personal meanings students applied to and generated about function and geometric transformations as they interacted with a sequence of dynamic geometry tasks either in pairs or in a small group that was facilitated by their teacher. However, with a focus on the semiotic potential of the dragging tool in dynamic geometry software (DGS), we are interested in the ways in which various students reasoned about the notion of function and geometric transformations.

The Semiotic Potential of the Dragging Tool in Dynamic Geometry Software: the Notion of Function

Dynamic geometry programs (e.g., *The Geometer's Sketchpad*, *Cabri-Géomètre*) enable students to drag primitive objects—such as *freehand* points (also known as *basic* points or *free* points)—which automatically update global or local dynamic behavior of other objects that are dependent on them. Such dynamism and haptic experiences through dragging allow students to use perceptual data of the moving objects as evidence to make a conjecture or conclusion about the preserved features (Mariotti, 2014). Falcade et al. (2007) identified two types of motions through the utilization of the dragging tool in DGS: direct and indirect motion. The direct motion dragging scheme refers to dragging a freehand point that may influence the motion of some other geometric objects via the movement of the freehand point. The indirect motion scheme refers to the dependency of an object on another geometric object in a construction in DGS. Accordingly, the dependent object's motion is influenced by the motion of the independent object (Bartolini Bussi & Mariotti, 2008). Students may approach the hierarchy of dependency tasks in reverse order, attempting to grasp dependent objects to make a change in the independent objects (Talmon & Yerushalmy, 2004).

Prior research indicates that students' use of the dragging tool works as a semiotic mediator which progresses into construing mathematical meanings in the process in which an externally oriented tool becomes a mathematical sign (Falcade et al., 2007; Ng & Sinclair, 2015). In particular, under the supervision of a mathematics teacher, Falcade et al. (2007) and Mariotti (2009) both outline how tools in DGS (e.g., the dragging tool, the trace tool) enable students to shape new mathematical meanings, with a focus on covariation (e.g., dependency between dependent and independent motions), as well as the domain and range of functions. For example, Falcade et al. (2007) reported that students used the trace tool to reify the domain and range of a function as mathematical signs. The authors described the mediating role of dragging and trace tools to exploit mathematical signs as follows:

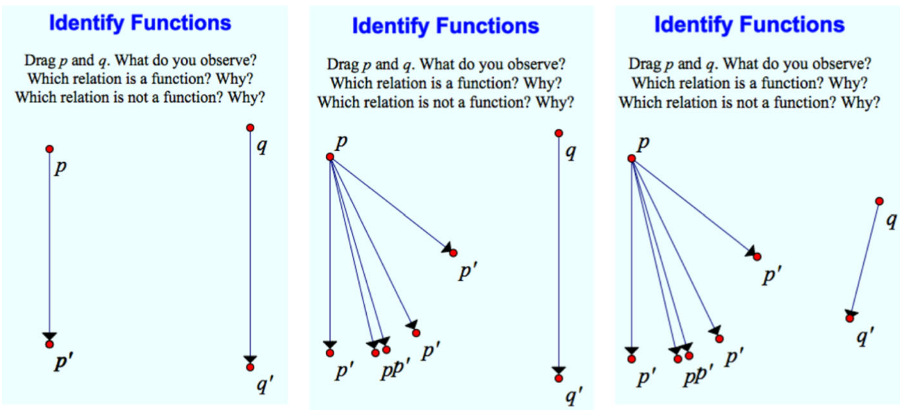
> The Dragging tool may be considered as a sign referring to the idea of function as covariation between dependent and independent variables. […] Personal meanings concerning the idea of variation and covariation as they emerge from students' activities in the Cabri environment, through the combined use of the

Dragging tool and the Trace tool, may evolve into the mathematical meaning of function. (pp. 321–322)

There are several task designs such as dynagraphs in DGS (e.g., Antonini, Baccaglini-Frank & Lisarelli, 2020) that elicit students' understanding of the notion of function. Since we are interested in characterizing students' idiosyncratic meanings about function and geometric transformations, we outline the semiotic potential of the dragging tool in a dynamic geometry task design that allows for applying geometric transformations to specific points (inputs) to create image points (outputs). As shown in Fig. 1a, this task design contains sets of points and entails dragging freehand points to identify functions. The graspable (arrow-selectable) freehand points are labeled with lowercase letters, and their mappings as non-graspable points are labeled with their corresponding letters with a prime, which gives a visual cue for students about dependent and independent objects.

Such a task design prompts students to drag freehand points and entails using specific dragging modalities in DGS to identify if the relations are functions. Inevitably, at first, the task requires students to drag freehand points randomly in the plane to decode information—that is known as *wandering dragging* (Arzarello, Olivero, Paola & Robutti, 2002). Working with moving points on the screen is expected to evolve into producing mathematical signs, namely, attending to the aspects of function (Carlson, 1998). For example, in Fig. 1b, it is evident that p is mapped to the multiple p-primes, which violates the definition of function. On the other hand, q is mapped to exactly one point (q') as shown in Fig. 1c. Then, the task design capitalizes on the aspect of distinguishing functions from non-functions.

The task design may lead students to reason about the domain and co-domain/range of functions. For example, from a transformational view, the student may notice that the freehand point can be moved anywhere in the plane, so the domain is the plane. Accordingly, the behavior of the dependent points can be characterized based on the movement of the freehand point. Alternatively, from a functional view, the student may view points and their mappings as ordered pairs. Then, the Cartesian plane is brought to the fore, in which the domain is $\mathbb{R}^2$.



(a) (b) (c)

Fig. 1 **a** The Identify Functions task; **b** point p is dragged; **c** point q is dragged

Bartolini Bussi and Mariotti (2008) emphasize that students most often produce artifact signs (or hybrid signs as a blend of artifact and mathematical signs) when they use the dragging tool of DGS to observe interrelated motions. For example, without referring to dependent and independent variables, the student may focus on the motion of the points such as “the point moves” (see Bartolini Bussi & Mariotti, 2008). The abandonment of the artifact signs that progresses into producing mathematical signs to accomplish the task does not occur spontaneously or immediately. In a didactical cycle, this meaning-making process is supported and guided by an expert (e.g., a mathematics teacher) who is aware of the semiotic potential of the artifact at stake.

Geometric Transformations and Understanding Transformations as Functions in DGS

Geometric transformations typically studied by secondary students include translations, reflections, rotations, and dilations. In a dynamic geometry environment, students can drag the input points and explore properties of geometric transformations. These properties might include whether the transformation preserves lengths of segments and measures of angles or how the input and output points behave relative to each other. For example, students may observe that each input point is mapped to exactly one output point. They may also notice that, for some transformations, the input and output points coincide at the same point (e.g., at the center of rotation), at infinite points (e.g., along the line of reflection as shown in Fig. 2 (a–d)) or at no points (e.g., when translated using a non-zero translation vector).

Much research has examined how the dragging tool is used by students as they reason about geometric transformations (Falcade et al., 2007; Hollebrands, 2003, 2007; Ng & Sinclair, 2015). For example, Ng and Sinclair (2015) characterized young children’s use of the dragging tool in *The Geometer’s Sketchpad* (GSP) as a sign accounting for functional dependency between pre-image and image figures created by geometric transformations. The dragging tool served as a dynamic mediator to help students relate the distance between the interrelated points from the line of reflection on a square grid. The children produced spoken texts to describe the movement between

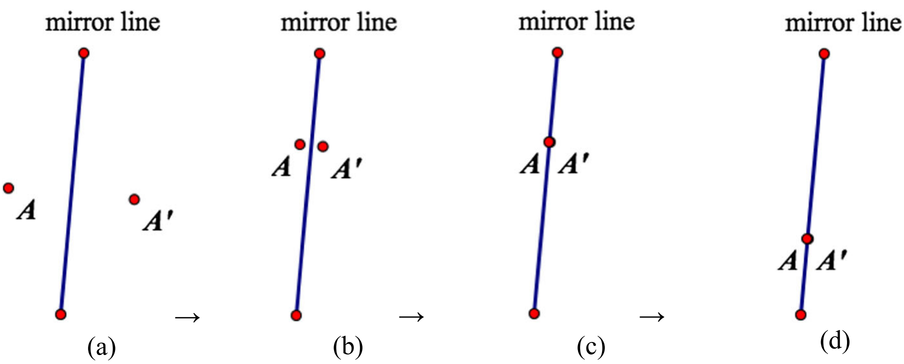


Fig. 2 (a) Point A is reflected over the mirror line; (b) the pre-image point is moved towards the mirror line; (c) the pre-image and image points coincide on the mirror line; (d) they coincide at a different location on the mirror line

pre-image and image figures and employed gestures to indicate how the objects were related. They produced diagrams denoting arrows that signified interrelated motions as an invariant property under reflection and employed gestures to indicate movement between pre-image and image figures. Ng and Sinclair emphasized that “the children’s gestures and arrows acted as their own use of signs to communicate the movement of the squares. These gestures and visual mediators operated effectively as signs in the absence of the computer to communicate the symmetric movement” (p. 430).

When students reason about a transformation, visual cues of geometric objects (e.g., orientation of shapes, preservation of lengths) help students understand how the pre-image and image figures are related (Hollebrands, 2003, 2007; Ng & Sinclair, 2015; Yao & Manouchehri, 2019). For example, Hollebrands (2007) found that high school students used a proactive strategy decoding the visual cues of the pre-image and image figures that was followed by testing and verifying a conjecture. On the other hand, some students used a reactive strategy making use of dragging objects (e.g., points) to observe how the given figures were related and then made a conjecture. Thinking about transformations using a formal definition of function may not be an easy task for students (Hegg, Papadopoulos, Katz & Fukawa-Connelly, 2018). As Hegg et al. (2018) stated, “thinking about a transformation as moving a figure in the plane may be intuitive and sensible, but to formalize that notion requires defining a function that has all of $\mathbb{R}^2$ as both the domain and range” (p. 57).

When students are engaged in a transformation task in DGS, they may focus on different aspects of function such as interpreting invariances with the use of the dragging tool. Dragging freehand points randomly in the plane, students may attend to what is fixed as the independent variable is manipulated and conjecture about the dynamic behavior of the dependent variable. For example, through wandering dragging, students may observe that the input and output points of a reflection coincide at one point (Fig. 2 (a–c)). From there, through *maintaining dragging* (Baccaglini-Frank & Mariotti, 2010), they may make an intentional dragging along the path of the mirror line to maintain the property (Fig. 2 (d)), namely, keeping the input and output points coincided. More broadly, different modes of dragging objects in dynamic geometry software (DGS) may help students to reason about interrelated motions (e.g., covariation) as a means to exploit mathematical signs concerning different aspects of function characterized by Carlson (1998).

Using the affordances of the dynamic capabilities of DGS, researchers have investigated how their features mediate students’ understanding of the underlying function concept in transformations (e.g., Falcade et al., 2007; Hollebrands, 2003). For instance, Falcade et al. (2007) examined how high school students reasoned about functional relationships and covariation in a Cabri environment. The researchers reported that the dragging and trace tools were utilized as psychological tools to accomplish the tasks. Students dragged objects whose functional dependency and covariational relationships were concealed in tasks and characterized how the dependent objects behaved upon dragging the freehand point.

They report that the use of the dragging and trace tools shifted towards identifying the nature of the dragging motion (e.g., its dependency, independency) and functions (e.g., its domain, range). The teacher played an important role in the emergence of mathematical meanings. In particular, she guided “students to abandon the reference to the artifact context, selecting specific qualities from the use of the artifact to be transferred to the mathematical context" (Mariotti, 2009, p. 437). The artifact context refers to the work students were completing with the technology (e.g., dragging, tracing, measuring). The teacher played an important role in assisting students in observing how a particular invariance they noticed in the artifact context (e.g., diagonals of a square are equal in measure) was related to the geometrical properties of the mathematical object.

Understanding the domain of the function is one of the key factors in understanding transformations as functions. As Steketee and Scher (2018) pointed out, transformational approaches to functions with sensorimotor experiences were found to be a helpful strategy in thinking about "what it means to apply a function all at once to an entire set of points (a polygon)" (p. 68). Hollebrands (2003) found that high school students developed a deeper understanding of transformations as functions through a 7-week instructional unit using GSP. While students learned about translations, reflections, rotations, dilations, and their compositions, they also used function terminology that included domain, range, parameter, input, and output. Yet, the researcher reported that students initially thought of the domain as containing geometric objects (labeled points, all points on a geometric shape, etc.) in transformation tasks. Also, Yanik and Flores (2009) found that describing a transformation as a function that includes all points in the plane was difficult for their interviewee (a college student). Students may not attend to geometric figures as parts of the plane in a transformation task and have difficulty conceiving a transformation, "as a mapping of all points in the plane to other points in the plane" (p. 54). This view of a transformation as a mapping requires dragging freehand points on the screen to feel motion dependency without being limited to the labeled points given in the task, which progresses into characterizing the transformation (preserving lengths, measures of angles, etc.), as well as attending to the different aspects of function (e.g., domain and range).

Design of the Instructional Activities

The instructional activity used in this study, *Identify Functions*, was designed by Steketee and Scher (n.d.) and was built to capitalize on students' use of dragging and tracing to make sense of function concepts, ultimately developing a robust definition of function. The pre-constructed sketches use examples of geometric relations—some functions and some not—as a context for students to explore, describe, and define characteristics of functions. The use of pre-constructed sketches (i.e., students do not creating the mathematical objects of investigation themselves) allow us to provide students with specific mathematical objects to explore.

The *Identify Functions* activity involves a series of 11 pages of pre-constructed GSP sketches and a handout with directions and questions. The first page introduces students to independent and dependent variables as points in the plane and asks students to explore relationships among them (see Fig. 3). Students are able to drag independent variables freely, but dependent variables can only be changed by dragging the corresponding independent variable. Here, each independent variable is mapped to a dependent variable using a geometric transformation allowing students to consider not only independent and dependent relationships but also ways in which dragging the independent variable affects the movement of the dependent variable (i.e., its relative direction and speed).

Q1 By dragging, determine which points are related. Then list the independent and dependent points, and describe the relationship.

Independent Point	Dependent Point(s)	Relationship
→		

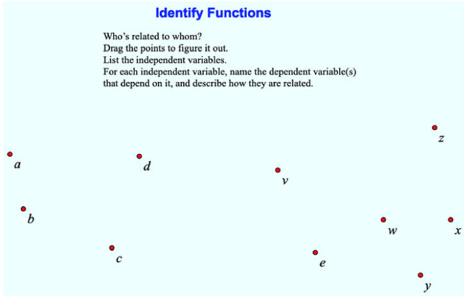


Fig. 3 First page of the *Identify Functions* activity

Next, there are two pages that each include two constructions, one labeled function and one labeled non-function (e.g., see Fig. 4). Students are asked to explore each example by dragging the independent variable and observing the dependent variable, describing similarities and differences in their behavior. Note that, when students first open each page, the independent and dependent variables are located along a vertical line on the plane with an arrow between them indicating the mapping, e.g., $x \rightarrow x'$. Figure 4 shows what students would see if they dragged x vertically towards x' and y to the left. On the fourth page, the examples are no longer labeled, and students are asked to determine which relation is a function, which is not, and to justify their decision. Finally, based on the examples they explored on pages 2 through 11, students are prompted to "describe in your own words what a function is."

The dynamic pre-constructed sketches were developed to promote an understanding of function that includes dynamic characteristics like speed, direction, relative position, and rate of change. We anticipated students would describe functions as those relations that map an independent variable to a unique dependent variable, notice how independent and dependent variables co-varied when dragged, and recognize a particular function as one that behaves in a predictable manner.

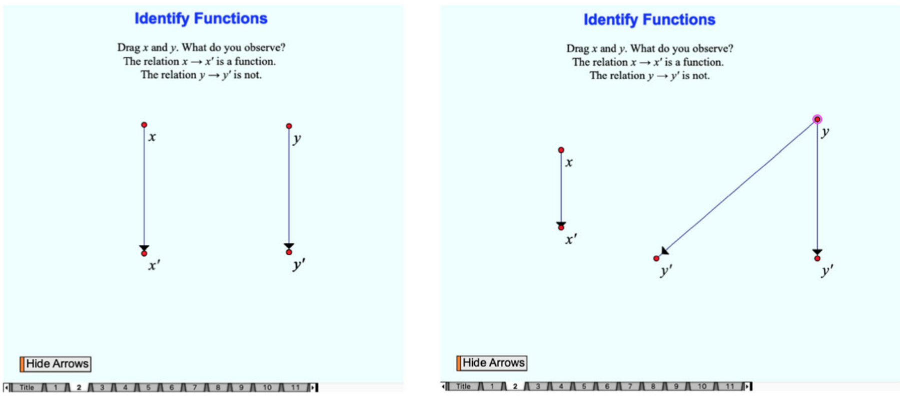


Fig. 4 Page two of the *Identify Functions* activity: On the left is how it looks when it is opened; on the right, x has been dragged vertically closer towards x', while y has been dragged to the left

Methods

Our goal was to understand the various ways in which students' interactions with pre-constructed dynamic sketches of geometric transformations related to the ways in which they reasoned about function. Using a similar semiotic analysis to that reported in Mariotti (2014), the roles of the teacher as a semiotic mediator are not foregrounded in this report. A collective case study design (Stake, 2005) was used, where each case represented students with different mathematical backgrounds, as we anticipated that we would learn something different about possible student interactions with geometric transformations as functions. The overarching research question that guided the current study was: How do students use the dragging feature in a dynamic geometry program as they attend to particular aspects of function? This encompassed two sub-questions:

a) How do students use the dragging feature as they engage with the *Identify Functions* tasks?
b) What aspects of function to students' attend to as they engage with the *Identify Functions* tasks?

Participants

Eleven students (seven males and four females), attended a 90-min, out-of-school, instructional session during which they engaged with the *Identifying Functions* tasks. Students who attended the session were from the same large school district in the Southeast USA, were all 15–16 years old, and varied from having just completed a first course in algebra (typically taken by 13–14-year-olds in 8th or 9th grade) to pre-calculus (typically taken by 16–17-year-olds in 11th or 12th grade). As a result, all of the students had previous experience with relations and function, but the depth of those experiences varied. The curriculum within the school district aligned with the *Common Core State Standards for Mathematics* (CCSS-M, 2010), meaning that all students had been taught a correspondence definition of function—"understand that a function is a rule that assigns each input exactly one output" (p.55). None of the students had much experience using dynamic geometry.

The students were placed into groups of two to three, with those having similar mathematical backgrounds working together and representing a case. Each group had a laptop computer that they shared. Three groups were selected to serve as case studies. One case included two students who had just completed a first algebra course (typically taken when 13–14 years old), the second case included two students who had completed a second-year algebra or geometry course (typically taken when 14–15 years old), and the third case included three students who had just completed a pre-calculus course (typically taken when 16–17 years old). The three cases (see Table 1) illustrate the ways that students with different experiences with function might make sense of function concepts introduced from a transformational perspective. Further details about each case are presented in the results.

Table 1 Description of cases

Case	Most recent mathematics course completed and experiences with functions and transformations
Caleb and Stan	First-year algebra course; had been taught a correspondence definition of function and the key characteristics of linear and quadratic function families
Jayden and Mason	Geometry/second-year algebra course; had been taught a correspondence definition of function, geometric transformations, and key characteristics of various polynomial and exponential function families
Chloe, Tamara, and Sophia	Pre-calculus; familiar with the definition of function, key characteristics of various function families, geometric transformations, and key characteristics in relationships between functions and their derivatives

Context

The *Identify Function* tasks were designed to be implemented as an open investigation followed by a carefully orchestrated whole-class discussion. In the context of this study, which took place in an out-of-school setting, the session instructor (one of the designers of the *Identify Function* activity) carefully launched the task. Students were sitting in groups of two or three, sharing a laptop computer and an external mouse. In launching the task, the instructor addressed some of the key contextual features related to the technology itself, making sure that the students knew how to turn the pages at the bottom, that the independent points were draggable, and that students could and should drag them all over the plane to look for relationships.

The instructor noted that students should pay attention to the ways in which the points move in relation to each other and introduced the terms “fixed point,” “relation,” “independent,” “dependent,” “rate of change,” and “x-prime” while one student dragged the variable a on page 1 on the teacher’s computer, in order to provide some language to use as they discussed their investigation with their peers and to make sure they would understand the directions they encountered as they moved through the activity. Finally, the instructor provided explicit directions about the kinds of things to pay attention to and that the goal of the task is to eventually determine a definition of function. In doing so, he maintained the cognitive demand of the task and provided them just enough language to use in their descriptions of the mathematical relationships they should be paying attention to as they explored. Once the task was launched, the instructor monitored the small groups as they worked and asked probing questions to ensure they explained what they explored.

Data Collection

The data for this study included students’ written work, video recordings, and field notes from the 90-min, out-of-school, instructional session. One video camera was set up at the rear of the classroom to capture the whole-class instruction. This camera followed the instructor and captured his interactions with each student group, as well as the whole-class discussions. Since our goal was to understand the ways in which the students reasoned about function, we utilized the data from their small groups (not the whole-group discussions that occurred at the end of the session) as they worked through the pages in the pre-constructed sketch.

To capture the work each group of students did on the *Identifying Functions* tasks, we recorded their interactions with the dynamic sketches using a screen-casting software. In addition, a stationary camera was set up to the side of the students to capture their gestures as they discussed (and pointed to) the objects on their computer screens. All of the students' written work was also collected. This included their written responses to prompts on the activity sheet, as well as what they drew on scratch paper. Finally, a member of the research team was seated behind each group of students taking detailed field notes.

Analysis

All of the data was first processed by creating picture-in-picture videos, so both the screen-captured video and the stationary video could be seen at the same time (with the computer screen being full screen and the stationary camera view placed in the lower right corner). Next the video was transcribed, and transcriptions were added to the video file as captions. The final case video compilations were then uploaded to *Transana*, a qualitative data analysis tool (Woods & Fassnacht, 2018) for coding. Using *Transana* allowed us to examine the video recorded and transcribed data simultaneously, applying codes directly to the compiled video file.

In our coding process, we first utilized Powell, Francisco, and Maher's (2003) framework for identifying critical episodes and then within those episodes coded both for the ways in which the students were interacting with the technology and for the aspects of function to which they were attending. We created our codebook in an iterative fashion (DeCuir-Gunby, Marshall & McCulloch, 2011), drawing on both theory-driven and data-driven codes.

When attending to the ways in which the students interacted with the technology, we drew upon the literature related to modes of dragging (Arzarelloet al., 2002; Baccaglini-Frank &Mariotti, 2010; Laborde, 2005). In attending to the aspects of function for which the students were making sense, we used Carlson's (1998) description of the important aspects of function. Coding these provided us with a way to connect students' articulated reasoning about functions to their interactions (dragging modalities) with the dynamic representations. When codes emerged from the data (i.e., data-driven), we used a constant comparative method to ensure it was considered for all data. Our final codebook is shown in Table 2.

Once the codebook was complete, the team used the cases that were not included here to establish reliability in our code application. Once reliability was determined, all remaining data was coded independently by all members of the research team, and then any discrepancies were discussed until agreement was met (DeCuir-Gunby, Marshall, & McCulloch, 2011). Once coding was complete, in addition to examining the coded video data, we created multiple representations of the code applications and identified themes within and across code groups (i.e., dragging modes and aspects of function), as well as within and across cases.

Table 2 Description of codes

	Code	Description
Interaction with the technology	Wandering dragging	Randomly dragging points on the screen (Arzarello et al., 2002)
	Maintaining dragging	Drag in such a way as to maintain a particular property or invariant (Baccaglini-Frank, 2010)
	Dragging test	Move a draggable point to test whether or not sketch keeps the initial properties (Arzarello et al., 2002)
	Guided dragging	Dragging a point of a figure to make a particular shape (Arzarello et al., 2002)
	Drag non-draggable point	Repeatedly attempt to drag a non-draggable point
	Variable speed dragging	Dragging a point at variable speed to test its effect on dependent points
Aspect of function	Independent and dependent variables	Identifies, interprets, and/or characterizes independent and dependent variables (Carlson, 1998)
	Domain/range	Identifies and/or describes the domain and range of the function (Carlson, 1998)
	Function/non-function	Distinguish functions from non-functions (Carlson, 1998)
	Rate of chance/covariation	Reason about rates of change and covariation (Carlson, 1998)
	Multiple representations	Use one representation to make sense of another (Carlson, 1998)
	Local/global behavior	Consider local and global function behavior (Carlson, 1998)
	Function families	Recognize and make distinctions between families of functions (Carlson, 1998)
	Interpret invariances	Interpret invariances related to independent/dependent variables or rates of change (Carlson, 1998)
	Function notation	Use and make sense of function notation to create and interpret function rules (Carlson, 1998)

Results

Here we report on three cases: Caleb and Stan; Chloe, Tamara, and Sophia; and Jayden and Mason (all names are pseudonyms). Each of these cases represent groups of students with different prior experiences of function. Together, they provide insight into the ways students with different mathematical backgrounds might use different modes of dragging to make sense of function within the context of geometric transformations.

Caleb and Stan

During the 21-min exploration phase of the task, Caleb and Stan spent over 9 min working on the first page (Fig. 3), where they were prompted to identify and describe independent and dependent relationships among the points. This was considerably more time than the other groups spent on this part of the task. Influenced by the written prompts given in the tasks, they relied on the artifact context in the beginning of their exploration of the independent and dependent variables. The dragging that they used to complete this page of the task consisted primarily of wandering.

When their goal switched from identifying independent/dependent relationships to interpreting invariances, their dragging also changed from wandering to a drag test. For example, when using a wandering action to drag point *z*, Stan stated, "Actually, this one's a bit different, cause, uh, it moves around a fix … Caleb …[it] moves around a fixed point". Just before Stan showed Caleb how point *z* was different, his dragging switched from "wandering" to a "drag test" because he tested whether or not a fixed points *z* and *b* would coincide.

Caleb and Stan's work on this page of the task went beyond simply identifying the independent and dependent relationships and focused on interpreting invariances as well. For example, as Stan moved point *x*, he said, "Do [drag], the *x* one [prime], cause when you put them up and down or something. The *x*, like they would move the same way apart, like the same distance from a like the middle of the line." In this case, Stan observed how the two points were related by a reflection and wanted to drag them in such a way as to observe the equal distances the points were from the hidden mirror line.

In addition to invariances, Stan and Caleb also noticed and discussed covariation with respect to the independent/dependent point pairs on the first page. Their discussion around covariation was associated with a drag test action. For example, when testing the relationship between points *c* and *d*, Caleb stated, "*c* is the independent, *d* is the dependent…They stay the same distance apart…but move together" (see Fig. 5).

Although Stan and Caleb became purposeful in using the dragging tool to formulate and test their conjectures with a focus on invariances and covariation, they continued to make extensive use of the technology. When they moved on to the remaining pages of the task, they spent more time dragging than discussing the mathematics, and their dragging was still mostly wandering, although they did switch to use of the drag test when interpreting invariances and covariation. For example, when discussing the relation *aa'* on page 3 (Fig. 6), while performing a drag test, they had the following discussion with the teacher.

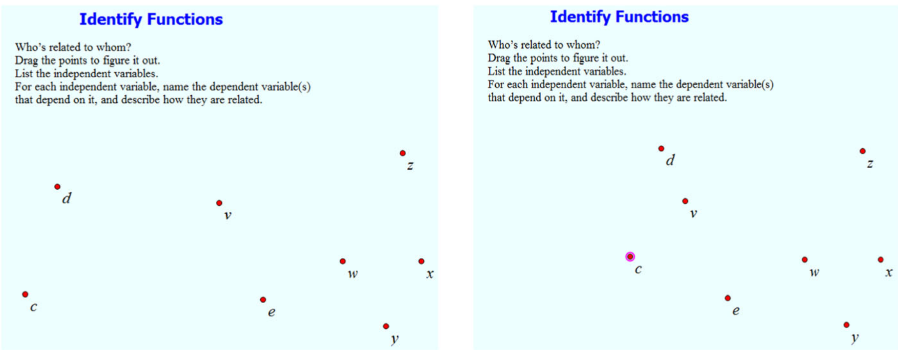


Fig. 5 Picture depicting the movement of point *c* and the change in location of point *d*

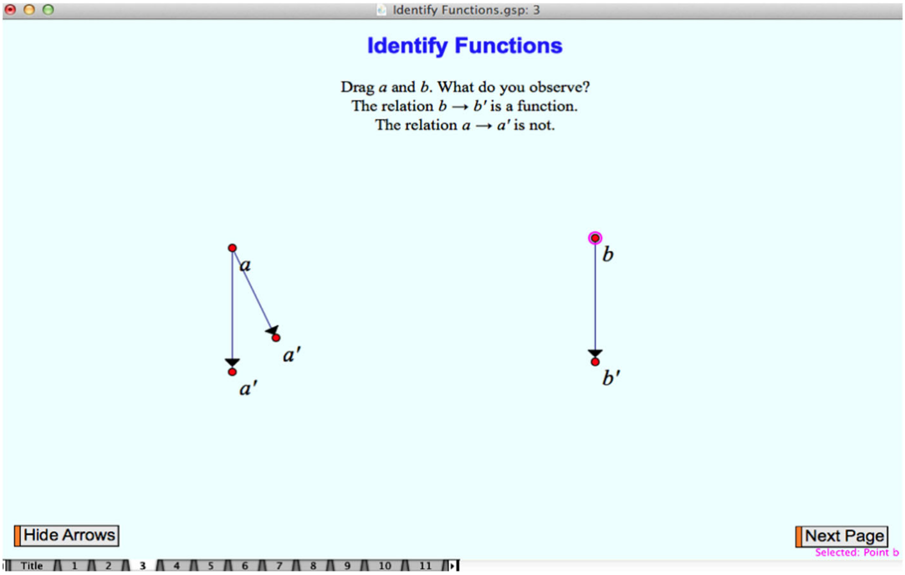


Fig. 6 Page 3 of the 'Identify Functions' activity

Stan: I'm trying to think of how to explain this like what is happening, cause one of the *a* ones is moving slower than the other *a* one when you move the top a.

Instructor: The *a*-primes you mean?

Stan: Yeah.

Instructor: Yeah, okay. So, one moves faster than the other.

Stan: Oh, wait, no, no. This one's fixed and this one's stuck.

Caleb: It's not stuck there.

Stan: That one's not is it…

Caleb:That's not stuck, it can move... But you wrote at first that this one moves slower...

Stan: But this one does not move at all, like it does not have any effect on it.

Caleb: Oh, you are right.

Stan: It's just stuck to it. Like it's fixed.

In this excerpt, Stan and Caleb experienced some difficulty at first with the mathematical language. They referred to *a'* as "*a* one," until the teacher introduced the term "*a*-prime." The fact that they were paying attention to how quickly one of the points moved relative to the other is evidence of their attention to covariation, or some beginning ideas of rate of change. In addition, they also discussed invariance. When they dragged *a*, the *a*' that is directly below *a* stayed the same distance from *a* and directly below it; this is what they were referring to when they said that *a'* was "stuck."

While Stan and Caleb were very focused on describing invariances they saw, they also correctly identified the function and non-function relations on each of the pages that they completed. They noticed differences among functions (based on behavior) but had trouble specifying the differences. Also, they described function aspects with reference to the artifact context, such as "fixed points" and "points that were stuck." The students used informal language to describe their observations of the points and their behavior. Another example of this comes from their discussion of the points on page 5 compared with those on page 4. Stan noticed that, for the function *qq'* on page 4, there was a fixed point that remained at the mid-point of the segment that connected *q* and *q'*. They compared this with the function *bb'* on page 5, where the fixed point is not at the mid-point of the segment but always located at *b'*.

Looking across the entire episode (see Fig. 7), we see that their dragging was primarily coded as "wandering," with "drag test" being their second most-often used dragging type. Looking at their mathematical codes, we see that interpreting invariances occurred most often, with identifying and interpreting independent/dependent relationships and covariation being the most frequent aspect of function to which they attended. They focused on invariances more than twice as often as they did independent/dependent relationships and covariation.

In summary, Stan and Caleb identified the independent and dependent variables at ease using wandering dragging and brought up the invariance aspect of function through the drag test modality. However, they had difficulty describing how these variables were related, so their work was drawn upon the artifact context. In this respect, their personal meanings of covariation with the use of the dragging tool were not associated with the nature of functions or transformations corresponding to dependent and independent variables.

Chloe, Tamara, and Sophia

Chloe, Tamara, and Sophia had all just completed a pre-calculus course. As such, this group had more experience than the other groups with different function families and drew on that experience in making sense of the mappings presented. During the activity, the most prevalent uses of dragging were wandering and drag test. However, two new types of dragging were also identified from analysis of their work on the task: drag non-draggable points and variable-speed dragging.

Dragging a non-draggable point occurred during the independent/dependent variable activity. The students correctly associated a non-draggable point with a dependent variable. They frequently varied the speed with which they dragged a point to analyze relationships between points. When discussing function, they focused on distinguishing functions from non-functions, using one representation to make sense of another, and interpreting invariances. They also spent time discussing rates of change and covariation, but little time was spent attending to domain and range and independent and

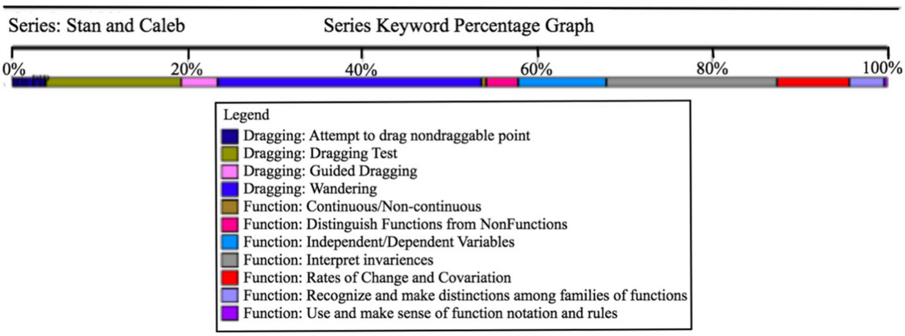


Fig 7 Codes applied to Stan and Caleb

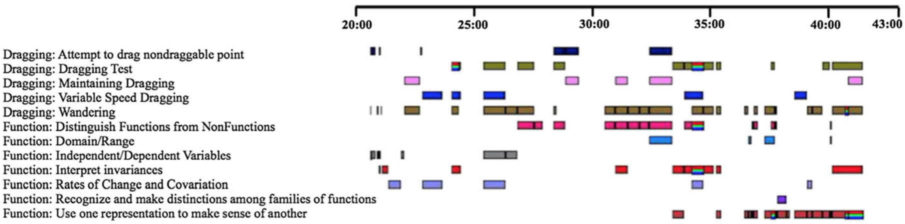


Fig. 8 Code frequency map

dependent variables. Only one instance of recognizing and making distinctions among families of functions was noted (see Fig. 8).

When posed with the task of identifying independent and dependent variables, Chloe, Tamara, and Sophia worked quickly using wandering dragging and attempted to drag non-draggable points as strategies for correct identification of each point presented in the sketch (see Fig. 3 above). They seemed to draw upon previous knowledge to help determine relationships between variables as illustrated through their attention to rates of change and covariation. One particular instance when these ideas were present was when they dragged *w*. Instantly, they knew that *w* was the independent variable and that *e* and *x* were the dependent variables. However, when trying to describe the relationship between *w* and *e*, the students made use of variable-speed dragging to discuss how "*e* moves faster" than *w*: they determined that *e* moves twice as fast as *w*. In this episode, we saw a combination of dragging with attention to rate of change.

While the stated purpose of the task was to determine which mappings were functions and which were not, Chloe, Tamara, and Sophia spent much of their time using wandering dragging to determine how the behavior of the function differed from that of the non-function. They also frequently (19 out of 47 coded episodes) used one representation to make sense of another when attempting to describe what they were seeing. For example, when the instructor asked them "What did you think about page 5?" (an example in which *b* mapped to a stationary point, *b'*), Tamara dragged *a* using wandering dragging and saw that it was mapped to multiple *a*-primes. She then dragged *b* similarly. Sophia said, "So, that would basically be like a straight line" (referring to $b \rightarrow b'$), potentially thinking about a line $y = k$ for which all input values are mapped to the same output value, *k*.

The students used what they knew about a Cartesian representation of function to describe what they observed when acting on the mappings they were exploring. This was a strategy they continued to draw upon as the mappings became more complex. For example, on tab 8, when *s* is dragged, it has only one output, *s'* (Fig. 9a), but this function acts differently from what the students had seen up until this point, because, as *s* is dragged, *s'* remains the at the same location for some values of *s* but for other values of *s, s'* jumps around the page (see Fig. 9b).

Tamara: (dragging *s*)

Sophia: *s* is the function.

Tamara: Oh. I do not know if I like that...(dragging *s*)

Sophia: Wait, do it again.

Sophia: Wait. That's weird.

Tamara: Look, look, there it stays.

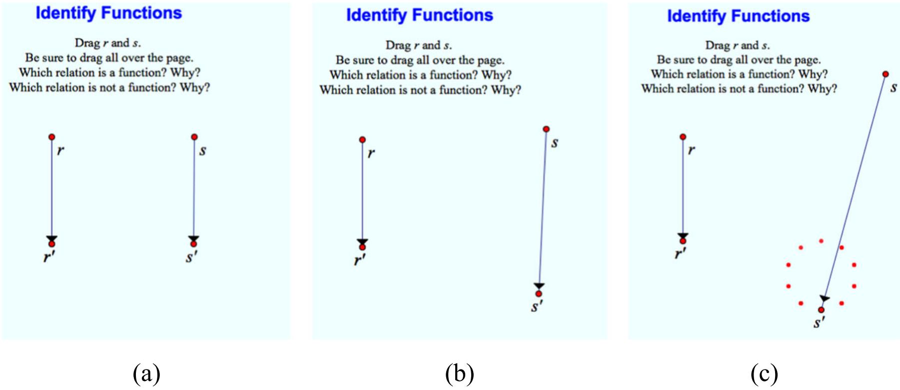


Fig. 9 **a** Point *s* is mapped onto point *s'*; **b** *s'* jumps when *s* is moved; **c** point *s'* traced (the range)

Sophia: It's a step function.

Chloe: Yeah, it would be the greatest.

Sophia: Yeah, yeah.

Chloe: Integer.

Tamara: Greatest integer?

Sophia: It's like one.

Chloe: Yeah.

Sophia: The closed circle, open circle. Something like that, right? (see Figure 10)

Tamara: Oh. But then why is it the same *s* prime for all these locations? (continuing to drag *s*)

Chloe: So, it's saying like if *x* is 1.5, the greatest integer would be like, it'd be like... You'd have to round.

This particular example also shows the only instance where Chloe, Tamara, and Sophia distinguished between families of functions. Sophia made a reference to the behavior of the freehand point and sketched out a greatest integer function (Fig. 10) to describe the relations between dependent and independent points. Also, the emergence of mathematical meaning to characterize the relations between dependent and

Fig. 10 A student sketch of a greatest integer function

independent points contained aspects of the motion. In other words, Sophia's graph contained aspects of the function where she capitalized on the jumps with open and closed circles and characterized the points of the function in the range that stayed the same until the next jump occurred upon dragging the freehand point. Also, they implicitly referred to the domain and range of the functions. For example, Tamara asked, "Why is it the same *s*-prime for all these locations?" when she noticed the freehand point was mapped onto specific points (the range) that jumped around a hidden circle (see Fig. 9c). In response to Tamara's question, Chloe capitalized on the jumps that implied that the range of the function would be some integers that were also mappings of non-integers in the domain.

There were also two instances where Chloe, Tamara, and Sophia used the drag test to interpret invariances when examining the mappings. Attending to invariances was helpful to them when making sense of one of the more complex mapping pairs. First, the students dragged w and z and determined that $w \rightarrow w'$ was a function and $z \rightarrow z'$ was not, because "it has two *z*-primes." They conjectured through a drag test and using one representation to make sense of another that when z is "less than zero or something," it has two *z*-primes. Chloe also noted that "*z*-prime is a, um, the same value, it doesn't change." While they made a claim, the students seemed unsure of their response. As a result, they chose to examine the sketch further by hiding the arrows and dragging w and z again. As Tamara dragged w and z, Sophia realized that, "one of the *z*-primes moves" and Chloe noted that "one *z*-prime stays the same". Here it seems the decision to hide the arrows facilitated the students in recognizing the invariance in z.

Once the students were confident that there were two *z*-primes once z was dragged past a certain point, they made use of maintaining dragging to further discuss the behavior of $z \rightarrow z'$. As mentioned above, they referred to the emergence of the second *z*-prime when z was "less than zero or something." Beyond that location, the independent point was mapped to a single *z*-prime. This location was another aspect of the motion between dependent and independent variables that enabled them to distinguish between a function and a non-function. Sophia reified this relation by stating that it was a non-function that had two mappings when x was less than or equal to zero and a function "when x is greater than zero," as she stated in the vignette below. Then, the point where she characterized the motion, as well as the relation between dependent and independent points, was the origin. To do this, they also used of one representation to make sense of another. In this respect, their personal meaning emerged from interactions with the dragging tool evolved into a mathematical representation that signified the relation between the freehand point and its corresponding mappings.

Sophia: This one's weird.

Tamara: So, what I do not get about this one is how can you have two *z*-primes on the same, like, axis as z. But, when z goes below the first z-prime

Sophia: It's like a graph where…

Tamara: Woah, see, look at this, that *z*-prime (dragging z in a parabolic motion)

Chloe: Look, look, look, that *z*-prime never moves up and down, but once it gets below a certain...

Sophia: It's like a graph where the, uh, when x is greater than zero, then it's like a function, but then, I do not know, something like this possibly? (drawing on paper) (see Figure 11)

Fig. 11 Sophia's sketch describing a function that behaves similar to z → z'

In summary, Chloe, Tamara, and Sophia's work on these tasks drew heavily on their knowledge of a variety of function families to describe the interrelated motions of points. Dragging, especially wandering dragging, allowed for them to attend to the characteristics of the movement, and they described this movement using graphs of functions they were familiar with. In addition, in both their exploration of independent/dependent variables and function/non-function relationships, the use of maintaining dragging and variable speed dragging drew their attention to invariances and covariation, characteristics of the mappings they might have otherwise overlooked. When they characterized the motions with a function, they attended to the aspects of the basic point and capitalized on the jumps of the dependent point. Also, they described the nature of mappings by sketching functions and non-functions. As the students dragged, they produced graphs (mathematical signs) to describe the relations and functions. In this respect, their approaches to the tasks showed a detachment process from the artifact context.

Jayden and Mason

Jayden had just completed a Geometry class and Mason an Algebra II course. In each of these courses, they had had the most recent experience with geometric transformations. Perhaps as a result, more than other students, they drew upon their understanding of geometric transformations to describe and compare the functions represented in this task. Most of their dragging was categorized as wandering or drag test with some instances of dragging a non-draggable point, maintaining dragging, and guided dragging. When discussing characteristics of functions, they focused much of their attention on invariances, rates of change, and independent/dependent variables and made use of one representation to make sense of another representation.

Jayden and Mason spent about 10 min of the first task describing which points are related to each other by systematically dragging points. They quickly associated the language of independent and dependent variables with draggable and non-draggable points. For example, they began the task by selecting point *d* and then point *b*, each of which were non-draggable points, and Jayden stated "These, uh, *b* and d are both dependent." They selected point *c* and were able to drag it and *d* also moved. Jayden claimed, "*c* is independent." They used wandering dragging as they moved point *z* and noted that it was related to point *b*. However, when they wanted to better describe how *z* and *b* were related, they used a drag test and Jayden stated, "Both directions are opposite…*x*- and *y*-axis" which suggested they were attending to the relative directions of the two points.

Jayden's personal meaning to the interrelated motions was linked to the artifact context with a focus on the direction of the points upon dragging. Mason responded, "Oh wow. So, *x* equals *y*. You know the line $x=y$?" Jayden agreed and added, "But the other one, the x- and y-axis are totally equal. They're exactly...perfect." It is likely that they were contrasting the behavior of points *z* and *b*, which were created using a 180-degree rotation with the behavior of points *c* and *d*, which were created using a translation. They appeared to be focusing on the direction the points moved and whether they were the same or opposite and whether the distance between the two points varied or remained the same.

Their way of attending to geometric objects when dragging the freehand point was important for the emergence of their personal meanings of covariation, which triggered thinking of the interrelated motions with a focus on the dynamic distances between the points. They became more interested in seeking point to point correspondence to describe the relations between the variables. In this process, their dragging practices switched from wandering to a purposeful drag test to describe how the points co-varied and used another representation, the *x*- and *y*-axes, to make sense of that behavior. This pattern was repeated when they examined point *w*.

Jayden: Ooh, this is a tricky one.

Mason: What? So, *w*, *e*, and *x*? Try clicking on the other two.

Jayden: I did.

Mason: Oh, you did?

Jayden: Watch. You cannot.

Mason: That's weird.

Jayden: So, I guess *w* is independent in relation to *e* and *x*.

Mason: Um...

Jayden: *w* is equal to *x* on the *x*- and...no wait, wait... *k*, in relation *w* to *e*...I do not know what the relationship is between *w* and *e*. Here, let me see something. How about the relation to the x-axis is two...it's, uh…

Mason: It's weird.

Jayden: The *x* is equal to two. I mean *x* is equal to *2x* in relationship between *w*. See, look in comparison how it moves twice as far from both directions.

Mason: Yeah.

As Jayden and Mason considered relationships between the points, they brought in a symbolic relationship, *2x*, to assist them with their sense making. The exploitation of the dragging test yielded to the emergence of a new approach to describe the nature of the relationships between the independent and dependent points. A translation was used to create *x* from *w*, and the translation vector was horizontal, and a dilation was used to create *e* from *w* with a scale factor of two (see Fig. 12).

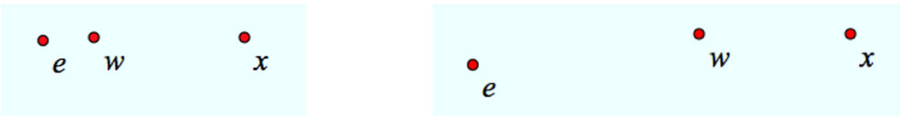


Fig. 12 A picture of how points *e*, *w*, and *x* were related in the task

Although Jayden and Mason were not able to articulate that the linked motion was a specific transformation, their ways of attending to the distances between the points *e* and *w* implied a dilation. Jayden used the expression, *2x*, to characterize the distance between the two points. The students struggled in characterizing the relationship between *e* and *w*, perhaps because the distance between the pre-image and the hidden center point and the image and the hidden center point varied while the ratio of these distances remained constant. In this sense, the detachment from the artifact context was still in progress for Jayden and Mason's solutions to the tasks.

We observed another emergence of an approach to identify relations between points where the dragging tool was used not only for exploration but also for using transformations to describe the linked points. When the instructor asked the students to explain how their written responses that included references to the *x*- and *y*-axes were related to the sketch, Jayden and Mason showed the instructor how they were thinking about a reflection that related point *a* and *y*. They reasoned about fixed points by dragging point *a* to coincide with point *y* and determined where the points coincide is the line of reflection.

In this case, they used the drag test to determine the location of one fixed point along the line of reflection. They used maintaining dragging to determine other locations where the two points coincided. Their point to point mappings with a variety of dragging modalities where the dependent and independent points coincided along the line of reflection lead to an evolution into the mathematical meaning of reflection as a function. In other words, they used the aspect of function, namely, a set of points along the line of reflection as a reference in characterizing the behavior of the interrelated points globally. When the instructor visited this group, the following discussion occurred:

Instructor: Um, could you show me where that line would be?

Mason: Put 'em right on top of each other.

Instructor: Ok. So, there's a point...

Mason: And you could put like a number...

Instructor: So, you just showed me a point where they come together.

Mason: But the line would be like...

Jayden: Where the points meet.

Mason: Diagonal.

Jayden: No. Yeah. Well...

Mason: It... The line would go through where the points meet.

Mason: It'd go...

Instructor: Ah…

Mason: Like right there would be another point.

Instructor: There's another point. OK.

Mason: And you go, right there would be a line.

Instructor: Oh, you are showing me a line, aren't you?

Mason: Yeah.

Instructor: Oh, that's cool.

Mason: Go like that, somewhere down there.

The students dragged the points so that they coincided along the line of reflection. The language of transformations and functions was prominent in their work on these tasks. As they worked on other tabs to determine whether a relation is a function or not, they also focused on identifying the transformation that could be used to describe the relation between the independent and dependent variables. We observed a change in signifier when they described the nature of functions with a focus on transformations linked with point by point correspondence between interrelated motions. Their use of the dragging tool became more purposeful in characterizing the interrelated motions. In other words, they reified the nature of functions with transformations in which their use of dragging tool shifted towards drag test and maintaining dragging.

Cross-Case Discussion

A summary of the ways in which each case of students interacted with the technology is shown in Table 3. In all groups, wandering dragging was the most common interaction used. This is not surprising, considering the design of the task that required students to use dragging to investigate how points were related.

The immediate goal of the task was for students was to identify the interrelated behavior of the points through wandering dragging (e.g., identify dependent and independent points), which was a precursor activity that progressed into describing how the dependent and independent variables were related. Previous research has shown that if students are given geometric objects (e.g., polygons), students tend to use visual cues of the objects (e.g., orientation of shapes, preservation of lengths) along with dragging test and guiding dragging to make sense of how the two objects are related (Hollebrands, 2003, 2007). In this activity, students were unable to use the strategy of attending to visual cues of the objects as they were only provided with points and had to reason about behavior and variation to make inferences about the underlying function.

Students found a variety of ways to describe how the dependent and independent variables were related through dragging. For example, the drag test was used when students investigated invariance or reasoned about covariation. Students made use of maintaining dragging to further describe variable behaviors. We associate their approaches to the tasks with their mathematical background, considering they described the relation of the variables using the mathematical lens they were bringing to the problems. In other words, semiotic processes of students' solutions to the tasks were influenced by their mathematical background.

Table 3 Interaction with the technology: cross-case code frequencies

Code	Caleb and Stan	Jayden and Mason	Chloe, Tamara, and Sophia
Dragging test	10	16	13
Guided dragging	2	4	0
Maintaining dragging	0	1	5
Variable-speed dragging	0	0	5
Wandering dragging	34	18	25
Attempt to drag a non-draggable point	11	6	6

For example, the pair of students who had recently completed a first-year algebra course and had the least experience with function and transformations (Caleb and Stan) relied on the artifact context with a focus on invariances. They used the language of invariances (e.g., "fixed points") to characterize similarities and differences of the functions with which they were interacting. However, their personal meanings for covariation were not transformed into the characterization of the nature of functions or transformations corresponding to dependent and independent variables. In other words, their difficulty in detaching from the artifact context with a focus on point to point correspondence seemed to impede the evolution of knowledge towards a more formal mathematical meaning of function.

Students who had recently completed a pre-calculus course and had the most previous experience with function (Chloe, Tamara, and Sophia) used their knowledge of function families to describe the interrelated motions that resulted from their dragging. In other words, they used one representation to make sense of another. Their accomplishment of the tasks and the semiotic meanings in the evolution process of knowledge into characterizing the relations by means of functions were influenced by their use of the dragging artifact as a psychological tool.

As Mariotti (2000) stressed, the internalization of a tool of semiotic mediation entails validating a conjecture that emerged from dragging objects in DGS such that "validation is sought within the Geometric theory, i.e. in principle, conjectures ask for a proof" (p. 45). In this respect, the dragging was used as a sign concerning the idea of reifying the covariational motions as functions/non-functions in a global way. They attended to the aspects of the function with a reference to the artifact context and described how the points behaved upon dragging. These aspects were transformed into global characterization of functions that led to a description of how the interrelated motions were related. Accordingly, students frequently spent time on distinguishing functions and non-functions, sketched out graphs to reify the relations between independent and dependent variables. They capitalized on the discontinuities that occurred and locations where the relation was rendered no longer a function.

The students who had just completed a Geometry or an Algebra II course and hence had had the most recent experience with geometric transformations (Jayden and Mason) characterized the relations between the independent and dependent variables with transformations. They approached the problems using the rate of change and covariation aspects and focused on the direction of the points upon dragging the independent point, as well as the distance between the interrelated points. They assigned new variables to the dynamic distances as a means to characterize how the dependent and independent variables were related using a drag test.

Jayden and Mason also made use of maintaining dragging when they articulated the transformation that took place between the variables was a reflection. They purposefully dragged the independent point in such a way that it coincided with the dependent point. They articulated that the locations where these points coincided would be the line of reflection. In this sense, they used the aspect of the transformation (i.e., distance from the line of reflection) with reference to the artifact context and made a global claim drawing upon their point to point correspondence approach to communicate about covariation.

Falcade et al. (2007) reported a similar situation in which the tools in DGS are used as a tool of semiotic mediation. These researchers found that, building on a geometric transformation (reflection), high school students used the trace tool in a DGS to investigate the nature of a function with a focus on range and domain. From this view, they claimed, “the Trace tool is used not only in exploration (externally oriented), but also in the reasoning which leads to the solution of the problem: it has become an intellectual tool (internally oriented) used to answer the question concerning the nature of a function” (p. 326).

Conclusions

In this study, students were provided pre-image and image points and were prompted to drag points to identify functions without providing students with information about which transformation was used in the creation of the sketch. Different from prior research in which transformations and functions were explicitly connected with geometric figures (e.g., Hegg et al., 2018; Steketee & Scher, 2018; Yanik & Flores, 2009), our tasks only presented points. The purpose was to better understand students' use of the dragging tool to identify if the given relations were a function, applying all points in the plane through dragging freehand points.

Unlike most introductory function tasks, this one provided an opportunity for students to reason about function in sophisticated ways without the use of numbers. Students were required to drag points that encouraged them to attend to dynamic relationships among independent and dependent variables. By unfolding the semiotic potential of the artifact (the dragging tool in DGS), we were able to examine the ways in which students with different backgrounds reasoned about function and how the use of the dragging tool and semiotic mediation contributed to their descriptions of geometric transformations and functions.

By characterizing how students with different mathematical backgrounds connected functions and transformations, mathematics educators and curriculum designers may have some insights into how they might design other tasks to help students see these connections. Our research gives some information about what might happen that could inform one in anticipating what students will do and how mathematics educators and curriculum designers could create tasks and pose questions that elicit and support students' thinking. The results of this study indicated that when students are provided interactive tasks, the way they approached the tasks or mathematics differed. Regardless of mathematical background, students engaged with activities pertaining to the function concept.

Students' use of the dragging tool shaped their idiosyncratic meanings concerning aspects of the function that served as mathematical signs. They developed different utilization schemes and exploited the dragging tool, which progressed into attending to the aspects of function, observing perceptible invariants between pre-image and image points. For example, those who learned functions in pre-calculus tended to describe the relations with a focus on specific functions (e.g., the greatest-integer function) using formal language more often around domain and range moving from informal descriptions. Embracing a more analytic approach to function, they sketched out graphs to characterize the covariation between the dependent and independent variables.

Those who had studied Geometry or Algebra II connected the dynamic behavior of the pre-image points with transformations. Yet, some of their characterizations of the transformations comprised the interpreting invariances aspect of function. On the other hand, those who completed a first-year algebra course focused on interpreting invariances more often (e.g., attending to the preserved distance between the variables without referring to the transformations or functions).

Although the groups of students attended to some aspects of function in the transformation tasks, we do not take for granted they fully accomplished the tasks. In a didactical design, the engagement with tasks with the use of an artifact necessitates individual and collective production of signs under the teacher's supervision exploiting the semiotic potential of the artifact (Bartolini Bussi & Mariotti, 2008). At least, mathematical discussion of individual and collective signs (e.g., signs produced in a group work) by the teacher is important in the process of guiding students' idiosyncratic meanings to progress into mathematical meanings. However, the tasks used in the current study could serve as an entry point to examine how students reason about functions and transformations. Because only those who studied geometry referred to the transformations in the dependency tasks, some refinements in tasks may help students make a better transition to transformations.

As Falcade et al. (2007) reported, students used the trace tool in DGS as a tool of semiotic mediation when they described the behaviors of pre-image and image points. In a similar vein, allowing students to use the trace tool may assist them in characterizing the dynamic behavior of points with a focus on transformations because the trace tool enables them to see what the path of the pre-image point makes and how the image point responds to this change. Then students may use the visual cues with the use of the trace tool as a tool of semiotic mediation to connect functions and transformations. As Steketee and Scher (2018) pointed out, transformational approaches to functions such as dragging a freehand point so as to move the dependent variable to a target destination with the use of the trace tool, students grasp "what it means to apply a function all at once to an entire set of points (a polygon)" (p. 68). For this purpose, a longer period of engagement with tasks is required for students to exploit the semiotic potentials of DGS tools.

Prior research has demonstrated ways students use dragging to make sense of geometric transformations (Falcade et al., 2007; Hollebrands, 2007). For example, Hollebrands (2007) reported that one student moved the pre-image points to the locations where they coincided the image points maintaining the invariants in DGS. From there, the student connected the coincided pair of points with a line segment that was characterized as the line of reflection. In a similar vein, tasks should guide students to make explicit connections between the investigated concepts. Therefore, follow-up tasks with explicit questions may guide students to connect functions and transformations. In this process, teachers may use different schemes of utilizations of tools as an opportunity and elicit a network of signs serving as a mediator. Also, teachers have a vital role in assisting students with abandoning the artifact context and guide students to progress into establishing a mathematical meaning within social interaction where the tool is used as a sign to conduct instrumented activity.

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