Finance Stoch (2020) 24:423–463
https://doi.org/10.1007/s00780-019-00414-2

Trading strategies generated pathwise by functions of market weights

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Received: 9 March 2019 / Accepted: 3 November 2019 / Published online: 17 December 2019

Abstract Twenty years ago, E.R. Fernholz introduced the notion of "functional generation" to construct a variety of portfolios solely in terms of the individual companies' market weights. I. Karatzas and J. Ruf recently developed another approach to the functional construction of portfolios which leads to very simple conditions for strong relative arbitrage with respect to the market. Here, both of these notions are generalized in a pathwise, probability-free setting; portfolio-generating functions, possibly less smooth than twice differentiable, involve the current market weights as well as additional bounded-variation functionals of past and present market weights. This leads to a wider class of functionally generated portfolios than was heretofore possible to analyze, to novel methods for dealing with the "size" and "momentum" effects, and to improved conditions for outperforming the market portfolio over suitable time horizons.

Keywords Stochastic portfolio theory · Pathwise Itô and Tanaka formulas · Trading strategies · Functional generation · Strong relative arbitrage

Mathematics Subject Classification (2010) 60G48 · 60H05 · 91G10

JEL Classification D52 · D53 · G12

1 Introduction

The concept of "functionally generated portfolios" was introduced by Fernholz [9, 11] and has been one of the essential components of stochastic portfolio theory;

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see Fernholz and Karatzas [12] for an overview. Portfolios generated by appropriate functions of the individual companies' market weights have wealth dynamics which can be expressed solely in terms of these weights and do not involve any stochastic integration. Constructing such portfolios does not require any statistical estimation of parameters, or any optimization. Completely observable quantities such as the current values of "market weights", whose temporal evolution is modelled in terms of continuous semimartingales, are the only ingredients needed for building these portfolios. Once this structure has been discerned, the underpinning mathematics involves just simple calculus. Then the goal is to construct such portfolios that outperform a reference portfolio, for example the market, under suitable structural conditions.

Karatzas and Ruf [14] recently discovered a new functional generation of trading strategies which they call "additive", as opposed to Fernholz's "multiplicative generation". This new methodology weakens the assumptions on the market: asset prices and market weights are continuous semimartingales, and trading strategies are constructed from "regular" functions of these semimartingales. Strategies generated in this additive manner require simpler conditions for strong relative arbitrage with respect to the market over appropriate time horizons; see also Fernholz et al. [10].

In a different but related development, Föllmer [13] showed almost 40 years ago that certain aspects of Itô calculus can be developed "path by path" without any probabilistic structure, and in particular, without any semimartingale assumption. Once a given function has finite quadratic variation/covariation along a given nested sequence of partitions over a fixed time interval of finite length, change-of-variable formulas can be proved by Taylor expansions in a surprisingly simple way. Then Würmli [25] introduced in this setting the concept of local times and the corresponding pathwise Tanaka formula. This allows the change-of-variable formula to be applied to less regular functions by involving appropriately defined pathwise local times. Such local times have been further developed recently; see Cont and Perkowski [3], Davis et al. [4], Kim [15] and Perkowski and Prömel [19].

In this paper, we generalize both additive and multiplicative functional generation of trading strategies in several ways. First, we use pathwise Itô calculus to show how to construct trading strategies, generated additively or multiplicatively from a given function, depending on the market weights and in a manner completely devoid of probability considerations. The only analytic structure we impose is that market weights admit continuous covariations in a pathwise sense. Second, we allow generating functions that depend on an additional argument of finite variation. Introducing new arguments other than the market weights provides extra flexibility for constructing portfolios; see Ruf and Xie [20], Schied et al. [21], Strong [23]. We present various types of such arguments so that a variety of new trading strategies can be generated from a function depending on them; these strategies yield new sufficient conditions for outperforming the market. We also apply the pathwise Tanaka formula to construct portfolios from generating functions rougher than heretofore possible. The classical Itô formula applies to functions which are at least twice differentiable, whereas the Tanaka formula only requires absolute continuity. Thus, usage of the latter broadens the class of portfolio-generating functions considerably.

We also provide new sufficient conditions for strong relative arbitrage via additively and multiplicatively generated trading strategies. The existing condition in

Karatzas and Ruf [14] requires the generating function to be "Lyapunov", or the corresponding "Gamma function" to be nondecreasing. By contrast, the new conditions in this paper depend on the intrinsic nondecreasing structure of the generating function itself. This shows that trading strategies outperforming the market can be generated from a much richer collection of functions depending on the market weights and an additional argument of finite variation. We provide some interesting examples of such strategies and an empirical analysis of them.

The paper is structured as follows. Section 2 presents elements of pathwise Itô–Tanaka calculus and the relevant notion of local time needed for our purposes. Section 3 defines trading strategies and regular functions, then discusses how to generate the former from the latter in ways both additive and multiplicative. Section 4 gives sufficient conditions for such strategies to generate strong arbitrage relative to the market. Section 5 provides examples of trading strategies generated from entropic functions, and corresponding sufficient conditions for strong arbitrage. Section 6 contains empirical results regarding the portfolios discussed in Sect. 5, and Sect. 7 offers some concluding remarks.

2 Pathwise Itô–Tanaka calculus

2.1 Multidimensional pathwise Itô formula

In what follows, we let $X = (X_1, \dots, X_d)'$ be an $\mathbb{R}^d$-valued continuous function, representing a vector of quantities defined on $[0, T]$, for a fixed $T > 0$, whose values change over time. We require the components of X to admit continuous covariations in the pathwise sense with respect to a given, refining sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions of the interval $[0, T]$. The sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ is such that each partition is of the form $\mathbb{T}_n = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{N(\mathbb{T}_n)}^{(n)} = T\}$ for $n \in \mathbb{N}$ as well as $\mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \cdots$, and the mesh size $\|\mathbb{T}_n\| := \max_{t_j\in\mathbb{T}_n} |t_{j+1} - t_j|$ decreases to zero as $n \to \infty$.

We fix such a sequence $\mathbb{T} = (\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions for the remainder of the paper. Here and below, t_j and t_{j+1} are consecutive points in the partition $\mathbb{T}_n$, i.e., $t_j < t_{j+1}$, $\mathbb{T}_n \cap (t_j, t_{j+1}) = \emptyset$. Also, when we write $t_j \in \mathbb{T}_n$ and $t_j \le t$ simultaneously, we set $t_{j+1} = t$ when j is the biggest index satisfying $t_j \le t$. With this notation, the notion of pathwise quadratic covariation of X along $(\mathbb{T}_n)_{n\in\mathbb{N}}$ is defined as follows.

Definition 2.1 A continuous function $X : [0, T] \to \mathbb{R}^d$ has *pathwise quadratic covariation* along $(\mathbb{T}_n)_{n\in\mathbb{N}}$ if for all $1 \le i, k \le d$, the limit as $n \to \infty$ of the sequence

$$\sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} \big(X_i(t_{j+1}) - X_i(t_j)\big)\big(X_k(t_{j+1}) - X_k(t_j)\big), \qquad n \in \mathbb{N},$$

exists in $\mathbb{R}$ for all $t \in [0, T]$ and the resulting mapping, denoted by $t \mapsto [X_i, X_k](t)$, is continuous. We call $[X_i, X_k]$ the *pathwise quadratic covariation* of X_i and X_k and denote the *pathwise quadratic variation* of X_i by $[X_i] := [X_i, X_i]$ as usual.

We stress that the existence of pathwise covariations and quadratic variations for the components of X depends heavily on the choice of the sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions. Cont [1, Example 5.3.2 and the arguments following it] illustrates this fact. We also note that the existence of pathwise covariations and quadratic variations is required for Itô's formula to hold in a pathwise sense.

We shall need a higher-dimensional pathwise Itô formula with an extra "input" as additional argument. For this purpose, we let $A = (A_1, \dots, A_m)' : [0, T] \to \mathbb{R}^m$ be a function of finite variation and consider $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ as well as the quantity $f(X_1(t), \dots, X_d(t), A_1(t), \dots, A_m(t))$ that depends on time $t \in [0, T]$. We say that such a function f is in $\mathbb{C}^{j,r}(\mathbb{R}^d \times \mathbb{R}^m; \mathbb{R})$ if it is j times continuously differentiable with respect to the first d and r times continuously differentiable with respect to the last m components. We also denote by $\partial_i f$, $\partial_{i,k} f$ the first- and second-order partial derivatives for the first d components of f $(1 \le i, k \le d)$ and by $D_\ell f$ the first partial derivative with respect to the $(d+\ell)$-th component of f, for the last m components of f $(1 \le \ell \le m)$.

We present now the following version of the pathwise Itô formula involving both components X and A. It can be proved using the Taylor expansion as in the proof of Föllmer's original result, which can be found in the Appendix of [22].

Proposition 2.2 *Fix a continuous function $X : [0, T] \to \mathbb{R}^d$ having pathwise quadratic covariations along $(\mathbb{T}_n)_{n\in\mathbb{N}}$ and a continuous function $A : [0, T] \to \mathbb{R}^m$, all of whose components have finite variation. Then for every $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ of class $\mathbb{C}^{2,1}$, the pathwise change of variable formula*

$$
\begin{aligned}
f\big(X(t), A(t)\big) - f\big(X(0), A(0)\big) &= \int_0^t \sum_{i=1}^d \partial_i f\big(X(s), A(s)\big) dX_i(s) \\
&\quad + \sum_{\ell=1}^m \int_0^t D_\ell f\big(X(s), A(s)\big) dA_\ell(s) \\
&\quad + \frac{1}{2} \sum_{i,k=1}^d \int_0^t \partial^2_{i,k} f\big(X(s), A(s)\big) d[X_i, X_k](s)
\end{aligned}
\tag{2.1}
$$

holds for all $t \in [0, T]$. Here, the last two integrals in (2.1) are of the Lebesgue–Stieltjes type, and the first, so-called "Föllmer–Itô" integral is defined as the pointwise limit

$$
\begin{aligned}
&\int_0^t \sum_{i=1}^d \partial_i f\big(X(s), A(s)\big) dX_i(s) \\
&:= \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \sum_{i=1}^d \partial_i f\big(X(t_j), A(t_j)\big)\big(X_i(t_{j+1}) - X_i(t_j)\big).
\end{aligned}
$$

2.2 Pathwise local time and Tanaka formula

For a given subset V of a Euclidean space, $C([0,T];V)$ denotes the space of continuous V-valued functions defined on $[0,T]$, whereas $CBV([0,T];V)$ stands for the space of those functions in $C([0,T],V)$ whose components are of bounded variation.

For $X \in C([0,T];\mathbb{R})$, we recall the notion of quadratic variation of X along $(\mathbb{T}_n)_{n\in\mathbb{N}}$ as introduced by Föllmer [13].

Definition 2.3 A function $X \in C([0,T],\mathbb{R})$ is said to have finite *quadratic variation along* $\mathbb{T} = (\mathbb{T}_n)_{n\in\mathbb{N}}$ if the sequence $\mu^n := \sum_{t_j\in\mathbb{T}_n} |X(t_{j+1}) - X(t_j)|^2 \delta_{t_j}$, $n \in \mathbb{N}$, of measures converges vaguely to a locally finite measure μ without atoms; here δ_t denotes the Dirac measure at $t \in [0,T]$. We write $Q(\mathbb{T})$ for the collection of all continuous functions having quadratic variation along $\mathbb{T}$ and denote by $[X](t) := \mu([0,t])$ the quadratic variation of X on $[0,T]$ for $t \in [0,T]$.

For a sequence of measures $(\mu^n)_{n\in\mathbb{N}}$ on $[0,T]$, vague convergence is equivalent to the pointwise convergence of their cumulative distribution functions at all continuity points of the limiting distribution function; and if the latter is continuous, the convergence is uniform. Thus we are led to the following remark which guarantees in particular that the quadratic variation $[X]$ of X as in Definition 2.3 coincides for $d=1$ with that of X in Definition 2.1.

Remark 2.4 $X \in C([0,T];\mathbb{R})$ belongs to the collection $Q(\mathbb{T})$ of Definition 2.3 if and only if there exists a continuous function $[X]$ such that

$$\sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\le t}} |X(t_{j+1}) - X(t_j)|^2 \xrightarrow{n\to\infty} [X](t) \tag{2.2}$$

for every $t \in [0,T]$. If this holds, the convergence in (2.2) is uniform.

Remark 2.5 The assumption on the sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions that the mesh size $\|\mathbb{T}_n\|$ goes to zero as $n \to \infty$ is stronger than the assumption usually imposed in other works involving pathwise local time. For example, in [3, 4, 19], the authors define the "oscillation" of the function X along a partition $\mathbb{T}_n$ as

$$\operatorname{osc}(X,\mathbb{T}_n) := \max_{t_j\in\mathbb{T}_n} \max_{r,s\in[t_j,t_{j+1}]} |X(s) - X(r)|$$

and require $\operatorname{osc}(X,\mathbb{T}_n) \to 0$ as $n \to \infty$ instead of the mesh size going to zero. This is because it is enough to work with Lebesgue partitions generated by X when defining the pathwise local time and deriving the pathwise Tanaka formula. Since the function X is uniformly continuous on the compact interval $[0,T]$, the decrease to zero of the mesh size implies that the oscillation of X also shrinks to zero.

One reason for imposing here the stronger condition on $(\mathbb{T}_n)_{n\in\mathbb{N}}$ is to follow the definition of pathwise quadratic covariation/variation in Definition 2.1. Another reason is that we are going to involve additional continuous functions A when generating trading strategies and the oscillation of A also has to shrink to zero along $(\mathbb{T}_n)_{n\in\mathbb{N}}$.

In other words, by using the "mesh" assumption instead of the "oscillation", we can get rid of such a "dependence" of the sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ on both X and A.

The very first definition of pathwise local time was given in the unpublished diploma thesis of Würmli [25]. This original local time is called "$\mathbb{L}^2$-local time" of a path X along a sequence $\mathbb{T} = (\mathbb{T}_n)_{n\in\mathbb{N}}$ of partitions. Using that notion, Würmli established the expression (2.6) below for f in $H^2(\mathbb{R};\mathbb{R})$, the Sobolev space of functions in $\mathbb{L}^2(\mathbb{R};\mathbb{R})$ which are twice weakly differentiable. Since then, many versions of pathwise Tanaka formulas (generalized Itô formulas) and different definitions of local times have been introduced and studied; they vary according to the regularity of the path X, the function f and the notion of "convergence for local time". Weaker convergence in defining a local time requires more regularity on the part of the function f. Some of these versions are in Perkowski and Prömel [19, Sect. 2] and Davis et al. [4] for continuous paths with quadratic variation. Similar results for rougher paths (with finite p-th variation, $p > 2$) can be found in Cont and Perkowski [3, Sect. 3] and in Kim [15]. We present here a version of local time and Tanaka's formula which is most appropriate in our setting.

We adopt throughout the notation

$$(\!|a, b]\!] = \begin{cases} (a, b], & a \le b, \\ (b, a], & b \le a, \end{cases} \tag{2.3}$$

used in [3, 4, 15, 19]. Then we have the following definition of continuous local time.

Definition 2.6 We say that $X \in C([0, T];\mathbb{R})$ has *a continuous local time along* $\mathbb{T} = (\mathbb{T}_n)_{n\in\mathbb{N}}$ if the "discrete local times"

$$x \mapsto L_t^{X,\mathbb{T}_n}(x) := \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \mathbb{1}_{(\!|X_{t_j}, X_{t_{j+1}}]\!]}(x)\,|X_{t_{j+1}} - x| \tag{2.4}$$

converge as $n \to \infty$ uniformly in x to a continuous limit $x \mapsto L_t^{X,\mathbb{T}}(x)$ for every fixed $t \in [0, T]$ and the resulting mapping $(t, x) \mapsto L_t^{X,\mathbb{T}}(x)$ is jointly continuous. We call this limit *continuous local time of X along* $\mathbb{T}$ and write $\mathcal{L}^c(\mathbb{T})$ for the collection of functions X in $C([0, T];\mathbb{R})$ which admit a continuous local time along $\mathbb{T}$.

The existence of a continuous local time for "typical price paths" is shown in [19, Theorem 3.5]. To simplify notation, we write $L_t^X(x)$, or simply $L_t(x)$, whenever the context is unambiguous. With this definition, we have the following version of the pathwise Tanaka formula, which is Theorem 2.6 of [19].

Proposition 2.7 *Let $X \in \mathcal{L}^c(\mathbb{T})$ and $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous with right-continuous Radon–Nikodým derivative f' of finite variation. Then the one-dimensional Föllmer–Itô integral is defined as the pointwise limit*

$$\int_0^t f'\big(X(s)\big)dX(s) := \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} f'\big(X(t_j)\big)\big(X(t_{j+1}) - X(t_j)\big), \tag{2.5}$$

and we have the generalized change-of-variable formula

$$f\big(X(t)\big) - f\big(X(0)\big) = \int_0^t f'\big(X(s)\big)dX(s) + \int_{\mathbb{R}} L_t(x)df'(x), \qquad 0 \le t \le T. \quad (2.6)$$

3 Trading strategies generated pathwise

We place ourselves from now on in a frictionless equity market with a fixed number $d \geq 2$ of companies. We consider $S = (S_1, \ldots, S_d)' \in C([0, T]; [0, \infty)^d)$, where $S_i(t)$ represents the capitalization of the i-th company at time $t \in [0, T]$. Here we take $S_i(0) > 0$ and allow $S_i(t)$ to vanish at some time $t > 0$ for each $i = 1, \ldots, d$; but we assume that the total capitalization $\Sigma(t) := S_1(t) + \cdots + S_d(t)$ does not vanish at any time $t \in [0, T]$.

With these ingredients, we define another vector $\mu = (\mu_1, \ldots, \mu_d)'$ of continuous functions that consists of the companies' market weights,

$$\mu_i(t) := \frac{S_i(t)}{\Sigma(t)} = \frac{S_i(t)}{S_1(t) + \cdots + S_d(t)}, \qquad t \in [0, T], i = 1, \ldots, d. \qquad (3.1)$$

We assume that the components of μ admit finite quadratic covariations $[\mu_i, \mu_j]$, $1 \leq i, j \leq d$, along $(\mathbb{T}_n)_{n\in\mathbb{N}}$.

We also consider $A = (A_1, \ldots, A_m)' \in CBV([0, T]; \mathbb{R}^m)$ along with the vector μ of market weights. For the purposes of this section, the components of A model the evolution of an observable, but non-tradable quantity related to the market weights. In what follows, we consider functions of the form $G(\mu(\cdot), A(\cdot))$. Examples of A appear in (4.2), (4.3) below. With this notation, we have the following definition of trading strategy with respect to the pair (μ, A) in the manner of [14].

Definition 3.1 For the market weights μ, suppose that $\vartheta = (\vartheta_1, \ldots, \vartheta_d)'$ is a d-dimensional function for which the "Föllmer–Itô integral"

$$\int_0^{\cdot} \vartheta(t)d\mu(t) := \int_0^{\cdot} \sum_{i=1}^d \vartheta_i(t)d\mu_i(t) := \lim_{n\to\infty} \sum_{\substack{t_j\in\mathbb{T}_n \\ t_j\leq\cdot}} \sum_{i=1}^d \vartheta_i(t_j)\big(\mu_i(t_{j+1}) - \mu_i(t_j)\big)$$

with respect to μ exists in $\mathbb{R}$. We write $\vartheta \in \mathcal{L}(\mu)$ to express this. We say that $\vartheta \in \mathcal{L}(\mu)$ is a *trading strategy with respect to* μ if it is "self-financing" in the sense that its value $V^{\vartheta}(\cdot) := \sum_{i=1}^d \vartheta_i(\cdot)\mu_i(\cdot)$ satisfies

$$V^{\vartheta}(\cdot) - V^{\vartheta}(0) = \int_0^{\cdot} \sum_{i=1}^d \vartheta_i(t)d\mu_i(t).$$

Above, $\vartheta_i(t)$ stands for the ratio of number of shares of asset i held at time t by the trading strategy θ, divided by the number of outstanding shares of asset i. Since $\mu_i(t)$ is the market weight of this asset, $\vartheta_i(t)\mu_i(t)$ is the (relative) dollar amount invested in asset i, divided by the total capitalization of the entire market at time t; and $V^{\vartheta}(t)$ is the relative value of ϑ to the market portfolio, that is, the total dollar amount of investment across all assets of ϑ divided by the total capitalization of the entire market. “Self-financing” means that there are neither infusions nor withdrawals of capital; gains are re-invested, losses have to be absorbed.

The pathwise Itô formula (2.1) suggests that integrands $\vartheta \in \mathcal{L}(\mu)$ of the special form $\vartheta(t) = \partial f(\mu(t), A(t))$, for some function $f \in \mathbb{C}^{2,1}(\mathbb{R}^{d+m}; \mathbb{R})$, play an important role for integrators $\mu \in C([0, T]; \mathbb{R}^d)$ that admit finite quadratic covariations $[\mu_i, \mu_j]$, $1 \le i, j \le d$, along $(\mathbb{T}_n)_{n\in\mathbb{N}}$. This gives rise to the following definition.

Definition 3.2 For the given pair (μ, A) of market weights $\mu \in C([0, T]; \mathbb{R}^d)$ and $A \in CBV([0, T]; \mathbb{R}^m)$, we say that $G \in C(\mathbb{R}^d \times \mathbb{R}^m; \mathbb{R})$ is *regular* if

(i) there exists a function $\nabla G = (\nabla_1 G, \dots, \nabla_d G)' : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ such that the vector $\vartheta = (\vartheta_1, \dots, \vartheta_d)'$ with components

$$\vartheta_i(t) := \nabla_i G\big(\mu(t), A(t)\big), \qquad i = 1, \dots, d, 0 \le t \le T, \tag{3.2}$$

is in $\mathcal{L}(\mu)$, and

(ii) the continuous function

$$\Gamma^G(\cdot) := G\big(\mu(0), A(0)\big) - G\big(\mu(\cdot), A(\cdot)\big) + \int_0^{\cdot} \nabla G\big(\mu(s), A(s)\big) d\mu(s), \tag{3.3}$$

which we call the *Gamma function* of G, has finite variation on $[0, T]$.

Example 3.3 As foretold in the discussion preceding Definition 3.2, any function G in $\mathbb{C}^{2,1}(\mathbb{R}^d \times \mathbb{R}^m; \mathbb{R})$ is regular for the pair (μ, A). If we then set

$$\vartheta_i(t) := \nabla_i G\big(\mu(t), A(t)\big) := \partial_i G\big(\mu(t), A(t)\big), \qquad i = 1, \dots, d, 0 \le t \le T,$$

the resulting $\vartheta = (\vartheta_1, \dots, \vartheta_d)'$ is in $\mathcal{L}(\mu)$ by Proposition 2.2. Furthermore, we can apply the pathwise Itô formula (2.1) to G to deduce that Γ^G in (3.3) can be cast in the notation of Proposition 2.2 as

$$\begin{aligned}\Gamma^G(\cdot) = &- \sum_{\ell=1}^{m} \int_0^{\cdot} D_\ell G\big(\mu(s), A(s)\big) dA_\ell(s) \\ &- \frac{1}{2} \sum_{i,k=1}^{d} \int_0^{\cdot} \partial^2_{i,k} G\big(\mu(s), A(s)\big) d[\mu_i, \mu_k](s).\end{aligned} \tag{3.4}$$

Regular functions as in Example 3.3 must be sufficiently smooth (at least $\mathbb{C}^{2,1}$) for the pathwise Itô formula to apply. However, usage of the pathwise Tanaka formula accommodates regular functions which are less smooth.

Example 3.4 To use the pathwise Tanaka formula of Proposition 2.7 in place of the pathwise Itô formula of Proposition 2.2, we need a more specific form of regular function G than that of Example 3.3. We assume that μ and A have the same dimension d and that each component μ_i belongs to $\mathcal{L}^c(\mathbb{T})$, i.e., admits a continuous local time, for $i = 1, \dots, d$. Then we set

$$X_i := \mu_i - A_i, \qquad i = 1, \dots, d, \tag{3.5}$$

and assume that each X_i is also in $\mathcal{L}^c(\mathbb{T})$. For any absolutely continuous functions f_i with right-continuous Radon–Nikodým derivatives f_i' of finite variation for every $i = 1, \dots, d$, we define the function $G(m, a) := \sum_{i=1}^d f_i(m_i - a_i)$ for $(m, a) \in \mathbb{R}^{2d}$ and evaluate it along the pair (μ, A) as

$$G\big(\mu(t), A(t)\big) = \sum_{i=1}^d f_i\big(X_i(t)\big) = \sum_{i=1}^d f_i\big(\mu_i(t) - A_i(t)\big), \qquad 0 \le t \le T. \tag{3.6}$$

We claim that the function G in (3.6) is regular for the pair (μ, A). To see this, we start by noting that we are only able to consider such G represented as the sum of individual functions f_i for $i = 1, \dots, d$ because there is no "multidimensional Tanaka formula" that can be applied to G directly. However, we can apply Proposition 2.7 to each component $f_i(X_i(\cdot))$ separately and sum up to obtain

$$\begin{aligned} G\big(\mu(t), A(t)\big) &= G\big(\mu(0), A(0)\big) \\ &\quad + \sum_{i=1}^d \left(\int_0^t f_i'\big(X_i(s)\big) dX_i(s) + \int_{\mathbb{R}} L_t^{X_i}(x) df_i'(x) \right). \end{aligned} \tag{3.7}$$

Furthermore, the Föllmer–Itô integral in (3.7) defined via (2.5) can be decomposed as

$$\int_0^t f_i'\big(X_i(s)\big) dX_i(s) = \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} f_i'\big(X_i(t_j)\big)\big(\mu_i(t_{j+1}) - \mu_i(t_j)\big) \tag{3.8}$$

$$- \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} f_i'\big(X_i(t_j)\big)\big(A_i(t_{j+1}) - A_i(t_j)\big) \tag{3.9}$$

because the limit (3.9) exists as A is in $CBV([0, T]; \mathbb{R}^d)$. Thus the limit (3.8) also exists and we denote (3.8) and (3.9) as $\int_0^t f_i'(X_i(s)) d\mu_i(s)$ and $\int_0^t f_i'(X_i(s)) dA_i(s)$, respectively. Then by setting

$$\vartheta_i(t) = \nabla_i G\big(\mu(t), A(t)\big) := f_i'\big(X_i(t)\big), \qquad i = 1, \dots, d, \tag{3.10}$$

for $0 \le t \le T$, we see that $\vartheta \in \mathcal{L}(\mu)$, and on account of (3.7) and (3.8), the function in (3.3) is seen to be of bounded variation as it takes the form

$$\Gamma^G(t) = \sum_{i=1}^d \int_0^t \vartheta_i(s) dA_i(s) - \sum_{i=1}^d \int_{\mathbb{R}} L_t^{(\mu_i - A_i)}(x) df_i'(x), \qquad 0 \le t \le T. \tag{3.11}$$

From now on, we only consider $\mathbb{C}^{2,1}$ regular functions as in Example 3.3, or regular functions G of the form (3.6) in Example 3.4.

3.1 Additively generated trading strategies

We should like now to introduce an additively generated trading strategy, starting from a regular function in the pathwise sense. This requires a result from [14]. For any function G which is regular for the pair (μ, A), where μ is the vector of market weights and A in $CBV([0, T]; \mathbb{R}^m)$, we consider the vector ϑ with components

$$\vartheta_i(\cdot) = \nabla_i G\big(\mu(\cdot), A(\cdot)\big), \qquad i = 1, \dots, d \tag{3.12}$$

as in (3.2) and the vector $\varphi = (\varphi_1, \dots, \varphi_d)'$ with

$$\varphi_i(t) := \vartheta_i(t) - Q^{\vartheta}(t) - C(0), \qquad i = 1, \dots, d, \quad 0 \le t \le T \tag{3.13}$$

as its components. Here,

$$Q^{\vartheta}(t) := V^{\vartheta}(t) - V^{\vartheta}(0) - \int_0^t \sum_{i=1}^{d} \vartheta_i(s) d\mu_i(s) \tag{3.14}$$

is the so-called "defect of self-financibility" at time $t \in [0, T]$ of the integrand ϑ in (3.12), and $V^{\varphi}(t)$ the "value" of the strategy φ as in Definition 3.1, whereas

$$C(0) := \sum_{i=1}^{d} \nabla_i G\big(\mu(0), A(0)\big)\mu_i(0) - G\big(\mu(0), A(0)\big) \tag{3.15}$$

is the so-called "defect of balance" at time $t = 0$ for the regular function G. By analogy with Proposition 2.3 of [14], the vector $\varphi = (\varphi_1, \dots, \varphi_d)'$ of (3.13), (3.12) defines a trading strategy with respect to μ.

Definition 3.5 We say that the trading strategy φ from (3.13), (3.12) is *additively generated* by $G : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, which is assumed to be regular for the pair (μ, A).

Proposition 3.6 *Consider the trading strategy φ generated additively as in (3.13) by a regular function G for the pair (μ, A). This strategy has value*

$$V^{\varphi}(t) = G\big(\mu(t), A(t)\big) + \Gamma^{G}(t), \qquad 0 \le t \le T, \tag{3.16}$$

as in Definition 3.2, and its components can be represented, for $i = 1, \dots, d$, as

$$\begin{aligned} \varphi_i(t) &= \nabla_i G\big(\mu(t), A(t)\big) + \Gamma^{G}(t) + G\big(\mu(t), A(t)\big) - \sum_{j=1}^{d} \mu_j(t) \nabla_j G\big(\mu(t), A(t)\big) \\ &= V^{\varphi}(t) + \nabla_i G\big(\mu(t), A(t)\big) - \sum_{j=1}^{d} \mu_j(t) \nabla_j G\big(\mu(t), A(t)\big). \end{aligned} \tag{3.17}$$

Proof The proof does not involve any usage of an Itô or Tanaka formula; it is exactly the same as that of Proposition 4.3 of [14] if we change $G(\mu(t))$ and $D_j G(\mu(t))$ there into $G(\mu(t), A(t))$ and $\nabla_j G(\mu(t), A(t))$ in our present context. $\square$

The decomposition (3.16) suggests that we can think of the quantity $\Gamma^G(\cdot)$ in (3.3), (3.4) or (3.11) as expressing the "cumulative earnings" of the strategy φ in (3.13) around the "baseline" $G(\mu(\cdot), A(\cdot))$.

Remark 3.7 (i) When the function G in Proposition 3.6 is "balanced", i.e., if

$$G\big(\mu(t), A(t)\big) = \sum_{j=1}^{d} \mu_j(t)\, \nabla_j G\big(\mu(t), A(t)\big), \qquad 0 \le t \le T, \tag{3.18}$$

the additively generated trading strategy φ of (3.17) takes the simpler form

$$\varphi_i(t) = \nabla_i G\big(\mu(t), A(t)\big) + \Gamma^G(t), \qquad i = 1, \dots, d. \tag{3.19}$$

(ii) For an additively generated trading strategy φ with positive value $V^\varphi > 0$, the corresponding portfolio weights are defined as

$$\pi_i(t) := \frac{\varphi_i(t)\mu_i(t)}{V^\varphi(t)} = \frac{\varphi_i(t)\mu_i(t)}{\sum_{i=1}^d \varphi_i(t)\mu_i(t)}, \qquad i = 1, \dots, d,$$

or, with the help of (3.16) and (3.17), as

$$\pi_i(t) = \mu_i(t)\left(1 + \frac{\nabla_i G(\mu(t), A(t)) - \sum_{j=1}^d \mu_j(t)\nabla_j G(\mu(t), A(t))}{G(\mu(t), A(t)) + \Gamma^G(t)}\right). \tag{3.20}$$

3.2 Multiplicatively generated trading strategies

Next we introduce the notion of a multiplicatively generated trading strategy. We suppose that $G : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ is regular as in Definition 3.2 for the pair (μ, A), where μ is the vector of market weights and A in $CBV([0, T]; \mathbb{R}^m)$, and that the scalar function $1/G(\mu(\cdot), A(\cdot))$ is locally bounded. This holds for example if G is bounded away from zero. We consider the vector $\eta = (\eta_1, \dots, \eta_d)'$ with components

$$\eta_i(\cdot) := \nabla_i G\big(\mu(\cdot), A(\cdot)\big) \exp\left(\int_0^\cdot \frac{d\Gamma^G(t)}{G(\mu(t), A(t))}\right), \qquad i = 1, \dots, d, \tag{3.21}$$

in the notation of (3.3), (3.12). The integral is well defined as $1/G(\mu(\cdot), A(\cdot))$ is assumed to be locally bounded. Moreover, η is in $\mathcal{L}(\mu)$ since $\vartheta = \nabla G(\mu, A) \in \mathcal{L}(\mu)$ from Definition 3.2 and the exponential term is a locally bounded function. We turn η into a trading strategy $\psi = (\psi_1, \dots, \psi_d)'$ as before by setting

$$\psi_i := \eta_i - Q^\eta - C(0), \qquad i = 1, \dots, d, \tag{3.22}$$

in the manner of (3.13) and with Q^η, $C(0)$ defined as in (3.14) and (3.15).

Definition 3.8 The trading strategy $\psi = (\psi_1, \dots, \psi_d)'$ of (3.22), (3.21) is said to be *multiplicatively generated* by the regular function G for the pair (μ, A).

The following result is a multiplicative counterpart of Proposition 3.6. The proof is in the Appendix.

Proposition 3.9 *Consider the trading strategy $\psi = (\psi_1, \dots, \psi_d)'$ generated as in (3.22) by $G \in \mathbb{C}^{2,1}(\mathbb{R}^d \times \mathbb{R}^m; \mathbb{R})$ which is regular for (μ, A), with a suitable function $A \in CBV([0, T]; \mathbb{R}^m)$ such that $1/G(\mu(\cdot), A(\cdot))$ is locally bounded. The value generated by this strategy is given by*

$$V^{\psi}(t) = G\big(\mu(t), A(t)\big) \exp\left(\int_0^t \frac{d\Gamma^G(s)}{G(\mu(s), A(s))} \right) > 0, \qquad 0 \le t \le T \tag{3.23}$$

in the notation of (3.3). *This strategy ψ can be represented for $i = 1, \dots, d$ as*

$$\psi_i(t) = V^{\psi}(t) \left(1 + \frac{\nabla_i G(\mu(t), A(t)) - \sum_{j=1}^d \mu_j(t) \nabla_j G(\mu(t), A(t))}{G(\mu(t), A(t))} \right). \tag{3.24}$$

For the trading strategy ψ multiplicatively generated from the less smooth function G in Example 3.4 and with the notation of (3.6), (3.10), we have a result similar to Proposition 3.9. The proof, given in the Appendix, requires additional attention and computation as there is no "product rule" that can be applied to such functions.

Theorem 3.10 *The trading strategy ψ generated multiplicatively as in* (3.22) *by the regular function G of the form* (3.6) *can be also represented as in* (3.24) *and has value V^{θ} as in* (3.23).

Remark 3.11 (i) When the function G of Theorem 3.10 is "balanced" as in (3.18), the strategy ψ in (3.24) takes the simpler form

$$\psi_i(t) = \vartheta_i(t) \exp\left(\int_0^t \frac{d\Gamma^G(s)}{G(\mu(s), A(s))} \right), \qquad i = 1, \dots, d. \tag{3.25}$$

(ii) The portfolio weights corresponding to ψ are similarly defined as

$$\Pi_i(t) := \frac{\psi_i(t)\mu_i(t)}{\sum_{i=1}^d \psi_i(t)\mu_i(t)} = \mu_i(t) \left(1 + \frac{\vartheta_i(t) - \sum_{j=1}^d \mu_j(t)\vartheta_j(t)}{G(\mu(t), A(t))} \right)$$

for $i = 1, \dots, d$. Here, the last expression follows with the help of (3.23) and (3.24). For a "balanced" function G as in (3.18), this last expression simplifies to

$$\Pi_i(t) = \frac{\mu_i(t)\vartheta_i(t)}{G(\mu(t), A(t))}, \qquad i = 1, \dots, d.$$

4 Sufficient conditions for strong relative arbitrage

We consider the vector $\mu = (\mu_1, \dots, \mu_d)'$ of market weights as in (3.1). For a given trading strategy φ with respect to μ, let us recall the value $V^{\varphi} = \sum_{i=1}^d \varphi_i \mu_i$ from

Definition 3.1. For some fixed $T_* \in (0, T]$, we say that φ is a *strong arbitrage relative to the market* over the interval $[0, T_*]$ if we have

$$V^\varphi(t) \geq 0, \quad \forall\, t \in [0, T_*], \qquad \text{along with } V^\varphi(T_*) > V^\varphi(0). \tag{4.1}$$

The value $V^\varphi(\cdot)$ of a trading strategy generated functionally, either additively or multiplicatively, admits a simple representation in terms of the generating function G and the derived function Γ^G as in (3.16), (3.23). This simple representation provides in turn sufficient conditions for strong relative arbitrage with respect to the market, for example as in [14, Theorems 5.1 and 5.2]. In this section, we find such conditions on trading strategies generated by a regular function $G(\mu(\cdot), A(\cdot))$, which depends not only on the vector of market weights μ, but also on an additional finite-variation process A related to μ. We also give new sufficient conditions leading to strong relative arbitrage for strategies generated both additively and multiplicatively, which are different from [14, Theorems 5.1 and 5.2].

We have not yet specified the function $A \in CBV([0, T]; \mathbb{R}^m)$, so it is time to consider some plausible candidates. A first one would be the vector

$$A = [\mu] = ([\mu_1], [\mu_2], \ldots, [\mu_d])' \tag{4.2}$$

of quadratic variations for the market weights. We can also think of a more general candidate, namely the S_d^+-valued covariation process of market weights. Here, S_d^+ is the collection of symmetric, nonnegative definite $d \times d$ matrices, and we use double brackets [[ ]] to distinguish this d^2-dimensional vector from (4.2), i.e.,

$$A = [[\mu]], \quad A_{i,j} = [\mu_i, \mu_j], \qquad 1 \leq i, j \leq d. \tag{4.3}$$

Choosing A as in (4.3), we can match the integrators of the two integrals in (3.4) and the resulting expression for $\Gamma^G(\cdot)$ can then be cast as one integral.

There are many other functions of finite variation which can be candidates for the process A. We list some examples below:

1) The moving average $\bar{\mu}$ of μ defined for $i = 1, \ldots, d$ by

$$\bar{\mu}_i(t) := \begin{cases} \frac{1}{\delta}\int_0^t \mu_i(s)ds + \frac{1}{\delta}\int_{t-\delta}^0 \mu_i(0)ds, & t \in [0, \delta), \\ \frac{1}{\delta}\int_{t-\delta}^t \mu_i(s)ds, & t \in [\delta, T]. \end{cases}$$

2) The running maximum μ^* and the running minimum μ_* of the market weights, with the components $\mu_i^*(t) := \max_{0 \leq s \leq t} \mu_i(s)$, $\mu_{*i}(t) := \min_{0 \leq s \leq t} \mu_i(s)$, respectively, for $i = 1, \ldots, d$.

3) The vector $L_{\cdot}^{\mu}(\alpha) = (L_{\cdot}^{\mu_1}(\alpha), \ldots, L_{\cdot}^{\mu_d}(\alpha))$ of pathwise local times $L_{\cdot}^{\mu_i}(\alpha)$ of μ_i at some real number $\alpha \in (0, 1)$, for $i = 1, \ldots, d$, as defined in Sect. 2.2.

Since the vectors $\bar{\mu}$, μ^*, μ_* and $L_{\cdot}^{\mu}(\alpha)$ are d-dimensional, $m = d$ holds for these choices of A. Empirical results using the moving average $\bar{\mu}$ are given in [21, Sect. 3]. The running maximum μ^* and minimum μ_* appear in Sects. 5 and 6. The vector $L_{\cdot}^{\mu}(\alpha)$ of pathwise local times plays an important role in Examples 4.4, 4.9.

We first consider conditions leading to strong relative arbitrage with respect to the market with a general A as the second input of the generating function G. Then we present some examples of G with specific finite-variation functions A chosen from among the above candidates, and provide empirical results for these examples.

4.1 Additively generated strong relative arbitrage

We start with a condition leading to additively generated strong arbitrage, which is similar to [14, Theorem 5.1].

Proposition 4.1 *Fix a function* $G : \mathbb{R}^d \times \mathbb{R}^m \to [0, \infty)$ *which is regular for the pair* (μ, A), *and such that the function* $\Gamma^G(\cdot)$ *in* (3.3) *is nondecreasing. For some real number* $T_* > 0$, *suppose that*

$$\Gamma^G(T_*) > G\big(\mu(0), A(0)\big). \tag{4.4}$$

Then the trading strategy φ *additively generated by* G *as in Definition* 3.5 *is a strong arbitrage relative to the market over every interval* $[0, t]$ *with* $T_* \le t \le T$.

Proof Since $\Gamma^G(\cdot)$ is nondecreasing, we obtain from (3.16) that

$$V^\varphi(t) = G\big(\mu(t), A(t)\big) + \Gamma^G(t) \ge \Gamma^G(0) = 0 \qquad \text{for every } t \in [0, T_*].$$

We also have

$$V^\varphi(t) = G\big(\mu(t), A(t)\big) + \Gamma^G(t) \ge \Gamma^G(T_*) > G\big(\mu(0), A(0)\big) = V^\varphi(0)$$

for every $t \in [T_*, T]$. The last equality holds because $\Gamma^G(0) = 0$. □

Remark 4.2 With $A = [[\mu]]$ as in (4.3), the function $\Gamma^G(\cdot)$ of (3.4) is nondecreasing when

$$-\sum_{i,k=1}^{d} \int_0^{\cdot} \left(D^1_{(i,k)} + \frac{1}{2}\partial^2_{i,k}\right) G\big(\mu(s), [[\mu]](s)\big) d[\mu_i, \mu_j](s)$$

is nondecreasing. Here, $D^1_{(i,k)}$ denotes the first-order partial derivative operator with respect to the (i, k)-th entry of $[[\mu]]$. We can substitute from (4.3), (3.4) into (4.4) to obtain the more explicit form

$$-\sum_{i,k=1}^{d} \int_0^{T_*} \left(D^1_{(i,k)} + \frac{1}{2}\partial^2_{i,k}\right) G\big(\mu(s), [[\mu]](s)\big) d[\mu_i, \mu_j](s) > G\big(\mu(0), [[\mu]](0)\big)$$

of the condition (4.4) for strong relative arbitrage. Thus unlike the situation of [14, Theorem 3.7], we can have a nondecreasing Γ^G and a chance for achieving strong relative arbitrage even without "concavity" of G in μ.

Remark 4.3 Suppose the arguments μ and A are "additively separated" in the smooth regular function $G \in \mathbb{C}^{2,1}(\mathbb{R}^d \times \mathbb{R}^m; \mathbb{R})$ of Example 3.3. This means that there exist two functions K and H such that K depends only on $\mu(t)$ and H only on $A(t)$ and

$$G\big(\mu(t), A(t)\big) = K\big(\mu(t)\big) + H\big(A(t)\big), \qquad \forall\, t \in [0, T]. \tag{4.5}$$

Then $\partial^2_{i,k} G(\mu(t), A(t)) = \partial^2_{i,k} K(\mu(t))$ and $D_\ell G(\mu(t), A(t)) = D_\ell H(A(t))$ hold. Substituting these expressions into (3.4), we obtain

$$\begin{aligned}\Gamma^G(T_*) = &- \sum_{\ell=1}^{m} \int_0^{T_*} D_\ell H\big(A(s)\big) dA_\ell(s) \\ &- \frac{1}{2} \sum_{i,k=1}^{d} \int_0^{T_*} \partial^2_{i,k} K\big(\mu(s)\big) d[\mu_i, \mu_k](s),\end{aligned} \tag{4.6}$$

and from (3.16) of Proposition 3.6, the relative value process of the additively generated trading strategy φ from G can be expressed as

$$V^\varphi(T_*) = K\big(\mu(T_*)\big) + H\big(A(T_*)\big) + \Gamma^G(T_*), \quad V^\varphi(0) = K\big(\mu(0)\big) + H\big(A(0)\big). \tag{4.7}$$

After inserting (4.6), (4.7) into (4.1) and rearranging terms in such a manner that the left-hand side contains only terms involving K, the strong arbitrage condition (4.1) takes the form

$$K\big(\mu(T_*)\big) - K\big(\mu(0)\big) - \frac{1}{2} \sum_{i,k=1}^{d} \int_0^{T_*} \partial^2_{i,k} K\big(\mu(s)\big) d[\mu_i, \mu_k](s) > B_H\big(A(T_*)\big), \tag{4.8}$$

where

$$B_H\big(A(T_*)\big) := -H\big(A(T_*)\big) + H\big(A(0)\big) + \sum_{\ell=1}^{m} \int_0^{T_*} D_\ell H\big(A(s)\big) dA_\ell(s).$$

When we apply the pathwise Itô formula of Proposition 2.2 to the function $H(A(t))$, $0 \le t \le T$, the right-hand side of the above expression vanishes. Hence the requirement (4.8) becomes

$$K\big(\mu(T_*)\big) - \frac{1}{2} \sum_{i,k=1}^{d} \int_0^{T_*} \partial^2_{i,k} K\big(\mu(s)\big) d[\mu_i, \mu_k](s) > K\big(\mu(0)\big),$$

and we are in very similar situation as in [14, Theorem 5.1].

To be more precise, if K takes nonnegative values and is a "Lyapunov function" in the sense that $\Gamma^K(t) := -\frac{1}{2} \sum_{i,k=1}^{d} \int_0^t \partial^2_{i,k} K(\mu(s)) d[\mu_i, \mu_k](s)$ is nondecreasing, then the requirement $\Gamma^K(T_*) > K(\mu(0))$ ensures strong relative arbitrage over every interval $[0, t]$ with $T_* \le t \le T$. Thus in this "separated" case, we cannot achieve more than the result in [14, Theorem 5.1], as all terms on the right-hand side of (4.8) that involve H vanish. This is because when we generate additively the trading strategy φ in (3.13) from a regular function $G \in \mathbb{C}^{2,1}(\mathbb{R}^d \times \mathbb{R}^m; \mathbb{R})$, only the partial derivatives of G with respect to the market weights in (3.12) are involved in φ, and this makes the H-term in (4.5) meaningless in generating φ. Therefore, in order to be able to find new sufficient conditions for strong relative arbitrage, we need forms of G more sophisticated than (4.5). All the examples of G we develop in this paper from now onwards are of such more elaborate forms.

Concave functions such as $x \mapsto -x^2$ and $x \mapsto -x \log x$, when used to generate trading strategies, produce nondecreasing functions Γ^G in (3.4); this is because such functions have negative semidefinite Hessians $\partial^2 G = (\partial_{i,k} G)_{1 \le i,k \le d}$, which play the role of integrand in the last integral of (3.4). Such concavity is known to lead to "diversity-weighted" investment strategies as explained in [9, Definition 3.4.1]. However, these concave functions had to be twice differentiable to apply Itô's rule. Now we can use concave but not differentiable functions, while still being able to generate portfolios with the help of the Tanaka formula. Typical such examples are $x \mapsto -x^+ := -\max(x, 0)$ and $x \mapsto -x^- := -\min(x, 0)$.

Example 4.4 Consider a constant $\alpha \in (0, 1)$ and the function

$$f(x) := \frac{1}{d} - (x - \alpha)^+, \qquad x \in \mathbb{R},$$

which satisfies the conditions in Proposition 2.7. Then for the pair (μ, A) with $A \equiv 0$, we have $X \equiv \mu$ in (3.5) and can set $f = f_i$ for $i = 1, \ldots, d$ to obtain the analogue

$$G\big(\mu(t)\big) = 1 - \sum_{i=1}^{d} \big(\mu_i(t) - \alpha\big)^+ \tag{4.9}$$

of (3.6), which is nonnegative by construction. Here, α plays the role of a threshold on the market weights: we only include in our generating function those stocks whose market weights exceed the level α. From (3.10) and (3.11), for $0 \le t \le T$,

$$\vartheta_i(t) = -\mathbb{1}_{\{\mu_i(t) \ge \alpha\}}, \quad i = 1, \ldots, d, \qquad \text{and} \qquad \Gamma^G(t) = \sum_{i=1}^{d} L_t^{\mu_i}(\alpha). \tag{4.10}$$

Note that $\Gamma^G(\cdot)$ is nondecreasing and increases whenever a market weight hits the threshold α. The strategy φ additively generated as in (3.13) can be represented as

$$\varphi_i(t) = -\mathbb{1}_{\{\mu_i(t) \ge \alpha\}} + \sum_{j=1}^{d} \mathbb{1}_{\{\mu_j(t) \ge \alpha\}} \mu_j(t) + V^\varphi(t), \qquad i = 1, \ldots, d, \tag{4.11}$$

by Proposition 3.6 and has value

$$V^\varphi(t) = 1 - \sum_{i=1}^{d} \big(\mu_i(t) - \alpha\big)^+ + \sum_{i=1}^{d} L_t^{\mu_i}(\alpha), \qquad 0 \le t \le T.$$

Since Γ^G is nondecreasing, we can use the condition in Proposition 4.1: Strong arbitrage relative to the market exists over every interval $[0, t]$ with $T_* \le t \le T$ satisfying

$$\Gamma^G(T_*) = \sum_{i=1}^{d} L_{T_*}^{\mu_i}(\alpha) > G\big(\mu(0)\big) = 1 - \sum_{i=1}^{d} \big(\mu_i(0) - \alpha\big)^+.$$

In the expression of $\varphi_i(t)$ in (4.11), the sum $\sum_{j=1}^d \mathbb{1}_{\{\mu_j(t)\geq\alpha\}} + V^\varphi(t)$ is a universal term, the same for all indices $i = 1, \dots, d$. Thus φ invests one currency unit less to this universal baseline amount for those "big-capitalization stocks" whose market weights exceed the threshold α. Therefore we can interpret the strategy φ of (4.11) as outperforming the market by investing more in "small-capitalization stocks". This is in broad agreement with results in stochastic portfolio theory to the effect that "tilting" in favor of small capitalization stocks, as opposed to their larger brethren, can lead to superior results under appropriate conditions.

Example 4.5 In Example 4.4, we compared the individual market weights $\mu_i(t)$ with a fixed constant α to determine whether to include them in the generating function or not. Now we extend this idea by comparing current with past market weights. To be specific, we want our trading strategy to depend on the difference between $\mu(t)$ and $\mu(t-\delta)$ for some fixed $\delta > 0$. To do this, we enlarge the domain of each μ_i from $[0, T]$ to $[-\delta, T]$. This can be done because even before we start investing in our strategy at time $t = 0$, there must be past stock prices and past market weights. We simply attach these past data to the left of the time line so as to extend its domain.

Furthermore, since the evolution of $\mu(t-\delta)$ is as rough as the original path $\mu(t)$, we need to make it smoother. Thus we take the moving average of market weights over a very small time interval $[t-\delta,\ t-\delta+\theta]$ for some small θ satisfying $0 < \theta < \delta$, and we use this moving average instead of $\mu(t-\delta)$. Therefore we introduce for each $i = 1, \dots, d$ as an estimate of $\mu_i(t-\delta)$ the function of finite variation

$$A_i(t) := \frac{1}{\theta}\int_{t-\delta}^{t-\delta+\theta} \mu_i(s)ds, \qquad 0 \leq t \leq T.$$

Now we introduce the nonnegative quantity

$$G\big(\mu(t)\big) = 1 - \sum_{i=1}^d \big(\mu_i(t) - A_i(t)\big)^+, \qquad 0 \leq t \leq T.$$

This includes those stocks whose current market weight $\mu_i(t)$ is bigger than or equal to its (estimate of) past market weight $\mu_i(t-\delta)$. This G is also very similar to that of (4.9), with the difference that the threshold α is replaced by the stock-specific level $A_i(t)$, capturing the "momentum effect". In this manner, we compute the quantities of (3.10), (3.11) as

$$\vartheta_i(t) = -\mathbb{1}_{\{\mu_i(t)\geq A_i(t)\}}, \qquad i = 1, \dots, d,$$
$$\Gamma^G(t) = -\sum_{i=1}^d \int_0^t \mathbb{1}_{\{\mu_i(s)\geq A_i(s)\}} dA_i(s) + \sum_{i=1}^d L_t^{(\mu_i - A_i)}(0), \tag{4.12}$$

with the continuous local time $L_t^{(\mu_i-A_i)}(0)$ of $\mu_i - A_i$ at the origin as in Definition 2.6. In the integral above, the integrand $\mathbb{1}_{\{\mu_i(s)\geq A_i(s)\}}$ is a quantity observable at time s, whereas the integrator $dA_i(s)$ represents the increment of the moving average of μ_i over the time interval $[s-\delta,\ s-\delta+\theta]$, also an observable value at time s. Therefore this integral can be computed at any time between 0 and T, even though integrand and integrator are from different times. The last term in (4.12) is nondecreasing, but the integral term is generally not monotone.

The trading strategy φ additively generated in the manner of (3.13) and its value V^φ can now be represented by Proposition 3.6 as

$$\begin{aligned}\varphi_i(t) &= -\mathbb{1}_{\{\mu_i(t)\ge A_i(t)\}} \\ &\quad + \sum_{j=1}^d \mathbb{1}_{\{\mu_j(t)\ge A_j(t)\}}\mu_j(t) + V^\varphi(t), \qquad i=1,\dots,d, \end{aligned} \tag{4.13}$$

$$\begin{aligned}V^\varphi(t) &= 1 - \sum_{i=1}^d \big(\mu_i(t) - A_i(t)\big)^+ \\ &\quad - \sum_{i=1}^d \int_0^t \mathbb{1}_{\{\mu_i(s)\ge A_i(s)\}} dA_i(s) + \sum_{i=1}^d L_t^{(\mu_i - A_i)}(0).\end{aligned}$$

Since the function $\Gamma^G(\cdot)$ of (4.12) is no longer monotone, it is hard to formulate appropriate conditions for strong relative arbitrage in this context. We note, however, that the strategy φ in (4.13) invests one unit of currency less in stocks whose current market weight is bigger than or equal to its (estimate of) past value.

From (3.16), the value $V^\varphi(t)$ at time t of the additively generated trading strategy φ in (3.13) has two additive components, $G(\mu(t), A(t))$ and $\Gamma^G(t)$. In Proposition 4.1, we derived the strong arbitrage condition from the "nondecrease" of $\Gamma^G(\cdot)$, but there is no a priori reason to differentiate between $G(\mu(t), A(t))$ and $\Gamma^G(t)$. If $t \mapsto G(\mu(t), A(t))$ is nondecreasing, it is possible derive a strong arbitrage condition like Proposition 4.1, switching the roles of $G(\mu(t), A(t))$ and $\Gamma^G(t)$. However, it is difficult to find functions G for which $t \mapsto G(\mu(t), A(t))$ is monotone, because G must depend on the market weights $\mu(\cdot)$ and these fluctuate all the time. Thus we have to "extract a nondecreasing structure" from the generating function $G(\mu(\cdot), A(\cdot))$ and use this structure, instead of G itself, to derive a new strong arbitrage condition. This is done as follows.

Theorem 4.6 *Fix a regular function $G : \mathbb{R}^d \times \mathbb{R}^m \to [0,\infty)$ for the pair (μ, A) such that the following conditions are satisfied*:

(i) $V^\varphi(\cdot) = G(\mu(\cdot), A(\cdot)) + \Gamma^G(\cdot) \ge 0$, *with* $\Gamma^G(\cdot)$ *as in* (3.3) *or* (3.4);
(ii) *there exists a function* $F(\mu(\cdot), A(\cdot))$ *satisfying* $G(\mu(t), A(t)) \ge F(\mu(t), A(t))$ *for all* $t \in [0,T]$ *and the mapping* $t \mapsto F(\mu(t), A(t))$ *is nondecreasing*;
(iii) $\Gamma^G(\cdot) \ge -\kappa$ *holds for some constant* κ.

For some real number $T_* > 0$, *suppose that*

$$F\big(\mu(T_*), A(T_*)\big) > G\big(\mu(0), A(0)\big) + \kappa. \tag{4.14}$$

Then the additively generated strategy φ of Definition 3.5 is a strong arbitrage relative to the market over every interval $[0,t]$ with $T_* \le t \le T$.

Proof The first inequality of (4.1) is satisfied due to (i). From (ii), (iii) and (3.16), (4.14), we obtain also the second inequality of (4.1) since for every $t \in [T_*, T]$,

$$\begin{aligned} V^{\varphi}(t) = G\big(\mu(t), A(t)\big) + \Gamma^G(t) &\geq F\big(\mu(t), A(t)\big) - \kappa \\ &\geq F\big(\mu(T_*), A(T_*)\big) - \kappa > G\big(\mu(0), A(0)\big) \\ &= V^{\varphi}(0). \end{aligned}$$

□

In Theorem 4.6, the function $F(\mu(\cdot), A(\cdot))$ can be seen as the "nondecreasing structure extracted from G". This result states that the generating function G can lead to strong arbitrage relative to the market without necessarily being "Lyapunov" as in [14, Theorem 5.1]. There can be a strong relative arbitrage even if $\Gamma^G(\cdot)$ is nonincreasing. This is intuitively plausible already on the basis of the representation (3.16) when $G(\mu(\cdot), A(\cdot))$ grows faster than $\Gamma^G(\cdot)$ decays. Some applications of Theorem 4.6 appear in Sect. 5 (Examples 5.4 and 5.6).

4.2 Multiplicatively generated strong relative arbitrage

To simplify arguments, **we assume in this subsection** that the regular function G has nonnegative values and satisfies $G(\mu(0), A(0)) = 1$. This can be achieved by replacing G by $G/G(\mu(0), A(0))$ if $G(\mu(0), A(0)) > 0$, or by $G+1$ if $G(\mu(0), A(0)) = 0$.

Theorem 4.7 *Let us fix* $G : \mathbb{R}^d \times \mathbb{R}^m \to [\alpha, \beta]$ *with* $0 < \alpha < 1 < \beta < \infty$ *which is regular for the pair* (μ, A) *and for which* Γ^G *in* (3.3) *is nondecreasing. For some real number* $T_* > 0$, *suppose that*

$$\Gamma^G(T_*) > \beta \log \frac{1}{\alpha}. \tag{4.15}$$

Then the multiplicatively generated strategy ψ *of Definition* 3.8 *is a strong arbitrage relative to the market over every interval* $[0, t]$ *with* $T_* \leq t \leq T$.

Proof First, we note that $V^{\psi}(\cdot) > 0$ from (3.23). Taking logarithms on both sides of (3.23), we obtain for all $T_* \leq t \leq T$ that

$$\begin{aligned} \log V^{\psi}(t) = \log G\big(\mu(t), A(t)\big) + \int_0^t \frac{d\Gamma^G(s)}{G(\mu(s), A(s))} \geq \log \alpha + \frac{1}{\beta}\Gamma^G(t) \\ \geq \log \alpha + \frac{1}{\beta}\Gamma^G(T_*) > 0 = \log G\big(\mu(0), A(0)\big) = \log V^{\psi}(0), \end{aligned}$$

and the result follows. Here $G(\mu(0), A(0)) = 1$ by the normalization imposed on G. □

Remark 4.8 Since the market weights μ_i, $i = 1, \dots, d$, and the continuous function A are bounded on the compact interval $[0, T]$, a regular function G as in Example 3.3 or Example 3.4, depending on the pair (μ, A), is also bounded. Thus the boundedness condition in Theorem 4.7 just requires the lower bound α to be strictly positive. Also, in (4.15), tighter bounds α, β for G yield smaller T_* satisfying the arbitrage condition (4.15). See Remark 5.2 for further discussion regarding the bounds on G in the case of specific entropy functions.

Example 4.9 Recall the generating function G of (4.9) in Example 4.4 and add a small constant $\epsilon > 0$ to have

$$G\big(\mu(t)\big) = (1+\epsilon) - \sum_{i=1}^{d} \big(\mu_i(t) - \alpha\big)^+,$$

with the same ϑ and the same Gamma function as in (4.10). The reason for inserting the constant $\epsilon > 0$ is to ensure the uniform bounds $\epsilon \le G(\mu(\cdot)) \le 1 + \epsilon$ regardless of the choice of $\alpha \in (0, 1)$, so that $1/G$ is locally bounded. The trading strategy ψ multiplicatively generated by this G as in Definition 3.8 can be represented as

$$\psi_i(t) = -K(t)\mathbb{1}_{\{\mu_i(t)\ge\alpha\}} + \sum_{j=1}^{d} K(t)\mathbb{1}_{\{\mu_j(t)\ge\alpha\}}\mu_j(t) + V^{\varphi}(t) \tag{4.16}$$

for $i = 1, \dots, d$ by Theorem 3.10, and its value is given as

$$V^{\psi}(t) = \left((1+\epsilon) - \sum_{i=1}^{d} \big(\mu_i(t) - \alpha\big)^+\right) K(t),$$

$$K(t) := \exp\left(\int_0^t \sum_{i=1}^{d} \frac{dL_s^{\mu_i}(\alpha)}{1+\epsilon - \sum_{j=1}^{d}(\mu_j(s)-\alpha)^+}\right).$$

From Theorem 4.7, strong arbitrage with respect to the market exists over every interval $[0, t]$ with $T_* \le t \le T$ satisfying the inequality

$$\Gamma^G(T_*) = \sum_{i=1}^{d} L_{T_*}^{\mu_i}(\alpha) > (1+\epsilon)\log\frac{1+\epsilon - \sum_{i=1}^{d}(\mu_i(0)-\alpha)^+}{\epsilon}.$$

In the manner of Example 4.4, the strategy ψ in (4.16) invests $K(t)$ units of currency less than the "baseline amount" $\sum_{j=1}^{d} K(t)\mathbb{1}_{\{\mu_j(t)\ge\alpha\}}\mu_j(t) + V^{\varphi}(t)$ in those "big-capitalization stocks" whose market weight exceeds the threshold α at time t. Because $K(\cdot)$ is nondecreasing, ψ keeps investing less and less money to those "big-capitalization stocks" as time goes by, and the "size effect" increases gradually.

The conditions of Theorem 4.7 resemble those of Proposition 4.1. We also have the following formulation, which is similar to Theorem 4.6.

Theorem 4.10 *Fix a regular function $G : \mathbb{R}^d \times \mathbb{R}^m \to (0, \infty)$ for the pair (μ, A) such that the following conditions hold*:

(i) *there exists an $F : \mathbb{R}^{d+m} \to (0, \infty)$ such that $G(\mu(t), A(t)) \ge F(\mu(t), A(t))$ for all $t \in [0, T]$ and the mapping $t \mapsto F(\mu(t), A(t))$ is nondecreasing*;
(ii) *$\Gamma^G(\cdot)$ is nonincreasing and $\Gamma^G(\cdot) \ge -\kappa$ holds for some positive constant κ.*

For some real number $T_* > 0$, *suppose that*

$$\log F\big(\mu(T_*), A(T_*)\big) > \frac{\kappa}{F(\mu(0), A(0))}. \tag{4.17}$$

Then the multiplicatively generated strategy ψ *of Definition* 3.8 *is a strong arbitrage relative to the market over every interval* $[0, t]$ *with* $T_* \le t \le T$.

Proof First note that $\Gamma^G(\cdot)$ is nonpositive due to (ii) and $\Gamma^G(0) = 0$. Again, from (3.23), we have

$$\begin{aligned}\log V^{\psi}(t) &= \log G\big(\mu(t), A(t)\big) + \int_0^t \frac{d\Gamma^G(s)}{G(\mu(s), A(s))} \\ &\ge \log F\big(\mu(t), A(t)\big) - \frac{\kappa}{\min_{0\le s\le t} G(\mu(s), A(s))} \\ &\ge \log F\big(\mu(t), A(t)\big) - \frac{\kappa}{\min_{0\le s\le t} F(\mu(s), A(s))} \\ &\ge \log F\big(\mu(T_*), A(T_*)\big) - \frac{\kappa}{F(\mu(0), A(0))} > 0 = \log V^{\psi}(0)\end{aligned}$$

for all $T_* \le t \le T$ by (i), (ii) and (4.17). □

The example that follows provides a condition for strong relative arbitrage more general than [14, Example 5.5] by deploying an additional function A into the generating function G. We specifically use $A = \mu^* = (\mu_1^*, \dots, \mu_d^*)$, the vector consisting of the running maxima of the market weights $\mu_i^*(t) := \max_{0\le s\le t} \mu_i(s)$, $i = 1, \dots, d$.

Example 4.11 For fixed constants $c \in \mathbb{R}$ and $p > 0$, consider

$$\begin{aligned}G^{(c,p)}\big(\mu(t), \mu^*(t)\big) &:= c - \sum_{i=1}^{d} \big(\mu_i(t)\big)^2 - p\sum_{i=1}^{d} \mu_i(t)\mu_i^*(t) \\ &= c - \sum_{i=1}^{d} \big(\mu_i(t)\big)^2 - p\sum_{i=1}^{d} \mu_i(t) \max_{0\le s\le t} \mu_i(s).\end{aligned}$$

This is the same as $Q^{(c)}$ in [14, Example 5.5] except for the last term. Note that $G^{(c,p)}$ takes values in the interval $[c - (1 + p),\ c - \frac{1}{d}(1 + p)]$. Straightforward computation of partial derivatives gives for $i = 1, \dots, d$ that $D_i G^{(c,p)}(\mu(t), \mu^*(t)) = -p\mu_i(t)$, $\partial_i G^{(c,p)}(\mu(t), \mu^*(t)) = -2\mu_i(t) - p\mu_i^*(t)$, $\partial^2_{i,i} G^{(c,p)}(\mu(t), \mu^*(t)) = -2$, and using these expressions along with (3.4), we obtain

$$\Gamma^{G^{(c,p)}}(t) = \sum_{i=1}^{d} \int_0^t p\mu_i(s)d\mu_i^*(s) + \sum_{i=1}^{d} [\mu_i](t).$$

As $\mu_i^*(\cdot)$ is nondecreasing and $p\mu_i(\cdot) \ge 0$, the integral is always nonnegative and nondecreasing in t, which makes $\Gamma^{G^{(c,p)}}(\cdot)$ nondecreasing and nonnegative. Also, the nondecreasing $\mu_i^*(\cdot)$ is flat off the set $\{s \geq 0 : \mu_i(s) = \mu_i^*(s)\}$ so that

$$\int_0^t \mu_i(s) d\mu_i^*(s) = \int_0^t \mu_i^*(s) d\mu_i^*(s) = \frac{1}{2}\Big(\big(\mu_i^*(t)\big)^2 - \big(\mu_i^*(0)\big)^2\Big),$$

$$\Gamma^{G^{(c,p)}}(t) = \frac{p}{2}\sum_{i=1}^d \Big(\big(\mu_i^*(t)\big)^2 - \big(\mu_i^*(0)\big)^2\Big) + \sum_{i=1}^d [\mu_i](t).$$

Since $G^{(1+p,p)} \geq 0$, we consider the case $c = 1 + p$ from now on. Arguing as in the proof of Proposition 4.1, the condition

$$\frac{p}{2}\sum_{i=1}^d \Big(\big(\mu_i^*(T)\big)^2 - \big(\mu_i^*(0)\big)^2\Big) + \sum_{i=1}^d [\mu_i](T) > G^{(1+p,p)}\big(\mu(0), \mu^*(0)\big), \tag{4.18}$$

where $G^{(1+p,p)}(\mu(0), \mu^*(0)) = (1 + p)(1 - \sum_{i=1}^d (\mu_i(0))^2) > 0$, yields a strategy which is a strong arbitrage relative to the market on $[0, T]$. If we compare the condition (4.18) with the condition (5.4) of [14, Example 5.5], i.e.,

$$\sum_{i=1}^d [\mu_i](T) > 1 - \sum_{i=1}^d \big(\mu_i(0)\big)^2, \tag{4.19}$$

there is a trade-off between the left- and right-hand sides. The presence of the extra nondecreasing term $(p/2)\sum_{i=1}^d ((\mu_i^*(T))^2 - (\mu_i^*(0))^2)$ in (4.18) guarantees that its left-hand side grows faster than the left-hand side of (4.19) as T increases; but we also have a bigger constant on the right-hand side of (4.18), namely

$$(1 + p)\bigg(1 - \sum_{i=1}^d \big(\mu_i(0)\big)^2\bigg) > 1 - \sum_{i=1}^d \big(\mu_i(0)\big)^2.$$

Thus by choosing the value of p wisely, we can obtain bounds for the times T for which there is a strong relative arbitrage with respect to the market over $[0, T]$ which are better than those of [14, Example 5.5].

Additional interesting applications of Theorems 4.7 and 4.10 appear in Sect. 5.

5 Examples of entropic functions

In this section, we present some examples of trading strategies additively and multiplicatively generated from variants of the "entropy function", and the corresponding conditions for strong relative arbitrage introduced in Sect. 4. Empirical results regarding these examples are given in the next section.

Consider the Gibbs entropy function

$$H(x) = -\sum_{i=1}^d x_i \log x_i, \qquad x \in (0, 1)^d, \tag{5.1}$$

with values in $(0, \log d)$. Being nonnegative, twice differentiable and concave, this is one of the most frequently used functions in stochastic portfolio theory. See [9, 12, 14] for its usage in generating portfolios and also [20, 21] for some variants of portfolios generated by this function.

Example 5.1 In order to compare the trading strategy generated by the original entropy function with those generated from variants of functions related to it, we first derive and summarize the former strategy. Consider the "shifted entropy"

$$G\big(\mu(t)\big) := -\sum_{i=1}^{d} \mu_i(t) \log\big(p\mu_i(t)\big) = -\log p - \sum_{i=1}^{d} \mu_i(t) \log \mu_i(t) \tag{5.2}$$

for some given constant $p \geq 1$. This coincides with the original entropy $H(\mu(t))$ in (5.1) when $p = 1$; the reason for inserting the additive constant is explained in the following remark. From (3.4), (3.17) and (3.24), the additively generated trading strategy φ and the multiplicatively generated ψ from G can be represented as

$$\begin{aligned} \varphi_i(t) &= -\log\big(p\mu_i(t)\big) + \Gamma^G(t), \\ \psi_i(t) &= -\exp\left(\int_0^t \frac{d\Gamma^G(s)}{G(\mu(s))}\right) \log\big(p\mu_i(t)\big), \end{aligned} \tag{5.3}$$

for $i = 1, \ldots, d$, where

$$\Gamma^G(t) = \sum_{i=1}^{d} \int_0^t \frac{d[\mu_i](s)}{2\mu_i(s)}$$

is nondecreasing in t. The values of these trading strategies are given via (3.16) and (3.23). Note that φ and ψ in (5.3) have relatively simple forms because G in (5.2) is "almost balanced", in the sense that

$$G\big(\mu(\cdot)\big) - 1 = \sum_{j=1}^{d} \mu_j(\cdot) \partial_j G\big(\mu(\cdot)\big)$$

holds; compare this with (3.18), and (5.3) with (3.19) and (3.25). Then the condition (4.4) for additively generated strong arbitrage in Proposition 4.1 becomes

$$\sum_{i=1}^{d} \int_0^{T_*} \frac{d[\mu_i](s)}{2\mu_i(s)} > -\sum_{i=1}^{d} \mu_i(0) \log\big(p\mu_i(0)\big), \tag{5.4}$$

whereas the condition (4.15) for multiplicatively generated strong arbitrage in Theorem 4.7 is

$$\sum_{i=1}^{d} \int_0^{T_*} \frac{d[\mu_i](s)}{2\mu_i(s)} > \beta \log \frac{-\sum_{i=1}^{d} \mu_i(0) \log(p\mu_i(0))}{\alpha}.$$

Here, the constants α, β are the lower and upper bounds on G in Theorem 4.7.

Remark 5.2 The construction of trading strategies described in the previous sections does not require any optimization or statistical estimation of parameters. However, the relative performance of trading strategies with respect to the market can be improved by introducing a parameter, or set of parameters, in the generating function G. To achieve strong relative arbitrage faster, or to find the smallest T_* satisfying (5.4), or more generally (4.4), it helps to be able to make smaller the "threshold" value $G(\mu(0), A(0))$ on the right-hand side of the inequality, while keeping the "growth rate" of $\Gamma^G(\cdot)$ fixed.

It is in this spirit that we introduced the parameter p in (5.2). Inserting $p > 1$ inside the logarithm makes the initial value $G(\mu(0))$ smaller by the amount $\log p$; at the same time, this does not affect $\Gamma^G(\cdot)$, as subtracting a constant from G does not change any derivatives of G. However, if we pick p so large that $-\sum_{i=1}^d \mu_i(t) \log \mu_i(t) < \log p$ holds at some time t, then $G(\mu(t), A(t))$ has a negative value. Theoretically, $-\sum_{i=1}^d \mu_i(t) \log \mu_i(t)$ has the minimum value of 0 only when one of the market weights is equal to 1 and all the other weights vanish, which does not happen in the real world. Empirically, $-\sum_{i=1}^d \mu_i(t) \log \mu_i(t)$ is always bounded away from zero, and we can guarantee this condition theoretically by imposing a weak condition on the market weights. For example, restricting the maximum value of the market weights, say $\max_i \mu_i(\cdot) \le 0.5$, yields an additional condition on the market weights: there must be an index $j \in \{1, \dots, d\}$ such that

$$\mu_j(t) \ge \frac{0.5}{d-1}, \qquad 0 \le t \le T,$$

as $\sum_{i=1}^d \mu_i \equiv 1$. Then the value of $-\sum_{i=1}^d \mu_i(t) \log \mu_i(t)$ is at least $-\frac{0.5}{d-1} \log \frac{0.5}{d-1}$ and hence bounded away from 0 at all times. Finding a suitable value of $p > 1$ while maintaining $G(\mu(\cdot))$ bounded away from 0 by α should be done statistically and depends on d, the number of stocks. It is quite straightforward that $G(\mu(\cdot))$ is bounded from above by some constant β as $x \mapsto -x \log x$ has the maximum value $1/e$. An empirical estimation of such p can be found in the next section.

Making the initial value of $G(\mu(0), A(0))$ small while keeping the growth rate of $\Gamma^G(\cdot)$ is also beneficial for calculating the "excess return rate" of a trading strategy φ with respect to the market. This quantity can be defined as

$$R^\varphi(t) := \frac{V^\varphi(t) - V^\varphi(0)}{V^\varphi(0)}, \qquad t \in (0, T], \tag{5.5}$$

and from (3.16), this can be represented as

$$R^\varphi(t) = \frac{G(\mu(t), A(t)) + \Gamma^G(t) - G(\mu(0), A(0))}{G(\mu(0), A(0))}$$

for an additively generated strategy. So if we make the denominator $G(\mu(0), A(0))$ smaller while keeping the value of $\Gamma^G(t)$ in the numerator, we can obtain larger excess return rates for φ. In the following examples, we use this method to decrease $G(\mu(0), A(0))$ by inserting an appropriate constant p in G whenever possible.

The following two examples use for A two "polar opposite" functions of finite variation, the running maximum μ_i^* and minimum μ_{*i}, respectively, of the market weights.

Example 5.3 Consider an entropic function of the type

$$G\big(\mu(t), A(t)\big) := G\big(\mu(t), \mu^*(t)\big) := -\log p - \sum_{i=1}^d \mu_i(t) \log \mu_i^*(t). \tag{5.6}$$

As before, $p \geq 1$ is a constant and the initial value

$$G\big(\mu(0), \mu^*(0)\big) = -\log p - \sum_{i=1}^d \mu_i(0) \log \mu_i(0)$$

is the same as in Example 5.1. We then obtain the derivatives, for $1 \leq i, j \leq d$,

$$\partial_i G\big(\mu(t), \mu^*(t)\big) = -\log \mu_i^*(t), \qquad \partial^2_{i,j} G\big(\mu(t), \mu^*(t)\big) = 0,$$
$$D_i G\big(\mu(t), \mu^*(t)\big) = -\frac{\mu_i(t)}{\mu_i^*(t)}.$$

From (3.4), and the fact that $\mu_i^*(s)$ increases only when $\mu_i(s) = \mu_i^*(s)$, we also have

$$\Gamma^G(t) = \sum_{i=1}^d \int_0^t \frac{\mu_i(s)}{\mu_i^*(s)} d\mu_i^*(s) = \sum_{i=1}^d \big(\mu_i^*(t) - \mu_i(0)\big) = \sum_{i=1}^d \mu_i^*(t) - 1.$$

As G is linear in $\mu_i(\cdot)$, the second-order partial derivatives with respect to μ_i of G vanish and the nondecreasing structure of $\Gamma^G(\cdot)$ comes solely from $\mu_i^*(\cdot)$. Also from (3.16) and (3.17), the trading strategy φ generated additively from G in (5.6) is expressed as

$$\varphi_i(t) = -\log\big(p\mu_i^*(t)\big) + \sum_{j=1}^d \mu_j^*(t) - 1, \qquad i = 1, \ldots, d,$$

and the value of φ is given as

$$V^\varphi(t) = -\sum_{i=1}^d \mu_i(t) \log\big(p\mu_i^*(t)\big) + \sum_{i=1}^d \mu_i^*(t) - 1.$$

The strong relative arbitrage condition (4.4) in Proposition 4.1 takes the form

$$\sum_{i=1}^d \mu_i^*(T_*) > 1 - \sum_{i=1}^d \mu_i(0) \log\big(p\mu_i(0)\big).$$

On the other hand, from (3.23) and (3.24), the trading strategy ψ generated multiplicatively by G in (5.6) is given as

$$\psi_i(t) = -K(t)\log\big(p\mu_i^*(t)\big), \qquad i = 1, \dots, d,$$

and the associated value is

$$V^\psi(t) \;= -K(t)\sum_{i=1}^d \mu_i(t)\log\big(p\mu_i^*(t)\big),$$

$$K(t) := \exp\Bigg(-\int_0^t \sum_{i=1}^d \frac{d\mu_i^*(s)}{\sum_{j=1}^d \mu_j(s)\log(p\mu_j^*(s))}\Bigg).$$

The strong relative arbitrage condition (4.15) in Theorem 4.7 takes the form

$$\sum_{i=1}^d \mu_i^*(T_*) > 1 + \beta\log\frac{-\sum_{i=1}^d \mu_i(0)\log(p\mu_i(0))}{\alpha}.$$

Here α, β are again lower and upper bounds on G and depend on p and the condition imposed on the market weights. Empirical results regarding this example can be found in the next section.

The function $\Gamma^G(\cdot)$ which represents the "cumulative earnings" of the next example is nonincreasing; but surprisingly, the empirical values $V^\varphi(\cdot)$ and $V^\psi(\cdot)$ of trading strategies grow asymptotically in the long run as the value of G grows, as indicated in the empirical results of the next section. Thus in this case, it is more appropriate to apply Theorems 4.6 and 4.10 regarding the strong arbitrage condition.

Example 5.4 Consider the function

$$G\big(\mu(t), A(t)\big) := G\big(\mu(t), \mu_*(t)\big) := -\log p - \sum_{i=1}^d \mu_i(t)\log\mu_{*i}(t). \tag{5.7}$$

As before, p is a constant and the initial value $G(\mu(0), \mu_*(0))$ is the same as in previous examples. Similarly as before, (3.4) gives

$$\Gamma^G(t) = \sum_{i=1}^d \int_0^t \frac{\mu_i(s)}{\mu_{*i}(s)} d\mu_{*i}(s) = \sum_{i=1}^d \int_0^t 1\, d\mu_{*i}(s) = \sum_{i=1}^d \mu_{*i}(t) - 1, \tag{5.8}$$

which is a nonpositive and nonincreasing function of t.

We first consider the trading strategy φ additively generated from G, which is

$$\varphi_i(t) = -\log\big(p\mu_{*i}(t)\big) + \sum_{j=1}^d \mu_{*j}(t) - 1, \qquad i = 1, \dots, d, \tag{5.9}$$

by (3.17). Note that $\varphi_i(t)$ admits the lower bound

$$\varphi_i(t) = -\log p - \log \mu_{*i}(t) + \mu_{*i}(t) + \sum_{\substack{j=1\\ j\neq i}}^{d} \mu_{*j}(t) - 1$$

$$\geq -\log p - \log \mu_i(0) + \mu_i(0) - 1, \tag{5.10}$$

because $x \mapsto -\log x + x$ is decreasing for $x \in (0, 1)$, and thus $\varphi_i(t)$ is positive if $\log(p\mu_i(0)) < \mu_i(0) - 1$. By (3.16), the value of φ is given as

$$V^{\varphi}(t) = -\log p - \sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t) + \Bigg(\sum_{i=1}^{d} \mu_{*i}(t) - 1\Bigg). \tag{5.11}$$

While $\Gamma^G(t) = \sum_{i=1}^d \mu_{*i}(t) - 1$, the last term on the right-hand side of (5.11), is nonincreasing, the second term $-\sum_{i=1}^d \mu_i(t) \log \mu_{*i}(t)$ asymptotically increases as $t \mapsto -\log \mu_{*i}(t)$ is nondecreasing. Actually, as we can see in the next section, the value of φ grows in the long run. We apply Theorem 4.6, rather than Proposition 4.1, to find a strong arbitrage condition, because $\Gamma^G(\cdot)$ here is not nondecreasing.

To apply Theorem 4.6, we first need to show that $V^{\varphi}(\cdot) \geq 0$. From (5.10), we get

$$-\log \mu_{*i}(t) \geq -\sum_{j=1}^{d} \mu_{*i}(t) - \log \mu_i(0) + \mu_i(0) \geq -1 - \log \mu_i(0) + \mu_i(0)$$

$$\geq -1 - \log \max_{j=1,\dots,d} \mu_j(0) + \max_{j=1,\dots,d} \mu_j(0)$$

for $i = 1, \dots, d$. The last inequality follows from the fact that $x \mapsto -\log x + x$ is decreasing for $x \in [0, 1]$. Then we also obtain

$$-\sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t) \geq -1 - \log \max_{j=1,\dots,d} \mu_j(0) + \max_{j=1,\dots,d} \mu_j(0),$$

because $-\sum_{i=1}^d \mu_i(t) \log \mu_{*i}(t)$ is the weighted average of $(-\log \mu_{*i}(t))_{1\leq i\leq d}$ with weights $\mu_i(t)$ with $\sum_{i=1}^d \mu_i(t) = 1$. Thus $V^{\varphi}(t)$ in (5.11) admits the lower bound

$$V^{\varphi}(t) \geq -\log p - 2 - \log \max_j \mu_j(0) + \max_j \mu_j(0)$$

for any $t \in [0, T]$, and $V^{\varphi}(\cdot) \geq 0$ is guaranteed when

$$p \leq e^{-2-\log \max_j \mu_j(0) + \max_j \mu_j(0)} \tag{5.12}$$

holds. Regarding the second condition of Theorem 4.6, we have

$$\begin{aligned} G\big(\mu(t), \mu_*(t)\big) &= -\log p - \sum_{i=1}^{d} \mu_i(t) \log \mu_{*i}(t) \\ &\geq -\log p - \sum_{i=1}^{d} \mu_i(t) \log \max_{i=1,\dots,d} \mu_{*i}(t) \\ &= -\log p - \max_{i=1,\dots,d} \log \mu_{*i}(t) := F\big(\mu(t), \mu_*(t)\big). \end{aligned} \tag{5.13}$$

Now the mapping $t \mapsto \mu_{*i}(t)$ is nonincreasing so that $F(\mu(t), \mu_*(t))$ is nondecreasing in t. Finally, the last condition of Theorem 4.6 follows easily from (5.8), as

$$\Gamma^G(t) \geq -1 =: -\kappa. \tag{5.14}$$

Thus Theorem 4.6 shows that the additively generated strategy φ in (5.9) is a strong arbitrage relative to the market over every interval $[0, t]$ with $T_* \leq t \leq T$ satisfying the condition

$$\sum_{i=1}^{d} \mu_i(0) \log \mu_i(0) - \max_{i=1,\dots,d} \log \mu_{*i}(T_*) > 1.$$

Next, from (3.24), the trading strategy ψ multiplicatively generated by G in (5.7) is represented as

$$\psi_i(t) = -K(t) \log\big(p\mu_{*i}(t)\big), \qquad i = 1, \dots, d,$$

with the value

$$\begin{aligned} V^\psi(t) &= -K(t) \sum_{i=1}^{d} \mu_i(t) \log\big(p\mu_{*i}(t)\big), \\ K(t) &:= \exp\left(-\int_0^t \sum_{i=1}^{d} \frac{d\mu_{*i}(s)}{\sum_{j=1}^{d} \mu_j(s) \log(p\mu_{*j}(s))}\right). \end{aligned}$$

For the strong arbitrage condition, we apply Theorem 4.10. Since $F(\mu(t), \mu_*(t))$ and κ from (5.13), (5.14) satisfy the conditions (i), (ii) (with an appropriate choice of p to make F positive), the strong relative arbitrage condition (4.17) becomes

$$\log\left(-\max_{i=1,\dots,d} \log p\mu_{*i}(T_*)\right) > \frac{-1}{\log p + \max_{i=1,\dots,d} \log \mu_i(0)}.$$

Remark 5.5 In Remark 5.2, we need to find a suitable value for p satisfying an inequality, for instance $-\sum_{i=1}^{d} \mu_i(t) \log \mu_i(t) \geq \log p$ for all $t \in [0, T]$ in Example 5.1, to ensure $G \geq 0$. This inequality usually depends on the values $\mu_i(t)$, $t \in [0, T]$, which are not observable at time 0. Thus we need to impose some condition on the market weights, or statistically analyse historical market data to find an appropriate value for p, before we can construct the trading strategy.

However, in Example 5.4, due to its unique structure, we can analytically find a suitable value of p without any statistical estimation at time $t = 0$. Indeed, from (5.13), we have that

$$G\big(\mu(t), \mu_*(t)\big) \geq -\log p - \max_{i=1,\ldots,d} \log \mu_{*i}(t) \geq -\log p - \max_{i=1,\ldots,d} \log \mu_i(0),$$

and setting

$$p = \frac{1}{\max_{i=1,\ldots,d} \mu_i(0)}$$

guarantees the condition $G(\mu(t), \mu_*(t)) \geq 0$ for all $t \in [0, T]$. Note that this p can be calculated from quantities observable at time 0. Actually, p satisfying (5.12) also guarantees the nonnegativity of G because

$$G\big(\mu(\cdot), \mu_*(\cdot)\big) \geq V^{\varphi}(\cdot) = G\big(\mu(\cdot), \mu_*(\cdot)\big) + \Gamma^G(\cdot) \geq 0$$

due to the nonpositivity of $\Gamma^G(\cdot)$. Of course, one can perform a statistical estimation of p using past market data to obtain a better value of p while satisfying both $G(\mu(\cdot), \mu_*(\cdot)) \geq 0$ and $V^{\varphi}(\cdot) \geq 0$.

The next example provides yet another application of Theorem 4.6.

Example 5.6 Fix a positive constant r such that the initial market weights satisfy

$$\mu_i(0) \leq \frac{1}{re}, \qquad i = 1, \ldots, d. \tag{5.15}$$

As the $\mu_i(0)$ are observable before we construct a trading strategy, we can find such an r at the moment we start investing. Then we consider the function

$$\begin{aligned} G\big(\mu(t), A(t)\big) &:= G\big(\mu(t), \mu_*(t)\big) \\ &:= -p - \sum_{i=1}^{d} \mu_i(t) \log\Big(-r\mu_{*i}(t) \log\big(r\mu_{*i}(t)\big)\Big). \end{aligned} \tag{5.16}$$

As in Remark 5.5, we can pre-determine the value of the constant p without any statistical estimation because of the inequalities

$$\begin{aligned} G\big(\mu(t), \mu_*(t)\big) &\geq -p - \sum_{i=1}^{d} \mu_i(t) \log \max_{j=1,\ldots,d} \Big(-r\mu_{*j}(t) \log\big(r\mu_{*j}(t)\big)\Big) \\ &\geq -p - \log \max_{j=1,\ldots,d} \Big(-r\mu_{*j}(t) \log\big(r\mu_{*j}(t)\big)\Big) \\ &=: F\big(\mu(t), \mu_*(t)\big) \qquad (5.17) \\ &\geq -p - \log \max_{j=1,\ldots,d} \Big(-r\mu_j(0) \log\big(r\mu_j(0)\big)\Big), \qquad \forall\, t \in [0, T]. \end{aligned}$$

The first inequality uses that $-\log$ is decreasing, the second that $\sum_{i=1}^{d}\mu_i(t)=1$. The last inequality holds because $x\mapsto -rx\log(rx)$ is increasing on $[0,\frac{1}{re}]$ and

$$0\le \mu_{*i}(\cdot)\le \mu_i(0)\le \frac{1}{re} \tag{5.18}$$

holds due to (5.15). Note that $F(\mu(t),\mu_*(t))$ defined in (5.17) is nondecreasing in t as $t\mapsto \mu_{*i}(t)$ and $t\mapsto -r\mu_{*i}(t)\log(r\mu_{*i}(t))$ are nonincreasing. Then the choice

$$p\le -\log \max_{i=1,\dots,d}\Big(-r\mu_i(0)\log\big(r\mu_i(0)\big)\Big), \tag{5.19}$$

which is completely observable at time 0, guarantees that $G(\mu(\cdot),\mu_*(\cdot))$ is always nonnegative. Next, after some computation, we obtain the partial derivatives

$$\partial_i G\big(\mu(t),\mu_*(t)\big)=-\log\Big(-r\mu_{*i}(t)\log\big(r\mu_{*i}(t)\big)\Big)\ge 1, \tag{5.20}$$

$$\partial^2_{i,k} G\big(\mu(t),\mu_*(t)\big)=0,\qquad D_i G\big(\mu(t),\mu_*(t)\big)=-\frac{\mu_i(t)\log(r\mu_{*i}(t))+\mu_i(t)}{\mu_{*i}(t)\log(r\mu_{*i}(t))}$$

for $1\le i,k\le d$. We note that $\partial_i G(\mu(t),\mu_*(t))\ge 1$ holds again because the mapping $x\mapsto -rx\log(rx)$ is increasing from 0 to $\frac{1}{e}$ in the interval $[0,\frac{1}{re}]$. From (3.4) and the fact that $\mu_{*i}(s)$ increases only when $\mu_i(s)=\mu_{*i}(s)$, we obtain

$$\Gamma^G(t)=\sum_{i=1}^{d}\int_0^t\bigg(1+\frac{1}{\log(r\mu_{*i}(s))}\bigg)d\mu_{*i}(s); \tag{5.21}$$

this is a nonincreasing function of t because $0\le 1+\frac{1}{\log(r\mu_{*i}(\cdot))}\le 1$ holds due to (5.18). The function Γ^G admits the lower bound

$$\begin{aligned}\Gamma^G(t)&=\sum_{i=1}^{d}\int_0^t 1\,d\mu_{*i}(s)+\sum_{i=1}^{d}\int_0^t\frac{1}{\log(r\mu_{*i}(s))}d\mu_{*i}(s)\\&=\sum_{i=1}^{d}\mu_{*i}(t)-\sum_{i=1}^{d}\mu_{*i}(0)+\sum_{i=1}^{d}Li_r\big(\mu_{*i}(t)\big)-\sum_{i=1}^{d}Li_r\big(\mu_i(0)\big)\\&\ge -1-\sum_{i=1}^{d}Li_r\big(\mu_i(0)\big)=:-\kappa,\end{aligned} \tag{5.22}$$

with the notation

$$Li_r(x):=\int_0^x\frac{du}{\log(ru)}=\frac{1}{r}\int_0^{rx}\frac{dv}{\log v}=\frac{1}{r}Li(rx).$$

Here, $Li(x)=\int_0^x\frac{du}{\log u}$ represents the logarithmic integral function. Note that $Li_r(x)$ takes negative values and decreases from 0 to $-\infty$ for $x\in[0,\frac{1}{r})$. The inequality in

(5.22) holds because of the inequality

$$\mu_{*i}(\cdot) + Li_r\big(\mu_{*i}(\cdot)\big) \geq 0 \tag{5.23}$$

which holds under (5.18). We also note that κ defined in (5.22) satisfies

$$-1 + \sum_{i=1}^{d} \mu_i(0) = 0 \leq \kappa < 1$$

from the same inequality (5.23). On the other hand, by (3.17), the trading strategy φ additively generated from G is expressed as

$$\begin{aligned}\varphi_i(t) = -p - \log\Big(-r\mu_{*i}(t)\log\big(r\mu_{*i}(t)\big)\Big)& \\ + \sum_{i=1}^{d}\int_0^t \bigg(1 + \frac{1}{\log(r\mu_{*i}(s))}\bigg) d\mu_{*i}(s).&\end{aligned} \tag{5.24}$$

Finally, by (3.16), the value of φ is given as

$$\begin{aligned}V^{\varphi}(t) = -p - \sum_{i=1}^{d}\mu_i(t)\log\Big(-r\mu_{*i}(t)\log\big(r\mu_{*i}(t)\big)\Big)& \\ + \sum_{i=1}^{d}\int_0^t \bigg(1 + \frac{1}{\log(r\mu_{*i}(s))}\bigg) d\mu_{*i}(s)&\end{aligned}$$

and admits the lower bound

$$V^{\varphi}(t) \geq -p - \log \max_{i=1,\dots,d}\Big(-r\mu_i(0)\log\big(r\mu_i(0)\big)\Big) - \kappa$$

from (5.17) and (5.22). Consequently, the choice

$$p = -\log \max_{i=1,\dots,d}\Big(-r\mu_i(0)\log\big(r\mu_i(0)\big)\Big) - \kappa \tag{5.25}$$

guarantees $V^{\varphi}(\cdot) \geq 0$ and also satisfies (5.19). We emphasize here again that p defined in (5.25) depends only on the initial market weights $\mu_i(0)$; thus no statistical estimation of p is required. Using the same technique as in (5.17), we obtain

$$\varphi_i(t) \geq -p - \log \max_i \Big(-r\mu_i(0)\log\big(r\mu_i(0)\big)\Big) - \kappa = 0, \qquad i = 1, \dots, d,$$

from (5.24) and (5.25), so this trading strategy is "long-only".

As we showed above that all conditions of Theorem 4.6 are satisfied, the additively generated strategy φ in (5.24) is a strong arbitrage relative to the market over every interval $[0,t]$ with $T_* \le t \le T$ satisfying, with κ as in (5.22), the condition

$$\log \max_{i=1,\dots,d} \Big(-r\mu_i(T_*) \log \big(r\mu_i(T_*) \big) \Big) < \sum_{i=1}^{d} \mu_i(0) \log \Big(-r\mu_i(0) \log \big(r\mu_i(0) \big) \Big) - \kappa.$$

We move on to the trading strategy ψ multiplicatively generated by G in (5.16). From (3.24), (3.23) as well as (5.20) and (5.21), we have

$$\psi_i(t) = -K(t) \bigg(p + \log \Big(-r\mu_{*i}(t) \log \big(r\mu_{*i}(t) \big) \Big) \bigg), \qquad i = 1, \dots, d,$$

with the value function

$$V^{\psi}(t) = -K(t) \bigg(p + \sum_{i=1}^{d} \mu_i(t) \log \Big(-r\mu_{*i}(t) \log \big(r\mu_{*i}(t) \big) \Big) \bigg),$$

$$K(t) := \exp \left(-\sum_{i=1}^{d} \int_0^t \frac{1 + \frac{1}{\log(r\mu_{*i}(s))}}{p + \sum_{j=1}^{d} \mu_j(t) \log(-r\mu_{*j}(t) \log(r\mu_{*j}(t)))} d\mu_{*i}(s) \right).$$

As Γ^G in (5.21) is nonincreasing, we again use Theorem 4.10. We already have $F(\mu(t), \mu_*(t))$ and κ defined in (5.17) and (5.22) which satisfy the conditions (i), (ii). Thus the strong arbitrage condition (4.17) becomes

$$\log \bigg(-p - \log \max_j \Big(-r\mu_{*j}(T_*) \log \big(r\mu_{*j}(T_*) \big) \Big) \bigg) > \frac{1 + \sum_{i=1}^{d} Li_r(\mu_i(0))}{-p - \log \max_j (-r\mu_j(0) \log(r\mu_j(0)))}.$$

6 Empirical results

We present now some empirical results regarding the behavior of the additively generated portfolios in Sect. 5, using historical data. We first analyse the values $V^{\varphi}(\cdot)$ of these portfolios with respect to the market by decomposing them with generating functions G and corresponding Gamma functions Γ^G in (3.16). In particular, we demonstrate empirically that all the portfolios in Sect. 5 outperform the market. We show also that the different choices of the parameter p, explained in Remark 5.2, indeed significantly influence the performance of the resulting portfolios.

6.1 Data description and notation

To simulate a perfect "closed market", we constructed a "universe" with $d = 1085$ stocks which had been continuously traded during 4528 consecutive trading days between 2000 January 1st and 2017 December 31st. These 1085 stocks were chosen from those listed at least once among the constituents of the S&P 1500 index in this period and did not undergo mergers, acquisitions, bankruptcies, etc.

Remark 6.1 This selection of 1085 stocks is somewhat biased, in the sense that we are looking ahead into the future at time $t = 0$ by blocking out those stocks which will go bankrupt in the future. However, the reason for this biased selection is to keep the number d of stocks constant all the time which is the essential assumption of our "closed" market model. If we compose our portfolio from $d = 1500$ stocks included in the S&P 1500 index at the beginning, remove one stock whenever it goes bankrupt, or take in a new stock whenever it is newly added to the index, the number d of stocks in our portfolio fluctuates over time and the generating function G would be discontinuous whenever d changes.

One possible solution to this problem is to consider an "open market". We first fix the value of d, say $d = 1500$ at the beginning, keep track of price dynamics of all stocks in the market (which should be composed of more than d stocks, say D stocks with $D > d$), rank them by market capitalization and construct our portfolio using the top $d = 1500$ among D stocks. In this way, we keep the same number d of companies all the time, but considering ranked market weights always involves a "leakage" issue. As explained in [9, Chaps. 4.2, 4.3] and [14, Example 6.2], this refers to the loss incurred when we have to sell a stock that has been relegated from the top d to the lower capitalization index. Even worse, as we want to invest only in the top d among D companies in this open market, our trading strategy $\varphi = (\varphi_1, \dots, \varphi_D)$ should satisfy $\varphi_i(t) = 0$ whenever the i-th company fails to be included in the top d at time t. However, we do not know yet how to construct such trading strategies.

Thus it is not easy to set up a perfect empirical model, and we decided to select $d = 1085$ stocks in a biased manner which fits better our theoretical model described in the previous sections.

We obtained daily closing prices and total number of outstanding shares of these stocks from the CRSP and Compustat data sets. The data can be found at https://wrds-web.wharton.upenn.edu/wrds/. We used *R* and *C++* to program our portfolios.

As we used daily data for $N = 4528$ days, we discretized the time interval via $0 = t_0 < t_1 < \cdots < t_{N-1} = T$. For $\ell \in \{1, 2, \dots, N\}$, we summarize our notations:

1) $S_i(t_\ell)$: the capitalization (daily closing price multiplied by total number of outstanding shares) of i-th stock at the end of day t_ℓ.

2) $\Sigma(t_\ell) := \sum_{i=1}^{d} S_i(t_\ell)$: the total capitalization of d stocks at the end of day t_ℓ. This quantity also represents the dollar value of the market portfolio at the end of day t_ℓ with the initial wealth $\Sigma(0)$.

3) $\mu_i(t_\ell) := \frac{S_i(t_\ell)}{\Sigma(t_\ell)}$: the i-th market weight at the end of day t_ℓ.

4) $\pi_i(t_\ell)$: the additively generated portfolio weight of the i-th stock at the end of day t_ℓ; this can be computed using (3.20). Note that $\sum_{i=1}^{d} \pi_i(t_\ell) = 1$.

5) $W(t_\ell)$: the total value of the portfolio at the end of day t_ℓ calculated as in (6.1) below. Then $W(t_\ell)\pi_i(t_\ell)$ represents the amount of money invested by our portfolio in the i-th stock at the end of day t_ℓ.

As the capitalization of the i-th stock at the beginning of day t_ℓ should be equal to $S_i(t_{\ell-1})$, the capitalization of the same stock at the end of the last trading day $t_{\ell-1}$, we also deduce that $\Sigma(t_{\ell-1})$, $\mu_i(t_{\ell-1})$, $\pi_i(t_{\ell-1})$ and $W(t_{\ell-1})$ represent the total capitalization, the i-th market weight, the i-th additively generated portfolio weight and the monetary value of the portfolio at the beginning of day t_ℓ, respectively.

The transaction, or rebalancing, of our portfolio on day t_ℓ is made at the beginning of day t_ℓ, using the market weights $\mu_i(t_{\ell-1})$ at the end of the last trading day. We compute $\pi_i(t_{\ell-1})$ from $\mu_i(t_{\ell-1})$ via (3.20) and redistribute the generated value $W(t_{\ell-1})$ according to these weights $\pi_i(t_{\ell-1})$. Then the monetary value of the portfolio $W(t_\ell)$ at the end of day t_ℓ can be calculated as

$$W(t_\ell) = \sum_{i=1}^{d} W(t_{\ell-1})\pi_i(t_{\ell-1})\frac{S_i(t_\ell)}{S_i(t_{\ell-1})}. \tag{6.1}$$

In order to compare the performance of our portfolios to the market portfolio, we set our initial wealth as $W(0) = \Sigma(0)$ and compare the evolutions of $\Sigma(\cdot)$ and $W(\cdot)$. Once the initial amount $W(0)$ invested in our portfolio is determined, its monetary value can be obtained recursively by (6.1). However, $W(\cdot)$ can also be defined with the trading strategy $\varphi_i(\cdot)$ in (3.13) or (3.17) as

$$W(\cdot) = \sum_{i=1}^{d} S_i(\cdot)\varphi_i(\cdot). \tag{6.2}$$

Then the value $V^\varphi(\cdot)$ of Definition 3.1 and (3.16) has another representation as the ratio between the money value of our portfolio and total market capitalization, i.e.,

$$V^\varphi(\cdot) = \sum_{i=1}^{d} \varphi_i(\cdot)\mu_i(\cdot) = \sum_{i=1}^{d} \varphi_i(\cdot)\frac{S_i(\cdot)}{\Sigma(\cdot)} = \frac{W(\cdot)}{\Sigma(\cdot)}.$$

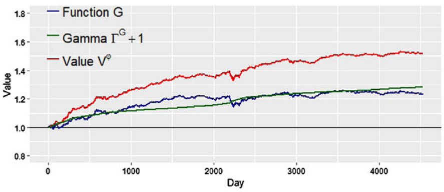
Thus, the expression "value of a trading strategy (or portfolio) with respect to the market" makes sense. Furthermore, the excess return rate $R^\varphi(\cdot)$ of the portfolio defined in (5.5) can be represented as

$$R^\varphi(\cdot) = \frac{V^\varphi(\cdot) - V^\varphi(0)}{V^\varphi(0)} = \frac{\frac{W(\cdot)}{\Sigma(\cdot)} - 1}{1} = \frac{W(\cdot) - \Sigma(\cdot)}{\Sigma(\cdot)} \ \big(= V^\varphi(\cdot) - 1\big),$$

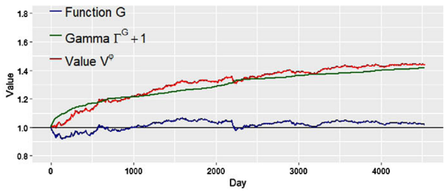
and the expression "excess return rate with respect to the market" also makes sense. Here, $V^\varphi(0) = 1$ because we set $W(0) = \Sigma(0)$. In the last part of the following subsection, we show the evolution of $W(\cdot)$ for several portfolios and compare their performance.

6.2 Empirical results

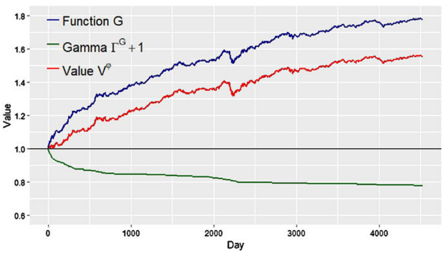
We first decompose the value functions $V^\varphi(\cdot)$ of the trading strategies additively generated from the functions G in the entropic examples (Examples 5.1, 5.3, 5.4 and 5.6)



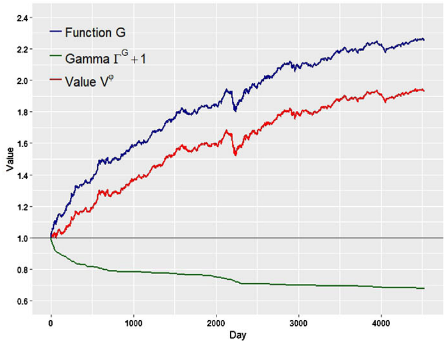
(a) Example 5.1, original entropy



(b) Example 5.3, entropy with running maximum



(c) Example 5.4, entropy with running minimum



(d) Example 5.6, iterated entropy

Fig. 1 Decomposition of value function of additively generated trading strategies

into the generating function component $G(\mu(\cdot), A(\cdot))$ and the corresponding Gamma function $\Gamma^G(\cdot)$. For ease of comparison, we normalize all generating functions so that $G(\mu(0), A(0)) = 1$ and shift up the Gamma functions by 1 in Fig. 1.

Figure 1 confirms that all trading strategies additively generated in Sect. 5 outperform the market, as the values V^φ (red lines in the figure) gradually increase. In panels (a) and (b), the growth of V^φ comes from the growth of the Gamma function. In contrast, the values of the strategies grow in panels (c) and (d) as the function G increases substantially, even though the corresponding Gamma function decreases. In panel (d), we set the parameter $r = 5$ as this is the largest integer satisfying (5.15); the initial market weight data give us $\max_i \mu_i(0) = 0.065 < 1/(5e)$. We chose the same parameter $p = 9$ (see Remark 5.2) in all panels for fair comparison, but this is a very sloppy choice for (a), (b) and (c). If we choose the value of p elaborately by using statistical estimation in each of these examples, the portfolio performance is improved, as Fig. 2 demonstrates in the case of Example 5.1.

Figure 2 shows the values of additively generated portfolios in Example 5.1 with different choices of the parameter p. We observe that strategies with bigger values of p perform better, as described in Remark 5.2. From the data, the Gibbs entropy $-\sum_{i=1}^{1085} \mu_i(t) \log \mu_i(t)$ of the market weights ranged from 4.954 to 5.726 during 4528 days. Thus $p = 90$ is a safe estimation for p which guarantees the nonnegativity of G in (5.2), as $\log 90 < 4.5 < 4.954$.

Finally, the "dollar values" $W(\cdot)$ of the portfolios from the four examples of Sect. 5, defined as in (6.2), are illustrated in Fig. 3, along with the total market value

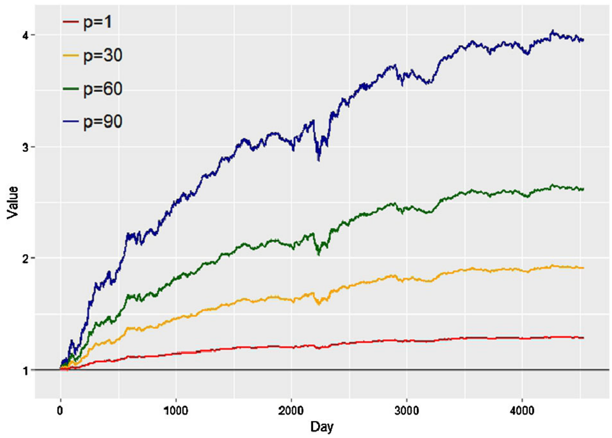


Fig. 2 Value of additively generated trading strategies from Example 5.1 with different values of p

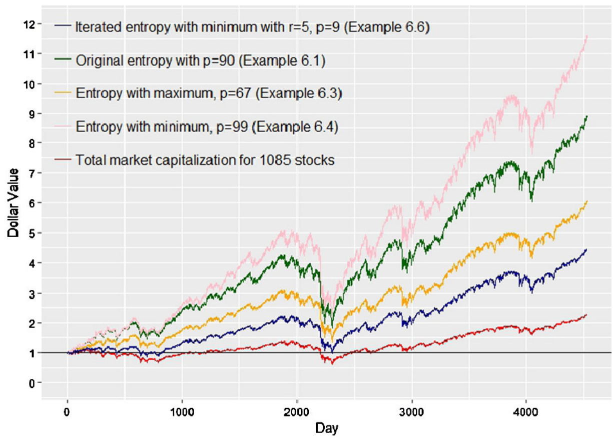


Fig. 3 (Normalized) Dollar values of portfolios over 18 years

$\Sigma(\cdot)$ of $d = 1085$ stocks from the start of 2000 to the end of 2017. Dollar values are normalized, replacing $W(\cdot)$ by $W(\cdot)/W(0)$. In Fig. 3, while the market capitalization approximately doubles during 18 years, the dollar values of all other portfolios grow by more than 4.5 times. Parameters are appropriately chosen using statistical estimation for each portfolio.

7 Conclusion

Karatzas and Ruf [14] introduced an additive functional generation of trading strategies as an alternative to the original multiplicative functional portfolio generation initiated by E.R. Fernholz. That new approach weakens the assumption on asset prices from Itô processes to continuous semimartingales, characterizes the class of functions called Lyapunov functions which generate trading strategies that can outperform the market, and gives a very simple sufficient condition for such outperformance.

The present paper generalizes these two approaches to functional generation even further. Its results can be summarized as follows:

1) We show how to generate, both additively and multiplicatively, trading strategies without any probabilistic assumptions on the market model. This is done by using pathwise Itô calculus, and the only analytical assumption we impose is that the market weights admit continuous covariations in a pathwise sense.

2) We extend the class of functions which generate trading strategies by introducing as an input, in addition to the vector of market weights, an argument of finite variation. Inserting this in the generating function gives extra flexibility for portfolio construction. While this has already been done in the existing literature by Ruf and Xie [20], Schied et al. [21], we present several new examples demonstrating what this extra argument can accomplish when it comes to simple sufficient conditions leading to strong arbitrage relative to the market.

3) We also extend the class of functions which generate additive and multiplicative strong relative arbitrage by presenting new sufficient conditions. These allow the function to be not Lyapunov, or not concave with respect to the market weights. We also offer empirical results of portfolios which outperform the market.

4) We extend the class of portfolio-generating functions from twice differentiable to absolutely continuous functions with the help of the pathwise Tanaka formula. This involves the concept of local time and yields new interesting types of portfolios and strong relative arbitrage conditions.

While we generalize here the functional generation of portfolios in several respects, we suggest some new questions and directions. First, this paper assumes a "closed market"; in other words, the number d of stocks is fixed. In this respect, it fails to represent or resemble the real market. As explained in Remark 6.1, an "open market" models the real world better. To construct trading strategies in such an open market, the notion of "piecewise semimartingales of stochastic dimension" in Strong [24] might be an appropriate starting point. Secondly, the path-dependent functional Itô calculus of Dupire [5] or Cont and Fournié [2] can be used to generate portfolios, as has been done in Schied et al. [21]. However, all examples of such portfolios generated from path-dependent functionals we can think of can also be constructed in the manner we described in this paper, by introducing an additional functional argument A of bounded variation with the help of Föllmer's pathwise formula. *Are there additional novel examples of portfolios generated from a functional depending on the entire history of market weights, where the use of Dupire's path-dependent functional calculus is essential?* It might be useful here to work in the framework of Ekren et al. [6, 7, 8], or Leão and Ohashi [16], Leão et al. [17, 18]; these are avenues we intend to explore.

Acknowledgements We are very grateful to the referees, the Associate Editor and the Co-Editor for calling our attention to a serious conceptual error in an earlier version of this paper, as well as for their many and constructive comments which helped us improve drastically the quality of this work. We thank also Drs. Robert Fernholz, Johannes Ruf and Rama Cont for many discussions and suggestions. Research supported in part by the National Science Foundation under grant NSF-DMS-14-05210.

Appendix: Proofs

Proof of Proposition 3.9 We follow the argument in [14, Proposition 4.8], using the pathwise Itô formula instead of the standard Itô formula for semimartingales. With

$$K(t) := \exp\left(\int_0^t \frac{d\Gamma^G(s)}{G(\mu(s), A(s))}\right) \tag{A.1}$$

in (3.23), the pathwise Itô formula (Proposition 2.2) yields

$$\begin{aligned}
G\big(\mu(t), A(t)\big)K(t) &= G\big(\mu(0), A(0)\big)K(0) + \int_0^t \sum_{i=1}^d \partial_i G\big(\mu(s), A(s)\big)K(s)d\mu_i(s) \\
&\quad + \int_0^t K(s)d\Gamma^G(s) + \int_0^t \sum_{i=0}^m D_i G\big(\mu(s), A(s)\big)K(s)dA_i(s) \\
&\quad + \frac{1}{2}\int_0^t \sum_{i=1}^d \sum_{j=1}^d \partial^2_{i,j} G\big(\mu(s), A(s)\big)K(s)d[\mu_i, \mu_j](s) \\
&= G\big(\mu(0), A(0)\big)K(0) + \int_0^t \sum_{i=1}^d \partial_i G\big(\mu(s), A(s)\big)K(s)d\mu_i(s) \\
&= G\big(\mu(0), A(0)\big)K(0) + \int_0^t \sum_{i=1}^d \eta_i(s)d\mu_i(s) \\
&= G\big(\mu(0), A(0)\big)K(0) + \int_0^t \sum_{i=1}^d \psi_i(s)d\mu_i(s).
\end{aligned}$$

Here, the second equality uses the expression in (3.4) and the last equality relies on [14, Proposition 2.3]. Since (3.23) holds at time zero, it follows that (3.23) holds at any time $t \in [0, T]$. The justification for (3.24) is exactly the same as that of [14, Proposition 4.8]. □

Proof of Theorem 3.10 For any absolutely continuous function f with a right-continuous Radon–Nikodým derivative f' of finite variation and any two real numbers a and b, applying integration by parts with the notation (2.3) gives

$$\begin{aligned}
f(b) - f(a) &= \int_a^b f'(x)dx \\
&= \begin{cases} \int_a^b f'(x)(b-x)^0 dx = -f'(x)(b-x)\big|_{x=a}^{x=b} + \int_{(a,b]}(b-x)df'(x), & \text{if } a \le b \\ -\int_b^a f'(x)(b-x)^0 dx = f'(x)(b-x)\big|_{x=b}^{x=a} - \int_{(b,a]}(b-x)df'(x), & \text{if } b < a \end{cases} \\
&= \begin{cases} f'(a)(b-a) + \int_{(a,b]}(b-x)df'(x), & \text{if } a \le b \\ f'(a)(b-a) - \int_{(b,a]}(b-x)df'(x), & \text{if } b < a \end{cases} \\
&= f'(a)(b-a) + \int_{\mathbb{R}} \mathbb{1}_{(\!(a,b]\!]}(x)|b-x|df'(x).
\end{aligned} \tag{A.2}$$

We then recall (3.5), (3.6), (A.1) and consider the telescoping expansion

$$
\begin{aligned}
&G\big(\mu(t), A(t)\big)K(t) - G\big(\mu(0), A(0)\big)K(0) \\
&= \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big(f_i\big(X_i(t_{j+1})\big)K(t_{j+1}) - f_i\big(X_i(t_j)\big)K(t_j) \Big) \\
&= \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} f_i\big(X_i(t_{j+1})\big)\big(K(t_{j+1}) - K(t_j)\big) \qquad \text{(A.3)} \\
&\quad + \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big(f_i\big(X_i(t_{j+1})\big) - f_i\big(X_i(t_j)\big) \Big) K(t_j). \qquad \text{(A.4)}
\end{aligned}
$$

Then we further expand the last double sum (A.4) as

$$
\begin{aligned}
&\sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big(f_i\big(X_i(t_{j+1})\big) - f_i\big(X_i(t_j)\big) \Big) K(t_j) \\
&= \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} \Big(f_i'\big(X_i(t_j)\big)K(t_j)\big(X_i(t_{j+1}) - X_i(t_j)\big) \\
&\qquad\qquad + \int_{\mathbb{R}} \mathbb{1}_{(\!| X_{t_j}, X_{t_{j+1}}]\!]}(x) |X_{t_{j+1}} - x| K(t_j) df_i'(x) \Big) \\
&= \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} f_i'\big(X_i(t_j)\big)K(t_j)\big(\mu_i(t_{j+1}) - \mu_i(t_j)\big) \qquad \text{(A.5)} \\
&\quad - \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} f_i'\big(X_i(t_j)\big)K(t_j)\big(A_i(t_{j+1}) - A_i(t_j)\big) \qquad \text{(A.6)} \\
&\quad + \sum_{i=1}^{d} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} K(t_j) \int_{\mathbb{R}} \big(L_{t_{j+1}}^{X_i, \mathbb{T}_n}(x) - L_{t_j}^{X_i, \mathbb{T}_n}(x) \big) df_i'(x), \qquad \text{(A.7)}
\end{aligned}
$$

where the first equation is from (A.2), and the second follows from (3.5) and (2.4).

Next, we show that the sum of (A.3), (A.6) and (A.7) vanishes as $n \to \infty$. First, since $\lim_{n\to\infty} \|\mathbb{T}_n\| = 0$, the limit of (A.3) is a Lebesgue–Stieltjes integral

$$
\sum_{i=1}^{d} \int_0^t f_i\big(X_i(s)\big) dK(s) = \int_0^t G\big(\mu(s), A(s)\big) dK(s),
$$

because $f_i(X_i(\cdot))$ is bounded on the compact interval $[0, T]$ for each $i = 1, \dots, d$. From (3.11), the change-of-variable formula for Lebesgue–Stieltjes integrals gives

$$\int_0^t G\big(\mu(s), A(s)\big) dK(s) = \int_0^t K(s) d\Gamma^G(s) = \int_0^t K(s)\big(d\Gamma_1^G(s) - d\Gamma_2^G(s)\big), \quad \text{(A.8)}$$

where

$$\Gamma_1^G(t) := \sum_{i=1}^d \int_0^t \vartheta_i(s) dA_i(s), \qquad \Gamma_2^G(t) := \sum_{i=1}^d \int_{\mathbb{R}} L_t^{X_i}(x) df_i'(x).$$

It follows that the limit of (A.6) is $-\sum_{i=1}^d \int_0^t K(s)\vartheta_i(s) dA_i(s)$. On the other hand, the last integral of (A.8) can be expressed as the limit

$$\int_0^t K(s) d\Gamma_2^G(s) = \lim_{n\to\infty} \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} K(t_j)\big(\Gamma_2^G(t_{j+1}) - \Gamma_2^G(t_j)\big)$$

$$= \lim_{n\to\infty} \sum_{i=1}^d \sum_{\substack{t_j \in \mathbb{T}_n \\ t_j \le t}} K(t_j) \int_{\mathbb{R}} \big(L_{t_{j+1}}^{X_i}(x) - L_{t_j}^{X_i}(x)\big) df_i'(x),$$

which coincides with the limit of (A.7). Therefore, the limits of (A.3), (A.6) and (A.7) are zero, whereas the remaining term of (A.4) is (A.5), whose limit we denote as

$$\sum_{i=1}^d \int_0^t f_i'\big(X_i(s)\big) K(s) d\mu_i(s) = \sum_{i=1}^d \int_0^t \eta_i(s) d\mu_i(s),$$

from (3.6) and (3.21). We obtain in this way that

$$G\big(\mu(t), A(t)\big)K(t) - G\big(\mu(0), A(0)\big)K(0) = \sum_{i=1}^d \int_0^t \eta_i(s) d\mu_i(s)$$

$$= \sum_{i=1}^d \int_0^t \psi_i(s) d\mu_i(s),$$

where the last equality follows from $\sum_{i=1}^d \mu_i(\cdot) \equiv 1$ and (3.22). The result (3.23) then follows from the self-financibility of ψ and the relationship

$$V^\psi(0) = \sum_{i=1}^d \psi_i(0)\mu_i(0) = \sum_{i=1}^d \big(\vartheta_i(0) - C(0)\big)\mu_i(0) = G\big(\mu(0), A(0)\big)K(0).$$

Finally, (3.24) can be justified in the same manner as Proposition 3.9. $\square$

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