

The Subelliptic Heat Kernel of the Octonionic Anti-De Sitter Fibration

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Abstract. In this note, we study the sub-Laplacian of the 15-dimensional octonionic anti-de Sitter space which is obtained by lifting with respect to the anti-de Sitter fibration the Laplacian of the octonionic hyperbolic space $\mathbb{O}H^1$. We also obtain two integral representations for the corresponding subelliptic heat kernel.

Key words: sub-Laplacian; 15-dimensional octonionic anti-de Sitter space; the anti-de Sitter fibration

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1 Introduction and results

In this note we study the sub-Laplacian and the corresponding sub-Riemannian heat kernel of the octonionic anti-de Sitter fibration

$$\mathbb{S}^7 \hookrightarrow \text{AdS}^{15}(\mathbb{O}) \rightarrow \mathbb{O}H^1.$$

This paper follows the previous works [2, 3, 10] which respectively concerned:

1. *The complex anti-de Sitter fibrations:*

$$\mathbb{S}^1 \hookrightarrow \text{AdS}^{2n+1}(\mathbb{C}) \rightarrow \mathbb{C}H^n.$$

2. *The quaternionic anti-de Sitter fibrations:*

$$\mathbb{S}^3 \hookrightarrow \text{AdS}^{4n+3}(\mathbb{H}) \rightarrow \mathbb{H}H^n.$$

The 15-dimensional anti-de Sitter fibration is the last model space that remained to be studied of a sub-Riemannian manifold arising from a H -type semi-Riemannian submersion over a rank-one symmetric space, see the Table 3 in [4].

Similarly to the complex and quaternionic case, the sub-Laplacian is defined as the lift on $\text{AdS}^{15}(\mathbb{O})$ of the Laplace–Beltrami operator of the octonionic hyperbolic space $\mathbb{O}H^1$. However, in the complex and quaternionic case the Lie group structure of the fiber played an important role that we can not use here, since the fiber $\mathbb{S}^7$ is not a group. Instead, we make use of some algebraic properties of $\mathbb{S}^7$ that were already pointed out and used by the authors in [1] for the study of the octonionic Hopf fibration:

$$\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \rightarrow \mathbb{O}P^1.$$

Let us briefly describe our main results. Due to the cylindrical symmetries of the fibration, the heat kernel of the sub-Laplacian only depends on two variables: the variable r which is the

Riemannian distance on $\mathbb{O}H^1$ (the starting point is specified with inhomogeneous coordinate in Section 3) and the variable η which is the Riemannian distance starting at a pole on the fiber $\mathbb{S}^7$. We prove in Proposition 3.1 that in these coordinates, the radial part of the sub-Laplacian $\tilde{L}$ writes

$$\tilde{L} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \tanh^2 r \left(\frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right).$$

As a consequence of this expression for the sub-Laplacian, we are able to derive two equivalent formulas for the heat kernel. The first formula, see Proposition 4.1, reads as follows: for $r \geq 0$, $\eta \in [0, \pi)$, $t > 0$

$$p_t(r, \eta) = \int_0^\infty s_t(\eta, iu) q_{t,15}(\cosh r \cosh u) \sinh^6 u \, du,$$

where s_t is the heat kernel of the Jacobi operator

$$\tilde{\Delta}_{\mathbb{S}^7} = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}$$

with respect to the measure $\sin^6 \eta \, d\eta$, and where $q_{t,15}$ is the Riemannian heat kernel on the 15-dimensional real hyperbolic space $\mathbb{H}^{15}$ given in (4.1). The second formula, see Proposition 4.2, writes as follows:

$$p_t(r, \eta) = \int_0^\pi \int_0^\infty G_t(\eta, \varphi, u) q_{t,9}(\cosh r \cosh u) \sin^5 \varphi \, du \, d\varphi,$$

where $q_{t,9}$ is Riemannian heat kernel on the 9-dimensional hyperbolic space $\mathbb{H}^9$ and $G_t(\eta, \varphi, u)$ is given in (4.3).

Similarly to [2, 3, 10], it might be expected that explicit integral representations of the heat kernel might be used to study small-time asymptotics, inside and outside of the cut-locus. Integral representations of heat kernels can also be used to obtain sharp heat kernel estimates, see [7]. Those applications of the heat kernel representations we obtain will possibly be addressed in a future research project.

2 The octonionic anti-de Sitter fibration

Let

$$\mathbb{O} = \left\{ x = \sum_{j=0}^7 x_j e_j, x_j \in \mathbb{R} \right\},$$

be the division algebra of octonions (see [9] for explicit representations of this algebra). We recall that the multiplication rules are given by

$$\begin{aligned} e_i e_j &= e_j & \text{if } i = 0, \\ e_i e_j &= e_i & \text{if } j = 0, \\ e_i e_j &= -\delta_{ij} e_0 + \epsilon_{ijk} e_k & \text{otherwise,} \end{aligned}$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} is the completely antisymmetric tensor with value 1 when $ijk = 123, 145, 176, 246, 257, 347, 365$ (also see [1]). The octonionic norm is defined for $x \in \mathbb{O}$ by

$$||x||^2 = \sum_{j=0}^7 x_j^2.$$

The octonionic anti-de Sitter space $\text{AdS}^{15}(\mathbb{O})$ is the quadric defined as the pseudo-hyperbolic space by:

$$\text{AdS}^{15}(\mathbb{O}) = \{(x, y) \in \mathbb{O}^2, \|(x, y)\|_{\mathbb{O}}^2 = -1\},$$

where

$$\|(x, y)\|_{\mathbb{O}}^2 := \|x\|^2 - \|y\|^2.$$

In real coordinates we have $x = \sum_{j=0}^7 x_j e_j$, $y = \sum_{j=0}^7 y_j e_j$, and the pseudo-norm can be written as

$$x_0^2 + \cdots + x_7^2 - y_0^2 - \cdots - y_7^2.$$

As such, $\text{AdS}^{15}(\mathbb{O})$ is embedded in the flat 16-dimensional space $\mathbb{R}^{8,8}$ endowed with the Lorentzian real signature $(8, 8)$ metric

$$ds^2 = dx_0^2 + \cdots + dx_7^2 - dy_0^2 - \cdots - dy_7^2.$$

Consequently, $\text{AdS}^{15}(\mathbb{O})$ is naturally endowed with a pseudo-Riemannian structure of signature $(8, 7)$.

Let $\mathbb{O}H^1$ denote the octonionic hyperbolic space. The map $\pi: \text{AdS}^{15}(\mathbb{O}) \rightarrow \mathbb{O}H^1$, given by $(x, y) \mapsto [x : y] = y^{-1}x$ is a pseudo-Riemannian submersion with totally geodesic fibers isometric to the seven-dimensional sphere $\mathbb{S}^7$. Notice that, as a topological manifold, $\mathbb{O}H^1$ can therefore be identified with the unit open ball in $\mathbb{O}$. The pseudo-Riemannian submersion π yields the octonionic anti-de Sitter fibration

$$\mathbb{S}^7 \hookrightarrow \text{AdS}^{15}(\mathbb{O}) \rightarrow \mathbb{O}H^1.$$

For further information on semi-Riemannian submersions over rank-one symmetric spaces, we refer to [6].

3 Cylindrical coordinates and radial part of the sub-Laplacian

The sub-Laplacian L on $\text{AdS}^{15}(\mathbb{O})$ we are interested in is the horizontal Laplacian of the Riemannian submersion $\pi: \text{AdS}^{15}(\mathbb{O}) \rightarrow \mathbb{O}H^1$, i.e., the horizontal lift of the Laplace–Beltrami operator of $\mathbb{O}H^1$. It can be written as

$$L = \square_{\text{AdS}^{15}(\mathbb{O})} + \triangle_{\mathcal{V}}, \tag{3.1}$$

where $\square_{\text{AdS}^{15}(\mathbb{O})}$ is the d'Alembertian, i.e., the Laplace–Beltrami operator of the pseudo-Riemannian metric and $\triangle_{\mathcal{V}}$ is the vertical Laplacian. Since the fibers of π are totally geodesic and isometric to $\mathbb{S}^7 \subset \text{AdS}^{15}(\mathbb{O})$, we note that $\square_{\text{AdS}^{15}(\mathbb{O})}$ and $\triangle_{\mathcal{V}}$ are commuting operators, and we can identify

$$\triangle_{\mathcal{V}} = \triangle_{\mathbb{S}^7}. \tag{3.2}$$

The sub-Laplacian L is associated with a canonical sub-Riemannian structure on $\text{AdS}^{15}(\mathbb{O})$ which is of H -type, see [4].

To study L , we introduce a set of coordinates that reflect the cylindrical symmetries of the octonionic unit sphere which provides an explicit local trivialization of the octonionic anti-de Sitter fibration. Consider the coordinates $w \in \mathbb{O}H^1$, where w is the inhomogeneous coordinate on $\mathbb{O}H^1$ given by $w = y^{-1}x$, with $x, y \in \text{AdS}^{15}(\mathbb{O})$. Consider the north pole $p \in \mathbb{S}^7$ and take

$Y_1, \dots, Y_7$ to be an orthonormal frame of $T_p \mathbb{S}^7$. Let us denote $\exp_p$ the Riemannian exponential map at p on $\mathbb{S}^7$. Then the cylindrical coordinates we work with are given by

$$(w, \theta_1, \dots, \theta_7) \mapsto \left(\frac{\exp_p \left(\sum_{i=1}^7 \theta_i Y_i \right) w}{\sqrt{1 - \rho^2}}, \frac{\exp_p \left(\sum_{i=1}^7 \theta_i Y_i \right)}{\sqrt{1 - \rho^2}} \right) \in \text{AdS}^{15}(\mathbb{O}),$$

where $\rho = \|w\|$ and $\|\theta\| = \sqrt{\theta_1^2 + \dots + \theta_7^2} < \pi$.

A function f on $\text{AdS}^{15}(\mathbb{O})$ is called radial cylindrical if it only depends on the two coordinates $(\rho, \eta) \in [0, 1) \times [0, \pi]$ where $\eta = \sqrt{\sum_{i=1}^7 \theta_i^2}$. More precisely f is radial cylindrical if there exists a function g so that

$$f \left(\frac{\exp_p \left(\sum_{i=1}^7 \theta_i Y_i \right) w}{\sqrt{1 - \rho^2}}, \frac{\exp_p \left(\sum_{i=1}^7 \theta_i Y_i \right)}{\sqrt{1 - \rho^2}} \right) = g(\rho, \eta).$$

We denote by $\mathcal{D}$ the space of smooth and compactly supported functions on $[0, 1) \times [0, \pi]$. Then the radial part of L is defined as the operator $\tilde{L}$ such that for any $f \in \mathcal{D}$, we have

$$L(f \circ \psi) = (\tilde{L}f) \circ \psi. \quad (3.3)$$

We now compute $\tilde{L}$ in cylindrical coordinates.

Proposition 3.1. *The radial part of the sub-Laplacian on $\text{AdS}^{15}(\mathbb{O})$ is given in the coordinates (r, η) by the operator*

$$\tilde{L} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \tanh^2 r \left(\frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right),$$

where $r = \tanh^{-1} \rho$ is the Riemannian distance on $\mathbb{O}H^1$ from the origin.

Proof. Note that the radial part of the Laplace–Beltrami operator on the octonionic hyperbolic space $\mathbb{O}H^1$ is

$$\tilde{\Delta}_{\mathbb{O}H^1} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r},$$

and the radial part of the Laplace–Beltrami operator on $\mathbb{S}^7$ is

$$\tilde{\Delta}_{\mathbb{S}^7} = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}. \quad (3.4)$$

Since the octonionic anti-de Sitter fibration defines a totally geodesic submersion with base space $\mathbb{O}H^1$ and fiber $\mathbb{S}^7$, the semi-Riemannian metric on $\text{AdS}^{15}(\mathbb{O})$ is locally given by a warped product between the Riemannian metric of $\mathbb{O}H^1$ and the Riemannian metric on $\mathbb{S}^7$. Hence the radial part of the d'Alembertian becomes

$$\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + g(r) \left(\frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right), \quad (3.5)$$

for some smooth function g to be computed.

On the other hand, from the isometric embedding $\text{AdS}^{15}(\mathbb{O}) \subset \mathbb{O} \times \mathbb{O}$, the d'Alembertian on $\text{AdS}^{15}(\mathbb{O})$ is a restriction of the d'Alembertian on $\mathbb{O} \times \mathbb{O} \simeq \mathbb{R}^{8,8}$ in the sense that for a smooth $f: \text{AdS}^{15}(\mathbb{O}) \rightarrow \mathbb{R}$

$$\square_{\text{AdS}^{15}(\mathbb{O})} f = \square_{\mathbb{O} \times \mathbb{O}} f^*_{/\text{AdS}^{15}(\mathbb{O})},$$

where $\square_{\mathbb{O} \times \mathbb{O}} = \sum_{i=0}^7 \left(\frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial y_i^2} \right)$ and for $x, y \in \mathbb{O}$ such that $\|y\|^2 - \|x\|^2 > 0$, $f^*(x, y) = f\left(\frac{x}{\sqrt{\|y\|^2 - \|x\|^2}}, \frac{y}{\sqrt{\|y\|^2 - \|x\|^2}}\right)$. For the specific choice of the function $f(x, y) = y_1$, one easily computes that $\square_{\mathbb{O} \times \mathbb{O}} f^*_{/\text{AdS}^{15}(\mathbb{O})}(x, y) = 15y_1$, thus

$$\square_{\text{AdS}^{15}(\mathbb{O})} f(x, y) = 15y_1.$$

For the point with coordinates

$$\left(\frac{\exp_p \left(\sum_{i=1}^7 \theta_i Y_i \right) w}{\sqrt{1 - \rho^2}}, \frac{\exp_p \left(\sum_{i=1}^7 \theta_i Y_i \right)}{\sqrt{1 - \rho^2}} \right) \in \text{AdS}^{15}(\mathbb{O})$$

one has

$$y_1 = \frac{\cos \eta}{\sqrt{1 - \rho^2}} = \cosh r \cos \eta.$$

We therefore deduce that

$$\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})} (\cosh r \cos \eta) = 15 \cosh r \cos \eta.$$

Using the formula (3.5), after a straightforward computation, this yields $g(r) = -\frac{1}{\cosh^2 r}$ and therefore

$$\begin{aligned} \tilde{\square}_{\text{AdS}^{15}(\mathbb{O})} &= \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} - \frac{1}{\cosh^2 r} \left(\frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right) \\ &= \tilde{\Delta}_{\mathbb{O}H^1} - \frac{1}{\cosh^2 r} \tilde{\Delta}_{\mathbb{S}^7}. \end{aligned}$$

Finally, to conclude, one notes that the sub-Laplacian L is given by the difference between the Laplace–Beltrami operator of $\text{AdS}^{15}(\mathbb{O})$ and the vertical Laplacian. Therefore by (3.1) and (3.2),

$$\tilde{L} = \tilde{\square}_{\text{AdS}^{15}(\mathbb{O})} + \tilde{\Delta}_{\mathbb{S}^7} = \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \tanh^2 r \left(\frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta} \right). \quad \blacksquare$$

Remark 3.2. As a consequence of the previous result, we can check that the Riemannian measure of $\text{AdS}^{15}(\mathbb{O})$ in the coordinates (r, η) , which is the symmetric and invariant measure for $\tilde{L}$ is given by

$$d\bar{\mu} = \frac{\pi^7}{90} \sinh^7 r \cosh^7 r \sin^6 \eta \, dr \, d\eta. \quad (3.6)$$

(See also Remark 2 in [1], which corresponds to the case of the octonionic Hopf fibration.)

4 Integral representations of the subelliptic heat kernel

In this section, we give two integral representations of the subelliptic heat kernel associated with $\tilde{L}$. We denote by $p_t(r, \eta)$ the heat kernel of $\tilde{L}$ issued from the point $r = \eta = 0$ with respect to the measure (3.6). We remark that studying the subelliptic heat kernel associated with $\tilde{L}$ is enough to study the heat kernel of L , because due to (3.3) the heat kernel $h_t(w, \theta)$ of L issued from the point with cylindric coordinates $w = 0$, $\theta = 0$ is then given by

$$h_t(w, \theta) = p_t(\tanh^{-1} \|w\|, \|\theta\|).$$

4.1 First integral representation

We denote by s_t the heat kernel of the operator

$$\tilde{\Delta}_{\mathbb{S}^7} = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}$$

with respect to the reference measure $\sin^6 \eta d\eta$. The operator $\tilde{\Delta}_{\mathbb{S}^7}$ belongs to the family of Jacobi diffusion operators which have been extensively studied in the literature, see for instance the appendix in [5] and the references therein. In particular, the spectrum of $\tilde{\Delta}_{\mathbb{S}^7}$ is given by

$$\mathbf{Sp}(-\tilde{\Delta}_{\mathbb{S}^7}) = \{m(m+6), m \in \mathbb{N}\},$$

and the eigenfunction corresponding to the eigenvalue $m(m+6)$ is $P_m^{5/2,5/2}(\cos \eta)$ where $P_m^{5/2,5/2}$ is the Jacobi polynomial

$$P_m^{5/2,5/2}(x) = \frac{(-1)^m}{2^m m! (1-x^2)^{5/2}} \frac{d^m}{dx^m} (1-x^2)^{5/2+m}.$$

As a consequence, one has the following spectral decomposition for the heat kernel:

$$s_t(\eta, u) = \frac{1}{\pi} \sum_{m=0}^{+\infty} \frac{2^{4m+7} m! (m+5)! [(m+3)!]^2}{(2m+6)! (2m+5)!} e^{-m(m+6)t} P_m^{5/2,5/2}(\cos \eta) P_m^{5/2,5/2}(\cos u).$$

Proposition 4.1. *For $r \geq 0$, $\eta \in [0, \pi]$, and $t > 0$ we have*

$$p_t(r, \eta) = \int_0^\infty s_t(\eta, iu) q_{t,15}(\cosh r \cosh u) \sinh^6 u du,$$

where

$$q_{t,15}(\cosh s) := \frac{e^{-49t}}{(2\pi)^7 \sqrt{4\pi t}} \left(-\frac{1}{\sinh s} \frac{d}{ds} \right)^7 e^{-s^2/4t} \quad (4.1)$$

is the Riemannian heat kernel on the 15-dimensional real hyperbolic space $\mathbb{H}^{15}$.

Proof. Since $\pi: \text{AdS}^{15}(\mathbb{O}) \rightarrow \mathbb{O}H^1$ is a (semi-Riemannian) totally geodesic submersion, the operators $\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})}$ and $\tilde{\Delta}_{\mathbb{S}^7}$ commute. Thus

$$e^{t\tilde{L}} = e^{t(\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})} + \tilde{\Delta}_{\mathbb{S}^7})} = e^{t\tilde{\Delta}_{\mathbb{S}^7}} e^{t\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})}}.$$

We deduce that the heat kernel of $\tilde{L}$ can be written as

$$p_t(r, \eta) = \int_0^\pi s_t(\eta, u) p_t^{\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})}}(r, u) \sin^6 u du, \quad (4.2)$$

where s_t is the heat kernel of (3.4) with respect to the measure $\sin^6 \eta d\eta$, $\eta \in [0, \pi]$, and $p_t^{\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})}}(r, u)$ the heat kernel at $(0, 0)$ of $\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})}$ with respect to the measure in (3.6), i.e.,

$$d\mu(r, u) = \frac{\pi^7}{90} \sinh^7 r \cosh^7 r \sin^6 u dr du, \quad r \in [0, \infty), \quad u \in [0, \pi].$$

In order to write (4.2) more precisely, let us consider the analytic change of variables $\tau: (r, \eta) \rightarrow (r, i\eta)$ that will be applied on functions of the type $f(r, \eta) = h(r)e^{-i\lambda\eta}$, with h smooth and

compactly supported on $[0, \infty)$ and $\lambda > 0$. Then as we saw in the proof of Proposition 3.1 one can see that

$$\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})}(f \circ \tau) = (\tilde{\Delta}_{\mathbb{H}^{15}} f) \circ \tau,$$

where

$$\tilde{\Delta}_{\mathbb{H}^{15}} = \tilde{\Delta}_{\mathbb{O}H^1} + \frac{1}{\cosh^2 r} \tilde{\Delta}_P, \quad \tilde{\Delta}_P = \frac{\partial^2}{\partial \eta^2} + 6 \coth \eta \frac{\partial}{\partial \eta}.$$

Then, one deduces

$$e^{t\tilde{L}}(f \circ \tau) = e^{t\tilde{\Delta}_{S^7}} e^{t\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})}}(f \circ \tau) = e^{t\tilde{\Delta}_{S^7}}((e^{t\tilde{\Delta}_{\mathbb{H}^{15}}} f) \circ \tau) = (e^{-t\tilde{\Delta}_P} e^{t\tilde{\Delta}_{\mathbb{H}^{15}}} f) \circ \tau.$$

Now, since for every $f(r, \eta) = h(r)e^{-i\lambda\eta}$,

$$(e^{t\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})}} f)(0, 0) = (e^{t\tilde{\Delta}_{\mathbb{H}^{15}}} f)(0, 0),$$

one deduces that for a function h depending only on u ,

$$\int_0^\pi h(u) p_t^{\tilde{\square}_{\text{AdS}^{15}(\mathbb{O})}}(r, u) \sin^6 u \, du = \int_0^\pi h(-iu) q_{t,15}(\cosh r \cosh u) \sinh^6 u \, du.$$

Therefore, coming back to (4.2), one infers that using the analytic extension of s_t one must have

$$\int_0^\pi s_t(\eta, u) p_t^{\square_{\text{AdS}^{15}(\mathbb{O})}}(r, u) \sin^6 u \, du = \int_0^\pi s_t(\eta, -iu) q_{t,15}(\cosh r \cosh u) \sinh^6 u \, du,$$

where $q_{t,15}$ is the Riemannian heat kernel on the real hyperbolic space $\mathbb{H}^{15}$ given in (4.1). ■

4.2 Second integral representation

Proposition 4.2. *For $r \geq 0$, $\eta \in [0, \pi]$, and $t > 0$ we have*

$$p_t(r, \eta) = \int_0^\pi \int_0^\infty G_t(\eta, \varphi, u) q_{t,9}(\cosh r \cosh u) \sin^5 \varphi \, du \, d\varphi.$$

where $q_{t,9}$ is the 9-dimensional Riemannian heat kernel on the hyperbolic space $\mathbb{H}^9$:

$$q_{t,9}(\cosh s) := \frac{e^{-16t}}{(2\pi)^4 \sqrt{4\pi t}} \left(\frac{1}{\sinh s} \frac{d}{ds} \right)^4 e^{-s^2/4t},$$

and

$$G_t(\eta, \varphi, u) = \frac{15}{8} \sum_{m \geq 0} e^{-(m(m+6)+33)t} (\cos \eta + i \sin \eta \cos \varphi)^m \cosh((m+3)u). \quad (4.3)$$

Proof. The strategy of the following method appeals to some results proved in [8]. Firstly, we decompose the subelliptic heat kernel in the η variable with respect to the basis of normalized eigenfunctions of $\tilde{\Delta}_{S^7} = \frac{\partial^2}{\partial \eta^2} + 6 \cot \eta \frac{\partial}{\partial \eta}$. Accordingly,

$$p_t(r, \eta) = \sum_{m \geq 0} f_m(t, r) h_m(\eta),$$

where for each m , h_m is given by

$$h_m(\eta) = \frac{15}{16} \int_0^\pi (\cos \eta + i \sin \eta \cos \varphi)^m \sin^5 \varphi \, d\varphi$$

and $f_m(t, \cdot)$ solves the following heat equation

$$\begin{aligned} \frac{\partial}{\partial t} f_m(t, r) &= \left(\frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} - m(m+6) \tanh^2 r \right) f_m(t, r) \\ &= \left(\frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \frac{m(m+6)}{\cosh^2 r} - m(m+6) \right) f_m(t, r). \end{aligned}$$

We consider then the operator

$$L_m := \frac{\partial^2}{\partial r^2} + (7 \coth r + 7 \tanh r) \frac{\partial}{\partial r} + \frac{m(m+6)}{\cosh^2 r} + 49,$$

which was studied in [8, p. 229]. From [8, Theorem 2], with $\alpha = 3 + \frac{m}{2}, \beta = -\frac{m}{2}$, we deduce that the solution to the wave Cauchy problem associated with the subelliptic Laplacian is given $f \in C_0^\infty(\mathbb{O}H^1)$ by

$$\cos(s\sqrt{-L_m})(f)(w) = \frac{-\sinh s}{(2\pi)^4} \left(\frac{1}{\sinh s} \frac{d}{ds} \right)^4 \int_{\mathbb{O}H^1} K_m(s, w, y) f(y) \frac{dy}{(1 - \|y\|^2)^8},$$

where

$$\begin{aligned} K_m(s, w, y) &= \frac{(1 - \langle w, y \rangle)^{3+m/2}}{(1 - \langle w, y \rangle)^{m/2}} \frac{1}{\cosh^3(d(w, y)) \sqrt{\cosh^2(s) - \cosh^2(d(w, y))}} \\ &\quad \times {}_2F_1 \left(m+3, -m-3, \frac{1}{2}; \frac{\cosh(d(w, y)) - \cosh(s)}{2 \cosh(d(w, y))} \right), \end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function and dy stands for the Lebesgue measure in $\mathbb{R}^8$. Using the spectral formula

$$e^{tL} = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-s^2/(4t)} \cos(s\sqrt{-L}) ds,$$

which holds for any non positive self-adjoint operator, we deduce that the solution to the heat Cauchy problem associated with L_m :

$$\begin{aligned} e^{tL_m}(f)(w) &= \frac{e^{-m(m+6)t-7^2t}}{\sqrt{4\pi t}(2\pi)^4} \int_{\mathbb{R}} ds (-\sinh s) e^{-s^2/(4t)} \\ &\quad \times \left(\frac{1}{\sinh s} \frac{d}{ds} \right)^4 \int_{\mathbb{O}H^1} K_m(s, w, y) f(y) \frac{dy}{(1 - \|y\|^2)^8}. \end{aligned}$$

Performing integration by parts 4-times,

$$\begin{aligned} &\int_{\mathbb{R}} ds (-\sinh s) \left(\frac{1}{\sinh s} \frac{d}{ds} \right)^4 e^{-s^2/(4t)} \int_{\mathbb{O}H^1} K_m(s, w, y) f(y) \frac{dy}{(1 - \|y\|^2)^8} \\ &= \int_{\mathbb{O}H^1} f(y) \frac{dy}{(1 - \|y\|^2)^8} \int_{\mathbb{R}} ds (-\sinh s) K_m(s, w, y) \left(\frac{1}{\sinh s} \frac{d}{ds} \right)^4 e^{-s^2/(4t)} \\ &= 2 \int_{\mathbb{O}H^1} f(y) \frac{dy}{(1 - \|y\|^2)^8} \int_{d(w, y)}^\infty d(\cosh(s)) K_m(s, w, y) \left(\frac{1}{\sinh s} \frac{d}{ds} \right)^4 e^{-s^2/(4t)}. \end{aligned}$$

Thus we get

$$e^{tL_m}(f)(0) = 2e^{-(m(m+6)+33)t} \int_{\mathbb{O}H^1} f(y) \frac{dy}{(1 - \|y\|^2)^8} \int_{d(0, y)}^\infty d(\cosh s) K_m(s, 0, y) q_{t,9}(\cosh s).$$

As a result, the subelliptic heat kernel of L_m reads

$$\begin{aligned} & \frac{dy}{(1 - \|y\|^2)^8} \int_{d(0,y)}^{\infty} d(\cosh s) K_m(s, 0, y) q_{t,9}(\cosh s) \\ &= dr \sinh^7 r \cosh^7 r \int_r^{\infty} d(\cosh s) K_m(s, 0, y) q_{t,9}(\cosh s). \end{aligned}$$

By changing the variable $\cosh s = \cosh r \cosh u$ for $u \geq 0$, the last expression becomes

$$dr \sinh^7 r \cosh^7 r \int_0^{\infty} {}_2F_1\left(m+3, -m-3, \frac{1}{2}; \frac{1-\cosh u}{2}\right) q_{t,9}(\cosh r \cosh u) du.$$

Therefore $p_t(r, \eta)$ has the integral representation

$$2 \sum_{m \geq 0} e^{-(m(m+6)+33)t} h_m(\eta) \int_0^{\infty} {}_2F_1\left(m+3, -m-3, \frac{1}{2}; \frac{1-\cosh u}{2}\right) q_{t,9}(\cosh r \cosh u) du.$$

Now, notice that ${}_2F_1\left(m+3, -m-3, \frac{1}{2}; \frac{1-\cosh u}{2}\right)$ is simply the Cheybyshev polynomial of the first kind

$$T_{m+3}(x) = {}_2F_1\left(m+3, -m-3, \frac{1}{2}; \frac{1-x}{2}\right),$$

for all $x \in \mathbb{C}$. Therefore, one has

$${}_2F_1\left(m+3, -m-3, \frac{1}{2}; \frac{1-\cosh u}{2}\right) = T_{m+3}(\cosh u) = \cosh((m+3)u),$$

and the proof is over. ■

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