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Brownian Motions and Heat Kernel Lower Bounds on Kähler and Quaternion Kähler Manifolds

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We study the radial parts of the Brownian motions on Kähler and quaternion Kähler manifolds. Thanks to sharp Laplacian comparison theorems, we deduce as a consequence a sharp Cheeger–Yau-type lower bound for the heat kernels of such manifolds and also sharp Cheng's type estimates for the Dirichlet eigenvalues of metric balls.

1 Introduction

It is by now well established that on Riemannian manifolds the study of the radial parts of the Brownian motions allows to prove the sharp Cheeger–Yau lower bound [7] for the heat kernel, and as a consequence the sharp Cheng's estimate [8] for the eigenvalues of metric balls, see the paper [10] and the book [9]. Those methods were then extended in the framework of RCD spaces in [12] and adapted to sub-Riemannian manifolds in [4]. The goal of the present paper is to use similar probabilistic techniques to prove a sharp Cheeger–Yau heat kernel lower bound on Kähler and quaternion Kähler manifolds. In Kähler manifold such techniques are available due to a recent Laplacian comparison theorem proved by Ni–Zheng [13]. In quaternion Kähler manifolds, we prove a sharp Laplacian comparison theorem that allows us to apply those

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techniques. Concerning the sharp lower bounds for the heat kernels, our results are then the following.

In Kähler manifolds we obtain

Theorem 1.1 (Cheeger–Yau estimate on Kähler manifolds, see Theorem 4.3). Let $\mathbb{M}$ be a Kähler manifold with complex dimension m (i.e., the real dimension is $2m$). Assume that $H \geq 4k$ and that $\mathrm{Ric}^{\perp} \geq (2m-2)k$ for some $k \in \mathbb{R}$, where H denotes the holomorphic sectional curvature and $\mathrm{Ric}^{\perp}$ the orthogonal Ricci curvature. Then, denoting by $p_t^R(x,y)$ the Dirichlet heat kernel of $\mathbb{M}$ on a metric ball of radius $R>0$ one has for every $t>0$ and x,y inside of the ball,

$$p_t^R(x,y) \geq q_t^{k,R}(0, d(x,y)),$$

where $q_t^{k,R}$ is the Dirichlet heat kernel of a metric ball of radius R in the Kähler model of holomorphic sectional curvature $4k$.

The Kähler model for $k=0$ is the complex flat space $\mathbb{C}^m$, for $k=1$ it is the complex projective space $\mathbb{C}P^m$ and for $k=-1$, it is the complex hyperbolic space $\mathbb{C}H^m$.

In quaternion Kähler manifolds, we obtain

Theorem 1.2 (Cheeger–Yau estimate on quaternion Kähler manifolds, see Theorem 4.5). Let $\mathbb{M}$ be a quaternion Kähler manifold with quaternionic dimension m (i.e., the real dimension is $4m$). Assume that $Q \geq 12k$ and that $\mathrm{Ric}^{\perp} \geq (4m-4)k$ for some $k \in \mathbb{R}$, where Q denotes the quaternionic sectional curvature and $\mathrm{Ric}^{\perp}$ the orthogonal Ricci curvature. Then, denoting by $p_t^R(x,y)$ the Dirichlet heat kernel of $\mathbb{M}$ on a metric ball of radius $R>0$ one has for every $t>0$ and x,y inside of the ball,

$$p_t(x,y) \geq q_t^{k,R}(0, d(x,y)),$$

where $q_t^{k,R}$ is now the Dirichlet heat kernel of a metric ball of radius R in the quaternion Kähler model of quaternionic sectional curvature $12k$.

The quaternion Kähler model for $k=0$ is the quaternionic flat space $\mathbb{H}^m$, for $k=1$ it is the quaternionic projective space $\mathbb{H}P^m$ and for $k=-1$, it is the quaternionic hyperbolic space $\mathbb{H}H^m$.

We note that since Kähler or quaternionic Kähler manifolds are Riemannian manifolds, the classical Cheeger–Yau lower bound [7] is available. However, the Riemannian model spaces spheres and hyperbolic spaces are not Kähler or quaternionic Kähler models (except for $m = 1$), therefore the two above theorems are sharper.

The paper is organized as follows. In Section 2, we introduce the basic definitions and notations used throughout the paper. We also study the Brownian motions on the Kähler and quaternion Kähler models. Such study is important, since those Brownian motions provide the model processes with respect to which we aim to develop a comparison theory. In particular, the radial parts of those Brownian motions are one-dimensional diffusions whose generators can explicitly be computed. A summary of those generators is given in Section 2.3. In Section 3 we establish sharp Laplacian comparison theorems on Kähler and quaternionic Kähler manifolds. The Kähler case is known and due to Ni–Zheng [13]. We give a slightly different and self-contained proof that is easy to adapt to the quaternion Kähler case. The quaternion Kähler case is new. Both of those Laplacian comparison theorems are sharp in the sense that we obtain an equality for the model spaces. Section 4 is devoted to the proof of the comparison theorems. Using the approach by Ichihara [10] we prove, thanks to the results proved in the previous sections, the sharp Cheeger–Yau lower bounds for the heat kernels. As an easy consequence we deduce a sharp Cheng's type estimate for the 1st eigenvalue of metric balls.

2 Brownian Motion on Kähler and Quaternion Kähler Model Manifolds

In this section we fix notations and give some reminders about Kähler and quaternion Kähler manifolds and study the Brownian motions on the model spaces of those geometries. Brownian motions on Kähler models and quaternion Kähler models have already been studied in disparate places in the literature, so that the present section is essentially a survey of known results. However, our goal is a unified presentation that has interest on its own. We refer to [2], [3], and [5] and the references therein for further details.

2.1 Basic definitions

Kähler and quaternion Kähler manifolds are Riemannian manifolds equipped with some invariant $(1, 1)$ tensors preserving the metric and inducing a complex or quaternionic structure. In this paper, we will take the point of view of real Riemannian geometry to study those structures. A detailed presentation of this viewpoint about Kähler and quaternion Kähler manifolds is given in Chapters 2 and 14 of the book by Besse [6] to which we refer for further references.

Throughout the paper, let $(\mathbb{M}, g)$ be a smooth complete Riemannian manifold. Denote by ∇ the Levi-Civita connection on $\mathbb{M}$.

2.1.1 *Kähler manifolds*

Definition 2.1. The manifold $(\mathbb{M}, g)$ is called a Kähler manifold, if there exists a smooth $(1,1)$ tensor J on $\mathbb{M}$ that satisfies the following:

- For every $x \in \mathbb{M}$, and $X, Y \in T_x\mathbb{M}$, $g_x(J_xX, Y) = -g_x(X, J_xY)$;
- For every $x \in \mathbb{M}$, $J_x^2 = -\mathbf{Id}_{T_x\mathbb{M}}$;
- ∇J=0.

The map J is called a complex structure.

On Kähler manifolds, we will be considering the following type of curvatures. Let

$$R(X, Y, Z, W) = g((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z, W)$$

be the Riemannian curvature tensor of $(\mathbb{M}, g)$. The holomorphic sectional curvature of the Kähler manifold $(\mathbb{M}, g, J)$ is defined as

$$H(X) = \frac{R(X, JX, JX, X)}{g(X, X)^2}.$$

The orthogonal Ricci curvature (see [14]) of the Kähler manifold $(\mathbb{M}, g, J)$ is defined for a vector field X such that $g(X, X) = 1$ by

$$\mathrm{Ric}^{\perp}(X, X) = \mathrm{Ric}(X, X) - H(X),$$

where Ric is the usual Riemannian Ricci tensor of $(\mathbb{M}, g)$.

2.1.2 *Quaternion Kähler manifolds*

In the paper we shall use the following definition of quaternion Kähler manifold, see [6, Chapter 14].

Definition 2.2. The manifold $(\mathbb{M}, g)$ is called a quaternion Kähler manifold, if there exists a covering of $\mathbb{M}$ by open sets U_i and, for each i, 3 smooth $(1,1)$ tensors I, J, K on U_i such that

- For every $x \in U_i$, and $X, Y \in T_x\mathbb{M}$, $g_x(I_xX, Y) = -g_x(X, I_xY)$, $g_x(J_xX, Y) = -g_x(X, J_xY)$, $g_x(K_xX, Y) = -g_x(X, K_xY)$;
- For every $x \in U_i$, $I_x^2 = J_x^2 = K_x^2 = I_xJ_xK_x = -\mathbf{Id}_{T_x\mathbb{M}}$;

- For every $x \in U_i$, and $X \in T_x\mathbb{M}$ $\nabla_X I, \nabla_X J, \nabla_X K \in \mathbf{span}\{I, J, K\}$;
- For every $x \in U_i \cap U_j$, the vector space of endomorphisms of $T_x\mathbb{M}$ generated by I_x, J_x, K_x is the same for i and j.

It is worth noting that in some cases like the quaternionic projective spaces for topological reasons the tensors I, J, K may not be defined globally. However, $\mathbf{span}\{I, J, K\}$ may always be defined globally according to the last bullet point.

On quaternion Kähler manifolds, we will be considering the following curvatures. As above, let

$$R(X, Y, Z, W) = g((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z, W)$$

be the Riemannian curvature tensor of $(\mathbb{M}, g)$. We define the quaternionic sectional curvature of the quaternionic Kähler manifold $(\mathbb{M}, g, J)$ as

$$Q(X) = \frac{R(X, IX, IX, X) + R(X, JX, JX, X) + R(X, KX, KX, X)}{g(X, X)^2}.$$

We define the orthogonal Ricci curvature of the quaternionic Kähler manifold $(\mathbb{M}, g, I, J, K)$ for a vector field X such that $g(X, X) = 1$ by

$$\mathrm{Ric}^{\perp}(X, X) = \mathrm{Ric}(X, X) - Q(X),$$

where Ric is the usual Riemannian Ricci tensor of $(\mathbb{M}, g)$.

2.2 Model spaces and their Brownian motions

The constant curvature model spaces of Riemannian geometry are the Euclidean spaces, the spheres and the hyperbolic spaces. Euclidean spaces are Kähler if the dimension is even and quaternion Kähler if the dimension is a multiple of 4. The only spheres and hyperbolic spaces that are Kähler are the 2D ones. The only spheres and hyperbolic spaces that are quaternion Kähler are the 4D ones. In order to develop a comparison geometry for the Brownian motion in higher dimensional Kähler or quaternion Kähler geometry, one therefore needs to first study the Brownian motion on the models of those geometries. In this section, we review the Kähler and quaternion Kähler model spaces and their Brownian motions. All of those model spaces are rank one Riemannian symmetric spaces. As such, see [1], the radial parts of the Brownian motions are diffusion processes.

2.2.1 *Kähler models*

Flat model. The flat model of a Kähler manifold is

$$\mathbb{C}^m = \left\{(z_1, \cdots, z_m),\, z_1, \cdots, z_m \in \mathbb{C}\right\}$$

equipped with its standard Hermitian inner product. The complex structure J in that case is just the component-wise multiplication by i. The Brownian motion $(W_t)_{t\geq 0}$ on $\mathbb{C}^m$ is the diffusion process associated with the Laplace operator

$$\Delta_{\mathbb{C}^m} = 4\sum_{i=1}^m \frac{\partial^2}{\partial z_i \partial \bar{z}_i} = \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2},$$

where x_i is the real part of z_i, y_i its imaginary part and

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right).$$

One has

$$W_t = \left(Z_t^1, \cdots, Z_t^m\right),$$

where the Z^i's are independent complex Brownian motions on $\mathbb{C}$. The radial part of W defined by

$$r_t = |W_t| = \sqrt{\sum_{i=1}^m |Z_t^i|^2}$$

is itself a diffusion process with Bessel generator

$$L_{\mathbb{C}^m} = \frac{\partial^2}{\partial r^2} + \frac{2m-1}{r}\frac{\partial}{\partial r}.$$

We note that the radial part of the Lebesgue measure on $\mathbb{C}^m$ then writes the following:

$$d\mu_{\mathbb{C}^m} = 2\frac{\pi^m}{(m-1)!} r^{2m-1} dr, \quad r \geq 0.$$

Positively curved model. The positively curved model of a Kähler manifold is the complex projective space $\mathbb{C}P^m$. It can be constructed as follows. Consider the unit sphere

$$\mathbb{S}^{2m+1} = \{z = (z_1, \cdots, z_{m+1}) \in \mathbb{C}^{m+1}, \|z\| = 1\}.$$

There is an isometric group action of $\mathbb{S}^1 = \mathbf{U}(1)$ on $\mathbb{S}^{2m+1}$, which is defined by

$$e^{i\theta} \cdot (z_1, \cdots, z_{m+1}) = (e^{i\theta} z_1, \cdots, e^{i\theta} z_{m+1}).$$

The quotient space $\mathbb{S}^{2m+1}/\mathbf{U}(1)$ is defined as $\mathbb{C}P^m$ and the projection map $\pi : \mathbb{S}^{2m+1} \to \mathbb{C}P^m$ is a Riemannian submersion with totally geodesic fibers. The Kähler structure on $\mathbb{C}P^m$ is inherited from the one in $\mathbb{C}^{m+1}$ through this construction.

To parametrize points in $\mathbb{C}P^m \setminus \{\infty\}$, it is convenient to use the local inhomogeneous coordinates given by $w_j = z_j/z_{m+1}$, $1 \le j \le m$, $z \in \mathbb{C}^{n+1}$, $z_{m+1} \neq 0$. The point ∞ on $\mathbb{C}P^m$ corresponds to $z_{m+1} = 0$.

The submersion π allows one to construct the Brownian motion on $\mathbb{C}P^m$ from the Brownian motion on $\mathbb{S}^{2m+1}$. Indeed, let $(Z_t)_{t\ge 0}$ be a Brownian motion on the Riemannian sphere $\mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1}$ started at the north pole. We call north pole the point with complex coordinates $z_1 = 0, \cdots, z_{m+1} = 1$. Since $\mathbb{P}(\exists t \ge 0, Z^{m+1}(t) = 0) = 0$, one can use the local description of the submersion π in inhomogeneous coordinates to deduce that

$$W_t = \left(\frac{Z_t^1}{Z_t^{m+1}}, \cdots, \frac{Z_t^m}{Z_t^{m+1}} \right), \quad t \ge 0, \tag{1}$$

is a Brownian motion on $\mathbb{C}P^m$, that is, is a diffusion process with generator

$$\Delta_{\mathbb{C}P^m} = 4(1+|w|^2) \sum_{k=1}^m \frac{\partial^2}{\partial w_k \partial \overline{w_k}} + 4(1+|w|^2)\mathcal{R}\overline{\mathcal{R}},$$

where

$$\mathcal{R} = \sum_{j=1}^m w_j \frac{\partial}{\partial w_j}.$$

The radial part of W defined by

$$r_t = \arctan |W_t| = \arctan \sqrt{\sum_{i=1}^m \frac{|Z_t^i|^2}{|Z_t^{m+1}|^2}} = \arctan\left(\frac{1}{|Z_t^{m+1}|} \sqrt{1 - |Z_t^{m+1}|^2} \right)$$

is a diffusion process with Jacobi generator

$$L_{\mathbb{C}P^m} = \frac{\partial^2}{\partial r^2} + ((2m-2)\cot r + 2\cot 2r)\frac{\partial}{\partial r}.$$

We note that $L_{\mathbb{C}P^m} = \mathcal{L}^{m-1,0}$ where $\mathcal{L}^{m-1,0}$ is the Jacobi operator studied in the appendix of [5]. In particular, the spectrum of $\mathbb{C}P^m$ is given by

$$\mathrm{Sp}(\mathbb{C}P^m) = \{4k(k+m), k \geq 1\}.$$

Finally, we note that the radial part of the Riemannian volume measure writes

$$d\mu_{\mathbb{C}P^m} = \frac{\pi^m}{(m-1)!} (\sin r)^{2m-2} \sin(2r)\, dr, \quad 0 \leq r \leq \frac{\pi}{2}.$$

Negatively curved model. The negatively curved model of a Kähler manifold is the complex hyperbolic space $\mathbb{C}H^m$. It can be constructed as follows. Let us consider the complex hyperboloid

$$\mathcal{H}^{2m+1} = \{z \in \mathbb{C}^{m+1}, |z_1|^2 + \cdots + |z_m|^2 - |z_{m+1}|^2 = -1\} \subset \mathbb{C}^{m+1}.$$

The group $\mathbf{U}(1)$ acts isometrically on $\mathcal{H}^{2m+1}$. The quotient space of $\mathcal{H}^{2m+1}$ by this action is defined to be $\mathbb{C}H^m$ and the projection map $\pi : \mathcal{H}^{2m+1} \to \mathbb{C}H^m$ is a Riemannian submersion with totally geodesic fibers. Thus, as a differential manifold, the complex hyperbolic space $\mathbb{C}H^m$ is simply the open unit ball in $\mathbb{C}^m$ with a Riemannian metric inherited from the previous submersion. The Kähler structure on $\mathbb{C}H^m$ is inherited from the one in $\mathbb{C}^{m+1}$ through the above construction.

To parametrize $\mathbb{C}H^m$, one can use the global inhomogeneous coordinates given by $w_j = z_j/z_{m+1}$ where $(z_1, \dots, z_{m+1}) \in \mathcal{H}^{2m+1}$. In those coordinates, the Laplace operator of $\mathbb{C}H^m$ can be written as follows:

$$\Delta_{\mathbb{C}H^m} = 4(1-|w|^2) \sum_{k=1}^{m} \frac{\partial^2}{\partial w_k \partial \overline{w_k}} + 4(1-|w|^2)\mathcal{R}\overline{\mathcal{R}},$$

where

$$\mathcal{R} = \sum_{j=1}^{m} w_j \frac{\partial}{\partial w_j}.$$

The Brownian motion $(W_t)_{t\geq 0}$ on $\mathbb{C}H^m$ is the diffusion with generator $\Delta_{\mathbb{C}H^m}$. As for the case of $\mathbb{C}P^m$, it may be represented in inhomogeneous coordinates as

$$W_t = \left(\frac{Z_t^1}{Z_t^{m+1}}, \cdots, \frac{Z_t^m}{Z_t^{m+1}} \right), \quad t \geq 0,$$

where $(Z_t^1, \cdots, Z_t^{m+1})$ is a Brownian motion on $\mathcal{H}^{2m+1}$. The radial part of W defined by

$$r_t = \operatorname{arctanh}|W_t| = \operatorname{arctanh}\sqrt{\sum_{i=1}^m \frac{|Z_t^i|^2}{|Z_t^{m+1}|^2}} = \operatorname{arctanh}\left(\frac{1}{|Z_t^{m+1}|}\sqrt{|Z_t^{m+1}|^2 - 1}\right)$$

is a diffusion process with hyperbolic Jacobi generator

$$L_{\mathbb{C}H^m} = \frac{\partial^2}{\partial r^2} + ((2m-2)\coth r + 2\coth 2r)\frac{\partial}{\partial r}.$$

Finally, we note that the radial part of the Riemannian volume measure writes

$$d\mu_{\mathbb{C}H^m} = \frac{\pi^m}{(m-1)!}(\sinh r)^{2m-2}\sinh(2r)\,dr, \quad r \geq 0.$$

2.2.2 *Quaternion Kähler models*

Flat model. Let $\mathbb{H}$ be the non-commutative field of quaternions

$$\mathbb{H} = \{q = t + xI + yJ + zK, (t, x, y, z) \in \mathbb{R}^4\},$$

where I, J, K satisfy $I^2 = J^2 = K^2 = IJK = -1$. For $q = t + xI + yJ + zK \in \mathbb{H}$, we denote by $\overline{q} = t - xI - yJ - zK$ its conjugate, $|q|^2 = t^2 + x^2 + y^2 + z^2$ its squared norm and $\mathrm{Im}(q) = (x, y, z) \in \mathbb{R}^3$ its imaginary part.

The quaternionic structure I, J, K in that case is the component-wise multiplication by I, J, K, respectively. The Brownian motion $(W_t)_{t\geq 0}$ on $\mathbb{H}^m$ is the diffusion process associated with the Laplace operator

$$\Delta_{\mathbb{H}^m} = \sum_{i=1}^m \frac{\partial^2}{\partial t_i^2} + \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2}.$$

One can represent

$$W_t = \left(Q_t^1, \cdots, Q_t^m\right),$$

where the Q^i's are independent complex Brownian motions on $\mathbb{H}$. The radial part of W defined by

$$r_t = |W_t| = \sqrt{\sum_{i=1}^m |Q_t^i|^2}$$

is a diffusion process with Bessel generator

$$L_{\mathbb{H}^m} = \frac{\partial^2}{\partial r^2} + \frac{4m-1}{r}\frac{\partial}{\partial r}.$$

We note that the radial part of the Lebesgue measure on $\mathbb{H}^m$ then writes

$$d\mu_{\mathbb{H}^m} = 2\frac{\pi^{2m}}{(2m-1)!} r^{4m-1} dr, \quad r \geq 0.$$

Positively curved model. The positively curved model of a Kähler manifold is the quaternionic projective space $\mathbb{H}P^m$. It can be constructed as follows. Consider the unit sphere

$$\mathbb{S}^{4m+3} = \{q = (q_1, \cdots, q_{m+1}) \in \mathbb{H}^{m+1}, \|q\| = 1\}.$$

The group of unit quaternions is isomorphic to the Lie group $\mathbf{SU}(2)$. Thus, there is an isometric group action of $\mathbf{SU}(2)$ on $\mathbb{S}^{4m+3}$, which is defined by

$$q \cdot (q_1, \cdots, q_{m+1}) = (qq_1, \cdots, qq_{m+1}).$$

The quotient space $\mathbb{S}^{4m+3}/\mathbf{SU}(2)$ is defined as the quaternionic projective space $\mathbb{H}P^m$ and the projection map $\pi : \mathbb{S}^{4m+3} \to \mathbb{H}P^m$ is a Riemannian submersion with totally geodesic fibers. The quaternion Kähler structure on $\mathbb{H}P^m$ is inherited from the one in $\mathbb{H}^{m+1}$ through this construction.

To parametrize points in $\mathbb{H}P^m \setminus \{\infty\}$, we use the local inhomogeneous coordinates given by $w_j = q_{m+1}^{-1} q_m$, $1 \leq j \leq m$, $q \in \mathbb{H}^{n+1}$, $q_{m+1} \neq 0$. The point ∞ on $\mathbb{H}P^m$ corresponds to $q_{m+1} = 0$ and one can identify $\mathbb{H}P^m$ with $\mathbb{H}^m \cup \{\infty\}$.

As before, the submersion π allows to construct the Brownian motion on $\mathbb{H}P^m$ from the Riemannian Brownian motion on $\mathbb{S}^{4m+3}$. Indeed, let $(Q_t)_{t\geq 0}$ be a Brownian motion on the Riemannian sphere $\mathbb{S}^{4m+3} \subset \mathbb{H}^{m+1}$ started at the north pole. We call here north pole the point with quaternionic coordinates $q_1 = 0, \cdots, q_{m+1} = 1$. Since $\mathbb{P}(\exists t \geq 0, Q^{m+1}(t) = 0) = 0$, one deduces that

$$W_t = \left((Q_t^{m+1})^{-1} Q_t^1, \cdots, (Q_t^{m+1})^{-1} Q_t^m\right), \quad t \geq 0, \tag{2}$$

is a Brownian motion on $\mathbb{H}P^m$, that is, is a diffusion process with generator

$$\Delta_{\mathbb{H}P^m} = 4(1+|w|^2)^2 \sum_{k=1}^{m} \mathrm{Re}\left(\frac{\partial^2}{\partial w_k \partial \overline{w_k}}\right) - 8(1+|w|^2)\mathrm{Re}\left(\sum_{j=1}^{m} w_j \frac{\partial}{\partial w_j}\right).$$

In real coordinates, we have $w_i = t_i + x_i I + y_i J + z_i K$ and

$$\frac{\partial}{\partial w_i} := \frac{1}{2}\left(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial x_i} I - \frac{\partial}{\partial y_i} J - \frac{\partial}{\partial z_i} K\right).$$

The radial part of W defined by

$$r_t = \arctan|W_t| = \arctan\left(\frac{1}{|Q_t^{m+1}|}\sqrt{1-|Q_t^{m+1}|^2}\right)$$

is a diffusion process with Jacobi generator

$$L_{\mathbb{H}P^m} = \frac{\partial^2}{\partial r^2} + ((4m-4)\cot r + 6\cot 2r)\frac{\partial}{\partial r}.$$

We note that $L_{\mathbb{H}P^m} = \mathcal{L}^{2m-1,1}$ where $\mathcal{L}^{2m-1,1}$ is the Jacobi operator studied in the appendix of [5]. In particular, the spectrum of $\mathbb{H}P^m$ is given by

$$\mathrm{Sp}(\mathbb{H}P^m) = \{4k(k+2m+1), k \geq 1\}.$$

Finally, we note that the radial part of the Riemannian volume measure writes

$$d\mu_{\mathbb{H}P^m} = \frac{\pi^{2m}}{4(2m-1)!}(\sin r)^{4m-4}\sin(2r)^3\,dr, \quad 0 \leq r \leq \frac{\pi}{2}.$$

Negatively curved model. The positively curved model of a Kähler manifold is the quaternionic hyperbolic space $\mathbb{H}H^m$. It can be constructed as follows. Let us consider the quaternionic hyperboloid

$$\mathcal{Q}^{4m+3} = \{q \in \mathbb{H}^{m+1}, |q_1|^2 + \cdots + |q_m|^2 - |q_{m+1}|^2 = -1\} \subset \mathbb{H}^{m+1}.$$

The group $\mathbf{SU}(2)$ acts isometrically on $\mathcal{Q}^{4m+3}$. The quotient space of $\mathcal{Q}^{4m+3}$ by this action is defined to be $\mathbb{H}H^m$ and the projection map $\pi : \mathcal{Q}^{4m+3} \to \mathbb{H}H^m$ is a Riemannian submersion with totally geodesic fibers. The quaternion Kähler structure on $\mathbb{H}H^m$ is inherited from the one in $\mathbb{H}^{m+1}$.

To parametrize $\mathbb{H}H^m$, we use the global inhomogeneous coordinates given by $w_j = q_{m+1}^{-1} q_j$ where $(q_1, \ldots, q_{m+1}) \in \mathcal{Q}^{4m+3}$. In those coordinates, the Laplace operator of $\mathbb{H}H^m$ can be written as follows:

$$\Delta_{\mathbb{H}H^m} = 4(1-|w|^2)^2 \sum_{k=1}^{m} \mathrm{Re}\left(\frac{\partial^2}{\partial w_k \partial \overline{w_k}}\right) + 8(1+|w|^2)\mathrm{Re}\left(\sum_{j=1}^{m} w_j \frac{\partial}{\partial w_j}\right).$$

The Brownian motion $(W_t)_{t\geq 0}$ on $\mathbb{H}H^m$ is the diffusion with generator $\Delta_{\mathbb{H}H^m}$. It can be represented as

$$W_t = \left((Q_t^{m+1})^{-1} Q_t^1, \cdots, (Q_t^{m+1})^{-1} Q_t^m\right), \quad t \geq 0,$$

where $(Q_t)_{t\geq 0}$ is a Brownian motion on $\mathcal{Q}^{4m+3}$.

The radial part of W defined by

$$r_t = \operatorname{arctanh}|W_t| = \operatorname{arctanh}\left(\frac{1}{|Q_t^{m+1}|}\sqrt{|Q_t^{m+1}|^2 - 1}\right)$$

is a diffusion process with hyperbolic Jacobi generator

$$L_{\mathbb{H}H^m} = \frac{\partial^2}{\partial r^2} + ((4m-4)\coth r + 6\coth 2r)\frac{\partial}{\partial r}.$$

Finally, we note that the radial part of the Riemannian volume measure writes

$$d\mu_{\mathbb{H}H^m} = \frac{\pi^{2m}}{4(2m-1)!}(\sinh r)^{4m-4}\sinh(2r)^3\, dr, \quad r \geq 0.$$

We refer to [2] and [3] and references therein for complementary details.

2.3 Summary of the model spaces

For later use, and as a summary, we collect the results about the model spaces that will be used later. Additionally, in those model spaces the holomorphic/quaternionic sectional curvatures and orthogonal Ricci curvatures defined earlier may be computed explicitly and yield the following results:

Table 1 Radial Laplacians in Kähler model spaces

$\mathbb{M}$	Radial Laplacian	Radial measure
$\mathbb{C}^m$	$L_{\mathbb{C}^m} = \frac{\partial^2}{\partial r^2} + \frac{2m-1}{r}\frac{\partial}{\partial r}$	$d\mu_{\mathbb{C}^m} = 2\frac{\pi^m}{(m-1)!} r^{2m-1} dr$
$\mathbb{C}P^m$	$L_{\mathbb{C}P^m} = \frac{\partial^2}{\partial r^2} + ((2m-2)\cot r + 2\cot 2r)\frac{\partial}{\partial r}$	$d\mu_{\mathbb{C}P^m} = \frac{\pi^m}{(m-1)!}(\sin r)^{2m-2}\sin(2r)\, dr$
$\mathbb{C}H^m$	$L_{\mathbb{C}H^m} = \frac{\partial^2}{\partial r^2} + ((2m-2)\coth r + 2\coth 2r)\frac{\partial}{\partial r}$	$d\mu_{\mathbb{C}H^m} = \frac{\pi^m}{(m-1)!}(\sinh r)^{2m-2}\sinh(2r)\, dr$

Table 2 Curvatures of Kähler model spaces

$\mathbb{M}$	H	$\mathrm{Ric}^{\perp}$
$\mathbb{C}^m$	0	0
$\mathbb{C}P^m$	4	$2m-2$
$\mathbb{C}H^m$	-4	$-(2m-2)$

Table 3 Radial Laplacians in quaternion Kähler model spaces

$\mathbb{M}$	Radial Laplacian	Radial measure
$\mathbb{H}^m$	$L_{\mathbb{H}^m} = \frac{\partial^2}{\partial r^2} + \frac{4m-1}{r}\frac{\partial}{\partial r}$	$d\mu_{\mathbb{H}^m} = 2\frac{\pi^{2m}}{(2m-1)!} r^{4m-1} dr$
$\mathbb{H}P^m$	$L_{\mathbb{H}P^m} = \frac{\partial^2}{\partial r^2} + ((4m-4)\cot r + 6\cot 2r)\frac{\partial}{\partial r}$	$d\mu_{\mathbb{H}P^m} = \frac{\pi^{2m}}{4(2m-1)!}(\sin r)^{4m-4}\sin(2r)^3\, dr$
$\mathbb{H}H^m$	$L_{\mathbb{H}H^m} = \frac{\partial^2}{\partial r^2} + ((4m-4)\coth r + 6\coth 2r)\frac{\partial}{\partial r}$	$d\mu_{\mathbb{H}H^m} = \frac{\pi^{2m}}{4(2m-1)!}(\sinh r)^{4m-4}\sinh(2r)^3\, dr$

Table 4 Curvatures of the quaternion Kähler model spaces

$\mathbb{M}$	Q	$\mathrm{Ric}^{\perp}$
$\mathbb{H}^m$	0	0
$\mathbb{H}P^m$	12	$4m-4$
$\mathbb{H}H^m$	-12	$-(4m-4)$

3 Laplacian Comparison Theorems

This subsection is devoted to the proofs of the sharp Laplace comparison theorems in Kähler and quaternion Kähler manifolds. The main technical tool is the classical index lemma. In the Kähler case, the comparison theorem is due to Ni–Zheng [13] but seems to be new in the quaternion Kähler case.

We introduce the comparison function.

$$F(k,r) = \begin{cases} \sqrt{k}\cot\sqrt{k}r & \text{if } k>0, \\ \frac{1}{r} & \text{if } k=0, \\ \sqrt{|k|}\coth\sqrt{|k|}r & \text{if } k<0. \end{cases} \tag{3}$$

3.1 Kähler case

Let $(\mathbb{M}, g, J)$ be a complete Kähler with complex dimension m (i.e., the real dimension is $2m$). We denote by $d(x,y)$ the Riemannian distance between $x, y \in \mathbb{M}$ and by Δ the Laplace–Beltrami operator on $\mathbb{M}$. The following Laplacian comparison theorem was proved in [13]. As before, we denote by H the holomorphic sectional curvature of $\mathbb{M}$ and by $\mathrm{Ric}^{\perp}$ its orthogonal Ricci curvature.

Theorem 3.1 (Ni–Zheng [13]). Let $k \in \mathbb{R}$. Assume that $H \geq 4k$ and that $\mathrm{Ric}^{\perp} \geq (2m-2)k$. Let $x_0 \in \mathbb{M}$ and denote $r(x) = d(x_0, x)$. Then, pointwise outside of the cut-locus of x_0, and everywhere in the sense of distributions, one has

$$\Delta r \leq (2m-2)F(k,r) + 2F(k,2r).$$

Proof. The result can be found in [13]. We provide here a self-contained proof not only for completeness but also because the structure of our proof will be generalized to the quaternionic Kähler case which is new.

We can assume $m \geq 2$ since for the case $m = 1$, the statement reduces to the classical Laplacian comparison theorem in Riemannian geometry. Let $x_0 \in \mathbb{M}$ and $x \neq x_0$, which is not in the cut-locus of x_0. Let $\gamma : [0, r(x)] \to \mathbb{M}$ be the unique length parametrized geodesic connecting x_0 to x. At x, we consider an orthonormal frame $\{X_1(x), \cdots, X_{2m}(x)\}$ such that

$$X_1(x) = \gamma'(r(x)),\ X_2(x) = J\gamma'(r(x)).$$

We then have

$$\Delta r(x) = \sum_{i=1}^{2m} \nabla^2 r(X_i(x), X_i(x)).$$

We divide the above sum into three parts: $\nabla^2 r(X_1(x), X_1(x))$, $\nabla^2 r(X_2(x), X_2(x))$, and $\sum_{i=3}^{2m} \nabla^2 r(X_i(x), X_i(x))$. The 1st term $\nabla^2 r(X_1(x), X_1(x))$ is zero because $X_1(x) = \gamma'(r(x))$.

We now estimate the 2nd term. Note that the vector field defined along γ by $J\gamma'$ is parallel because J is parallel and γ is a geodesic, thus satisfies $\nabla_{\gamma'}\gamma' = 0$. We consider then the vector field defined along γ by

$$\tilde{X}(\gamma(t)) = \frac{\mathfrak{s}(4k,t)}{\mathfrak{s}(4k,r(x))} J\gamma'(t),$$

where

$$\mathfrak{s}(k,t) = \begin{cases} \sin\sqrt{k}t & \text{if } k > 0, \\ t & \text{if } k = 0, \\ \sinh\sqrt{|k|}t & \text{if } k < 0. \end{cases} \tag{4}$$

From the index lemma we have

$$\begin{aligned}
\nabla^2 r(X_2(x), X_2(x)) &\le \int_0^{r(x)} \left(\langle \nabla_{\gamma'}\tilde{X}, \nabla_{\gamma'}\tilde{X} \rangle - \langle R(\gamma', \tilde{X})\tilde{X}, \gamma' \rangle \right) \mathrm{d}t \\
&\le \frac{1}{\mathfrak{s}(4k,r(x))^2} \int_0^{r(x)} \left(\mathfrak{s}'(4k,t)^2 - \mathfrak{s}(4k,t)^2 \langle R(\gamma', J\gamma')J\gamma', \gamma' \rangle \right) \mathrm{d}t \\
&\le \frac{1}{\mathfrak{s}(4k,r(x))^2} \int_0^{r(x)} \left(\mathfrak{s}'(4k,t)^2 - 4k\mathfrak{s}(4k,t)^2 \right) dt \\
&\le 2F(k, 2r(x)).
\end{aligned}$$

Finally, we estimate the last term $\sum_{i=3}^{2m} \nabla^2 r(X_i(x), X_i(x))$. In order to proceed, we denote by $\{X_3, \cdots, X_{2m}\}$ the vector fields along γ obtained by parallel transport of $\{X_3(x), \cdots, X_{2m}(x)\}$. We observe that everywhere along γ, the family

$$\{\gamma', J\gamma', X_3, \cdots, X_{2m}\}$$

is an orthonormal frame. We consider then the vector field defined along γ by

$$\tilde{X}_i(\gamma(t)) = \frac{\mathfrak{s}(k,t)}{\mathfrak{s}(k,r(x))} X_i(\gamma(t)),\ i = 3, \cdots, 2m.$$

From the index lemma we obtain

$$\begin{aligned}\sum_{i=3}^{2m}\nabla^2 r(X_i(x),X_i(x)) &\le \sum_{i=3}^{2m}\int_0^{r(x)}\left(\langle\nabla_{\gamma'}\tilde{X}_i,\nabla_{\gamma'}\tilde{X}_i\rangle-\langle R(\gamma',\tilde{X}_i)\tilde{X}_i,\gamma'\rangle\right)\mathrm{d}t\\
&\le \frac{1}{\mathfrak{s}(k,r(x))^2}\sum_{i=3}^{2m}\int_0^{r(x)}\left(\mathfrak{s}'(k,t)^2-\mathfrak{s}(k,t)^2\langle R(\gamma',\tilde{X}_i')\tilde{X}_i,\gamma'\rangle\right)\mathrm{d}t\\
&\le \frac{1}{\mathfrak{s}(k,r(x))^2}\int_0^{r(x)}\left((2m-2)\mathfrak{s}'(k,t)^2-\mathfrak{s}(k,t)^2\sum_{i=3}^{2m}\langle R(\gamma',\tilde{X}_i')\tilde{X}_i,\gamma'\rangle\right)\mathrm{d}t\\
&\le \frac{1}{\mathfrak{s}(k,r(x))^2}\int_0^{r(x)}\left((2m-2)\mathfrak{s}'(k,t)^2-\mathfrak{s}(k,t)^2\mathrm{Ric}^{\perp}(\gamma',\gamma')\right)\mathrm{d}t\\
&\le \frac{2m-2}{\mathfrak{s}(k,r(x))^2}\int_0^{r(x)}\left(\mathfrak{s}'(k,t)^2-k\mathfrak{s}(k,t)^2\right)\mathrm{d}t\\
&\le (2m-2)F(k,r(x)).\end{aligned}$$

Therefore, we conclude

$$\Delta r(x)\le(2m-2)F(k,r(x))+2F(k,2r(x)).$$

Finally, proving that everywhere in the sense of distributions, one has

$$\Delta r\le(2m-2)F(k,r)+2F(k,2r),$$

is similar to the corresponding proof in the Riemannian case (which relies on Calabi lemma), so we skip the details. ■

It is remarkable that the theorem is sharp on the model spaces $\mathbb{C}^m,\mathbb{C}P^m$ and $\mathbb{C}H^m$. On $\mathbb{C}^m$, one has $k=0$ and

$$(2m-2)F(k,r)+2F(k,2r)=\frac{2m-1}{r}.$$

On $\mathbb{C}P^m$, one has $k=1$ and

$$(2m-2)F(k,r)+2F(k,2r)=(2m-2)\cot r+2\cot 2r,$$

and on $\mathbb{C}H^m$, one has $k = -1$ and

$$(2m-2)F(k,r) + 2F(k,2r) = (2m-2)\coth r + 2\coth 2r.$$

3.2 Quaternion Kähler case

Let now $(\mathbb{M}, g, I, J, K)$ be a complete quaternion Kähler with quaternionic dimension m (i.e., the real dimension is $4m$). We also denote by $d(x,y)$ the Riemannian distance between $x, y \in \mathbb{M}$ and by Δ the Laplace–Beltrami operator on $\mathbb{M}$. As before, we denote by Q the quaternionic sectional curvature of $\mathbb{M}$ and by $\mathrm{Ric}^{\perp}$ its orthogonal Ricci curvature.

Theorem 3.2. Let $k \in \mathbb{R}$. Assume that $Q \geq 12k$ and that $\mathrm{Ric}^{\perp} \geq (4m-4)k$. Let $x_0 \in \mathbb{M}$ and denote $r(x) = d(x_0, x)$. Then, pointwise outside of the cut-locus of x_0, and everywhere in the sense of distributions, one has

$$\Delta r \leq (4m-4)F(k,r) + 6F(k,2r).$$

Proof. The proof proceeds as in the Kähler case but is slightly more involved. As before, we can assume $m \geq 2$ since for the case $m = 1$, the statement reduces to the classical Laplacian comparison theorem in Riemannian geometry. Let $x_0 \in \mathbb{M}$ and $x \neq x_0$, which is not in the cut-locus of x. Let $\gamma : [0, r(x)] \to \mathbb{M}$ be the unique length parametrized geodesic connecting x_0 to x At x, we consider an orthonormal frame $\{X_1(x), \cdots, X_{4m}(x)\}$ such that

$$X_1(x) = \gamma'(r(x)),\ X_2(x) = I\gamma'(r(x)),\ X_3(x) = J\gamma'(r(x)),\ X_4(x) = K\gamma'(r(x)).$$

We then have

$$\Delta r(x) = \sum_{i=1}^{4m} \nabla^2 r(X_i(x), X_i(x)).$$

We divide the above sum into three parts: $\nabla^2 r(X_1(x), X_1(x))$, $\sum_{i=2}^{4} \nabla^2 r(X_i(x), X_i(x))$, and finally $\sum_{i=5}^{4m} \nabla^2 r(X_i(x), X_i(x))$. The 1st term $\nabla^2 r(X_1(x), X_1(x))$ is zero because $X_1(x) = \gamma'(r(x))$. Estimating the 2nd term requires more work than in the Kähler case because the vectors $I\gamma', J\gamma'$ and $K\gamma'$ might not be parallel along γ. Let us denote by X_2, X_3, and X_4 the vector fields along γ obtained by parallel transport along γ of $X_2(x), X_3(x)$, and $X_4(x)$. Since along γ one has

$$\nabla_{\gamma'} I, \nabla_{\gamma'} J, \nabla_{\gamma'} K \in \mathbf{span}\{I, J, K\}$$

we deduce that along γ one has

$$\mathbf{span}\{X_2, X_3, X_4\} = \mathbf{span}\{I\gamma', J\gamma', K\gamma'\}.$$

Moreover, $\{X_2, X_3, X_4\}$ and $\{I\gamma', J\gamma', K\gamma'\}$ are both orthonormal along γ. One deduces

$$\begin{aligned}
&R(\gamma', X_2, X_2, \gamma') + R(\gamma', X_3, X_3, \gamma') + R(\gamma', X_4, X_4, \gamma') \\
=&R(\gamma', I\gamma', I\gamma', \gamma') + R(\gamma', J\gamma', J\gamma', \gamma') + R(\gamma', K\gamma', K\gamma', \gamma') \\
=&Q(\gamma').
\end{aligned}$$

As a consequence, if we consider the vector field defined along γ by

$$\tilde{X}_i(\gamma(t)) = \frac{\mathfrak{s}(4k, t)}{\mathfrak{s}(4k, r(x))} X_i(\gamma(t)),\ i = 2, 3, 4,$$

we obtain by the same computation as in the proof of theorem 3.1

$$\sum_{i=2}^{4} \nabla^2 r(X_i(x), X_i(x)) \leq 6F(k, 2r(x)).$$

The estimate of the term $\sum_{i=5}^{4m} \nabla^2 r(X_i(x), X_i(x))$ is similar as in the proof of Theorem 3.1, so we skip the details for conciseness. ■

As in the Kähler case, it is remarkable that the theorem is sharp on the model spaces $\mathbb{H}^m, \mathbb{H}P^m$ and $\mathbb{H}H^m$.

4 Comparison Theorems for Radial Processes and Applications

4.1 Itô formula for radial processes on Riemannian manifolds

To fix notations, we 1st recall the well-known Kendall theorem [11] about the Itô formula for the radial parts of Brownian motions on a Riemannian manifold. Throughout this subsection, $(\mathbb{M}, g)$ is a complete Riemannian manifold and Δ denotes the Laplace–Beltrami operator. Let $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{M}})$ be the diffusion process generated by Δ, that is, the Brownian motion on $\mathbb{M}$. Take $x_0 \in \mathbb{M}$ and set $r(x) := d(x_0, x)$. We denote by $\mathrm{Cut}(x_0)$ the cut-locus of x_0. Let ζ be the life time of X.

Theorem 4.1 (Kendall [11]). For each $x_1 \in \mathbb{M}$, there exist a non-decreasing continuous process l_t, which increases only when $X_t \in \mathrm{Cut}(x_0)$, and a Brownian motion β_t on $\mathbb{R}$ with $\langle \beta \rangle_t = 2t$ such that

$$r(X_{t\wedge\zeta}) = r(X_0) + \beta_t + \int_0^{t\wedge\zeta} \Delta r(X_s)\mathrm{d}s - l_{t\wedge\zeta} \tag{5}$$

holds $\mathbb{P}_{x_1}$-almost surely.

4.2 Comparison theorems on Kähler manifolds

Let $(\mathbb{M}, g, J)$ be a complete Kähler with complex dimension m. Let $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{M}})$ be the Brownian motion on $\mathbb{M}$. As before, we fix a point $x_0 \in \mathbb{M}$. For $x_1 \in \mathbb{M}$, we consider the solution of the stochastic equation

$$\rho_t^k = d(x_0, x_1) + \int_0^t \Big((2m-2)F(k, \rho_s^k) + 2F(k, 2\rho_s^k) \Big)\, \mathrm{d}s + \sqrt{2}\beta_t,$$

where β is a standard Brownian motion under $\mathbb{P}_{x_1}$.

With Laplacian comparison theorems and Itô's formula (5) in hands, it is possible to apply *mutatis mutandis* the general available comparison methods developed in the Riemannian case for instance by Ichihara [10]. We also refer to Sections 3.5, 3.6, and 4.5 in the book [9] by Hsu. This yields the following basic comparison result.

Theorem 4.2. Let $k \in \mathbb{R}$. Assume that $H \geq 4k$ and that $\mathrm{Ric}^{\perp} \geq (2m-2)k$. Then, for $x_1 \in \mathbb{M}$, $R > 0$, and $s \leq R$

$$\mathbb{P}_{x_1}\big\{ d(x_0, X_t) < s,\, t \leq \tau_R \big\} \geq \mathbb{P}_{x_1}\big\{ \rho_t^k < s,\, t \leq \tau_R^k \big\},$$

where τ_R is the hitting time of the geodesic ball in $\mathbb{M}$ with center x_0 and radius R and τ_R^k the hitting time of the level R by ρ^k.

4.2.1 *Cheeger–Yau-type lower bound for the heat kernel*

A 1st corollary of Theorem 4.2 is a Cheeger–Yau-type lower bound for the heat kernel. It gives a sharp lower bound for the Dirichlet heat kernel on balls in terms of the heat kernel of a corresponding Kähler model space.

We introduce the following notation. For $k \in \mathbb{R}$, let L_k be the diffusion operator given by

$$L_k = \begin{cases} \frac{\partial^2}{\partial r^2} + ((2m-2)\sqrt{k}\cot\sqrt{k}r + 2\sqrt{k}\cot 2\sqrt{k}r)\frac{\partial}{\partial r} & \text{if } k > 0 \\ \frac{\partial^2}{\partial r^2} + \frac{2m-1}{r}\frac{\partial}{\partial r} & \text{if } k = 0 \\ \frac{\partial^2}{\partial r^2} + ((2m-2)\sqrt{|k|}\coth\sqrt{|k|}r + 2\sqrt{|k|}\coth 2\sqrt{|k|}r)\frac{\partial}{\partial r} & \text{if } k < 0 \end{cases}$$

and let μ_k be the measure

$$d\mu_k = \begin{cases} \frac{\pi^m}{(m-1)!k^{m-1/2}}(\sin\sqrt{k}r)^{2m-2}\sin(2\sqrt{k}r)\,dr & \text{if } k > 0 \\ 2\frac{\pi^m}{(m-1)!}r^{2m-1}dr & \text{if } k = 0 \\ \frac{\pi^m}{(m-1)!|k|^{m-1/2}}(\sinh\sqrt{|k|}r)^{2m-2}\sinh(2\sqrt{|k|}r)\,dr & \text{if } k < 0. \end{cases}$$

Note that the operator L_k is symmetric with respect to the measure μ_k. With the notations of Section 2.3, we have

$$(L_{-1}, \mu_{-1}) = (L_{\mathbb{C}H^m}, \mu_{\mathbb{C}H^m}), \quad (L_0, \mu_0) = (L_{\mathbb{C}^m}, \mu_{\mathbb{C}^m}), \quad (L_1, \mu_1) = (L_{\mathbb{C}P^m}, \mu_{\mathbb{C}P^m}).$$

Moreover, depending on the sign of k, (L_k, μ_k) is obtained from (L_1, μ_1), (L_0, μ_0) or (L_{-1}, μ_1) by a simple rescaling by $\sqrt{|k|}$.

Theorem 4.3 (Cheeger–Yau-type heat kernel lower bound). Let $k \in \mathbb{R}$. Assume that $H \geq 4k$ and that $\mathrm{Ric}^{\perp} \geq (2m-2)k$. Let $R > 0$. Let $((X_t^R)_{t\geq 0}, (\mathbb{P}_x)_{x\in B(x_0,R)})$ be a Brownian motion on $B(x_0, R)$ with Dirichlet boundary condition. Let $p^R(t, x, y)$ be its heat kernel with respect to the Riemannian volume measure μ. Let now $q_k^R(t, r_1, r_2)$ be the heat kernel with respect to μ_k of the diffusion on $[0, R]$ with generator L_k and Dirichlet boundary condition at R. Then, for every $t > 0$ and $x_1 \in B(x_0, R)$

$$p^R(t, x_0, x_1) \geq q_k^R(t, 0, d(x_0, x_1)).$$

Proof. From theorem 4.2, one has

$$\int_{B(x_0,s)} p^R(t, x_1, y)\mathrm{d}\mu(y) \geq \int_0^s q_k^R(t, d(x_0, x_1), r)\mathrm{d}\mu_k(r).$$

When $s \to 0^+$, one has

$$\mu(B(x_0, s)) \sim \frac{\pi^m}{m!}s^{2m} \sim \mu_k([0, s]).$$

On the other hand, from the Lebesgue differentiation theorem one has

$$\lim_{s\to 0^+} \frac{1}{\mu(B(x_0,s))}\int_{B(x_0,s)} p^R(t,x_1,y)\mathrm{d}\mu(y) = p^R(t,x_1,x_0) = p^R(t,x_0,x_1)$$

and

$$\lim_{s\to 0^+} \frac{1}{\mu_k([0,s])}\int_0^s q_k^R(t,d(x_0,x_1),r)\mathrm{d}\mu_k(r) = q_k^R(t,d(x_0,x_1),0) = q_k^R(t,0,d(x_0,x_1)).$$

The conclusion follows. ■

4.2.2 *Cheng's estimates for Dirichlet eigenvalues on metric balls*

A nice corollary of the Cheeger–Yau-type heat kernel lower bound is a Cheng's type upper bound for the Dirichlet eigenvalues of Riemannian balls in terms of the eigenvalues of Riemannian balls in the corresponding Kähler model.

Proposition 4.4 (Cheng's type estimates). Let $k \in \mathbb{R}$. Assume that $H \geq 4k$ and that $\mathrm{Ric}^{\perp} \geq (2m-2)k$. Let $R > 0$. For $x_0 \in \mathbb{M}$ let $\lambda_1(B_0(x_0,R))$ denote the 1st Dirichlet eigenvalue of the Riemannian ball $B(x_0,R)$ and let $\lambda_1(m,k,R)$ denote the 1st Dirichlet eigenvalue of the operator L_k on the interval $[0,R]$ with Dirichlet boundary condition at R. Then, for every $x_0 \in \mathbb{M}$ and $R > 0$

$$\lambda_1(B(x_0,R)) \leq \lambda_1(m,k,R).$$

Proof. From spectral theory, one has

$$p^R(t,x_1,y) = \sum_{j=1}^{+\infty} e^{-\lambda_j t}\phi_j(x_1)\phi_j(y),$$

where the λ_j's are the Dirichlet eigenvalues of $B(x_0,R)$ and the ϕ_j's the eigenfunctions. One has a similar spectral expansion for $q_k^R(t,r_0,r)$. Thus, from Corollary 4.3, when $t\to+\infty$ one must have $\lambda_1 \leq \lambda_1(m,k,R)$. ■

4.3 Comparison theorems on quaternion Kähler manifolds

In the quaternionic Kähler framework the comparison theorems of Cheeger–Yau-type and of Cheng's type might be obtained in a similar way as in the Kähler case. The difference is the model diffusion with respect to which the comparison is made.

Let $(\mathbb{M}, g, I, J, K)$ be a complete quaternion Kähler with quaternionic dimension m and for $k \in \mathbb{R}$ consider the following diffusion operator

$$\tilde{L}_k = \begin{cases} \frac{\partial^2}{\partial r^2} + ((4m-4)\sqrt{k}\cot\sqrt{k}r + 6\sqrt{k}\cot 2\sqrt{k}r)\frac{\partial}{\partial r} & \text{if } k > 0 \\ \frac{\partial^2}{\partial r^2} + \frac{4m-1}{r}\frac{\partial}{\partial r} & \text{if } k = 0 \\ \frac{\partial^2}{\partial r^2} + ((4m-4)\sqrt{|k|}\coth\sqrt{|k|}r + 6\sqrt{|k|}\coth 2\sqrt{|k|}r)\frac{\partial}{\partial r} & \text{if } k < 0 \end{cases}$$

and measure

$$d\tilde{\mu}_k = \begin{cases} \frac{\pi^{2m}}{4(2m-1)!k^{2m-1/2}} (\sin\sqrt{k}r)^{4m-4}\sin(2\sqrt{k}r)^3\, dr & \text{if } k > 0 \\ 2\frac{\pi^{2m}}{(2m-1)!} r^{4m-1} dr & \text{if } k = 0 \\ \frac{\pi^m}{(m-1)!|k|^{2m-1/2}} (\sinh\sqrt{|k|}r)^{4m-4}\sinh(2\sqrt{|k|}r)^3\, dr & \text{if } k < 0. \end{cases}$$

Note that the operator $\tilde{L}_k$ is symmetric with respect to the measure $\tilde{\mu}_k$ and that with the notations of Section 2.3, we therefore have

$$(\tilde{L}_{-1}, \tilde{\mu}_{-1}) = (L_{\mathbb{H}H^m}, \mu_{\mathbb{H}H^m}), \quad (\tilde{L}_0, \tilde{\mu}_0) = (L_{\mathbb{H}^m}, \mu_{\mathbb{H}^m}), \quad (\tilde{L}_1, \tilde{\mu}_1) = (L_{\mathbb{H}P^m}, \mu_{\mathbb{H}P^m}).$$

As in the Kähler case, depending on the sign of k, $(\tilde{L}_k, \tilde{\mu}_k)$ is obtained from $(\tilde{L}_1, \tilde{\mu}_1)$, $(\tilde{L}_0, \tilde{\mu}_0)$ or $(\tilde{L}_{-1}, \tilde{\mu}_1)$ by a simple rescaling by $\sqrt{|k|}$

By applying the same methods as before, we obtain the following results.

Theorem 4.5 (Cheeger–Yau-type lower bound). Let $k \in \mathbb{R}$. Assume that $\Omega \geq 12k$ and that $\mathrm{Ric}^{\perp} \geq (4m-4)k$. Let $R > 0$. Let $((X_t^R)_{t\geq 0}, (\mathbb{P}_x)_{x\in B(x_0,R)})$ be a Brownian motion on $B(x_0, R)$ with Dirichlet boundary condition. Let $p^R(t, x, y)$ be its heat kernel with respect to the Riemannian volume measure μ. Let now $\tilde{q}_k^R(t, r_1, r_2)$ be the heat kernel with respect to $\tilde{\mu}_k$ of the diffusion on $[0, R]$ with generator $\tilde{L}_k$ and Dirichlet boundary condition at R. Then, for every $t > 0$ and $x_1 \in B(x_0, R)$

$$p^R(t, x_0, x_1) \geq \tilde{q}_k^R(t, 0, d(x_0, x_1)).$$

Proposition 4.6 (Cheng's type estimates). Let $k \in \mathbb{R}$. Assume that $\Omega \geq 12k$ and that $\mathrm{Ric}^{\perp} \geq (4m-4)k$. Let $R > 0$. For $x_0 \in \mathbb{M}$ let $\lambda_1(B_0(x_0, R))$ denote the 1st Dirichlet eigenvalue of the Riemannian ball $B(x_0, R)$ and let $\tilde{\lambda}_1(m, k, R)$ denote the 1st Dirichlet eigenvalue of the operator $\tilde{L}_k$ on the interval $[0, R]$ with Dirichlet boundary condition at

R. Then, for every $x_0 \in \mathbb{M}$ and $R > 0$

$$\lambda_1(B(x_0, R)) \leq \tilde{\lambda}_1(m, k, R).$$

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