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Gagliardo-Nirenberg, Trudinger-Moser and Morrey inequalities on Dirichlet spaces

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ABSTRACT

With a view towards Riemannian or sub-Riemannian manifolds, RCD metric spaces and specially fractals, this paper proves Sobolev embedding theorems in the general framework of Dirichlet spaces. Under suitable assumptions that are verified in a variety of settings, we obtain the whole family of Gagliardo-Nirenberg and Trudinger-Moser inequalities with optimal exponents. These turn out to depend not only on the Hausdorff and walk dimensions of the space but also on other invariants. In addition, we prove Morrey type inequalities and apply them to study the infimum of the exponents that ensure continuity of Sobolev functions. The results are illustrated in the case of fractals with the Vicsek set, whereas several conjectures are made for general nested fractals and the Sierpinski carpet.

1. Introduction

The theory of Sobolev spaces was first pushed forward in order to prove solvability of certain partial differential equations, see for example [31]. When X is a Riemannian manifold, a function $f \in L^p(X)$ is said to be in the Sobolev space $W^{1,p}(X)$ if its distributional gradient is given by a vector-valued function $\nabla f \in L^p(X : \mathbb{R}^n)$. In more general spaces, a distributional theory of derivatives relying on integration by parts may not be available, which makes necessary to find an alternative notion of derivative.

After the seminal paper of J. Cheeger [15], a variety of notions of a gradient were introduced in the general context of metric measure spaces; we refer for instance to the book by J. Heinonen [19] and the references therein. Those gradients naturally yield a rich theory of first order Sobolev spaces that was developed around stepstone works like the ones by N. Shanmugalingam [39]; see also the book [21] and the more recent papers by L. Ambrosio, M. Colombo and S. Di Marino [5], and G. Savaré [38].

The approach to Sobolev spaces undertaken in the above cited references crucially relies on a notion of a measure-theoretic gradient that requires the underlying space to admit enough "good" rectifiable curves,

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a property that may not be present in some singular, fractal-like, metric measure spaces. With the aim of including these, potential-theoretic based definitions have been introduced and studied at different levels of generality, see e.g. [40,24,35] and references therein. The present paper is set up in the framework of Dirichlet spaces that are general enough to also cover this type of fractals.

Dirichlet spaces are measure spaces equipped with a closed Markovian symmetric bilinear form $\mathcal{E}$, called Dirichlet form, whose domain is dense in L^2. Dirichlet spaces provide a unified framework to study doubling metric measure spaces supporting a 2-Poincaré inequality [28], fractals [25], infinite-dimensional spaces [13] and non-local operators [16]. An important tool available in any Dirichlet space is the heat semigroup. The latter is a priori an L^2 object, meaning that it is originally defined on L^2 by means of the Dirichlet form $\mathcal{E}$ itself using spectral theory of Hilbert spaces. However, the Markovian property of $\mathcal{E}$ and classical interpolation theory allow to define this semigroup as a family of operators acting on any L^p space, $1 \le p \le +\infty$.

Such an extension was used in [4] to develop a theory of L^p Besov type spaces that have systematically been studied in the context of strictly local spaces [1], strongly local spaces with sub-Gaussian heat kernel estimates [2] and non-local spaces [3]. While the papers [1,2] primarily dealt with the L^1 theory and the associated theory of bounded variation (BV) functions and sets of finite perimeter, the present paper focuses on the L^p theory for $p > 1$. The Sobolev spaces considered here arise as L^p Besov spaces at the critical exponent, cf. Definition 2.3, and coincide with their classical counterpart in the Riemannian and other often studied metric measure settings, see Section 3. This heat semigroup approach digresses from existing generalizations of the classical ideas of Mazy'a [31] to fractals, see e.g. [22,23].

Once Sobolev spaces have been identified, it is natural to investigate analogues of the famous Gagliardo-Nirenberg and Trudinger-Moser inequalities. Such inequalities classically play an important role in the study of partial differential equations and include as special cases the Sobolev embedding inequality, the Nash inequality and the Ladyzhenskaya's inequality to name but a few. Besides their applications to partial differential equations, Gagliardo-Nirenberg and Trudinger-Moser inequalities also carry geometric information and, in the context of Riemannian geometry, they have for instance been applied to the study of sets of finite perimeter, conformal geometry [14] and cohomology [34]. In the context of metric measure spaces, they have been closely related to the study of quasi-conformal or quasi-symmetric maps and invariants, see [20].

The paper is organized as follows: Section 2 introduces the Sobolev spaces $W^{1,p}(\mathcal{E})$, $p \ge 1$, associated with a general Dirichlet form $\mathcal{E}$. These are characterized in Section 3 for various specific classes of examples. In strictly local Dirichlet spaces, which admit a canonical gradient structure intrinsically associated to the form, it is shown in Theorem 3.3 that, under suitable conditions, $W^{1,p}(\mathcal{E})$ coincides with the Sobolev space defined by that gradient structure. Section 4 is devoted to the study of Gagliardo-Nirenberg and Trudinger-Moser inequalities in general Dirichlet spaces, cf. Theorem 4.1 and Corollary 4.6. The techniques rely on the general methods proposed by D. Bakry, T. Coulhon, M. Ledoux and L. Saloff-Coste in the paper [7]; besides the ultracontractivity of the semigroup, the main assumption is an L^p pseudo-Poincaré inequality that is related to a weak notion of curvature (in the Bakry-Émery sense) of the underlying space. The latter is shown to be satisfied in large classes of examples like RCD spaces or nested fractals. Finally, Section 5 investigates embedding of the Sobolev spaces into spaces of Hölder functions. Of particular interest is the infimum $\delta_{\mathcal{E}}$ of the exponents for which such embedding occurs. In strictly local spaces and under suitable assumptions it is possible to bound above this quantity by the Hausdorff dimension of the space, cf. Theorem 5.9. In the case of fractals, Theorem 5.10 shows that for the Vicsek set $\delta_{\mathcal{E}} = 1$. Moreover, it is conjectured that for the Sierpinski gasket also $\delta_{\mathcal{E}} = 1$, whereas for the Sierpinski carpet

$$\delta_{\mathcal{E}} = 1 + \frac{\log 2}{d_W \log 3 - 2 \log 2},$$

where $d_W \approx 2.097$ is the so-called walk dimension of the carpet.

Notations. If Λ_1 and Λ_2 are functionals defined on a class of functions $f \in \mathcal{C}$, the notation

$$\Lambda_1(f) \simeq \Lambda_2(f)$$

means that there exist constants $c, C > 0$ such that for every $f \in \mathcal{C}$

$$c\Lambda_1(f) \leq \Lambda_2(f) \leq C\Lambda_1(f).$$

Also, in proofs, c, C will generically denote positive constants whose values may change from one line to another.

2. Framework, basic definitions and preliminaries

Throughout the paper, X will denote a good measurable space (like a Polish or Radon space) as defined in [8, p. 54]. We assume that X is equipped with a σ-finite measure μ supported on X. In addition, the pair $(\mathcal{E}, \mathcal{F})$, where $\mathcal{F} = \operatorname{dom} \mathcal{E}$, will denote a Dirichlet form on $L^2(X, \mu)$. We refer to $(X, \mu, \mathcal{E}, \mathcal{F})$ as a Dirichlet space. Its associated heat semigroup $\{P_t\}_{t \geq 0}$ admits a heat kernel measure $p_t(y, dx)$ [8, Theorem 1.2.3] and we always assume the semigroup to be conservative, i.e. $P_t 1 = 1$. Further details about this setting can be found in [4].

2.1. Heat semigroup-based BV, Sobolev and Besov classes

Following [4], we define the (heat semigroup-based) Besov classes associated with a Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$.

Definition 2.1. For any $p \geq 1$ and $\alpha \geq 0$, define

$$\mathbf{B}^{p,\alpha}(X) := \left\{ f \in L^p(X, \mu) \,:\, \limsup_{t \to 0^+} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < +\infty \right\}.$$

The basic properties of the space $\mathbf{B}^{p,\alpha}(X)$ endowed with the semi-norm

$$\|f\|_{p,\alpha} = \sup_{t>0} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}$$

are studied in [4]. In the present paper, we shall also be interested in the localized semi-norms defined for $R > 0$ as

$$\|f\|_{p,\alpha,R} := \sup_{t \in (0,R)} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

The local theory is important for instance to treat the case when the underlying space is compact. Note that, in view of [4, Lemma 4.1], one has for every $R > 0$

$$\|f\|_{p,\alpha,R} \leq \|f\|_{p,\alpha} \leq \frac{2}{R^\alpha} \|f\|_{L^p(X,\mu)} + \|f\|_{p,\alpha,R}$$

and in particular all the norms $\|f\|_{L^p(X,\mu)} + \|f\|_{p,\alpha,R}$ are equivalent on $\mathbf{B}^{p,\alpha}(X)$ to the norm $\|f\|_{L^p(X,\mu)} + \|f\|_{p,\alpha}$.

The BV and Sobolev classes arise at the corresponding critical exponents as follows.

Definition 2.2. The class of heat semigroup based bounded variation (BV) functions is defined as

$$BV(\mathcal{E}) := \mathbf{B}^{1,\alpha_1}(X),$$

where

$$\alpha_1 = \sup\{\alpha > 0 : \ \mathbf{B}^{1,\alpha}(X) \text{ contains non a.e. constant functions}\}.$$

For any $f \in BV(\mathcal{E})$, its total variation is defined as

$$\mathbf{Var}_{\mathcal{E}}(f) := \liminf_{t\to 0^+} t^{-\alpha_1} \int_X P_t(|f - f(y)|)(y)d\mu(y).$$

As in the classical theory, the Sobolev classes are defined analogously for $p > 1$.

Definition 2.3. Let $p > 1$. The $(1, p)$ heat semigroup-based Sobolev class is defined as

$$W^{1,p}(\mathcal{E}) := \mathbf{B}^{p,\alpha_p}(X),$$

where

$$\alpha_p := \sup\{\alpha > 0 : \ \mathbf{B}^{p,\alpha}(X) \text{ contains non a.e. constant functions}\}.$$

For any $f \in W^{1,p}(\mathcal{E})$, its total p-variation is defined as

$$\mathbf{Var}_{p,\mathcal{E}}(f) := \liminf_{t\to 0^+} t^{-\alpha_p} \left(\int_X P_t(|f - f(y)|^p)(y)d\mu(y) \right)^{1/p}.$$

Remark 2.4. For consistency in the notation, we will write $\mathbf{Var}_{1,\mathcal{E}}(f) := \mathbf{Var}_{\mathcal{E}}(f)$ for $f \in BV(\mathcal{E})$.

Remark 2.5. From [4, Proposition 4.6], one has $\alpha_2 = \frac{1}{2}$, $W^{1,2}(\mathcal{E}) = \mathbf{dom}\,\mathcal{E} = \mathcal{F}$ and $\mathbf{Var}_{2,\mathcal{E}}(f) = 2\mathcal{E}(f, f)$.

The following lemma shows that the functionals $\mathbf{Var}_{p,\mathcal{E}}(f)$ behave nicely with respect to cut-off arguments. This is a crucial property that will allow us to use the techniques developed by D. Bakry, T. Coulhon, M. Ledoux and L. Saloff-Coste in [7].

Lemma 2.6. *For any nonnegative $f \in W^{1,p}(\mathcal{E})$, if $p > 1$, or $f \in BV(\mathcal{E})$ if $p = 1$, it holds that*

$$\left(\sum_{k\in\mathbb{Z}} \mathbf{Var}_{p,\mathcal{E}}(f_{\rho,k})^p \right)^{1/p} \le 2(p + 1)\mathbf{Var}_{p,\mathcal{E}}(f),$$

where $f_{\rho,k} := (f - \rho^k)_+ \wedge \rho^k(\rho - 1)$, $k \in \mathbb{Z}$ and $\rho > 1$.

Proof. Let $p_t(y,dx)$ denote the heat kernel measure of the semigroup P_t. We first observe that, once we prove

$$\sum_{k\in\mathbb{Z}}\int_X\int_X |f_{\rho,k}(x)-f_{\rho,k}(y)|^p p_t(y,dx)d\mu(y) \le 2(p+1)\int_X\int_X |f(x)-f(y)|^p p_t(y,dx)d\mu(y) \tag{1}$$

for any $\rho>0$, then

$$\liminf_{t\to 0^+}\left(\sum_{k\in\mathbb{Z}} t^{-p\alpha_p}\int_X\int_X |f_{\rho,k}(x)-f_{\rho,k}(y)|^p p_t(y,dx)d\mu(y)\right) \le 2(p+1)\liminf_{t\to 0^+} t^{-p\alpha_p}\int_X\int_X |f(x)-f(y)|^p p_t(y,dx)d\mu(y).$$

Using the superadditivity of the $\liminf$ one concludes

$$\sum_{k\in\mathbb{Z}}\liminf_{t\to 0^+} t^{-p\alpha_p}\int_X\int_X |f_{\rho,k}(x)-f_{\rho,k}(y)|^p p_t(y,dx)d\mu(y) \le 2(p+1)\liminf_{t\to 0^+} t^{-p\alpha_p}\int_X\int_X |f(x)-f(y)|^p p_t(y,dx)d\mu(y).$$

The inequality (1) can implicitly be found in the proof of [7, Lemma 7.1] with $a=p$. The details are left to the interested reader; keeping track of the constants in the aforementioned proof one sees in particular that the bound is independent of ρ. □

Remark 2.7. Lemma 2.6 corresponds to the condition (H_p), $p\ge 1$, introduced in [7, Section 2]; it will become relevant to obtain Trudinger-Moser inequalities.

2.2. L^p pseudo-Poincaré inequalities

Pseudo-Poincaré inequalities are a widely applicable tool to obtain Sobolev inequalities, see e.g. [36, Section 3.3]. In this paragraph we introduce and discuss two assumptions that are crucial to further analyze Gagliardo-Nirenberg and Trudinger-Moser inequalities.

The case $p>1$. The assumptions concern the validity of a L^p pseudo-Poincaré inequality, and the continuity of the heat semigroup in a suitable Sobolev space.

- **Condition** (PPI_p), $p\ge 1$. There exists a constant $C_p>0$ such that for every $t\ge 0$ and $f\in W^{1,p}(\mathcal{E})$ (or $BV(\mathcal{E})$ for $p=1$),

$$\|P_t f-f\|_{L^p(X,\mu)} \le C_p t^{\alpha_p}\mathbf{Var}_{p,\mathcal{E}}(f).$$

- **Condition** (G_q), $q>1$. There exists a constant $C_q>0$ such that for every $t>0$ and $f\in L^q(X,\mu)$,

$$\|P_t f\|_{q,\alpha_q} \le \frac{C_q}{t^{1-\alpha_p}}\|f\|_{L^q(X,\mu)}, \tag{2}$$

where p is the Hölder conjugate exponent of p, i.e. $\frac{1}{p}+\frac{1}{q}=1$.

Remark 2.8. It follows from spectral theory that $\alpha_2 = 1/2$ and that the assumptions (G_2) and (PPI_2) always hold.

Proposition 2.9. *Let $p > 1$ and let q be its Hölder conjugate. Under condition (G_q), for every $f \in W^{1,p}(\mathcal{E})$ and $t \geq 0$*

$$\|P_t f - f\|_{L^p(X,\mu)} \leq \frac{C_q}{2\alpha_p} t^{\alpha_p} \mathbf{Var}_{p,\mathcal{E}}(f),$$

where C_q is the same as in (2). In particular, condition (PPI_p) is satisfied.

Proof. Follow the p-version of the arguments proving [2, Proposition 3.10]. $\square$

The case $p = 1$. Recall that the semigroup $\{P_t\}_{t\geq 0}$ admits a measurable heat kernel $p_t(x, y)$, cf. [8, Theorem 1.2.3]. In addition, we consider the space (X, μ) to be endowed with a metric d. This metric d does not need to be intrinsically associated with the Dirichlet form but has to satisfy some conditions listed below.

- **Condition**. For any $\kappa \geq 0$, there exist constants $C, c > 0$ such that for every $t > 0$ and a.e. $x, y \in X$

$$d(x, y)^{\kappa} p_t(x, y) \leq C t^{\kappa/d_W} p_{ct}(x, y), \tag{3}$$

  where $d_W > 1$ is a parameter independent from κ, C and c.
- **Condition** (G_∞). There exists a constant $C > 0$ so that for every $t > 0$, $f \in L^\infty(X, \mu)$, and $x, y \in X$

$$|P_t f(x) - P_t f(y)| \leq C \frac{d(x, y)^{d_W(1-\alpha_1)}}{t^{1-\alpha_1}} \|f\|_{L^\infty(X,\mu)}. \tag{4}$$

We note that (3) is for instance satisfied if $p_t(x, y)$ satisfies sub-Gaussian heat kernel estimates, see [2, Lemma 2.3] and that the condition (G_∞) was called in [2] the weak Bakry-Émery estimate.

Remark 2.10. Since (G_2) always holds for every $t > 0$, using interpolation theory, one deduces as in the proof of [2, Theorem 3.9] that the assumption (G_∞) implies that for every $t > 0$, $q \geq 2$ and $f \in L^p(X, \mu)$,

$$\|P_t f\|_{q,\beta_q} \leq \frac{C_q}{t^{\beta_q}} \|f\|_{L^q(X,\mu)}, \tag{5}$$

where $\beta_q = \left(1 - \frac{2}{q}\right)(1 - \alpha_1) + \frac{1}{q}$. This is not quite the same as (G_q), unless $1 - \alpha_p = \beta_q$, i.e. $\alpha_p = \left(1 - \frac{2}{p}\right)(1 - \alpha_1) + \frac{1}{p}$. Note that for the Vicsek set (or direct products of it) one indeed has $\alpha_p = \left(1 - \frac{2}{p}\right)(1 - \alpha_1) + \frac{1}{p}$, see Remark 3.7.

Proposition 2.11. *If the Dirichlet space $(X, d, \mu, \mathcal{E})$ satisfies (G_∞) and (3), there exists a constant $C > 0$ such that for every $f \in BV(\mathcal{E})$ and $t \geq 0$,*

$$\|P_t f - f\|_{L^1(X,\mu)} \leq C t^{\alpha_1} \mathbf{Var}_{\mathcal{E}}(f).$$

In particular (PPI_1) is satisfied.

Proof. See [2, Proposition 3.10]. $\square$

To obtain the whole family of inequalities in the subsequent sections we will need the local counterparts of the previous conditions.

- **Condition** $(\mathrm{PPI}_p(R))$, $p \geq 1$. There exists a constant $C_p(R) > 0$ such that for every $t \in (0, R)$ and $f \in W^{1,p}(\mathcal{E})$ (or $BV(\mathcal{E})$ for $p = 1$),

$$\|P_t f - f\|_{L^p(X,\mu)} \leq C_p(R) t^{\alpha_p} \mathbf{Var}_{p,\mathcal{E}}(f).$$

- **Condition** $\mathrm{G}_q(R)$, $q > 1$, $R > 0$. There exists a constant $C_q(R) > 0$ such that for every $t \in (0, R)$ and $f \in L^q(X, \mu)$,

$$\|P_t f\|_{q,\alpha_q} \leq \frac{C_q}{t^{1-\alpha_p}} \|f\|_{L^q(X,\mu)}, \tag{6}$$

where as before p is the Hölder conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

The same proof as Proposition 2.9 yields the following result.

Proposition 2.12. *Let $p > 1$, $R > 0$ and assume that $\mathrm{G}_q(R)$ holds, where q is the Hölder conjugate of p. Then, for every $f \in W^{1,p}(\mathcal{E})$ and $t \in (0, R)$,*

$$\|P_t f - f\|_{L^p(X,\mu)} \leq \frac{C_q(R)}{2\alpha_p} t^{\alpha_p} \mathbf{Var}_{p,\mathcal{E}}(f)$$

with the same constant C_q as in (6). In particular, $(\mathrm{PPI}_p(R))$ is satisfied.

Similarly, to treat the case $p = 1$ one can introduce a localized version of (3) and of the condition $\mathrm{G}_\infty(R)$, $R > 0$ to prove the localized analogue of Proposition 2.11. We omit the details for conciseness.

2.3. Weak Bakry-Émery estimates

In this section, we investigate some self-improvement properties of the assumption $\mathrm{G}_\infty(R)$, $R > 0$.

Lemma 2.13. *Let d be a metric on X. Let $R > 0$ and assume that there exist constants $C, \kappa, d_W > 0$ such that for every $t \in (0, R)$, $f \in L^\infty(X, \mu)$ and $x, y \in X$,*

$$|P_t f(x) - P_t f(y)| \leq C \frac{d(x, y)^\kappa}{t^{\kappa/d_W}} \|f\|_{L^\infty(X,\mu)}. \tag{7}$$

Then, for any $R' \geq R$, (7) also holds for every $t \in (0, R')$ with a possibly different constant $C = C_{R'}$.

Proof. Let $f \in L^\infty(X, \mu)$ and $x, y \in X$. Applying (7) to the function $P_t f$ instead of f yields

$$|P_{2t} f(x) - P_{2t} f(y)| \leq C_R 2^{\kappa/d_W} \frac{d(x, y)^\kappa}{(2t)^{\kappa/d_W}} \|f\|_{L^\infty(X,\mu)}$$

and therefore (7) holds for $t \in (0, 2R)$ and $C = C_R 2^{\kappa/d_W}$. For any $R' > R$ we may choose $n > 0$ so that $R' < 2^n R$ and iterating the previous argument will give (7) for $t \in (0, R')$ with $C = C_R 2^{n\kappa/d_W}$. □

To extend (7) to all of $t > 0$ requires a better (uniform) control on the constants, which is possible under additional conditions.

Lemma 2.14. *Let d be a metric on X. Let $R > 0$ and assume that there exist constants $C, \kappa, d_W > 0$ such that for every $t \in (0, R)$, $f \in L^\infty(X, \mu)$ and $x, y \in X$,*

$$|P_tf(x) - P_tf(y)| \le C\frac{d(x,y)^\kappa}{t^{\kappa/d_W}}\|f\|_{L^\infty(X,\mu)}. \tag{8}$$

Moreover, assume that

(i) *the infinitesimal generator Δ of the Dirichlet form $(\mathcal{E},\mathcal{F})$ has a pure point spectrum,*
(ii) $1 \in \mathbf{dom}\,\Delta$,
(iii) *the Dirichlet space $(X,\mu,\mathcal{E},\mathcal{F})$ satisfies the Poincaré inequality*

$$\int_X \Big(f - \int_X f d\mu\Big)^2 d\mu \le \frac{1}{\lambda_1}\mathcal{E}(f,f)$$

for some $\lambda_1 > 0$ and all $f \in \mathcal{F}$,
(iv) *the heat kernel $p_t(x,y)$ of P_t satisfies the estimate $p_{t_0}(x,y) \le M$ for some $t_0, M > 0$ and μ-almost every $x,y \in X$.*

Then, (8) holds for all $t > 0$, possibly with a different constant $C > 0$.

Proof. By virtue of assumption (ii) one has $\mu(X) < +\infty$, so that without loss of generality we can assume $\mu(X) = 1$. Let $\{\lambda_j\}_{j\ge 0}$ denote the eigenvalues of Δ and $\{\phi_j\}_{j\ge 0}$ the associated eigenfunctions. Assumptions (ii) and (iii), see e.g. [8, Proposition 3.1.6], yield for any $f \in L^2(X,\mu)$

$$P_tf(x) = \int_X f d\mu + \sum_{j=1}^{+\infty} e^{-\lambda_j t}\phi_j(x)\int_X \phi_j(y)f(y)d\mu(y). \tag{9}$$

Now, since $P_{t_0}\phi_j = e^{-\lambda_j t_0}\phi_j$, applying Hölder's inequality and assumption (iv) we deduce for μ-a.e. $x \in X$

$$|\phi_j(x)| = e^{\lambda_j t_0}\left|\int_X p_{t_0}(x,y)\phi_j(y)d\mu(y)\right| \le e^{\lambda_j t_0}\left(\int_X p_{t_0}(x,y)^2 d\mu(y)\right)^{1/2} \le Me^{\lambda_j t_0}.$$

Next, using Lemma 2.13 if needed, we may assume $t_0 \le R$. Applying (8) to ϕ_j and the latter estimate we obtain

$$|e^{-\lambda_j t_0}\phi_j(x) - e^{-\lambda_j t_0}\phi_j(y)| \le CM\frac{d(x,y)^\kappa}{t_0^{\kappa/d_W}}e^{\lambda_j t_0},$$

and hence

$$|\phi_j(x) - \phi_j(y)| \le CM\frac{d(x,y)^\kappa}{t_0^{\kappa/d_W}}e^{2\lambda_j t_0}. \tag{10}$$

Finally, for any $f \in L^\infty(X,\mu)$ and $t > 2t_0$, (9) and (10) imply

$$\begin{aligned}|P_tf(x) - P_tf(y)| &\le \sum_{j=1}^{+\infty} e^{-\lambda_j t}|\phi_j(x) - \phi_j(y)|\int_X \phi_j(z)f(z)d\mu(z)\\ &\le CM\frac{d(x,y)^\kappa}{t_0^{\kappa/d_W}}\sum_{j=1}^{+\infty} e^{-\lambda_j(t-2t_0)}\int_X \phi_j(z)f(z)d\mu(z)\end{aligned}$$

$$\leq CM \frac{d(x,y)^\kappa}{t_0^{\kappa/d_W}} \|f\|_{L^\infty(X,\mu)} \sum_{j=1}^{+\infty} e^{-\lambda_j(t-2t_0)} \leq C' \frac{d(x,y)^\kappa}{t^{\kappa/d_W}} \|f\|_{L^\infty(X,\mu)},$$

where the constant C' depends on $M, C, \kappa, d_W, \lambda_j$ and t_0. $\square$

3. Examples of heat semigroup based BV and Sobolev classes

To illustrate the scope of our results we now present several classes of Dirichlet spaces that appear in the literature for which the heat semigroup based BV and Sobolev classes can be characterized. This generalizes previous results from [4,1,2].

3.1. Metric measure spaces with Gaussian heat kernel estimates

Further details to this particular framework can be found in [1]. We consider $(X, d, \mu, \mathcal{E}, \mathcal{F})$ to be a strictly local Dirichlet space, where d is the intrinsic metric associated to the Dirichlet form. The measure μ is assumed to be doubling and the space to supports a scale invariant 2-Poincaré inequality on balls; according to K.T. Sturm's results [43,44] these conditions are equivalent to the fact that there is a heat kernel with Gaussian estimates. In this setting, see [1, Lemma 2.11], $\mathcal{E}$ admits a carré du champ operator $\Gamma(f,f)$, $f \in \mathcal{F}$ and we denote $|\nabla f| = \sqrt{\Gamma(f,f)}$. Based on the ideas of M. Miranda [32], the following definitions were introduced in [1].

Definition 3.1 *(BV space).* We say that $f \in L^1(X,\mu)$ is in $BV(X)$ if there is a sequence of local Lipschitz functions $f_k \in L^1(X,\mu)$ such that $f_k \to f$ in $L^1(X,\mu)$ and

$$\|Df\|(X) := \liminf_{k\to\infty} \int_X |\nabla f_k| \, d\mu < \infty.$$

Definition 3.2 *(Sobolev space).* For $p \geq 1$, we define the Sobolev space

$$W^{1,p}(X) := \{f \in L^p(X,\mu) \cap \mathcal{F}_{\mathrm{loc}}(X) \,:\, |\nabla f| \in L^p(X)\} \tag{11}$$

whose norm is given by $\|f\|_{W^{1,p}(X)} = \|f\|_{L^p(X,d\mu)} + \|\,|\nabla f|\,\|_{L^p(X,\mu)}$.

Localization allows to extend the results appearing in [1, Theorem 4.4] to include compact spaces.

Theorem 3.3. *For each $R \in (0, +\infty]$ the following holds:*

(i) *Assume the weak Bakry-Émery estimate*

$$\|\,|\nabla P_t f|\,\|_{L^\infty(X,\mu)} \leq \frac{C}{\sqrt{t}} \|f\|_{L^\infty(X,\mu)} \qquad t \in (0,R) \tag{12}$$

for some constant $C > 0$ and any $f \in \mathcal{F} \cap L^\infty(X,\mu)$. Then, $(\mathrm{PPI}_1(R))$ is satisfied, $\alpha_1 = \frac{1}{2}$, $BV(\mathcal{E}) = BV(X)$ and

$$\mathbf{Var}_{\mathcal{E}}(f) \simeq \|f\|_{1,1/2,R} \simeq \liminf_{r\to 0^+} \int_X \int_{B(x,r)} \frac{|f(y)-f(x)|}{\sqrt{r}\mu(B(x,r))} \, d\mu(y) \, d\mu(x) \simeq \|Df\|(X).$$

(ii) *Assume the quasi Bakry-Émery condition estimate, cf. [1, Definition 2.15],*

$$|\nabla P_t f| \le C P_t |\nabla f| \qquad t \in (0, R) \tag{13}$$

μ-a.e. for some constant $C > 0$ and any $f \in \mathcal{F}$. Then, for every $p > 1$, condition $(\mathrm{PPI}_p(R))$ *is satisfied, $\alpha_p = \frac{1}{2}$, $W^{1,p}(\mathcal{E}) = W^{1,p}(X)$ and*

$$\mathbf{Var}_{p,\mathcal{E}}(f) \simeq \|f\|_{p,1/2,R} \simeq \left(\int_X |\nabla f|^p d\mu \right)^{1/p}.$$

Proof. It suffices to show the statements for non-negative functions.

(i) With the same proof as in [1, Lemma 4.3], condition (12) implies that

$$\|P_t f - f\|_{L^1(X,\mu)} \le C\sqrt{t} \int_X |\nabla f| \, d\mu$$

for any $t \in (0, R)$ and $f \in BV(X)$. Analogous to the proof of [1, Theorem 4.4], the latter inequality and the coarea formula [1, Theorem 3.11] yield $\mathbf{Var}_{\mathcal{E}}(f) \le \|f\|_{1,1/2,R} \le 2C\|Df\|(X)$. For $f \in BV(\mathcal{E})$, the Gaussian lower bound of the heat kernel and the second part of the proof of [30, Theorem 3.1] (which does not use 1-Poincaré inequality) give $\alpha_1 = 1/2$.

(ii) A local version of the arguments in the proof of [1, Theorem 4.11] yield

$$\| \, |\nabla f| \, \|_{L^p(X,\mu)} \le C \mathbf{Var}_{p,\mathcal{E}}(f) \le C\|f\|_{p,1/2,R}.$$

The reverse inequality follows as in the proof of [1, Theorem 4.17] with $t \in (0, R)$ and the quasi Bakry-Émery condition (13). $\square$

As one would expect, the quasi Bakry-Émery curvature condition (13) implies the weak one (12). Examples of spaces within the framework just discussed that satisfy (13) include Riemannian manifolds with Ricci curvature bounded from below and $RCD(K, +\infty)$ spaces; in that case for every $t \ge 0$, $|\nabla P_t f| \le e^{-Kt} P_t |\nabla f|$, and thus $|\nabla P_t f| \le C P_t |\nabla f|$ for $t \in (0, R)$ with $C = \max(1, e^{-KR})$, see [37]. On the other hand, Carnot groups [10] and complete sub-Riemannian manifolds with generalized Ricci curvature bounded from below in the sense of [11,12] are examples in this setting where the weak Bakry-Émery condition (12) is known but the stronger condition (13) unknown.

3.2. Fractal spaces

This paragraph summarizes and extends the results currently available that put some fractal spaces into our setting. In particular, Lemma 2.14 allows to treat the case of compact nested fractals by considering only *local* estimates.

Nested fractals. Nested fractals [29] are fractional metric spaces whose natural diffusion process is a fractional diffusion in the sense of Barlow [9, Definition 3.2]. For details about the following result we refer to Theorem 3.7, Theorem 4.9 and Theorem 5.1 of [2]. By an "infinite" fractal we mean its blow-up as introduced by R. S. Strichartz in [41].

Theorem 3.4. *Let (X, d, μ) be a compact or infinite nested fractal with $1 \le d_H \le d_W$. Then, it satisfies* (G_∞)*. In fact, the weak Bakry-Émery condition*

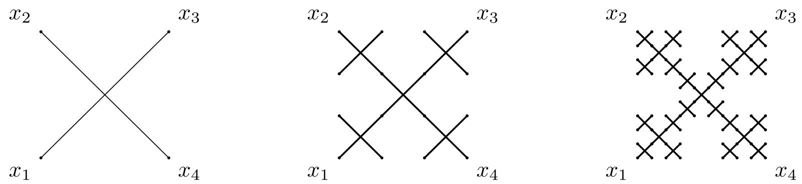

Fig. 1. Approximating graphs (V_m, E_m) for the Vicsek set.

$$|P_t f(x) - P_t f(y)| \le C \frac{d(x,y)^{d_W - d_H}}{t^{(d_W - d_H)/d_W}} \|f\|_{L^\infty(X,\mu)} \qquad t > 0 \tag{14}$$

holds for some $C > 0$ and any $f \in L^\infty(X, \mu)$. Moreover, $\alpha_1 = d_H/d_W$ and

$$\|f\|_{1,d_H/d_W} \simeq \mathbf{Var}_{\mathcal{E}}(f). \tag{15}$$

Remark 3.5. The condition (14) is satisfied in a more general class of fractals, cf. [2, Theorem 3.7], however the statement concerning α_1 and the equivalence of norms (15) is so far only valid for nested fractals. It is conjectured in [2, Conjecture 5.4] that for fractals like the Sierpinski carpet one has $\alpha_1 = (d_H - d_{tH} + 1)/d_W$, where d_{tH} denotes the topological Hausdorff dimension of the space.

Vicsek set. A specific example within this class of nested fractals is the Vicsek set in $\mathbb{R}^2$ equipped with its standard Dirichlet form $(\mathcal{E}, \mathcal{F})$, see e.g. [9, p. 26], for which $\alpha_1 = \frac{d_H}{d_W}$, cf. Theorem 3.4. In fact, it is possible to explicitly construct non-constant functions $h \in \mathcal{F}$ that belong to $\mathbf{B}^{p,\beta_p}(X)$ for any $p \ge 1$ and $\beta_p = \big(1 - \frac{2}{p}\big)(1 - \alpha_1) + \frac{1}{p}$ as in Remark 2.10. We shall see that such a function h is actually a *harmonic* function.

Denote by $\{\psi_i\}_{i=1}^5$ the contraction mappings that generate X and define for any $w \in \{1, \ldots, 5\}^m$ the mapping $\psi_w := \psi_{w_1} \circ \ldots \circ \psi_{w_m}$ that generates an m-level copy of X, so that $X = \bigcup_{w \in \{1,\ldots,5\}^m} \psi_w(X)$. One can approximate X by a sequence of metric graphs $\{(V_m, E_m)\}_{m \ge 0}$ as illustrated in Fig. 1. These are equipped with the Dirichlet form given by the standard graph energy that arises by treating each edge as an interval and adding the corresponding 1-dimensional energies in each of them. A function $h\colon X \to \mathbb{R}$ is said to be m-harmonic if it arises as the energy minimizing extension of a given function with values on the approximation level m, i.e.

$$\mathcal{E}(h, h) = \inf\{\mathcal{E}(g, g) :\ g|_{V_m} = f_m\}$$

for some $f_m\colon V_m \to \mathbb{R}$. Following the notation and the result in [9, Proposition 7.13], we write in this case $h = H_m f_m$ and know that $H_m f_m \in \mathcal{D} \cap C(X)$.

Theorem 3.6. *On the Vicsek set, the space $\mathbf{B}^{2,1/2}(X) \cap \mathbf{B}^{p,\beta_p}(X)$ contains non-trivial functions for any $p \ge 1$. In particular, for $1 \le p \le 2$,*

$$\alpha_p = \left(1 - \frac{2}{p}\right)\left(1 - \frac{d_H}{d_W}\right) + \frac{1}{p}$$

and (PPI_p) *is satisfied.*

Proof. Let us consider graph approximation (V_0, E_0) and $f_0\colon V_0 \to \mathbb{R}$ that takes the values a_1, a_2, a_3, a_4 on each vertex x_1, x_2, x_3, x_4 of V_0, respectively. For simplicity, we assume that the function is only non-zero at two connected vertices, say x_1 and x_3. The harmonic extension of f_0 to the Vicsek set X is defined as the function $h := H_0 f_0 \in \mathcal{F}$ such that $h|_{V_0} \equiv f_0$ and

$$\mathcal{E}(h,h) = \min\{\mathcal{E}(f,f) : f \in \mathcal{F} \text{ and } f_{|V_0} = f_0\}.$$

This 0-harmonic function h is obtained by linear interpolation on the diagonal that joins x_1 and the upper-right corner x_3. We call this the “distinguished” diagonal. On all branches intersecting it, including the other diagonal crossing lower-right to upper-left, h is constant according to its value on the distinguished diagonal. This harmonic extension is clearly non-constant, it is unique and belongs to $\mathcal{F} = \mathbf{B}^{2,1/2}(X)$, see e.g. [26, Lemma 8.2]. In order to prove that $\|h\|_{p,\beta_p} < \infty$ for any $p \geq 1$, we first fix $r \in (0, 1/6)$ and set $n := n_r \geq 0$ to be the largest such that $2r < 3^{-(n+1)}$. Note that X can be covered by 5^n squares of side length 3^{-n}, which we denote $\{Q_i^{(n)}\}_{i=1}^{5^n}$. By construction, the function h is constant on cells $B_i^{(n)} := X \cap Q_i^{(n)}$ for which $Q_i^{(n)}$ does not intersect the distinguished diagonal of X. In addition, h is also constant on the r-neighborhood of any such cell, i.e.

$$|h(x) - h(y)| = 0 \quad \text{for any } y \in B_i^{(n)} \text{ and } x \in B(y,r).$$

In other words, only in 3^n of the n-cells $\{B_i^{(n)}\}_{i=1}^{5^n}$ the latter difference is nonzero. Since h is by definition linear, on any of these 3^n cells it holds that

$$|h(x) - h(y)| \leq d(x,y) \quad \text{for all } y \in B_i^{(n)} \text{ and } x \in B(y,r).$$

Combining these two facts and using the Ahlfors regularity of the space we have for any $p \geq 1$

$$\begin{aligned} &\frac{1}{r^{p\beta_p d_W + d_H}} \int\limits_X \int\limits_{B(y,r)} |h(x) - h(y)|^p d\mu(x)\, d\mu(y) \qquad (16)\\ &\leq \frac{1}{r^{p\beta_p d_W + d_H}} \sum_{i=1}^{3^n} \int\limits_{B_i} \int\limits_{B(y,r)} r^p\, d\mu(x)\, d\mu(y) \leq \frac{C}{r^{p\beta_p d_W + d_H}} \sum_{i=1}^{3^n} r^{p+d_H} \mu(B_i)\\ &\leq \frac{C}{r^{p\beta_p d_W + d_H}} r^{-1+p+2d_H} = C r^{p+d_H-(1+p\beta_p d_W)}. \end{aligned}$$

From Theorem 3.4 we know that $\beta_1 = \frac{d_H}{d_W}$, which substituting above yields the exponent $p + d_H - (1 + p\beta_p d_W) = (p-1)(1 + d_H - d_W)$ which equals zero because the Vicsek set satisfies $d_W = 1 + d_H$, cf. [9, Theorem 8.18]. Therefore, (16) is bounded independent of r and

$$\sup_{r\in(0,1/6)} \frac{1}{r^{p\beta_p d_W + d_H}} \int\limits_X \int\limits_{B(y,r)} |h(x) - h(y)|^p d\mu(x)\, d\mu(y) \leq C$$

which in view of [2, Theorem 2.4] yields $\|h\|_{p,\beta_p} \leq C_{p,\beta_p}(C + 6^{\beta_p d_W}\|h\|_{L^p(X,\mu)})$. The space $\mathbf{B}^{p,\beta_p}(X)$ is thus non trivial and by definition of the critical exponent α_p we have $\alpha_p \geq \beta_p$. Finally, [2, Theorem 3.11] yields $\alpha_p = \beta_p$ and [2, Theorem 3.10] the property (PI$_p$). $\square$

Remark 3.7. It is actually possible to prove that any m-harmonic function on the Vicsek set belongs to $\mathbf{B}^{p,\beta_p}(X)$ for any $p \geq 1$. As a consequence, one can deduce that $\alpha_p = \beta_p$ for every $p \geq 2$.

Products of nested fractals. Higher dimensional examples of fractal spaces can be constructed by taking products [42]. In particular, as noticed in [2, Section 3.3], given a nested fractal X that satisfies sub-Gaussian estimates, its n-fold product X^n has Hausdorff dimension nd_H, while its walk dimension d_W remains unchanged. The next theorem puts these spaces into our setting.

Theorem 3.8 *(Proposition 3.8, Theorem 5.6 [2]). Let (X, d, μ) be a nested fractal with $1 \leq d_H \leq d_W$. Then, Theorem 3.4 holds with the same exponents for any n-fold product $(X^n, d_{X^n}, \mu^{\otimes n})$, $n \geq 1$.*

In the case of the Vicsek set, and in view of Theorem 3.6 and Remark 3.7 one has the following result.

Theorem 3.9. *Let (X, d, μ) denote the Vicsek set. For the n-fold product $(X^n, d_{X^n}, \mu^{\otimes n})$, $n \geq 1$, for any $p \geq 1$ it holds that*

$$\alpha_p = \left(1 - \frac{2}{p}\right)\left(1 - \frac{d_H}{d_W}\right) + \frac{1}{p}$$

and (PPI_p) is satisfied for any $1 \leq p \leq 2$, where d_H is the Hausdorff dimension of X and d_W the walk dimension of X.

We expect a similar result to hold in a more general framework within spaces with sub-Gaussian heat kernel estimates [2]; this will be the subject of future investigations.

4. Gagliardo-Nirenberg and Trudinger-Moser inequalities

We now turn to the core of the paper and show how the pseudo-Poincaré inequalities introduced in Section 2.2 can be applied to obtain the whole range of Gagliardo-Nirenberg and Trudinger-Moser inequalities for the Sobolev spaces $W^{1,p}(\mathcal{E})$. The techniques used rely on Lemma 2.6 in conjunction with general methods developed in [7].

4.1. Global versions

We start by recalling once again that, since (X, μ) is assumed to be a Radon space, the semigroup $\{P_t\}_{t \geq 0}$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ admits a measurable heat kernel $p_t(x, y)$, cf. [8, Theorem 1.2.3]. Throughout this section we will assume that the heat kernel satisfies

$$p_t(x, y) \leq C_h t^{-\beta} \tag{17}$$

for some $C_h > 0$ and $\beta > 0$, $\mu \times \mu$-a.e. $(x, y) \in X \times X$ and any $t > 0$. In addition, we will consider for each $p \geq 1$ the L^p pseudo-Poincaré inequality $(\mathrm{PPI_p})$ from Section 2.2: There exists a constant $C_p > 0$ such that for every $t \geq 0$ and $f \in W^{1,p}(\mathcal{E})$ (or $BV(\mathcal{E})$ for $p = 1$),

$$\|P_t f - f\|_{L^p(X,\mu)} \leq C_p t^{\alpha_p} \mathbf{Var}_{p,\mathcal{E}}(f).$$

The following result extends to the abstract Dirichlet space framework the classical Gagliardo-Nirenberg inequalities, see e.g. [6].

Theorem 4.1. *Assume that (PPI_p) is satisfied for some $p \geq 1$. Then, there exists a constant $c_p > 0$ such that for every $f \in W^{1,p}(\mathcal{E})$ (or $BV(\mathcal{E})$ for $p = 1$),*

$$\|f\|_{L^q(X,\mu)} \leq c_p C_p^{\frac{\beta}{\beta+\alpha_p}} C_h^{\frac{\alpha_p}{\beta+\alpha_p}} \mathbf{Var}_{p,\mathcal{E}}(f)^{\frac{\beta}{\beta+\alpha_p}} \|f\|_{L^1(X,\mu)}^{\frac{\alpha_p}{\beta+\alpha_p}}, \tag{18}$$

where $q = p\left(1 + \frac{\alpha_p}{\beta}\right)$.

Proof. For $p \ge 1$, we set $\theta := p/q$ and consider the semi-norm

$$\|f\|_{B^{\alpha_p\theta/(\theta-1)}_{\infty,\infty}} = \sup_{t>0} t^{-\alpha_p\theta/(\theta-1)} \|P_t f\|_{L^\infty(X,\mu)}. \tag{19}$$

Let $f \in W^{1,p}(\mathcal{E})$ (or $BV(\mathcal{E})$ if $p=1$) and assume first that $f \ge 0$ and also that, by homogeneity, $\|f\|_{B^{\alpha_p\theta/(\theta-1)}_{\infty,\infty}} \le 1$. For any $s>0$, set $t_s = s^{\frac{\theta-1}{\alpha_p\theta}}$ so that $|P_{t_s} f| \le s$. Then,

$$\begin{aligned} s^q \mu\big(\{x\in X:\ |f(x)|\ge 2s\}\big) &\le s^q\mu\big(\{x\in X:\ |f-P_{t_s}f|\ge s\}\big) \\ &\le s^{q-p}\|f-P_{t_s}f\|^p_{L^p(X,\mu)} \le s^{q-p} t_s^{p\alpha_p} C_p^p \mathbf{Var}_{p,\mathcal{E}}(f)^p = C_p^p\mathbf{Var}_{p,\mathcal{E}}(f)^p, \end{aligned} \tag{20}$$

where the last inequality follows from (PPI_p) and the last equality from $q-p+p(\theta-1)/\theta = 0$. Let us now define $f_k := \min\{(f-2^k)_+, 2^k\}$, $k\in\mathbb{Z}$. We note that $0\le f_k \le f$, so that

$$\|f_k\|_{B^{\alpha_p\theta/(\theta-1)}_{\infty,\infty}} \le \|f\|_{B^{\alpha_p\theta/(\theta-1)}_{\infty,\infty}} \le 1.$$

Applying (20) to f_k with $s=2^k$ yields

$$2^{kq}\mu\big(\{x\in X:\ |f_k(x)|\ge 2^{k+1}\}\big) \le C_p^p \mathbf{Var}_{p,\mathcal{E}}(f_k)^p$$

so that from Lemma 2.6 we deduce

$$\sum_{k\in\mathbb{Z}} 2^{kq}\mu\big(\{x\in X:\ |f_k(x)|\ge 2^{k+1}\}\big) \le C_p^p \sum_{k\in\mathbb{Z}} \mathbf{Var}_{p,\mathcal{E}}(f_k)^p \le 2^p(p+1)^p C_p^p \mathbf{Var}_{p,\mathcal{E}}(f)^p.$$

Further,

$$\begin{aligned} \|f\|^q_{L^q(X,\mu)} &= \sum_{k\in\mathbb{Z}} \int_{2^{k+1}}^{2^{k+2}} q s^{q-1}\mu\big(\{x\in X:\ |f(x)|\ge s\}\big)\,ds \\ &\le \sum_{k\in\mathbb{Z}} \int_{2^{k+1}}^{2^{k+2}} q s^{q-1}\mu\big(\{x\in X:\ |f(x)|\ge 2^{k+1}\}\big)\,ds \\ &\le (2^{2q}-2^q)\sum_{k\in\mathbb{Z}} 2^{kq}\mu\big(\{x\in X:\ |f_k(x)|\ge 2^k\}\big) \le 2^{3q}2^p(p+1)^p C_p^p\mathbf{Var}_{p,\mathcal{E}}(f)^p. \end{aligned}$$

One concludes that for every $f\in W^{1,p}(\mathcal{E})$ (or $BV(\mathcal{E})$ if $p=1$) such that $f\ge 0$

$$\|f\|_{L^q(X,\mu)} \le 2^3 2^\theta (p+1)^\theta C_p^\theta \mathbf{Var}_{p,\mathcal{E}}(f)^\theta \|f\|^{1-\theta}_{B^{\alpha_p\theta/(\theta-1)}_{\infty,\infty}}, \tag{21}$$

where $\theta = \frac{p}{q}$. On the other hand, the heat kernel upper bound (17) implies

$$\|P_t f\|_{L^\infty(X,\mu)} \le \frac{C_h}{t^\beta}\|f\|_{L^1(X,\mu)}$$

and by definition, see (19), it follows from (21) that

$$\|f\|_{L^q(X,\mu)} \le 2^3 2^\theta (p+1)^\theta C_p^\theta C_h^{1-\theta} \mathbf{Var}_{p,\mathcal{E}}(f)^\theta \|f\|^{1-\theta}_{L^1(X,\mu)}$$

for $\beta = \frac{\alpha_p\theta}{1-\theta} = \frac{\alpha_p p}{q-p}$, equivalently $\frac{1}{q} = \frac{1}{p} - \frac{\alpha_p}{q\beta}$. If one does not assume $f\ge 0$, the previous inequality applied to $|f|$ yields the expected result, since it is clear from the definition that $\mathbf{Var}_{p,\mathcal{E}}(|f|) \le \mathbf{Var}_{p,\mathcal{E}}(f)$. □

4.1.1. Gagliardo-Nirenberg

Thanks to general results proved in [7], Theorem 4.1 actually implies the full scale of Gagliardo-Nirenberg inequalities. We discuss them according to the value of $p\alpha_p$.

Corollary 4.2. *Assume that* (PPI_p) *is satisfied for some* $p \geq 1$ *such that* $p\alpha_p < \beta$*. Then, there exists a constant* $C_{p,r,s} > 0$ *such that for every* $f \in W^{1,p}(\mathcal{E})$ *(or* $BV(\mathcal{E})$ *for* $p = 1$*),*

$$\|f\|_{L^r(X,\mu)} \leq C_{p,r,s} \mathbf{Var}_{p,\mathcal{E}}(f)^\theta \|f\|^{1-\theta}_{L^s(X,\mu)}, \tag{22}$$

where $r, s \in [1, +\infty]$ *and* $\theta \in (0, 1]$ *are related by the identity*

$$\frac{1}{r} = \theta\Big(\frac{1}{p} - \frac{\alpha_p}{\beta}\Big) + \frac{1-\theta}{s}.$$

Proof. This follows from Theorem 4.1 and [7, Theorem 3.1]. □

Several special cases worth pointing out explicitly are described in Remark 4.9.

We now turn to the case $p\alpha_p > \beta$.

Corollary 4.3. *Assume that* (PPI_p) *is satisfied for some* $p \geq 1$ *such that* $p\alpha_p > \beta$*. Then, there exists a constant* $C_p > 0$ *such that for every* $f \in W^{1,p}(\mathcal{E})$ *(or* $BV(\mathcal{E})$ *for* $p = 1$*), and* $s \geq 1$*,*

$$\|f\|_{L^\infty(X,\mu)} \leq C_p \mathbf{Var}_{p,\mathcal{E}}(f)^\theta \|f\|^{1-\theta}_{L^s(X,\mu)}, \tag{23}$$

where $\theta \in (0, 1)$ *is given by* $\theta = \frac{p\beta}{p\beta + s(p\alpha_p - \beta)}$*.*

Proof. This follows from Theorem 4.1 and [7, Theorem 3.2]. □

Remark 4.4. For $s = 1$, we have that

$$\|f\|_{L^s(X,\mu)} = \|f\|_{L^1(X,\mu)} \leq \|f\|_{L^\infty(X,\mu)} \mu(\mathrm{Supp}(f)),$$

where $\mathrm{Supp}(f)$ denotes the support of f. Thus, (23) yields for any $f \in W^{1,p}(\mathcal{E})$ (or $BV(\mathcal{E})$)

$$\|f\|_{L^\infty(X,\mu)} \leq C_p \mathbf{Var}_{p,\mathcal{E}}(f) \mu(\mathrm{Supp}(f))^{\frac{\alpha_p}{\beta} - \frac{1}{p}}.$$

4.1.2. Trudinger-Moser

The case $p\alpha_p = \beta$ corresponds to Trudinger-Moser inequalities. We start with the case $p = 1$ that is particularly well-suited for applications to fractal spaces.

Corollary 4.5. *Assume that* (PPI_1) *is satisfied and that* $\alpha_1 = \beta$*. Then, there exists a constant* $C > 0$ *such that for every* $f \in BV(\mathcal{E})$*:*

$$\|f\|_{L^\infty(X,\mu)} \leq C \mathbf{Var}_{\mathcal{E}}(f).$$

Proof. By virtue of Lemma 2.6, the condition (H_1) from [7, Section 2] is satisfied, hence Theorem 4.1 and [7, Theorem 3.2] yield the result. □

We finally conclude with the Trudinger-Moser inequalities corresponding to $p > 1$.

Corollary 4.6. *Assume further that* (PPI_p) *is satisfied and that* $p\alpha_p = \beta$ *with* $p > 1$*. Then, there exist constants* $c, C > 0$ *such that*

$$\int_X \left(e^{c|f|^{\frac{p}{p-1}}} - 1 \right) d\mu \leq C\|f\|_{L^1(X,\mu)}$$

holds for every $f \in W^{1,p}(\mathcal{E})$ *with* $\mathbf{Var}_{p,\mathcal{E}}(f) = 1$.

Proof. Once again, Lemma 2.6 implies condition (H_p) from [7, Section 2] for $p > 1$, and the result follows from Theorem 4.1 and [7, Theorem 3.4]. □

4.2. Localized versions

In order to be able to treat spaces that lack global estimates, as for instance hyperbolic spaces, $RCD(K, +\infty)$ spaces with $K < 0$, or compact spaces where only the local time behavior is meaningful, in this section we adapt the previous ideas to obtain a local version of Theorem 4.1. In the spirit of [36, Section 3.3.2], Theorem 4.7 in fact provides a local inequality depending on a parameter R, which in the limit $R \to \infty$ recovers its global counterpart (18). The local version of the property (PPI_p) was introduced in Section 2.2 with the notation $(\mathrm{PPI}_p(R))$ for $p \geq 1$ and $R > 0$.

Theorem 4.7. *Fix* $R > 0$, $p \geq 1$ *and* $\alpha_p > 0$*. Assume that the space* $(X, d, \mu, \mathcal{E}, \mathcal{F})$ *satisfies:*

(i) *The heat semigroup* P_t *admits a measurable heat kernel* $p_t(x, y)$ *such that for some* $C_h > 0$ *and* $\beta > 0$*, for* $\mu \times \mu$*-a.e.* $(x, y) \in X \times X$ *and* $0 < t \leq R$

$$p_t(x, y) \leq C_h t^{-\beta}; \tag{24}$$

(ii) *The property* $(\mathrm{PPI}_p(R))$*, with constant* $C_p(R) > 0$.

Then, there exist $C_p > 0$ *such that for every* $f \in L^p(X, \mu)$,

$$\|f\|_{L^q(X,\mu)} \leq 4p(2p+2)^{\frac{\beta}{\beta+\alpha_p}} C_h^{\frac{\alpha_p}{\beta+\alpha_p}} \left(R^{-\alpha_p}\|f\|_{L^p(X,\mu)} + C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f) \right)^{\frac{\beta}{\beta+\alpha_p}} \|f\|_{L^1(X,\mu)}^{\frac{\alpha_p}{\beta+\alpha_p}},$$

where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha_p}{q\beta}$.

Proof. Modifying the arguments in Theorem 4.1 with the localized semi-norm

$$\|f\|_{B^{\alpha_p\theta/(\theta-1)}_{R,\infty,\infty}} = \sup_{t\in(0,R)} t^{-\alpha_p\theta/(\theta-1)}\|P_t f\|_{L^\infty(X,\mu)}, \quad \theta := \frac{p}{q} \in (0,1), \tag{25}$$

we obtain for $f \in L^p(X, \mu)$ non-negative and $s > R^{\frac{\alpha_p\theta}{\theta-1}} = (1/R)^{\frac{\alpha_p\theta}{1-\theta}}$

$$\sum_{k=k_0}^{\infty} 2^{kq}\mu\big(\{x \in X : |f_k(x)| \geq 2^{k+1}\}\big) \leq C_p(R)^p \sum_{k\in\mathbb{Z}} \mathbf{Var}_{p,\mathcal{E}}(f_k)^p \leq C_p(R)^p 2^p(p+1)^p \mathbf{Var}_{p,\mathcal{E}}(f)^p.$$

If $s < 2^{k_0}$, we write $s^q\mu\big(\{x \in X : |f(x)| > s\}\big) \leq s^{q-p}\|f\|^p_{L^p(X,\mu)}$. Using the previous two estimates, and setting $k_0 > 0$ so that $2^{k_0-1} < R^{\frac{\alpha_p\theta}{\theta-1}} \leq 2^{k_0}$,

$$
\begin{aligned}
\|f\|^q_{L^q(X\,u)} &= \int\limits_0^{2^{k_0+1}} qs^{q-1}\mu\big(\{x\in X:\ |f(x)|>s\}\big)\,ds + \int\limits_{2^{k_0+1}}^{\infty} qs^{q-1}\mu\big(\{x\in X:\ |f(x)|>s\}\big)\,ds \\
&\le \|f\|^p_{L^p(X,\mu)}\frac{q2^{(k_0+1)(q-p)}}{q-p} + \sum_{k=k_0}^{\infty}\int\limits_{2^{k+1}}^{2^{k+2}} qs^{q-1}\mu\big(\{x\in X:\ |f(x)|>2^{k+1}\}\big)\,ds \\
&\le \|f\|^p_{L^p(X,\mu)}\frac{q4^{q-p}}{q-p}R^{\frac{\alpha_p\theta(q-p)}{\theta-1}} + 2^q(2^q-1)\sum_{k=k_0}^{\infty}2^{qk}\mu\big(\{x\in X:\ |f(x)|>2^{k+1}\}\big) \\
&\le 2^{2q+p}(p+1)^p\big(\|f\|^p_{L^p(X,\mu)}R^{\frac{\alpha_p\theta(q-p)}{\theta-1}} + C_p(R)^p\mathbf{Var}_{p,\mathcal{E}}(f)^p\big).
\end{aligned}
$$

Since $\frac{\alpha_p\theta(q-p)}{\theta-1} = \frac{\alpha_p p(q-p)}{q(p/q-1)} = -\alpha_p p$, the latter inequality implies

$$
\|f\|^q_{L^q(X\,u)} \le 2^{2q+p}(p+1)^p p\Big(R^{-\alpha_p}\|f\|_{L^p(X,\mu)} + C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f)\Big)^p.
$$

Finally, we conclude by applying (24) to the norm (25). □

4.2.1. Gagliardo-Nirenberg

In the same lines as [36, Section 3.2.7], Theorem 4.7 extends to the full scale of Gagliardo-Nirenberg inequalities by noticing that for any $t,s>0$ the mapping $f\mapsto (f-t)_+\wedge s := f_t^s$ is a contraction and hence

$$
R^{-\alpha_p}\|f_t^s\|_{L^p(X,\mu)} + C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f_t^s) \le C\big(R^{-\alpha_p}\|f\|_{L^p(X,\mu)} + C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f)\big) \tag{26}
$$

for some constant $C>0$. As in the global case, we discuss in the following all these inequalities according to the value of $p\alpha_p$.

Corollary 4.8. *Assume that* $(\mathrm{PPI}_p(R))$ *is satisfied for some* $p\ge 1$ *such that* $p\alpha_p<\beta$. *Then, there exists a constant* $C_{p,r,s}>0$ *such that for every* $f\in W^{1,p}(\mathcal{E})$ *(or* $BV(\mathcal{E})$ *for* $p=1$*),*

$$
\|f\|_{L^r(X,\mu)} \le C_{p,r,s}\Big(R^{-\alpha_p}\|f\|_{L^p(X,\mu)} + C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f)\Big)^\theta \|f\|^{1-\theta}_{L^s(X,\mu)}, \tag{27}
$$

where $r,s\in[1,+\infty]$ *and* $\theta\in(0,1]$ *are related by the identity*

$$
\frac{1}{r} = \theta\Big(\frac{1}{p}-\frac{\alpha_p}{\beta}\Big) + \frac{1-\theta}{s}.
$$

Proof. The proof is the same as in Corollary 4.2 since (26) corresponds to the property (H^+_∞) from [7, Theorem 3.1]. □

We point out explicitly some particular cases whose global version corresponds to $R=\infty$.

Remark 4.9.

(i) If $r=s$, then $r=\frac{p\beta}{\beta-p\alpha_p}$ and (27) yields the global Sobolev inequality

$$
\|f\|_{L^r(X,\mu)} \le C_p\big(R^{-\alpha_p}\|f\|_{L^p(X,\mu)} + C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f)\big).
$$

(ii) If $r = p > 1$ and $s = 1$, then (27) yields with $\theta = \frac{(p-1)\beta}{p(\alpha_p+\beta)-\beta}$ the global Nash inequality

$$\|f\|_{L^p(X,\mu)} \leq C_p\big(R^{-\alpha_p}\|f\|_{L^p(X,\mu)} + C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f)\big)^{\theta}\|f\|^{1-\theta}_{L^1(X,\mu)}.$$

(iii) If $s = +\infty$, then (27) yields with $\theta = \frac{p\beta}{r(\beta-p\alpha_p)}$

$$\|f\|_{L^r(X,\mu)} \leq C_{p,r}\big(R^{-\alpha_p}\|f\|_{L^p(X,\mu)} + C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f)\big)^{\theta}\|f\|^{1-\theta}_{L^\infty(X,\mu)}.$$

We now turn to the case $p\alpha_p > \beta$.

Corollary 4.10. *Assume that* $(\mathrm{PPI}_p(R))$ *is satisfied for some* $p \geq 1$ *such that* $p\alpha_p > \beta$*. Then, there exists a constant* $C_p > 0$ *such that for every* $f \in W^{1,p}(\mathcal{E})$ *(or* $BV(\mathcal{E})$ *for* $p = 1$*), and* $s \geq 1$*,*

$$\|f\|_{L^\infty(X,\mu)} \leq C_p\big(R^{-\alpha_p}\|f\|_{L^p(X,\mu)} + C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f)\big)^{\theta}\|f\|^{1-\theta}_{L^s(X,\mu)},$$

where $\theta \in (0,1)$ *is given by* $\theta = \frac{p\beta}{p\beta+s(p\alpha_p-\beta)}$.

Proof. Analogously as Corollary 4.3, this follows by applying [7, Theorem 3.2] with (26) and Theorem 4.7. □

4.2.2. Trudinger-Moser

Trudinger-Moser inequalities correspond to the case $p\alpha_p = \beta$. To treat them, we observe first that Minkowski's inequality together with Lemma 2.6 implies

$$\Big(\sum_{k\in\mathbb{Z}}\Big(R^{-\alpha_p}\|f_{\rho,k}\|_{L^p(X,\mu)} + C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f_{\rho,k})\Big)^p\Big)^{1/p} \leq R^{-\alpha_p}\|f\|_{L^p(X,\mu)} + 2(p+1)C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f) \tag{28}$$

for any $p \geq 1$, $\rho > 1$ and $f_{\rho,k} := (f-\rho^k)_+ \wedge \rho^k(\rho-1)$.

Corollary 4.11. *Assume that* $(\mathrm{PPI}_1(R))$ *is satisfied and that* $\alpha_1 = \beta$*. Then, there exists a constant* $C > 0$ *such that for every* $f \in BV(\mathcal{E})$

$$\|f\|_{L^\infty(X,\mu)} \leq C\big(R^{-\alpha_1}\|f\|_{L^1(X,\mu)} + C_1(R)\mathbf{Var}_{1,\mathcal{E}}(f)\big).$$

Proof. By virtue of (28), the condition (H_1) from [7, Section 2] is satisfied, hence Theorem 4.7 and [7, Theorem 3.2] yield the result. □

We finish this section with the Trudinger-Moser inequalities that one obtains for $p > 1$.

Corollary 4.12. *Assume further that* $(\mathrm{PPI}_p(R))$ *is satisfied and that* $p\alpha_p = \beta$ *with* $p > 1$*. Then, there exist constants* $c, C > 0$ *such that*

$$\int_X \Big(e^{c|f|^{\frac{p}{p-1}}} - 1\Big)\,d\mu \leq C\|f\|_{L^1(X,\mu)}$$

for every $f \in W^{1,p}(\mathcal{E})$ *with* $\|f\|_{L^p(X,\mu)} = R^{\alpha_p}\big(1 - C_p(R)\mathbf{Var}_{p,\mathcal{E}}(f)\big)$.

Proof. In this case, (28) implies condition (H_p) from [7, Section 2] for $p > 1$, and the result follows from Theorem 4.7 and [7, Theorem 3.4]. □

4.3. Examples

The Gagliardo-Nirenberg and Trudinger-Moser inequalities proved in this section can be applied in large classes of examples. In particular, we mention the following:

- Metric measure spaces with Gaussian heat kernel estimates: Theorem 3.3 provides the class of strictly local spaces to which one can apply the results obtained in this paper, and in particular Gagliardo-Nirenberg and Trudinger-Moser inequalities. Note that a sufficient condition for condition (17) to hold is the volume growth condition $\mu(B(x,r)) \geq Cr^{d_H}$, in which case one has $\beta = \frac{d_H}{2}$.
- Metric measure spaces with sub-Gaussian heat kernel estimates: Theorem 3.4 yields another large set of examples, including unbounded nested fractals. These satisfy (PPI_p) for $1 \leq p \leq 2$ and condition (17) with $\beta = \frac{d_H}{d_W}$. In the case of the unbounded Vicsek fractal, its n-fold product satisfies (PPI_p) for $1 \leq p \leq 2$, cf. Theorem 3.8 and condition (17) with $\beta = \frac{d_H}{d_W}$. Compact nested fractals satisfy the corresponding localized versions.

5. Morrey's type inequalities

The classical Morrey's inequality implies that functions in the Sobolev space $W^{1,p}(\mathbb{R}^d)$ are Hölder continuous (after a possible modification on a set of measure zero) for all $p > d$. Besides of being an important inequality on its own, we are interested in the associated critical value

$$\delta_{\mathcal{E}} := \inf\{p \geq 1,\, W^{1,p}(\mathcal{E}) \subset C^0(X)\},$$

where $C^0(X)$ denotes the space of a.e. bounded functions which admit a continuous representative, and the connection of $\delta_{\mathcal{E}}$ to other dimensions studied in the metric measure setting [27]. The inequality that we prove in this section provides a general embedding of $\mathbf{B}^{p,\alpha}(X)$ into the space $C^\lambda(X)$, $\lambda > 0$, of bounded Hölder functions equipped with the norm

$$\|f\|_{C^\lambda(X)} := \|f\|_{L^\infty(X,\mu)} + \mu\text{-ess}\sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\lambda}.$$

Those types of embedding, however with weaker regularity, were already observed by Coulhon in [17] under volume doubling and (sub-)Gaussian heat kernel estimates. Here and throughout this section, we will work under the following additional assumptions:

- Condition 1. The underlying space is d_H-Ahlfors regular;
- Condition 2. The heat semigroup admits a heat kernel with Gaussian or sub-Gaussian estimates.

5.1. Metric approach

The proof of the following result is based on a generalization of the ideas in [18, Theorem 8.1]. Notice that Theorem 5.1 holds for any pair of exponents (p, α); Morrey's inequality will correspond to the specific pairs (p, α_p).

Theorem 5.1. *For any $p > \frac{d_H}{d_W \alpha}$ and $R > 0$, there exists $C_p > 0$ (independent from R) such that*

$$\mu\text{-}ess \sup_{0<d(x,y)<R/3} \frac{|f(x)-f(y)|}{d(x,y)^\lambda} \le C\|f\|_{p,\alpha,R} \tag{29}$$

for any $f \in \mathbf{B}^{p,\alpha}(X)$, where $\lambda = d_W\alpha - \frac{d_H}{p}$. In particular, if $\alpha p > \frac{d_H}{d_W}$, then $\mathbf{B}^{p,\alpha}(X) \subset C^\lambda(X)$, where $\lambda = d_W\alpha - \frac{d_H}{p}$.

Remark 5.2. When applied to the critical exponent $\alpha = \alpha_p$, the condition $\alpha_p p = \frac{d_H}{d_W}$ coincides with the critical exponent for Trudinger-Moser inequalities in the previous section.

Proof. Let first $0 < r < R/3$ and consider $x, y \in X$ with $d(x,y) \le r$. Define

$$f_r(x) := \frac{1}{\mu\big(B(x,r)\big)} \int\limits_{B(x,r)} u(z)\, d\mu(z)$$

and analogously one defines $f_r(y)$. Hölder's inequality yields

$$\begin{aligned}
|f_r(x) - f_r(y)| &= \frac{1}{\mu\big(B(x,r)\big)\mu\big(B(y,r)\big)} \Big| \int\limits_{B(x,r)} \int\limits_{B(y,r)} (u(z) - u(z'))\, d\mu(z')\, d\mu(z) \Big| \\
&\le \Bigg(\frac{1}{\mu\big(B(x,r)\big)\mu\big(B(y,r)\big)} \int\limits_{B(x,r)} \int\limits_{B(y,r)} |u(z) - u(z')|^p\, d\mu(z')\, d\mu(z) \Big| \Bigg)^{1/p}
\end{aligned}$$

and applying the d_H-Ahlfors regularity of the space we get

$$\begin{aligned}
|f_r(x) - f_r(y)|^p &\le \frac{C}{r^{2d_H}} \int\limits_X \int\limits_{B(z,3r)} |u(z) - u(z')|^p\, d\mu(z')\, d\mu(z) \\
&\le C r^{p\alpha d_W - d_H} \sup_{r\in(0,R/3)} \frac{1}{r^{d_H + p\alpha d_W}} \int\limits_X \int\limits_{B(z,3r)} |u(z) - u(z')|^p\, d\mu(z')\, d\mu(z) \\
&\le C r^{p\alpha d_W - d_H} \|f\|^p_{p,\alpha,R}.
\end{aligned}$$

The last inequality follows from the characterization of $\mathbf{B}^{p,\alpha}(X)$ as a Korevaar-Schoen class space, see e.g. [2, Theorem 2.4] for the sub-Gaussian case. Thus,

$$|f_r(x) - f_r(y)| \le C^{1/p} r^{\alpha d_W - \frac{d_H}{p}} \|f\|_{p,\alpha,R}$$

and an analogous bound for $|f_{2r}(x) - f_r(x)|$. As in the proof of [18, Theorem 8.1], for any pair of Lebesgue points of f we deduce

$$|f(x) - f(y)| \le C_p d(x,y)^{\alpha d_W - \frac{d_H}{p}} \|f\|_{p,\alpha,R}. \tag{30}$$

By virtue of [21, Theorem 3.4.3], the set of Lebesgue points of f is dense in X and so (30) implies (29). Finally, for any fixed $r > 0$ (e.g. $r = R/4$), Hölder's inequality yields $|f_r(x)| \le r^{-\frac{d_H}{p}} \|f\|_{L^p(X,\mu)}$, which implies

$$|f(x)| \le C_r (\|f\|_{L^p(X,\mu)} + \|f\|_{p,\alpha,R})$$

μ-a.e. $x \in X$. Thus, $L^\infty(X,\mu) \subseteq \mathbf{B}^{p,\alpha}(X)$. $\square$

Since the constant C_p in the previous theorem is independent of R, by letting $R \to +\infty$ one deduces the corresponding global inequality.

Corollary 5.3. *For any $p > \frac{d_H}{d_W \alpha}$, there exists $C_p > 0$ such that*

$$\mu\text{-}ess \sup_{d(x,y)>0} \frac{|f(x)-f(y)|}{d(x,y)^\lambda} \le C_p \|f\|_{p,\alpha}$$

for any $f \in \mathbf{B}^{p,\alpha}(X)$, where $\lambda = d_W\alpha - \frac{d_H}{p}$.

5.2. Heat semigroup approach

A drawback of Theorem 5.1 is that when applied to the pair (p,α_p), it would be sharper and more natural to get on the right hand side of (29) the p-variation $\mathbf{Var}_{p,\mathcal{E}}(f)$ instead of the Besov semi-norm $\|\cdot\|_{p,\alpha_p,R}$. This certainly requires more assumptions than just sub-Gaussian heat kernel estimates and Ahlfors regularity. So, in addition to the latter, we will also assume in this section the weak Bakry-Émery type estimate (G_∞) from (4).

- <u>Condition 3</u>. There exists a constant $C > 0$ so that for any $f \in L^\infty(X,\mu)$, $x,y \in X$ and all $t > 0$,

$$|P_t f(x) - P_t f(y)| \le C \frac{d(x,y)^{d_W(1-\alpha_1)}}{t^{1-\alpha_1}} \|f\|_{L^\infty(X,\mu)}.$$

We start by presenting the key estimate to obtain an *almost* optimal Morrey's type inequality. Its proof relies on ideas first developed by T. Coulhon [17] and E.M. Ouhabaz [33]. In the sequel, Δ will denote the infinitesimal generator of the Dirichlet form $(\mathcal{E},\mathcal{F})$.

Theorem 5.4. *Let $p > 1$ and $\frac{d_H}{pd_W} < \alpha < \frac{d_H}{pd_W} + \left(1-\frac{1}{p}\right)(1-\alpha_1)$. Then,*

$$|f(x)-f(y)| \le C d(x,y)^{\alpha d_W - \frac{d_H}{p}} \|(-\Delta)^\alpha f\|_{L^p(X,\mu)}$$

for $f \in \operatorname{dom}(-\Delta)^\alpha$, and μ-a.e. $x,y \in X$.

We decompose the proof into several lemmas; the first is a direct consequence of the heat kernel upper bound, and the second uses the fact that (G_∞) is equivalent to

$$|p_t(x,z) - p_t(y,z)| \le C \frac{d(x,y)^{d_W(1-\alpha_1)}}{t^{1-\alpha_1+\frac{d_H}{d_W}}}$$

for some $C > 0$ and every $t > 0$, $x,y,z \in X$, see [2, Lemma 3.4].

Lemma 5.5. *Let $p \ge 1$. There exists a constant $C > 0$ such that for every $f \in L^p(X,\mu)$, $t > 0$ and μ a.e. $x \in X$,*

$$|P_t f(x)| \le \frac{C}{t^{\frac{d_H}{pd_W}}} \|f\|_{L^p(X,\mu)}.$$

Lemma 5.6. *Let $p \geq 1$. There exists a constant $C > 0$ such that for every $f \in L^p(X,\mu)$, $t > 0$ and μ a.e. $x, y \in X$,*

$$|P_t f(x) - P_t f(y)| \leq C \frac{d(x,y)^{d_W(1-\alpha_1)(1-\frac{1}{p})}}{t^{\frac{d_H}{pd_W}+(1-\alpha_1)(1-\frac{1}{p})}} \|f\|_{L^p(X,\mu)}.$$

The third lemma is more involved and we provide its proof.

Lemma 5.7. *Let $\frac{d_H}{pd_W} < \alpha < \frac{d_H}{pd_W} + \left(1 - \frac{1}{p}\right)(1-\alpha_1)$. There exists a constant $C > 0$ such that for every $f \in L^2(X,\mu)$ and μ-a.e. $x, y \in X$,*

$$\int_0^{+\infty} t^{\alpha-1}|P_t f(x) - P_t f(y)| dt \leq C d(x,y)^{\alpha d_W - \frac{d_H}{p}} \|f\|_{L^p(X,\mu)}.$$

Proof. The idea is to split the integral into two parts,

$$\int_0^{+\infty} t^{\alpha-1}|P_t f(x) - P_t f(y)| dt = \int_0^{\delta} t^{\alpha-1}|P_t f(x) - P_t f(y)| dt + \int_\delta^{+\infty} t^{\alpha-1}|P_t f(x) - P_t f(y)| dt,$$

where $\delta > 0$ will be chosen later. First, by Lemma 5.5 we have

$$\begin{aligned}
\int_0^{\delta} t^{\alpha-1}|P_t f(x) - P_t f(y)| dt &\leq \int_0^{\delta} t^{\alpha-1}(|P_t f(x)| + |P_t f(y)|) dt \\
&\leq \int_0^{\delta} t^{\alpha-1} \frac{C}{t^{\frac{d_H}{pd_W}}} dt \|f\|_{L^p(X,\mu)} \leq C\delta^{\alpha-\frac{d_H}{pd_W}} \|f\|_{L^p(X,\mu)}.
\end{aligned}$$

As usual, the constant C in the previous inequalities may change from line to line. Secondly, applying Lemma 5.6 we get

$$\begin{aligned}
\int_\delta^{+\infty} t^{\alpha-1}|P_t f(x) - P_t f(y)| dt &\leq C \int_\delta^{+\infty} t^{\alpha-1} \frac{d(x,y)^{d_W(1-\alpha_1)(1-\frac{1}{p})}}{t^{\frac{d_H}{pd_W}+(1-\alpha_1)(1-\frac{1}{p})}} \|f\|_{L^p(X,\mu)} dt \\
&\leq C d(x,y)^{d_W(1-\alpha_1)(1-\frac{1}{p})} \int_\delta^{+\infty} t^{\alpha-1-\frac{d_H}{pd_W}-(1-\alpha_1)(1-\frac{1}{p})} dt \|f\|_{L^p(X,\mu)} \\
&\leq C d(x,y)^{d_W(1-\alpha_1)(1-\frac{1}{p})} \delta^{\alpha-\frac{d_H}{pd_W}-(1-\alpha_1)(1-\frac{1}{p})} \|f\|_{L^p(X,\mu)}.
\end{aligned}$$

Thus, one concludes

$$\int_0^{+\infty} t^{\alpha-1}|P_t f(x) - P_t f(y)| dt \leq C\left(\delta^{\alpha-\frac{d_H}{pd_W}} + d(x,y)^{d_W(1-\alpha_1)(1-\frac{1}{p})} \delta^{\alpha-\frac{d_H}{pd_W}-(1-\alpha_1)(1-\frac{1}{p})}\right) \|f\|_{L^p(X,\mu)}$$

and choosing $\delta = d(x,y)^{d_W}$ yields the result. $\square$

We are finally ready to prove Theorem 5.4.

Proof of Theorem 5.4. Let $f \in \operatorname{dom}(-\Delta)^{-\alpha}$. By virtue of Lemma 5.7,

$$|(-\Delta)^{-\alpha} f(x) - (-\Delta)^{-\alpha} f(y)| = C \left| \int_0^{+\infty} t^{\alpha-1}(P_t f(x) - P_t f(y)) dt \right|$$
$$\leq C \int_0^{+\infty} t^{\alpha-1} |P_t f(x) - P_t f(y)| \, dt \leq C d(x,y)^{\alpha d_W - \frac{d_H}{p}} \|f\|_{L^p(X,\mu)}.$$

Applying the inequality to $(-\Delta)^{\alpha} f$ instead of f yields the result. $\square$

As a consequence, we deduce a version of a Morrey's type inequality which is *almost* optimal. In addition to Ahlfors regularity, sub-Gaussian heat kernel estimates and condition (G_∞), it will be necessary to assume the property (PPI_p).

Theorem 5.8. *Let $p > 1$ and $\frac{d_H}{p d_W} < \alpha_p < \frac{d_H}{p d_W} + \left(1 - \frac{1}{p}\right)(1 - \alpha_1)$. Assuming (G_∞) and (PPI_p), for every $0 < \alpha < \alpha_p$ there exists a constant $C > 0$ such that*

$$|f(x) - f(y)| \leq C d(x,y)^{\alpha d_W - \frac{d_H}{p}} \|f\|_{L^p(X,\mu)}^{1 - \frac{\alpha}{\alpha_p}} \mathbf{Var}_{p,\mathcal{E}}(f)^{\frac{\alpha}{\alpha_p}}$$

for every $f \in W^{1,p}(\mathcal{E})$ and μ-a.e. $x, y \in X$.

Proof. Let $f \in W^{1,p}(\mathcal{E})$. For $\delta > 0$, applying (PPI_p) one has

$$\left\| \int_0^{\infty} t^{-s-1}(P_t f - f) \, dt \right\|_{L^p(X,\mu)} \leq \int_0^{\infty} t^{-s-1} \|P_t f - f\|_{L^p(X,\mu)} dt$$
$$\leq \mathbf{Var}_{p,\mathcal{E}}(f) \int_0^{\delta} t^{-s-1+\alpha_p} dt + 2\|f\|_{L^p(X,\mu)} \int_\delta^{\infty} t^{-s-1} dt$$
$$\leq \mathbf{Var}_{p,\mathcal{E}}(f) \frac{\delta^{\alpha-s}}{\alpha_p - s} + 2\|f\|_{L^p(X,\mu)} \frac{\delta^{-s}}{s}.$$

Finally, since

$$\|(-\Delta)^{\alpha} f\|_{L^p(X,\mu)} = C \left\| \int_0^{\infty} t^{-\alpha-1}(P_t f - f) \, dt \right\|_{L^p(X,\mu)},$$

the result follows from Theorem 5.4 by optimizing in δ. $\square$

5.3. Examples

As an illustration of the more concrete regularity results that can be obtained from the Morrey's inequality in Theorem 5.1, in this paragraph we apply that result to several settings covered by the general theory. In addition, we propose new conjectures for fractals in the case $p > 1$. Recall that we define the Sobolev continuity exponent of a Dirichlet form as

$$\delta_{\mathcal{E}} = \inf\{p \geq 1, \, W^{1,p}(\mathcal{E}) \subset C^0(X)\}.$$

Strictly local Dirichlet spaces. In the framework described in Section 3.1, we know from Theorem 3.3(ii) that under the quasi Bakry-Émery condition (13), the local Besov semi-norm $\|f\|_{\alpha_p,p,R}$ is equivalent to the L^p-norm of the gradient and $\alpha_p = 1/2$ for any $p \geq 2$. Hence, Theorem 5.1 recovers the classical Morrey inequality.

Theorem 5.9. *Let (X, d, μ) be a metric measure space that satisfies the volume doubling property and supports a 2-Poincaré inequality. Moreover, assume that it satisfies the quasi Bakry-Émery condition (13). Then, for any $p > d_H$, there exists $C > 0$ such that*

$$\sup_{0<d(x,y)\leq R} \frac{|f(x)-f(y)|}{d(x,y)^{1-\frac{d_H}{p}}} \leq C\||\nabla f|\|_{L^p(X,\mu)}.$$

In particular $\delta_{\mathcal{E}} \leq d_H$.

Nested fractals. Dealing with strongly local Dirichlet spaces with sub-Gaussian heat kernel estimates is more delicate due to the lack of an analogue to the quasi Bakry-Émery condition (13). Nevertheless, we would like to discuss several conjectures for nested fractals and the Sierpinski carpet that arise in the light of those presented in [2]. In view of recent developments, specially in the fractal setting [27, Section 19], it seems that the exponent $\delta_{\mathcal{E}}$ may be related to the so-called Ahlfors regular conformal dimension of the space. We leave this question open for possible future research.

Theorem 5.10. *For the Vicsek set, $\delta_{\mathcal{E}} = 1$. Moreover, $W^{1,p}(\mathcal{E}) \subset C^{1-1/p}(X)$ for any $p > 1$.*

Proof. The condition for the possible ranges of p is obtained as follows. Recall from Theorem 5.1 that we look for the infimum of the p's such that $\frac{d_H}{p} < d_W\alpha_p$. For Vicsek set, we know from Theorem 3.6 and [2, Theorem 3.11] that we always have

$$d_W\alpha_p \geq d_W\Big(1-\frac{d_H}{d_W}\Big)\Big(1-\frac{2}{p}\Big) + \frac{d_W}{p} = \frac{(d_W-d_H)(p-2)+d_W}{p}.$$

Thus, the condition for p becomes $d_H < (d_W - d_H)(p-2) + d_W$ which is equivalent to $p > 1$. Theorem 5.1 also yields $W^{1,p}(\mathcal{E}) \subset C^\lambda(X)$ with $\lambda = d_W\alpha_p - \frac{d_H}{p} \geq (d_W-d_H)\big(1-\frac{1}{p}\big) = 1-\frac{1}{p}$, where the last equality follows from the fact that on the Vicsek set $d_W - d_H = 1$. □

For a generic nested fractal X we can provide bounds for the critical exponent $\delta_{\mathcal{E}}$.

Theorem 5.11. *On nested fractals, $1 \leq \delta_{\mathcal{E}} \leq \frac{2d_H}{d_W}$. Moreover, $W^{1,p}(\mathcal{E}) \subset C^\lambda(X)$ for any $p \geq 2$ with*

$$\lambda = (d_W - d_H)\Big(1-\frac{1}{p}\Big).$$

Proof. From [2, Theorem 3.11], we know that $\alpha_p \geq \frac{1}{2}$ for $1 \leq p \leq 2$ and $\alpha_p \geq \big(1-\frac{d_H}{d_W}\big)\big(1-\frac{2}{p}\big) + \frac{d_W}{p}$ for $p \geq 2$. The result now follows as in the proof of Theorem 5.10. □

Since it is conjectured in [2, Section 5] that on all nested fractals one has $\alpha_p = \big(1-\frac{d_H}{d_W}\big)\big(1-\frac{2}{p}\big) + \frac{1}{p}$ for every $p \geq 1$, we can actually state the following more precise conjecture.

Conjecture 5.12. *On nested fractals, $\delta_{\mathcal{E}} = 1$ and for any $p > 1$, there exists $C > 0$ such that*

$$\mu\text{-}ess\sup_{x\neq y} \frac{|f(x)-f(y)|}{d(x,y)^\lambda} \leq C\,\mathbf{Var}_{p,\mathcal{E}}(f)$$

for every $f \in W^{1,p}(\mathcal{E})$ *with* $\lambda = (d_W - d_H)\big(1 - \frac{1}{p}\big)$.

In particular for the Sierpinski gasket, $\lambda = \frac{\log(5/3)}{\log 2}\big(1-\frac{1}{p}\big)$ and for the Vicsek set, $\lambda = 1-\frac{1}{p}$. The Sierpinski carpet is of different nature and it has been conjectured in [2, Conjecture 5.4] that $\alpha_1 = (d_H - d_{tH} + 1)/d_W$ and $\alpha_p = \big(1 - \frac{2}{p}\big)(1 - \alpha_1) + \frac{1}{p}$ for $p > 1$, where d_{tH} is the topological Hausdorff dimension of the carpet. After some elementary computations, this yields the following conjecture.

Conjecture 5.13. *For the Sierpinski carpet,* $\delta_{\mathcal{E}} = 2 - \frac{d_W - d_H}{d_W - d_H + d_{tH} - 1}$ *and for any* $p > \delta_{\mathcal{E}}$*, there exists* $C > 0$ *such that*

$$\mu\text{-}ess\sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x,y)^{\lambda}} \leq C\,\mathbf{Var}_{p,\mathcal{E}}(f)$$

for every $f \in W^{1,p}(\mathcal{E})$ *with* $\lambda = \frac{1}{p}\big((d_W - d_H + d_{tH} - 1)(p-2) + d_W\big) - \frac{d_H}{p}$.

Since for the Sierpinski carpet it is known that $d_H = \frac{\log 8}{\log 3} = \frac{3\log 2}{\log 3}$ and $d_{tH} = 1 + \frac{\log 2}{\log 3}$, $d_W \approx 2.097$, this gives $d_W - d_H + d_{tH} - 1 = d_W - \frac{2\log 2}{\log 3}$. The critical exponents thus read

$$\delta_{\mathcal{E}} = 1 + \frac{\log 2}{d_W \log 3 - 2\log 2} \quad \text{and} \quad \lambda = d_W\Big(1 - \frac{1}{p}\Big) - \frac{\log 2}{\log 3}\Big(2 - \frac{1}{p}\Big).$$

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