

Lattice deformations in the Heisenberg group

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Abstract. The space of deformations of the integer Heisenberg group under the action of $\text{Aut}(\mathbf{H}(\mathbb{R}))$ is a homogeneous space for a non-reductive group. We analyze its structure as a measurable dynamical system and obtain mean and variance estimates for Heisenberg lattice point counting in measurable subsets of $\mathbb{R}^3$; in particular, we obtain a random Minkowski-type theorem. Unlike the Euclidean case, we show there are necessary geometric conditions on the sets that satisfy effective variance bounds.

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1. Introduction

Minkowski's theorem in the geometry of numbers shows that in any sufficiently large convex centrally symmetric open set in $\mathbb{R}^n$ there are non-zero integral points. The asymptotic count of the number of such points is well understood in terms of the volume of the set, but the optimal error term is hard to obtain and can depend sensitively on the regularity of the boundary.

It is an old idea that to understand a particular instance of a complicated system, it is beneficial to understand its typical behavior. From the classical viewpoint of the 'metric theory of equidistribution', $\mathbb{Z}^n$ is one of many lattices in $\mathbb{R}^n$, and their average behavior is easier to grasp than the individual point counting stories each lattice has to tell. In the more modern conception of homogeneous dynamics, $\mathbb{Z}^n$ is a point in the space of unimodular lattices in $\mathbb{R}^n$, a finite volume homogeneous space of $\text{SL}(n, \mathbb{R})$. Thus, we can formulate questions about the average count of lattice points and their variance for a given set, with respect to the Haar probability measure on this space.

The first steps in that program were taken by Siegel [9] who considered averaged lattice point counting over unimodular lattices and gave a mean value formula. Subsequently, Rogers [7] studied higher moments of functions on the space of lattices and obtained a variance bound in $\mathbb{R}^n$ with $n \geq 3$.

Rogers's work was used by W. Schmidt in [8] to show that for a nested family of Borel sets in $\mathbb{R}^n$ with unbounded finite volumes, almost all lattices have the

expected number of lattice points with explicit discrepancy bounds. The hardest part in Schmidt's work was the case $n = 2$ where most of Rogers's identities were not applicable: there was no variance bound in $\mathbb{R}^2$ to rely upon and he had to work 'by hand' using classical estimates from analytic number theory and the action of $\mathrm{SL}(2, \mathbb{Z})$ on pairs of integer vectors.

The issue of a variance bound was also treated by Randol [6] who obtained Rogers-type variance estimates for primitive lattice point counting in disks using the spectral decomposition of $L_c^2(\mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z}))$. Using Randol's results and a deeper analysis of the Siegel operator (defined in [9]) for $\mathrm{SL}(2, \mathbb{R})$, Athreya-Margulis obtained in [1] a variance estimate for primitive lattice point counting allowing arbitrary Borel sets in $\mathbb{R}^2$.

The results above provide significant information on the average behavior of Euclidean lattices. A natural subsequent question that arises involves lattices in non-abelian groups. The recent work of Garg, Nevo and Taylor [3] addresses the lattice point count of $\mathbb{Z}^{2n+1}$ in large centered balls in certain norms homogeneous with respect to the dilations of the $(2n + 1)$ -dimensional Heisenberg group.

The present work provides variance bounds for random lattices in the case of the 3-dimensional Heisenberg group. We will see that even in this modest excursion outside the abelian world, the average behavior of lattices is much more erratic than the Euclidean case. In particular, we show in Proposition 5.1 that very simple sets $S \subset \mathbb{R}^3$ have no useful variance bound for the number of primitive points of a random Heisenberg lattice in S . On the other hand, we also provide a natural class of sets for which optimal variance bounds exist and discuss extensions (see eg. Corollary 4.14).

Our main technique is to realize the space of Heisenberg lattices as a fiber bundle over the space of Euclidean lattices, use the action of the automorphism group of $\mathbf{H}(\mathbb{R})$ on it and relate it to the action of $\mathrm{SL}(2, \mathbb{R})$ on the space of Euclidean lattices. Since the space of Heisenberg lattices embeds into the space X_3 of Euclidean lattices in $\mathbb{R}^3$, our results can also be seen as looking at orbits in X_3 of certain lower dimensional subgroups of $\mathrm{SL}(3, \mathbb{R})$.

1.1. Heisenberg group. The real Heisenberg group $\mathbf{H}(\mathbb{R})$ has $\mathbb{R}^3$ as the underlying manifold with the smooth addition law (see Section 2 for details)

$$\begin{pmatrix} r \\ s \\ t \end{pmatrix} + \begin{pmatrix} r' \\ s' \\ t' \end{pmatrix} = \begin{pmatrix} r + r' \\ s + r' \\ t + t' + rs' - sr' \end{pmatrix}.$$

With this law $\mathbb{Z}^3$ becomes a discrete subgroup denoted by $\mathbf{H}(\mathbb{Z})$ which will be taken to have co-volume 1 with respect to the Lebesgue (Haar) measure on $\mathbb{R}^3$. Let $X_{\mathbf{H}}$ be the orbit of $\mathbb{Z}^3$ under the action of the connected component of the identity of the automorphism group of $\mathbf{H}(\mathbb{R})$ preserving volume $\mathrm{Aut}_1^+(\mathbf{H}(\mathbb{R}))$ (this is a connected Lie group whose structure we describe in Section 2). This set has the

structure of a finite volume homogeneous space with the projected Haar measure from $\text{Aut}_1^+(\mathbf{H}(\mathbb{R}))$.

Our work involves the following class of sets: consider a Borel set $A \subset \mathbb{R}^2$ of finite measure greater than 1. The *epsilon-plate* over A at level z is the set

$$A_\epsilon^z := (z, 0, 0) + A \times [0, \epsilon) \subset \mathbb{R}^3.$$

A point $l = (m, n, k)$ (where k , the third coordinate, is the central direction) in a Heisenberg lattice $g\mathbb{Z}^3$ is called primitive if $\gcd(m, n) = 1$ (this is the correct analogue of primitivity in $\mathbf{H}(\mathbb{Z})$ as we shall see in Section 2).

The main result provides the following average deviation bound for primitive lattice point count.

Theorem 1.1 (See Corollary 4.13). *Let $\mu_{\mathbf{H}}$ be the projected Haar measure on the space of Heisenberg lattices $X_{\mathbf{H}}$ that are deformations of the standard lattice $\mathbf{H}(\mathbb{Z})$ and m the Lebesgue measure in $\mathbb{R}^3$. Suppose $0 < \epsilon < 1$. We have*

$$\mu_{\mathbf{H}}\left(\Lambda \in X_{\mathbf{H}}: \left|\#(A_\epsilon^z \cap \Lambda_{\text{prim}}) - \frac{m(A_\epsilon^z)}{\zeta(2)}\right| > r \sqrt{m(A_\epsilon^z)}\right) \leq \frac{C}{r^2}. \quad (1)$$

where C is an absolute constant.

Combining these results with standard analytic manipulations we get variance bounds for sets built up from a moderate number of plates. However, there is a gap between the sets for which we get bounds and those for which we prove there is no such bound. This reflects the limitations of our knowledge of Euclidean lattice point distribution in $\mathbb{R}^2$. Despite this, regarding features of Heisenberg lattices that come genuinely from the action of the Heisenberg group, the results of sections 3 and 4 provide a comprehensive picture and one that generalizes to higher dimensional Heisenberg groups.

In subsequent work we plan to treat those higher dimensional groups combining an analysis of lattice point distribution of symplectic Euclidean lattices in $\mathbb{R}^{2n}$ and the semi-direct product structure of the corresponding $\text{Aut}(\mathbf{H})$. We hope that the present paper will also serve as an accessible introduction to the more technical results to follow.

In the next section we provide all the relevant definitions and illustrate the differences between Euclidean and Heisenberg lattices.

2. The space of Heisenberg lattices

In this expository section we describe the structure of the space of Heisenberg lattices.

Definition 2.1. The three dimensional real Heisenberg group $\mathbf{H}(\mathbb{R})$ is defined to be the group with underlying set $\mathbb{R}^3$ (written as column vectors) and addition law

$$\begin{pmatrix} r \\ s \\ t \end{pmatrix} + \begin{pmatrix} r' \\ s' \\ t' \end{pmatrix} = \begin{pmatrix} r + r' \\ s + r' \\ t + t' + rs' - sr' \end{pmatrix}.$$

The integer Heisenberg group $\mathbf{H}(\mathbb{Z})$ is the discrete subgroup $\mathbf{H}(\mathbb{R}) \cap \mathbb{Z}^3$.

Remark 2.2. The standard symplectic form appears in the addition formula for $\mathbf{H}(\mathbb{R})$; any other bilinear form would give rise to a Heisenberg group whose structure would be determined by the antisymmetric part of the form. See [2] for the reduction and more general Heisenberg groups.

Proposition 2.3. *Lebesgue measure in $\mathbb{R}^3$ is a Haar measure for $\mathbf{H}(\mathbb{R})$. The group $\mathbf{H}(\mathbb{Z})$ is a lattice in $\mathbf{H}(\mathbb{R})$ with a fundamental domain $[0, 1)^3$. The group of automorphisms of $\mathbf{H}(\mathbb{R})$ that preserve volume and orientation is the group $\text{Aut}_1^+(\mathbb{R}) =: \text{Aut}$ consisting of matrices of the form*

$$\begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix} \simeq \text{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$$

where

$$g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \quad \text{and} \quad \vec{v} := \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

We abbreviate these elements by

$$\begin{pmatrix} g & \vec{v} \\ 0 & 1 \end{pmatrix}.$$

The group $\text{Aut}_1^+(\mathbb{R})$ acts on $\mathbf{H}(\mathbb{R})$ by

$$\begin{pmatrix} g & \vec{v} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} g^* \begin{pmatrix} r \\ s \end{pmatrix} \\ t - \vec{v}^t \cdot g^* \cdot \begin{pmatrix} r \\ s \end{pmatrix} \end{pmatrix} \quad (2)$$

where $g^* = (g^{-1})^t$ is the inverse transpose. In terms of matrix multiplication, the action is

$$\begin{pmatrix} g & \vec{v} \\ 0 & 1 \end{pmatrix}^* \begin{pmatrix} r \\ s \\ t \end{pmatrix}.$$

Proof. All the assertions can be found in [2, I.2 - I.3]. The action there is of the transpose of the group we have by simple matrix-column multiplication. Our group acts by taking inverse transpose (landing us in the group used by Auslander) and performing matrix-column multiplication. $\square$

Remark 2.4. The choice of group representation for Aut_1^+ may seem odd; we made this choice because we need to take quotients on the right and it is easier to see the fiber bundle structure of the automorphism group from the upper semidirect product when we take right quotients. Whenever no confusion can arise, we freely switch to the lower one and its matrix action on column vectors rather than passing through the inverse transpose.

We turn to our main object of study:

Definition 2.5. Normalize the Haar (Lebesgue) measure m of $\mathbf{H}(\mathbb{R})$ so that the co-volume of $\mathbf{H}(\mathbb{Z})$ is 1. The space of $\mathbf{H}(\mathbb{Z})$ -deformations is defined to be the orbit $\text{Aut}_1^+(\mathbb{R}) \cdot \mathbf{H}(\mathbb{Z})$. It can be identified with the quotient

$$X_{\mathbf{H}} = \text{Aut}_1^+(\mathbb{R}) / \text{Aut}_1^+(\mathbb{Z})$$

where the last group is the group of $\mathbb{Z}$ -points of the automorphism group (this is the stabilizer of $\mathbf{H}(\mathbb{Z})$).

This is essentially the space of all Heisenberg lattices of co-volume 1. A full description of all lattices in $\mathbf{H}(\mathbb{R})$ is given in [2, I.2]. Note that not all lattices in $\mathbf{H}(\mathbb{R})$ are isomorphic as groups to $\mathbf{H}(\mathbb{Z})$; those that are, are isomorphic through an ambient automorphism of $\mathbf{H}(\mathbb{R})$. This crucial rigidity result, among other important facts about nilpotent groups, can be found in [5].

The next two facts describe the topological and measure theoretic relations between $X_{\mathbf{H}}$ and the space of unimodular lattices in $\mathbb{R}^2$ (denoted by X).

Proposition 2.6. *The space $X_{\mathbf{H}}$ has an equivariant fiber bundle structure over X with compact fiber over $g \text{SL}(2, \mathbb{Z})$ equal to $\mathbb{R}^2 / g\mathbb{Z}^2$.*

Corollary 2.7. *Let $\mu_{\mathbf{H}}$ be the unique probability measure on $X_{\mathbf{H}}$ invariant under Aut . Then $d\mu_{\mathbf{H}}(g_{\mathbf{H}}) = d\mu_{\mathbf{E}}(g) \times d\mu_g(x)$ where $d\mu_{\mathbf{E}}(g)$ is the projection of Haar measure on the space of Euclidean lattices X that gives measure 1 to $\text{SL}(2, \mathbb{Z})$ and $d\mu_g(x)$ the probability Haar measure on the toral fiber.*

Remark 2.8. Finally observe that $X_{\mathbf{H}} \hookrightarrow X_3$ as topological spaces. This inclusion identifies $X_{\mathbf{H}}$ with a subset of Euclidean lattices, with the following caveat: when we consider the two spaces not simply as topological spaces but as orbit spaces with a common action on $\mathbb{R}^3$, we need to modify the inclusion as $X_{\mathbf{H}} \hookrightarrow X_3^*$ where the last space is given by the inverse transpose automorphism of $\text{SL}(3, \mathbb{R})$.

3. Lattice points in measurable sets

Let $A \subset \mathbf{H}(\mathbb{R})$ be measurable. Pick a Heisenberg lattice L at random from $X_{\mathbf{H}}$ using the Haar measure. What can we say about $A \cap L$? It turns out that the answer

has two parts: one involves the projection of L in the $\mathbb{R}^2$ plane orthogonal to the center of $\mathbf{H}(\mathbb{R})$ and the other the vertical distribution over each lattice point in the projection. There are no ‘slanted lines’ in the central direction for Heisenberg lattices.

The corresponding question for Euclidean lattices has a very elegant answer that we will use extensively. The following result is one of the main points of [1]:

Theorem 3.1 ([1, Theorem 2.2]). *Let $n \geq 2$. There exists $C_n > 0$ such that if $A \subset \mathbb{R}^n$ has $m(A) > 0$,*

$$\mu_{\mathbf{E}}(\Lambda \in X_n : A \cap \Lambda = \emptyset) \leq \frac{C_n}{m(A)}. \quad (3)$$

Here $\mu_{\mathbf{E}}$ is the probability Haar measure on Euclidean n -lattices X_n and m is Lebesgue measure. In fact, the computation in [1, Section 4.2] implies the following stronger statement for Euclidean lattices in $\mathbb{R}^2$:

Theorem 3.2. *Let $A \subset \mathbb{R}^2$ with $m(A) > 0$. Then for all $r > 0$,*

$$\mu_{\mathbf{E}}\left(\Lambda \in X_2 : \left| \#(A \cap \Lambda_{\text{prim}}) - \frac{m(A)}{\zeta(2)} \right| > r \sqrt{m(A)}\right) \leq \frac{C}{r^2}. \quad (4)$$

In order to derive these theorems, the authors made extensive use of the *Theta transform* of compactly supported functions.

Definition 3.3. Let $L = g\mathbb{Z}^n$ be a Euclidean lattice in $\mathbb{R}^n$. Given a function ϕ in $L^1(\mathbb{R}^n)$, the theta transform is

$$\Theta\phi(L) = \sum_{\lambda \in L_{\text{prim}}} f(\lambda)$$

where λ ranges over primitive points in L . When $\phi = \chi_A$, we write $\Theta\phi = \Theta_A$.

We next give a version of the theta transform adapted to our needs.

Definition 3.4. Let $L = g\mathbb{Z}^3$ be a Heisenberg lattice. An element of L is called *primitive* if $\lambda = g(k, l, m)$ with the greatest common divisor $\gcd(k, l) = 1$.

Definition 3.5. Given a function ϕ in $L^1(\mathbb{R}^3)$ let

$$\Theta^{\mathbf{H}}\phi(L) = \sum_{\lambda \in L_{\text{prim}}} f(\lambda)$$

where λ ranges over primitive points in L . The operator $\Theta^{\mathbf{H}}: \phi \rightarrow \Theta^{\mathbf{H}}\phi$ is called the nil-theta transform of ϕ . When $\phi = \chi_A$, we write $\Theta^{\mathbf{H}}\phi = \Theta_A^{\mathbf{H}}$.

For characteristic functions of Borel sets, the Theta transform has the following properties that we will use repeatedly:

Proposition 3.6. *Let $\phi = \chi_A$ with A a bounded Borel set such that there exists a neighborhood U of $\mathbb{R}^2$ containing 0 such that $A \cap (U \times \mathbb{R}) = \emptyset$. Then the Theta transform $\Theta^{\mathbf{H}}\phi$ is a bounded Borel function with compact support on $X_{\mathbf{H}}$.*

Proof. The Borel property is clear. For a Heisenberg lattice L to intersect A , its projection in $\mathbb{R}^2$ must intersect $A_0 = \pi_{\text{flat}}(A)$, i.e. we must have $\Theta\chi_{A_0} \neq 0$. A_0 being a bounded set that does not meet a neighborhood of the origin shows, via the Mahler Compactness Criterion ([4, Chapter XIII, §1]) that $\Theta\chi_{A_0}$ has compact support C in X .

Therefore the closure of the set of Heisenberg lattices giving a non-zero $\Theta_A^{\mathbf{H}}$ is a subset of a torus bundle over the compact set C and therefore is compact.

Finally, whenever $\phi \leq \psi$ we have $\Theta^{\mathbf{H}}\phi \leq \Theta^{\mathbf{H}}\psi$ so if ψ is a continuous function whose support is compact and contains A , we have $\Theta_A^{\mathbf{H}} \leq \Theta^{\mathbf{H}}\psi$. Since $\Theta^{\mathbf{H}}\psi$ is continuous (continuity is proven as in loc. cit.) with compact support, it is bounded, and therefore so is $\Theta_A^{\mathbf{H}}$. $\square$

The nil-theta transform of characteristic functions counts lattice points in sets like its Euclidean counterpart. However, a result as uniform as Theorem 3.1 cannot hold in the Heisenberg setting:

Proposition 3.7. *Let π_{flat} be the projection in $\mathbf{H}(\mathbb{R})$ to the first two coordinates. Let $A \subset \mathbf{H}(\mathbb{R})$ have positive measure. The following inequality holds:*

$$\mu_{\mathbf{H}}(L \in X_{\mathbf{H}} : A \cap L = \emptyset) \geq \mu_{\mathbf{E}}(\Lambda \in X : \pi_{\text{flat}}(A) \cap \Lambda = \emptyset). \quad (5)$$

Proof. The action (2) transforms the flat part of the lattice by an element of $\text{SL}(2, \mathbb{R})$ and then translates the third component of each lattice point accordingly along the central direction. Suppose the projection of A does not intersect a lattice Λ . Then A cannot intersect any lift of Λ , since lifts are determined by values of the third coordinate over the flat part. Thus the entire torus of lattices in $X_{\mathbf{H}}$ over Λ misses A , giving the inequality. $\square$

In particular, we can increase the measure of A indefinitely keeping $\pi_{\text{flat}}(A)$ fixed; in the extreme cases of Theorem 3.1 (which are attained), for some $C > 0$ we have

$$\mu_{\mathbf{H}}(\Lambda \in X : \pi_{\text{flat}}(A) \cap \Lambda = \emptyset) \sim \frac{C}{m(\pi_{\text{flat}}(A))}.$$

We see that even if $m(A)$ itself becomes large, the probability of missing a Heisenberg lattice remains bounded below by the inverse of the measure of the projection.

This discrepancy originates in the way Heisenberg lattices sit in the space of all Euclidean 3-lattices: all Heisenberg lattices project to 2-lattices in the plane without tilting across the central direction. This phenomenon can be illustrated by the following extreme example:

Example 3.8. For $\delta > 0$ and $N \geq 1$ large, define $T(\delta, N)$ to be the cylindrical punctured tube

$$T(\delta, N) = (D(0, \delta) \setminus (0, 0)) \times [-N, N]$$

in $\mathbb{R}^3$. Consider a compact subset $C \subset X$ of measure $\mu_{\mathbf{E}}(C) = 1 - \epsilon$. Then by the compactness criterion on the space of lattices, all lattices in C have vectors of length at least $2\delta = \sqrt{\epsilon}$. Then

$$\begin{aligned} \mu_{\mathbf{H}}(L \in X_{\mathbf{H}}: L \cap T(\delta, N) = \emptyset) \\ &\geq \mu_{\mathbf{E}}(\Lambda \in X: \Lambda \cap D(0, \delta) = \emptyset) \\ &\geq \mu_{\mathbf{E}}(C) = 1 - \epsilon. \end{aligned}$$

On the other hand, the measure of $T(\delta, N)$ can become arbitrarily large no matter how small δ is, by increasing N .

This is in stark contrast to the situation for Euclidean 3-lattices: starting from $\mathbb{Z}^3$, we can reach the tube with an infinitesimally small shear; the points $(0, 0, n) \in \mathbb{Z}^3$ will immediately tilt to intersect $T(\delta, N)$.

The following question arises from this discussion: given a set A in $\mathbf{H}(\mathbb{R})$ and knowledge of the statistics of lattice points on $\pi_{\text{flat}}(A)$, how can we deduce the statistics of Heisenberg lattice points on A ? The next two sections are devoted to the answer.

4. Level distribution of lattices

Definition 4.1. Let $A \subset \mathbb{R}^2$ be measurable with positive measure, $\epsilon > 0$ and $z := (0, 0, z)$ with $z \in \mathbb{R}$. Let

$$A_{\epsilon}^z = z + A \times [0, \epsilon)$$

be the level- z ϵ -plate over A . Consider a Heisenberg lattice $L = g_{\mathbf{H}}\mathbf{H}(\mathbb{Z})$. The set

$$\{\#(L \cap A_{\epsilon}^z): \epsilon \in (0, 1)\}$$

is called the z -level distribution of $L \in X_{\mathbf{H}}$. From now on we assume $0 < \epsilon < 1$.

Remark 4.2. The definition above captures significant information for characterizing lattice statistics given the statistics of the projection, because lattices above a given Euclidean lattice only differ in the vertical deviation of lattice points from the integer lattice.

Proposition 4.3 ([1, Proof of Theorem 2.2]). *For $A \subset \mathbb{R}^2$ of positive measure $m(A) = a$, we have the variance estimate*

$$\left\| \Theta_A - \frac{a}{\zeta(2)} \right\|_2^2 \leq 16a. \quad (6)$$

We now begin the study of moments of $\Theta_{A_\epsilon^z}^{\mathbf{H}}$. The following proposition gives the Siegel formula. Since some intermediate formulas will be used in the L^2 estimate, we give a quick proof.

Proposition 4.4. *We have*

$$\int_{X_{\mathbf{H}}} \Theta_{A_\epsilon^z}^{\mathbf{H}}(L) d\mu_{\mathbf{H}}(L) = \frac{m(A)\epsilon}{\zeta(2)}. \quad (7)$$

Proof. We have $\chi_{A_\epsilon^z} = \chi_A \chi_{z+[0,\epsilon)}$. If $g_{\mathbf{H}} = \begin{pmatrix} g & \vec{x} \\ 0 & 1 \end{pmatrix}$, write

$$\Theta_{A_\epsilon^z}^{\mathbf{H}}(g_{\mathbf{H}} \mathbf{H}(\mathbb{Z})_{\text{prim}}) = \sum_{(\vec{m}, k) \in \mathbf{H}(\mathbb{Z})_{\text{prim}}} \chi_A(g^* \vec{m}) \chi_{z+[0,\epsilon)}(k - \vec{x}^t \cdot g^* \cdot \vec{m})$$

which splits as

$$\begin{aligned} \Theta_{A_\epsilon^z}^{\mathbf{H}}(g_{\mathbf{H}} \mathbb{Z}^3) &= \sum_{\vec{m} \in \mathbb{Z}_{\text{prim}}^2} \chi_A(g^* \vec{m}) \sum_{k \in \mathbb{Z}} \chi_{z+[0,\epsilon)}(k - \vec{x}^t \cdot g^* \cdot \vec{m}) \\ &= \sum_{\vec{m} \in \mathbb{Z}_{\text{prim}}^2} \chi_A(g^* \vec{m}) \sum_{k \in \mathbb{Z}} \chi_{[0,\epsilon)}(k - z - \vec{x}^t \cdot g^* \cdot \vec{m}) \\ &= \sum_{\vec{m} \in \mathbb{Z}_{\text{prim}}^2} \chi_A(g^* \vec{m}) \mathbf{1}(\{z + \vec{x}^t \cdot g^* \cdot \vec{m}\} < \epsilon). \end{aligned} \quad (8)$$

We want to integrate the last expression over the fiber first, then over the base, using Corollary 2.7,

$$\begin{aligned} &\int_{X_{\mathbf{H}}} \Theta_{A_\epsilon^z}^{\mathbf{H}}(g_{\mathbf{H}} \mathbf{H}(\mathbb{Z})_{\text{prim}}) d\mu_{\mathbf{H}}(g_{\mathbf{H}}) \\ &= \int_X \sum_{\vec{m} \in \mathbb{Z}_{\text{prim}}^2} \chi_A(g^* \vec{m}) \left(\int_{\mathbb{R}^2/g\mathbb{Z}^2} \mathbf{1}(\{z + \vec{x}^t \cdot g^* \cdot \vec{m}\} < \epsilon) dx \right) d\mu_{\mathbf{E}}(g). \end{aligned} \quad (9)$$

Now perform the change of variables $\vec{x} = g\vec{u}$ in the inner integral and use $\det g = 1$ to simplify to

$$\int_{\mathbb{R}^2/\mathbb{Z}^2} \mathbf{1}(\{z + \vec{u}^t \cdot \vec{m}\} < \epsilon) du.$$

Since $\mathrm{SL}(2, \mathbb{Z})$ is transitive on primitive vectors, all the toral integrals must have the same value which we now compute.

The map $S(u_1) = z + u_1 m_1 + u_2 m_2 \pmod{g\mathbb{Z}^2}$ is measure preserving, so splitting the integral over the torus in (9), we get

$$\int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \mathbf{1}(S^{-1}([0, \epsilon])) du_1 du_2 = \epsilon.$$

The outer integral in (9) is

$$\int_X \Theta_A(\Lambda) d\mu_{\mathbf{E}}(\Lambda) = \frac{m(A)}{\zeta(2)}$$

using the Siegel formula [9, page 341] for Euclidean lattices, and our normalization of X . $\square$

Now we need some preparations for dealing with the second moment of $\Theta_{A_\epsilon}^{\mathbf{H}}$.

Definition 4.5. Fix an integer D . Consider the space of pairs of coprime integer vectors

$$M_D = \{\mathbf{m} = (\vec{m}, \vec{n}) \in \mathbb{Z}_{\mathrm{prim}}^2 \times \mathbb{Z}_{\mathrm{prim}}^2 : \det \mathbf{m} = D\}. \quad (10)$$

The diagonal action of $\mathrm{SL}(2, \mathbb{Z})$, $\gamma(\vec{m}, \vec{n}) = (\gamma\vec{m}, \gamma\vec{n})$ leaves invariant the determinant of $\mathbf{m}$, so it stabilizes each M_D . The space of $\mathrm{SL}(2, \mathbb{Z})$ orbits of the set $M_D \subset \mathbb{Z}_{\mathrm{prim}}^2 \times \mathbb{Z}_{\mathrm{prim}}^2$ of determinant D pairs is denoted by $M_D^\Theta = \mathrm{SL}(2, \mathbb{Z}) \backslash M_D$.

Definition 4.6. Let $\vec{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $z \in \mathbb{R}$. Define the map $S_{\vec{m}}: \mathbb{T}^2 \rightarrow \mathbb{T}$ by

$$S_{\vec{m}}(u_1, u_2) = z + m_1 u_1 + m_2 u_2 \pmod{1}.$$

Denote the pullback of $[0, \epsilon]$ by S into $\mathbb{T}^2$ by

$$C_{\vec{m}}(\epsilon) := S_{\vec{m}}^{-1}([0, \epsilon]) \subset \mathbb{T}^2.$$

Finally, for $\vec{m}, \vec{n}$ in $\mathbb{Z}_{\mathrm{prim}}^2$, define the correlation of

$$\mathrm{Cor}_{\vec{m}, \vec{n}}(\epsilon, z) = \mu_{\mathbb{R}^2/\mathbb{Z}^2}(C_{\vec{m}}(\epsilon) \cap C_{\vec{n}}(\epsilon))$$

where $\mu_{\mathbb{R}^2/\mathbb{Z}^2}$ is the probability Haar measure on the 2-torus.

Next, we turn to the second moment of $\Theta_{A_\epsilon}^{\mathbf{H}}$.

Proposition 4.7. *We have the following identity:*

$$\int_{X_{\mathbf{H}}} |\Theta_{A_\epsilon}^{\mathbf{H}}(L)|^2 d\mu_{\mathbf{H}}(L) = \int_X \sum_{\vec{m}, \vec{n} \in \mathbb{Z}_{\mathrm{prim}}^2} \chi_A(g^* \vec{m}) \chi_A(g^* \vec{n}) \mathrm{Cor}_{\vec{m}, \vec{n}}(\epsilon, z) d\mu_{\mathbf{E}}(g). \quad (11)$$

Proof. Using notation as in Proposition 4.4 and the expression (8) for the integrand, expand the square:

$$|\Theta_{A_\epsilon^z}^{\mathbf{H}}(L)|^2 = \sum_{\vec{m}, \vec{n}} \chi_A(g^* \vec{m}) \chi_A(g^* \vec{n}) \cdot \mathbf{1}(\{z + \vec{x}^t g^* \vec{m}\} < \epsilon) \mathbf{1}(\{z + \vec{x}^t g^* \vec{n}\} < \epsilon).$$

Let

$$w_{\vec{m}, \vec{n}}(g, \vec{x}) = \mathbf{1}(\{z + \vec{x}^t g^* \vec{m}\} < \epsilon) \mathbf{1}(\{z + \vec{x}^t g^* \vec{n}\} < \epsilon).$$

Once again integrating and changing variables, we get

$$\int_{\mathbb{R}^2/g\mathbb{Z}^2} w_{\vec{m}, \vec{n}}(g, \vec{x}) dx = \int_{\mathbb{R}^2/\mathbb{Z}^2} \mathbf{1}(\{z + \vec{u}^t \vec{m}\} < \epsilon) \mathbf{1}(\{z + \vec{u}^t \vec{n}\} < \epsilon) du.$$

Since this expression is invariant under $g \rightarrow g\gamma$, the result follows from the definition of $C_{\vec{m}}(\epsilon)$. $\square$

Remark 4.8. The correlation $\text{Cor}_{\vec{m}, \vec{n}}(\epsilon, z)$ is a ‘correction factor’ that weighs the double sum in (11). Therefore, the study now reduces to the distribution of the values of $\text{Cor}_{\vec{m}, \vec{n}}(\epsilon, z)$. Whenever $\mathbf{m} = \gamma \mathbf{n}$ are in the same $\text{SL}(2, \mathbb{Z})$ -orbit, the corresponding correlations must be the same. Thus one can write $\text{Cor}_{\vec{m}, \vec{n}}(\epsilon, z) = \text{Cor}_{\mathcal{O}}(\epsilon, z)$ for the common value of the correlation over an orbit $\mathcal{O} \in M_D^{\mathcal{O}}$. The idea is to look for a simple pair $\mathbf{m}$ in the orbit where the correlation is easy to compute.

The next lemma describes the structure of the above set; the proof can be found in [8, Lemma 6] or worked out by hand.

Lemma 4.9. *The structure of $M_D^{\mathcal{O}}$ is as follows:*

- (1) if $D = 0$, then $M_D^{\mathcal{O}}$ has two elements represented by $\pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;
- (2) if $D = \pm 1$, then $M_D^{\mathcal{O}}$ is a singleton represented by the identity matrix ($\text{diag}(-1, 1)$ respectively);
- (3) if $|D| > 1$, then $M_D^{\mathcal{O}}$ consists of $\phi(D)$ distinct orbits, represented by $\begin{pmatrix} D & 0 \\ k & 1 \end{pmatrix}$ where k runs through a complete set of residues modulo D .

Using this lemma, we can show:

Proposition 4.10. *The correlation $\text{Cor}_{\mathcal{O}}(\epsilon, z)$ is equal to ϵ^2 when $\mathcal{O} \in M_D$ with $D \neq 0$ and equal to ϵ when $D = 0$ and $\mathcal{O} = +1$, 0 when $\mathcal{O} = -1$.*

Proof. When $D = 0$ with equal parity, choose $\mathbf{m} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$; then

$$\text{Cor}_{\mathcal{O}}(\epsilon, z) = \mu_{\mathbb{R}^2/\mathbb{Z}^2}(S^{-1}([0, \epsilon])) = \epsilon$$

as in Proposition 4.4. For opposite parity, the two intervals $[0, \epsilon) + \mathbb{Z}$ and $(-\epsilon, 0] + \mathbb{Z}$ are disjoint so their pullback under S has empty intersection.

When $D \neq 0$, for any $\begin{pmatrix} D & 0 \\ k & 1 \end{pmatrix}$ compute

$$\begin{aligned} & \int_{\mathbb{R}^2/\mathbb{Z}^2} \mathbf{1}(\{z + \vec{u}^t \vec{m}\} < \epsilon) \mathbf{1}(\{z + \vec{u}^t \vec{n}\} < \epsilon) du \\ &= \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \mathbf{1}(\{z + Du_1 + ku_2\} < \epsilon) \mathbf{1}(\{z + u_2\} < \epsilon) du_1 du_2 \\ &= \int_{\mathbb{R}/\mathbb{Z}} \mathbf{1}(\{z + u_2\} < \epsilon) \int_{\mathbb{R}/\mathbb{Z}} \mathbf{1}(\{z + Du_1 + ku_2\} < \epsilon) du_1 du_2; \end{aligned}$$

the inner integrand is the pullback of $[0, \epsilon)$ by a (measure preserving) affine map of the form $Du_1 + \tau$ giving integral ϵ independent of u_2 , and then the remaining factor contributes another ϵ . $\square$

We now come to the main result of this section, the analog of Theorem 4.3 for Heisenberg lattices.

Theorem 4.11. *Let A_ϵ^z be an epsilon-plate over A of measure $m(A) > 1$. We have*

$$\left\| \Theta_{A_\epsilon^z}^{\mathbf{H}} - \frac{\epsilon m(A)}{\zeta(2)} \right\|_2^2 \leq \frac{\epsilon m(A)}{\zeta(2)} + 20\epsilon^2 m(A). \quad (12)$$

Proof. Write

$$\left\| \Theta_{A_\epsilon^z}^{\mathbf{H}} - \frac{\epsilon m(A)}{\zeta(2)} \right\|_2^2 = \left\| \Theta_{A_\epsilon^z}^{\mathbf{H}} \right\|_2^2 - \left(\frac{\epsilon m(A)}{\zeta(2)} \right)^2$$

and use (11) to expand

$$\left\| \Theta_{A_\epsilon^z}^{\mathbf{H}} \right\|_2^2 = \int_X \sum_{\vec{m}, \vec{n} \in \mathbb{Z}_{\text{prim}}^2} \chi_A(g^* \vec{m}) \chi_A(g^* \vec{n}) \text{Cor}_{\vec{m}, \vec{n}}(\epsilon, z) d\mu_{\mathbf{H}}(g);$$

by Proposition 4.10 the integrand on the right hand side becomes

$$\epsilon \sum_{\mathbf{m} \in M_0, +} \chi_A(g^* \vec{m}) \chi_A(g^* \vec{n}) + \epsilon^2 \sum_{\mathbf{m} \in M_D, D \neq 0} \chi_A(g^* \vec{m}) \chi_A(g^* \vec{n});$$

now add and subtract the sum over M_0 weighted by ϵ^2 to get

$$(\epsilon - \epsilon^2) \sum_{\mathbf{m} \in M_0, +} \chi_A(g^* \vec{m}) \chi_A(g^* \vec{m}) + \epsilon^2 \sum_{\mathbf{m} \in M} \chi_A(g^* \vec{m}) \chi_A(g^* \vec{n})$$

for the integrand; recall $\det \mathbf{m} = 0$ and parity preservation implies $\vec{m} = \vec{n}$. Integrating the two parts separately, we get

$$\|\Theta_{A_\epsilon^z}^{\mathbf{H}}\|_2^2 = (\epsilon - \epsilon^2) \int_X \Theta \chi_A^2(L) d\mu_{\mathbf{E}}(L) + \epsilon^2 \int_X |\Theta_A(L)|^2 d\mu_{\mathbf{E}}(L)$$

which simplifies to

$$\begin{aligned} \|\Theta_{A_\epsilon^z}^{\mathbf{H}}\|_2^2 &= (\epsilon - \epsilon^2) \int_X \Theta \chi_A(L) d\mu_{\mathbf{E}}(L) + \epsilon^2 \|\Theta_A\|_2^2 \\ &= \frac{(\epsilon - \epsilon^2)m(A)}{\zeta(2)} + \epsilon^2 \|\Theta_A\|_2^2 \end{aligned}$$

using $\chi_A = \chi_A^2$ and the Siegel formula [9] for Euclidean 2-lattices (recall our normalization of X). Now we subtract the constant term and absorb it into the second summand to get

$$\left\| \Theta_{A_\epsilon^z}^{\mathbf{H}} - \frac{\epsilon m(A)}{\zeta(2)} \right\|_2^2 = \frac{(\epsilon - \epsilon^2)m(A)}{\zeta(2)} + \epsilon^2 \left\| \Theta_A - \frac{m(A)}{\zeta(2)} \right\|_2^2. \quad (13)$$

Applying Theorem 4.3 and gathering ϵ^2 terms together, we obtain

$$\left\| \Theta_{A_\epsilon^z}^{\mathbf{H}} - \frac{\epsilon m(A)}{\zeta(2)} \right\|_2^2 \leq \frac{\epsilon m(A)}{\zeta(2)} + 20\epsilon^2 m(A). \quad (14)$$

□

Theorem 4.11 immediately implies the analogues of Theorems 3.1 and 3.2 for the plate distribution.

Corollary 4.12. *We have the following bound on the probability of a Heisenberg lattice missing a plate A_ϵ^z :*

$$\mu_{\mathbf{H}}(\Lambda \in X_{\mathbf{H}}: A_\epsilon^z \cap \Lambda = \emptyset) \leq \frac{C}{m(A_\epsilon^z)}. \quad (15)$$

Corollary 4.13. *The average discrepancy of lattice points in a plate A_ϵ^z satisfies the Chebyshev inequality*

$$\mu_{\mathbf{H}}\left(\Lambda \in X_{\mathbf{H}}: \left| \#(A_\epsilon^z \cap \Lambda_{\text{prim}}) - \frac{m(A_\epsilon^z)}{\zeta(2)} \right| > r \sqrt{m(A_\epsilon^z)}\right) \leq \frac{C}{r^2}. \quad (16)$$

From this information, we can obtain discrepancy estimates from sets built out of plates in a controlled number of steps. We illustrate this with the example of a stout cylinder:

Corollary 4.14. *Let $C = A \times I$ with $m(A) > 1$ and $|I| \leq m(A)^{\frac{1}{2}-\delta}$. Then*

$$\mu_{\mathbf{H}}\left(\Lambda \in X_{\mathbf{H}}: \left| \#(C \cap \Lambda_{\text{prim}}) - \frac{m(C)}{\zeta(2)} \right| > r m(C)^{1-\delta}\right) \leq \frac{C}{r^2}. \quad (17)$$

5. High discrepancy sets

Here we construct a wide variety of sets for which an estimate like the one in Corollary 4.13 cannot hold; the argument is an extension of Example 3.8. The point of the following proposition is to show that firstly high discrepancy sets need not be anchored to any specific point like the origin, and secondly do not need to have an approximately cylindrical shape (although they will be built out of cylinders).

Proposition 5.1. *Let $B(0, R)$ be a large ball in $\mathbb{R}^2$ centered at the origin and fix $\epsilon > 0$. Let*

$$Z_\circ = (\mathbb{Z}^2 + B(0, \epsilon)) \cap B(0, R)$$

be a thickening of the standard lattice inside the fixed ball. There exist Borel sets $S \subset \mathbb{R}^3$ such that:

- (1) *S has arbitrarily large measure,*
- (2) *for any cylinder C and $\epsilon' > 0$,*

$$m(C \triangle S) >_{\epsilon'} m(S)^{1-\epsilon'},$$

- (3) *$\pi_{\text{flat}}(S) = A$ is an arbitrary Borel set of positive measure in $B(0, R) \setminus Z_\circ$ and*
- (4) *the inequality*

$$\mu_{\mathbf{H}}(\Lambda \in X_{\mathbf{H}}: S \cap \Lambda = \emptyset) \geq \delta$$

holds, where δ depends only on ϵ .

Proof. Let U be a neighborhood of the identity in $\text{SL}(2, \mathbb{R})$ such that

$$(\bar{U}\mathbb{Z}^2) \cap B(0, R) \subset Z_\circ$$

and $\mu_{\mathbf{E}}(U) = \delta$ its measure.

Generally to meet requirements 1, 3 and 4 we can pick an arbitrary Borel $A \subset B(0, R) \setminus Z_\circ$ of positive measure, partition it into finitely or countably many disjoint Borel sets A_i of positive measure and over each set erect a cylinder $C_i = A_i \times I_i$ of any desired height to form S so that

$$\sum_i |I_i| m(A_i) = m(S)$$

is arbitrarily large and distributed in an arbitrary way over the bases A_i .

To meet the second requirement, observe that all that matters in the computation is the measures of the A_i and the placements of the cylinders over them. Thus we can visualize the A_i as disjoint horizontal intervals on the real line arranged in

decreasing size and the I_i as vertical intervals at prescribed level and of prescribed height.

Choose a large finite partition of A into $k = m(A)^N$ parts so that each A_i has measure approximately $\frac{1}{2^i}m(A)$ and choose segments $I_1 = [1, 2]$, $I_2 = [2, 4]$, $I_3 = [4, 8]$ and so on. Then $m(S) \sim km(A)$ and by controlling k we can make $m(S)$ as large as possible. Note that the top of the cylinder over A_i is at $t_i \sim 2^{i+1}$.

A cylinder that will minimize the difference $C \triangle S$ will necessarily have base inside A , so we may assume that $C = (\bigcup_{j \in J} A_j) \times I$ where the union is over some sub-collection of the partition. Assume C contains precisely l of the k cylinders, with $l \gg 1$.

Then C is based on at least l and at most $(l + 2)$ of the A_i . Let $r + 1$ be the smallest index so that the cylinder over A_{r+1} is contained in C . In order to minimize the difference, the optimal cylinder has to be based on all A_j for $j = r + 1, \dots, r + l$ (for each $j > r + 1$ skipped, the gains from removing C_j are offset by the fact that indices are shifted at least one place, doubling the cost of the cylinder over A_r ; this beats the gain because the measure of A_r is at least twice the measure of A_j).

For this configuration, the height of the cylinder is $t_{r+l} - t_r = 2^{r+1} - 2^r$ and the base has measure $m(A)2^{-r}(1 - 2^{-l})$; thus its measure is $m(A)2^l$. In contrast, the measure of $S \cap C$ is within $(l \pm 2)m(A)$. Therefore,

$$m(S \triangle C) \geq m(A)(2^l - l - 2).$$

If $l \geq \log_2(k)$ we get our result. If $l \leq \log_2(k)$, then

$$m(S \triangle C) \geq m(S) - m(C \cap S) \geq m(A)(k - l)$$

which again gives the result.

Finally, by the discussion preceding Example 3.8 we know that

$$\mu_{\mathbf{H}}(\Lambda \in X_{\mathbf{H}} : S \cap \Lambda = \emptyset) \geq \mu_{\mathbf{E}}(\Lambda \in X : A \cap \Lambda = \emptyset) \geq \mu_{\mathbf{E}}(U) = \delta.$$

Since our choice of U depended only on ϵ , we are done. $\square$

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