THE REGULARITY CONJECTURE FOR PRIME IDEALS IN POLYNOMIAL RINGS

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Abstract. This paper presents a survey on recent developments on regularity of prime ideals in polynomial rings.

1. Introduction

Throughout, we work over the field $\mathbb{C}$. We make this assumption for simplicity and because polynomial rings over $\mathbb{C}$ are our main case of interest; however, most results and open problems are valid over an algebraically closed field of characteristic zero, and several of them hold in greater generality. We work over the polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$, which is standard graded with $\deg(x_i) = 1$ for all i, and consider the projective space $\mathbb{P}^{n-1}$ over $\mathbb{C}$.

Castelnuovo-Mumford regularity is a numerical invariant which measures the complexity of the structure of a graded ideal or a coherent sheaf on projective space (regularity can be defined for other algebraic objects as well, but we will not pursue this direction). For a graded ideal I, its regularity shows how high we should truncate in order to make the homological properties of $I_{\geq r}$ as simple as possible (more precisely, to get an ideal with a linear minimal free resolution), and for a coherent sheaf on a projective space it shows how much one has to twist in order to make the cohomological properties simpler.

The concept of Castelnuovo-Mumford regularity, called regularity for short, was introduced by Mumford [Mu] generalizing ideas of Castelnuovo. The properties of $\mathrm{reg}(X)$ for a subvariety $X \subseteq \mathbb{P}^{n-1}$, which depend on the embedding, are discussed in Section 5. For more details on regularity, we refer the reader to the expository papers [BM, Ch, Ei] and the books [Ei2, La2]. The following fundamental problem stems from works of Castelnuovo, Mumford, and others:

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Problem 1.1. *Find explicit upper bound(s) for* $\operatorname{reg}(X)$ *of a subvariety* $X \subseteq \mathbb{P}^{n-1}$ *in terms of geometric invariants of* X*.*

This problem is partially motivated by the fact that $\operatorname{reg}(X)$ gives an upper bound for the degrees of the defining equations of X.

The definition of regularity of a graded ideal I in terms of its minimal free resolution is due to Eisenbud and Goto [EG]. We discuss it in Section 2 and use it later in the paper. The correspondence between the definitions of regularity of a graded ideal (or module) and regularity of a coherent sheaf is described in [Ei2, Proposition 4.16], for example. There is also a natural expression for regularity (of graded modules, not just of graded ideals) via local cohomology, which we will not use in this paper.

Papers of Bayer-Mumford, Bayer-Stillman, Koh [BM, BS, Ko] and others give examples of families of ideals attaining doubly exponential regularity in terms of the degrees of the minimal generators of the ideal and the number of variables of the ambient ring. Their examples are based on the Mayr-Meyer [MM2] construction. We discuss the doubly exponential behavior of regularity of non-prime ideals in Section 3. In contrast, Bertram-Ein-Lazarsfeld [BEL], Chardin-Ulrich [CU], and Mumford (published in [BM]) have proven that there are nice bounds on the regularity of the ideals of smooth (or nearly smooth) projective varieties. As discussed in an influential paper by Bayer-Mumford (1993) [BM], *the biggest missing link between the general case and the smooth case is to obtain a "decent bound on the regularity of all reduced equidimensional ideals*". For simplicity, in this paper we focus on regularity bounds for prime ideals – the ideals that define irreducible projective varieties. For such ideals, the long standing Eisenbud-Goto Regularity Conjecture (1984) [EG] predicts an elegant linear upper bound on regularity in terms of the degree of the variety (also called multiplicity). The conjecture is discussed in Section 4 and stated as the Regularity Conjecture 4.4. It was proven for curves by Gruson-Lazarsfeld-Peskine [GLP], and for smooth surfaces by Lazarsfeld and Pinkham [La, Pi]. Ran [Ra] studied regularity for most smooth 3-folds, and Kwak [Kw, Kw2] proved a slightly weaker bound for all smooth 3-folds. The (arithmetically) Cohen-Macaulay case was settled by Eisenbud-Goto [EG]. References for results in other special cases of the conjecture and for similar bounds are provided in Section 5. That section surveys some results on regularity in Algebraic Geometry.

This expository paper is an overview of the current state of results on regularity of prime ideals. Recently, in [MP], we produced many counterexamples to the Eisenbud-Goto conjecture. In fact, in Theorem 11.3 we show that the regularity of prime ideals is not bounded by any polynomial function of the degree. In the subsequent paper [CCMPV], joint with Caviglia, Chardin, and Varbaro, we answer several natural questions which arise from [MP]. Section 11 is devoted to counterexamples to the Regularity Conjecture 4.4. We present a small counterexample of dimension 3 and families of counterexamples, some of which rely on the Mayr-Meyer [MM2] construction. The latter lead to the above mentioned Theorem 11.3. In these examples, the degree (multiplicity) fails to bound the maximal degree of an element in a minimal system of generators of the ideal. We may ask if it is possible to find prime ideals generated in low degrees but with high regularity. Theorem 12.2 shows that such prime ideals exist, more precisely, there exist prime ideals for which the maximal degree of a minimal generator is 6 and the maximal degree of a minimal first syzygy is arbitrarily large.

Section 13 discusses some open questions. Using a result of Ananyan-Hochster, Theorem 13.3 shows that there exists an upper bound on regularity of prime ideals in terms of the multiplicity alone. Motivated by Stillman's Conjecture and Computational Algebra, one might wonder if that bound can be made less than doubly exponential. The main currently open conjecture on this topic seems to be Conjecture 14.4, raised by Bayer and Mumford in 1993 [BM, Comments after Theorem 3.12]. It conjectures a singly exponential bound on the regularity of non-degenerate homogeneous prime ideals. The bound has base the maximal degree of an element in a minimal system of generators and exponent in terms of the number of variables. In a different direction, it would be interesting to prove the Regularity Conjecture after imposing additional constraints; for example, extra tools are available in the smooth case and also for toric varieties.

One of the reasons why progress on the Regularity Conjecture was slow was the lack of techniques for constructing examples of prime ideals with high regularity, in particular, the lack of techniques for producing such ideals from non-prime examples. In [MP] we introduced a method which, starting from a homogeneous ideal I in S, produces a homogeneous prime ideal whose projective dimension, regularity, degree, dimension, depth, and codimension are expressed in terms of numerical invariants of I. Our method involves two new techniques:

(1) Rees-like algebras are described in Section 7. Their construction was inspired by an example of Hochster published in [Bec]. Rees algebras are of high interest in Commutative Algebra, but their properties are very intricate. The defining equations of Rees algebras are difficult to find in general, and usually we can only find bounds for their numerical invariants. In contrast, Theorem 7.4 provides simple explicit formulas for the generators and the numerical invariants of Rees-like algebras.

(2) We use a new homogenization technique for prime ideals, which is described in Section 6. Its key property is the preservation of the graded Betti numbers, which usually change after traditional homogenization (taking projective closure). In particular, note that traditional homogenization has to

be performed on a Gröbner basis, which is usually a much larger set than a set of minimal generators.

In Sections 7, 9, and 10 we describe and compare three methods for producing prime ideals – two versions of the well-known construction of Rees algebras and the new construction of Rees-like algebras. These constructions usually yield prime ideals which are homogeneous with respect to a non-standard grading, and thus need to be homogenized in order to make them standard graded; in Section 6 we discuss the new Step-by-step homogenization technique from [MP] and another version from [MMM], called Prime Standardization.

We close the introduction with some remarks about regularity over quotient rings. The definition of regularity via free resolutions works over graded quotients of a polynomial ring, but the situation there is usually considerably different than that over a polynomial ring. For example, by Serre's Theorem minimal free resolutions over graded quotients of polynomial rings are usually infinite, in contrast to Hilbert's Syzygy Theorem that every graded ideal in a polynomial ring has a finite minimal free resolution. Regularity is known to be finite over graded Koszul algebras, and several interesting results are known in this setting.

2. **Regularity via Minimal Free Resolutions**

This section provides background on the definition of regularity via minimal free resolutions. Throughout, we work over a polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$, which is standard graded with $\deg(x_i) = 1$ for all i.

In his famous paper in 1890 [Hi], Hilbert introduced the approach to use free resolutions in order to answer the following basic question.

Basic Question 2.1. *How can we describe the structure of a graded ideal?*

An initial guess is that perhaps a set of generators provides a lot of information in some simple way. The first issue to deal with is whether there exists a finite set of generators. This is resolved by Hilbert's Basis Theorem:

Theorem 2.2. (see for example, [Ei, Theorem 1.2]) *Every graded ideal in S has a finite set of homogeneous generators.*

However, the generators may give very little information about the structure of the ideal, because there are relations on the generators, relations on these relations, and so on, which we may need to understand. Hilbert's approach to Question 2.1 is to capture the structure of such relations by the concept of free resolution. He introduced this idea in a famous paper in 1890 [Hi] motivated by Invariant Theory; the idea can also be found in the work of Cayley [Ca]. The definition of a free resolution works a lot more broadly (for example, for modules over a possibly non-commutative ring); in this paper we will restrict the concept of resolution to our case of interest – graded ideals in the polynomial ring S.

Definition 2.3. A sequence

$$\mathbf{F}: \quad \cdots \to F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \tag{2.4}$$

of homomorphisms of finitely generated, graded free S-modules is a *graded free resolution* of a graded ideal N if:

(1) $\mathbf{F}$ is an exact complex, that is, $\mathrm{Ker}(\partial_i) = \mathrm{Im}(\partial_{i+1})$ for $i \geq 1$.
(2) $N \cong \mathrm{Coker}(\partial_1)$, that is, we have an exact sequence

$$\cdots \to F_{i+1} \xrightarrow{\partial_{i+1}} F_i \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} N \to 0\,. \tag{2.5}$$

(3) The *differential* $\partial = \{\partial_i\}$ of $\mathbf{F}$ is homogeneous, that is, it preserves degree.

Hilbert's key insight was that a graded free resolution of N can be interpreted as

$$\cdots \to F_2 \xrightarrow{\partial_2 = \begin{pmatrix}\text{relations}\\ \text{on the}\\ \text{relations}\\ \text{in } \partial_1\end{pmatrix}} F_1 \xrightarrow{\partial_1 = \begin{pmatrix}\text{relations}\\ \text{on the}\\ \text{generators}\\ \text{of } N\end{pmatrix}} F_0 \xrightarrow{\begin{pmatrix}\text{generators}\\ \text{of } N\end{pmatrix}} N \to 0,$$

and so it is a description of the structure of N. We illustrate this interpretation in the following simple example:

Example 2.6. Consider the ideal $N = (x^2, xy, y^2)$ in the polynomial ring $\mathbb{C}[x, y]$. It is generated by $f := x^2$, $g := xy$, and $h := y^2$. We have the relations

$$yf - xg = y(x^2) - x(xy) = 0 \qquad \text{and} \qquad yg - xh = y(xy) - x(y^2) = 0.$$

It can be shown that the ideal N has a free resolution

$$0 \to S^2 \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}} N \to 0\,, \tag{2.7}$$

which can be interpreted as

$$0 \to F_1 \xrightarrow{\partial_1 = \begin{pmatrix}\text{relations}\\ \text{on the}\\ \text{generators}\\ \text{of } N\end{pmatrix}} F_0 \xrightarrow{\begin{pmatrix}\text{generators}\\ \text{of } N\end{pmatrix}} N \to 0\,.$$

The key result in Hilbert's approach is Hilbert's Syzygy Theorem:

Theorem 2.8. (see for example, [Ei3, Corollary 19.7], [Pe, Theorem 15.2]) *Every graded ideal N in S has a finite free resolution.*

It is easy to see that a graded free resolution, as defined in Definition 2.3, is far from being unique. One can produce many different resolutions by simply adding extraneous short complexes $0 \longrightarrow S \xrightarrow{1} S \longrightarrow 0$ here and there:

Example 2.9. For example,

$$0 \to S \xrightarrow{\begin{pmatrix} -y \\ -x \\ 1 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} y & 0 & y^2 \\ -x & y & 0 \\ 0 & -x & -x^2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}} N \to 0$$

is a larger free resolution than (2.7) of the ideal N considered in Example 2.6. The differential ∂_1 contains the relation $\begin{pmatrix} y^2 \\ 0 \\ -x^2 \end{pmatrix}$, which is a linear combination of the other two relations, and so it could be omitted in order to produce the smaller resolution (2.7).

It is beneficial to construct a graded free resolution that is as small as possible since in this way we are not hindered by redundant information. This is captured in the concept of a *minimal* free resolution.

Definition 2.10. A graded free resolution $\mathbf{F}$ is *minimal* if

$$\partial_{i+1}(F_{i+1}) \subseteq (x_1, \ldots, x_n)F_i \quad \text{for all } i \geq 0.$$

This means that no invertible elements appear in the differential matrices.

Minimal resolutions are sufficient to deal with all graded ideals and are as small as possible:

Theorem 2.11. (see for example, [Ei, Theorem 20.2], [Pe, Theorem 7.5], [Pe, Theorem 3.5]) *Every graded ideal N in S has a minimal graded free resolution, which is unique up to isomorphism. Any graded free resolution of N contains its minimal graded free resolution as a direct summand.*

Minimal free resolutions are not only optimal in the sense that they are as small as possible, but they actually contain more information about the structure of the resolved ideal than non-minimal resolutions. For example, the Auslander-Buchsbaum Formula (see for example, [Pe, Theorem 15.3]) expresses depth in terms of the length (called projective dimension) of the minimal free resolution. The downside is that they are much harder to obtain than the non-minimal ones.

Definition 2.12. The ranks of the free modules in the minimal free resolution $\mathbf{F}$ of a graded ideal N are called the *Betti numbers* of N and denoted by

$$\beta_i(N) = \text{rank } F_i\,.$$

They can be expressed as

$$\beta_i(N) = \dim_{\mathbb{C}} \operatorname{Tor}_i^S(N, \mathbb{C}) = \dim_{\mathbb{C}} \operatorname{Ext}_S^i(N, \mathbb{C})\,,$$

since the differentials in the complexes $\mathbf{F} \otimes_S \mathbb{C}$ and $\operatorname{Hom}_S(\mathbf{F}, \mathbb{C})$ are zero. The connections between the structure of an ideal and the properties of its Betti numbers are a core topic in Commutative Algebra and have applications in Algebraic Geometry, Computational Algebra, Algebraic Topology, Invariant Theory, Number Theory, Non-Commutative Algebra, and Combinatorics. In this paper, we are interested in a more refined version, the graded Betti numbers, which takes into account the grading. For each i, we write

$$F_i = \bigoplus_{p \in \mathbb{Z}} S(-p)^{\beta_{i,p}}\,,$$

where $S(-p)$ is the rank-one graded free S-module generated in degree p. The ranks $\beta_{i,p}$ are called the *graded Betti numbers* of N and denoted by $\beta_{i,p}(N)$. The *Betti table* $\beta(N)$ has entry $\beta_{i,i+j} = \beta_{i,i+j}(N)$ in position i, j; its columns are indexed from left to right by homological degree starting with homological degree zero; its rows are indexed increasingly from top to bottom starting with degree zero. Thus, the Betti table has the form:

$$(\beta_{i,i+j}) = \begin{array}{r|cccc} & 0 & 1 & 2 & \cdots \\ \hline 0: & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots \\ 1: & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots \\ 2: & \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \cdots \\ 3: & \beta_{0,3} & \beta_{1,4} & \beta_{2,5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array} \tag{2.13}$$

In Example 2.6, the Betti table is

$$\begin{array}{r|cc} & 0 & 1 \\ \hline 0: & - & - \\ 1: & - & - \\ 2: & 3 & 2 \end{array}\,,$$

where "-" stands for zero.

The size of a Betti table is given by the projective dimension and the regularity: The *projective dimension*

$$\operatorname{pd}_S(N) = \max\left\{ i \,\middle|\, \beta_i^S(N) \neq 0 \right\}$$

is the index of the last non-zero column of the Betti table $\beta(N)$, and thus it measures its width. The height of the table is measured by the index of the last non-zero row, and is called the *(Castelnuovo-Mumford) regularity* of N, which is

$$\operatorname{reg}_S(N) = \max\left\{ j \;\middle|\; \text{there exists an } i \text{ such that } \beta^S_{i,\,i+j}(N) \neq 0 \right\}.$$

Furthermore, we say that N is r-regular for every $r \geq \operatorname{reg}(N)$. We often omit the subscripts, writing $\operatorname{pd}(N)$ and $\operatorname{reg}(N)$ instead, when the ring in question is clear. Note that

$$\operatorname{reg}(N) \geq \operatorname{gensdeg}(N)\,,$$

where $\operatorname{gensdeg}(N)$ is the maximal degree of an element in a minimal system of homogeneous generators of N, which we call the *generating degree*. In Example 2.6, $\operatorname{pd}(N) = 1$ and $\operatorname{reg}(N) = \operatorname{gensdeg}(N) = 2$.

Both projective dimension and regularity are very important and well-studied invariants. In the following sections, we discuss upper bounds for them.

One immediate consequence of the above definition of regularity is that it pinpoints how high we have to truncate an ideal in order to get a linear resolution. We say that a graded ideal has an *r-linear resolution* if the ideal is generated in degree r and the entries in the differential maps in its minimal free resolution are linear.

Theorem 2.14. (see for example, [Pe, Theorem 19.7]) *Let N be a graded ideal in S. If $r \geq \operatorname{reg}(N)$ then $N_{\geq r} := N \cap \left(\bigoplus_{i \geq r} S_i\right)$ has an r-linear minimal free resolution, equivalently,* $\operatorname{reg}(N_{\geq r}) = r$.

3. Doubly Exponential Regularity

In this section we discuss upper bounds on the projective dimension and regularity of a graded ideal in the standard graded polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$ that require no additional constraints on the ideal. Hilbert's Syzygy Theorem provides a nice upper bound on the projective dimension:

Theorem 3.1. (see for example, [Ei3, Corollary 19.7], [Pe, Theorem 15.2]) *Let N be a graded ideal in S. Then*

$$\operatorname{pd}(N) < n.$$

In contrast, the regularity bound is doubly exponential. Theorem 3.2 is proved by Bayer-Mumford [BM] using results of Giusti [Giu] and Galligo [Ga] over fields of characteristic 0, and by Caviglia-Sbarra [CS] in any characteristic.

Theorem 3.2. (Bayer-Mumford [BM], Caviglia-Sbarra [CS]) *Let N be a graded ideal in S. Then*

$$\operatorname{reg}(N) \leq \left(2\operatorname{gensdeg}(N)\right)^{2^{n-2}}.$$

This bound is nearly the best possible, due to examples based on the Mayr-Meyer construction [MM2]. There are several versions of such examples, listed in Theorems 3.3, 3.4, and 3.5. These examples make use of the fact that in order to produce high regularity it suffices to focus on the first step in the minimal resolution: the submodule

$$\mathrm{Syz}_1^S(N) := \mathrm{Im}(\partial_1) = \mathrm{Ker}(\partial_0) \cong \mathrm{Coker}(\partial_2)$$

of F_0 (in the notation of Definition 2.3), which is called the first *syzygy module* of N. We denote by $\mathrm{syzdeg}(N)$ the maximal degree of an element in a minimal system of homogeneous generators of $\mathrm{Syz}_1^S(N)$. Thus,

$$\mathrm{syzdeg}(N) = \max \left\{ j \,\middle|\, \beta_{1,j}^S(N) \neq 0 \right\} ,$$

and hence

$$\mathrm{reg}(N) \geq \mathrm{syzdeg}(N) - 1 .$$

Theorem 3.3. (Bayer-Stillman [BS, Theorem 2.6]) *For $r \geq 1$, there exists a homogeneous ideal I_r (using $d = 3$ in their notation) in a polynomial ring with $10r + 11$ variables for which*

$$\begin{aligned} \mathrm{gensdeg}(I_r) &= 5 \\ \mathrm{syzdeg}(I_r) &\geq 3^{2^{r-1}} . \end{aligned}$$

Theorem 3.4. (Bayer-Mumford, [BM, Proposition 3.11]) *For $r \geq 1$, there exists a homogeneous ideal I_r in $10r + 1$ variables for which*

$$\begin{aligned} \mathrm{gensdeg}(I_r) &= 4 \\ \mathrm{reg}(I_r) &\geq 2^{2^{r}} . \end{aligned}$$

Koh [Ko] achieved examples of high regularity of ideals generated by quadrics and one linear form. Applying Theorem 10.1 to Koh's examples we get:

Theorem 3.5. (Koh) *For $r \geq 1$, there exists a homogeneous I_r generated by $22r - 2$ quadrics in a polynomial ring with $22r$ variables for which*

$$\begin{aligned} \mathrm{gensdeg}(I_r) &= 2 \\ \mathrm{syzdeg}(I_r) &\geq 2^{2^{r-1}} . \end{aligned}$$

More examples of ideals with high regularity have been constructed by Beder et. al. [BMN], Caviglia [Ca], Chardin-Fall [CF], and Ullery [Ul].

4. **Regularity Conjecture for Prime Ideals**

In contrast to the doubly exponential regularity examples in Section 3, much better bounds on regularity are expected for geometrically nice ideals, for example for prime ideals. Consider a homogeneous prime ideal L in a standard graded polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$. First, one has to exclude some degenerate cases: the condition that $L \subset (x_1, \ldots, x_n)^2$ is equivalent to requiring that $X := V(L)$ does not lie on a hyperplane in $\mathbb{P}^{n-1}$; prime ideals that satisfy this condition are called *non-degenerate*. If L is non-degenerate, then

$$\deg(L) \geq 1 + \operatorname{codim}(L)\,, \tag{4.1}$$

(see for example, [EG, p. 112]), where $\deg(L)$ is the multiplicity of S/L (also called the degree of S/L, or the degree of X), and $\operatorname{codim}(L)$ is the codimension (also called height) of L. Geometrically $\deg(L)$ counts the number of points of intersection of the projective variety defined by L with a general linear space of complementary dimension. For a hypersurface defined by an irreducible form of degree c, the multiplicity $\deg(L)$ is precisely c, but for non-principal ideals L computation of $\deg(L)$ is not easy. Some bounds on regularity using two parameters (for example, using multiplicity and codimension) are discussed in Section 5. In view of (4.1), we can look for a bound in terms of a single parameter – the multiplicity – instead of using both multiplicity and codimension. The following elegant bound was conjectured by Eisenbud-Goto [EG], motivated by results in Algebraic Geometry (see Section 5), and has been very challenging:

Regularity Conjecture for Prime Ideals 4.2. (Eisenbud-Goto [EG], 1984) *If L is a homogeneous prime ideal in a standard graded polynomial ring over $\mathbb{C}$, then*

$$\operatorname{reg}(L) \;\leq\; \deg(L)\,. \tag{4.3}$$

Examples have shown that the hypotheses in the Regularity Conjecture cannot be weakened much. The hypothesis that we work over an algebraically closed field is necessary by [Ei2, Section 5C, Exercise 4]. The regularity of a reduced equidimensional ideal cannot be bounded by its degree by [EU, Example 3.1]. Furthermore, there is no bound on the regularity of non-reduced homogeneous ideals in terms of multiplicity, even for a fixed codimension, by [Ei, Example 3.11].

The Regularity Conjecture implies the following weaker conjecture, which provides a bound on the generating degree:

Conjecture 4.4. *If L is a homogeneous prime ideal in a standard graded polynomial ring over $\mathbb{C}$, then*

$$\operatorname{gensdeg}(L) \leq \deg(L),$$

that is, L is generated in degrees $\leq$ $\deg(L)$.

In order to make the Regularity Conjecture sharp for non-degenerate primes, the original form of the conjecture uses the following more refined version of inequality (4.3):

$$\text{reg}(L) \ \leq \ \deg(L) - \text{codim}(L) + 1 \,. \tag{4.5}$$

For instance, the rational normal curve that is the image of $\mathbb{P}^1 \longrightarrow \mathbb{P}^r$ defined by

$$[s:t] \mapsto [s^r : s^{r-1}t : \cdots : t^r] \,,$$

is defined by the ideal $I_2(M)$ generated by the (2×2)-minors of the $(2\times r)$-matrix

$$M = \begin{pmatrix} x_0 & x_1 & \dots & x_{r-1} \\ x_1 & x_2 & \dots & x_r \end{pmatrix} .$$

The ideal $I_2(M)$ has a linear free resolution. It follows that

$$\text{reg}(I_2(M)) = 2 = r - (r-1) + 1 = \deg(I_2(M)) - \text{codim}(I_2(M)) + 1 \,.$$

See [Ei2, Section 6A.1].

Eisenbud and Goto proved that the sharp inequality (4.5) holds in the Cohen-Macaulay case:

Theorem 4.6. (Eisenbud-Goto, [EG]) *The Regularity Conjecture 4.2 (in the sharp version (4.5)) holds if the prime ideal L is Cohen-Macaulay.*

5. Regularity Bounds and the Regularity Conjecture in Algebraic Geometry

Lazarsfeld's book [La2, Section 1.8] provides an overview of this topic; see also the introduction of the paper [KP] by Kwak-Park.

The concept of regularity was introduced by Mumford [Mu] and generalizes ideas of Castelnuovo. For an integer r, we say that a coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}^{n-1}$ is r-regular if $H^i(\mathbb{P}^{n-1}, \mathcal{F}(r-i)) = 0$ for all $i > 0$. The *regularity* of $\mathcal{F}$ is the least integer r such that $\mathcal{F}$ is r-regular, or $-\infty$ (if $\mathcal{F}$ is supported on a finite set). The *regularity* of a subscheme $X \subseteq \mathbb{P}^{n-1}$ is the regularity of its ideal sheaf $\mathcal{I}_X$, so

$$\text{reg}(X) := \text{reg}(\mathcal{I}_X) \,.$$

We give two simple examples: If X consists of k distinct reduced points then $\text{reg}(X) \leq k$. If $X \subseteq \mathbb{P}^{n-1}$ is a complete intersection of hypersurfaces of degrees $d_1, \dots, d_k$ then $\text{reg}(X) = (d_1 + \cdots + d_k - k + 1)$, which follows from Koszul's resolution.

The relation between the definitions of regularity of a coherent sheaf and regularity of a graded ideal (or module) is given in [Ei2, Proposition 4.16] and is due to Eisenbud-Goto [EG].

In this section, we outline some of the main results related to the Regularity Conjecture. In sharp contrast to the doubly exponential regularity behavior described in Section 3, a much better bound is expected for a geometrically nice projective variety $X \subset \mathbb{P}^{n-1}$. Recall from Section 4 that we say that X is *non-degenerate* if it does not lie on a hyperplane in $\mathbb{P}^{n-1}$.

Some of the best known bounds on regularity hold in the case when X is smooth. The following bound follows from a more general result by Bertram-Ein-Lazarsfeld [BEL]:

Theorem 5.1. (Bertram-Ein-Lazarsfeld, [BEL]) *Let $X \subset \mathbb{P}^{n-1}$ be a smooth irreducible projective variety. If X is cut out scheme-theoretically by hypersurfaces of degree $\leq s$, then*

$$\operatorname{reg}(X) \leq 1 + (s-1)\operatorname{codim}(X)\,.$$

For generalizations of this result, see [CU] and [DE]. See also [Ch2] for an overview. Furthermore, Mumford proved:

Theorem 5.2. (Mumford [BM, Theorem 3.12]) *If $X \subset \mathbb{P}^{n-1}$ is a non-degenerate smooth projective variety, then*

$$\operatorname{reg}(X) \leq (\dim(X)+1)(\deg(X)-2)+2\,.$$

The bound was improved by Kwak-Park:

Theorem 5.3. (Kwak-Park [KP, Theorem C]) *If $X \subset \mathbb{P}^{n-1}$ is a non-degenerate smooth projective variety with* $\operatorname{codim}(X) \geq 2$*, then*

$$\operatorname{reg}(X) \leq \dim(X)(\deg(X)-2)+1\,.$$

In [BM], Bayer and Mumford pointed out that *the main missing piece of information between the general case and the geometrically nice smooth case is that we do not have yet a reasonable bound on the regularity of all reduced equidimensional ideals*. Consider a non-degenerate subvariety $X \subset \mathbb{P}^{n-1}$. The following inequality was first considered in the smooth case:

$$\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 1\,. \tag{5.4}$$

It was conjectured by Eisenbud-Goto [EG] for any reduced and irreducible non-degenerate variety, and they expected that it might even hold for reduced equidimensional X which are connected in codimension 1 [BM]. This is called the Regularity Conjecture. The conjecture is different in flavor than the bounds in Theorems 5.1, 5.2, and 5.3: the bounds in these theorems are not linear in the degree (or the degree of the defining equations) since there is a coefficient involving the dimension or codimension.

The Regularity Conjecture was proved in several important cases. Castelnuovo [Cas] carried out fundamental work in this direction for smooth space curves; the case for curves was settled by Gruson-Lazarsfeld-Peskine:

Theorem 5.5. (Gruson-Lazarsfeld-Peskine [GLP]) *The Regularity Conjecture, i.e. inequality* (5.4)*, holds if* $X \subset \mathbb{P}^{n-1}$ *is a non-degenerate irreducible reduced curve.*

Lazarsfeld, using work of Pinkhum, proved the conjecture for smooth surfaces:

Theorem 5.6. (Lazarsfeld [La], Pinkham [Pi]) *The Regularity Conjecture, i.e. inequality* (5.4)*, holds for smooth irreducible non-degenerate projective surfaces.*

The case of mildly singular surfaces was considered by Niu [Ni]. The Regularity Conjecture was studied for smooth 3-folds by Ran [Ra].

Theorem 5.7. (Kwak [Kw, Kw2]) *Let* $X \subset \mathbb{P}^{n-1}$ *be a smooth irreducible projective non-degenerate threefold. The bound*

$$\mathrm{reg}(X) \leq \deg(X) - \mathrm{codim}(X) + 2$$

holds.

For smooth 3-folds, Kwak [Kw, Kw2] also proved other bounds. Regularity of 3-folds with rational singularities were studied by Niu-Park [NP]. Furthermore, nice bounds for lower dimensional smooth varieties were obtained by Kwak [Kw3].

Many other special cases of the Regularity Conjecture and also similar bounds in special cases are proved, for example, by Brodmann [Br], Brodmann-Vogel [BV], Derksen-Sidman [DS], Eisenbud-Ulrich [EU], Giaimo [Gia], Herzog-Hibi [HH], Hoa-Miyazaki [HM], Niu [Ni], Niu-Park [NP], Peeva-Sturmfels [PS2], and Stückrad-Vogel [SV], and several other authors.

In order to prove inequality (5.4), it is sufficient to show that the following two properties hold: X is $\big(\deg(X) - \mathrm{codim}(X)\big)$-normal and $\mathcal{O}_X$ is $\big(\deg(X) - \mathrm{codim}(X)\big)$-regular. Recent progress in the smooth case was made by Noma:

Theorem 5.8. (Noma [No, Corollary 5]) *Let* $X \subset \mathbb{P}^{n-1}$ *be a non-degenerate smooth projective variety of* $\dim(X) \geq 2$*. Then* $\mathcal{O}_X$ *is* $\big(\deg(X) - \mathrm{codim}(X)\big)$*-regular if* X *is not projectively equivalent to any scroll over a smooth projective curve.*

Furthermore, Kwak-Park [KP, Theorem B] obtained the following result; they also classified the extremal and nearly extremal cases. Their result reduces the Regularity Conjecture for the smooth case to the problem of finding a Castelnuovo-type bound for normality [KP].

Theorem 5.9. (Kwak-Park [KP, Theorem B]) *Let* $X \subset \mathbb{P}^{n-1}$ *be a non-degenerate smooth projective variety. Then* $\mathcal{O}_X$ *is* $\big(\deg(X) - \mathrm{codim}(X)\big)$*-regular.*

6. Homogenization

In this section we describe two versions of a new method for homogenizing (with respect to the standard grading in a polynomial ring) prime ideals that are homogeneous with respect to a non-standard grading. Such prime ideals appear for example as defining ideals of Rees algebras, Rees-like algebras, and toric rings. The first method, called Step-by-step homogenization was introduced in [MP, Section 4] and has the advantage that it needs fewer new variables. The second method, called Prime Standardization, was introduced in [MMM]. It needs more variables, but has the advantage that the codimension of the singular locus is preserved and that it is not limited to non-degenerate primes. Both versions have the key property that they preserve Betti numbers, in contrast to traditional homogenization (taking projective closure).

6.1. Traditional homogenization (taking projective closure).
We briefly review the construction of traditional homogenization used for taking the projective closure of an affine variety. Consider the map

$$Y = \mathbb{C}[y_1, \dots, y_p] \longrightarrow Y' := Y[y_0] = \mathbb{C}[y_0, y_1, \dots, y_p]$$

that sends a polynomial $f = \sum_i c_i m_i$, written as a $\mathbb{C}$-linear combination of monomials m_i with non-zero coefficients $c_i \in \mathbb{C}$, to

$$f' := \sum_i c_i m_i y_0^{\deg(f) - \deg(m_i)},$$

called the *homogenization* of f. The *homogenization* I' of an ideal $I \subset Y$ is the ideal

$$I' = \big(\{f' \mid f \in I\}\big) \subset Y'.$$

Usually, the homogenizations of the polynomials in a minimal set of generators of I fail to generate I'. In order to generate I' we need to homogenize a Gröbner basis of I (see for example, [CLO, Theorem 8.4.4]). Suppose that the polynomial ring Y is positively graded (but non-standard graded) and I is homogeneous with respect to that grading. Then we have graded Betti numbers $\beta^Y_{ij}(I)$. Now, ignore the grading of I which comes from the grading of Y and homogenize I with respect to the standard grading. The ideal I' is homogeneous with respect to the standard grading of the ring Y'. However, the graded Betti numbers $\beta^{Y'}_{ij}(I')$ are usually not equal to $\beta^Y_{ij}(I)$ (see Example 6.7 below).

6.2. Step-by-step homogenization.
As discussed in 6.1 above, traditional homogenization requires homogenizing a Gröbner basis. In contrast, for Step-by-step homogenization, we only need to homogenize a set of generators of the prime ideal. This makes it much easier to apply

(or compute). However, it needs the assumption that the ideal is homogeneous with respect to some non-standard (weighted) grading.

Consider a polynomial ring $Y = \mathbb{C}[y_1, \ldots, y_p]$ positively graded with integer degrees $\deg(y_i) \geq 1$ for every i. Suppose $\deg(y_i) > 1$ for $i \leq q$ and $\deg(y_i) = 1$ for $i > q$ (for some $q \leq p$). Consider the homogenous map (of degree 0)

$$\begin{aligned} \nu : Y = \mathbb{C}[y_1, \ldots, y_p] &\longrightarrow Y' := \mathbb{C}[y_1, \ldots, y_p, v_1, \ldots, v_q] \\ y_i &\longmapsto y_i v_i^{\deg_Y(y_i)-1} \quad \text{for } 1 \leq i \leq q\,, \\ y_i &\longmapsto y_i \quad \text{for } i > q\,, \end{aligned}$$

where $v_1, \ldots, v_q$ are new variables and the new polynomial ring Y' is standard graded. Given a homogeneous element $g \in Y$, the map ν homogenizes it with respect to the standard grading so that $g' := \nu(g) \in Y'$ has the same degree in Y' as g has in Y. This construction is called *Step-by-step-homogenization* (since one can homogenize one step at a time lowering the degree of one variable to 1). For example, if $Y = \mathbb{C}[x, y, z]$ with $\deg(x) = 3$, $\deg(y) = 2$, $\deg(z) = 1$ and $g = xyz - 3y^3$, which has degree 6, then g' is obtained by replacing x by xv_1^2 and replacing y by yv_2, so $g' = (xv_1^2)(yv_2)z - 3(yv_2)^3$ is a homogeneous degree 6 polynomial in the standard graded polynomial ring $Y = \mathbb{C}[x, y, z, v_1, v_2]$.

Theorem 6.3. ([MP, Theorem 4.5]) *Let M be a homogeneous non-degenerate prime ideal, and let* $\operatorname{gens}(M)$ *be a minimal set of homogeneous generators of M. The ideal $M' \subset Y'$ generated by the elements* $\nu(\operatorname{gens}(M))$ *is a homogeneous non-degenerate prime ideal in Y'. Furthermore, the graded Betti numbers of M' over Y' are the same as those of M over Y.*

We say that the ideal M' is the *Step-by-step-homogenization* of M.

6.4. Prime Standardization.

An alternate method to make standard graded homogenizations of non-standard graded prime ideals is introduced in [MMM] by Mantero-McCullough-Miller. The idea is based on the notion of a prime sequence introduced by Ananyan and Hochster [AH1]. A sequence of elements $g_1, \ldots, g_t \in S = \mathbb{C}[x_1, \ldots, x_n]$ is a *prime sequence* provided $(g_1, \ldots, g_t)$ is a proper ideal and $S/(g_1, \ldots, g_i)$ is a domain for all $1 \leq i \leq t$. Clearly any prime sequence is a regular sequence. Conversely, if $g_1, \ldots, g_t$ is a homogeneous regular sequence such that $S/(g_1, \ldots, g_i)$ is a domain for all $1 \leq i \leq t$, then $g_1, \ldots, g_t$ is a prime sequence and thus so is any permutation of $g_1, \ldots, g_t$. The usefulness of this idea is contained in the following result:

Proposition 6.5. (Ananyan and Hochster [AH1, Cor. 2.9, Prop. 2.10]) *Suppose $g_1, \ldots, g_t$ is a homogeneous prime sequence in S and set $R = \mathbb{C}[g_1, \ldots, g_t]$. Let $I \subset R$ be a homogeneous ideal.*

(1) *The ideals I and IS have the same graded Betti numbers.*

(2) *If I is prime, then IS is prime.*

Part (1) holds since the inclusion map $R \longrightarrow S$ is flat. However, if P is prime and ϕ is a flat map, it is not always the case that $\phi(P)$ is prime. (Consider e.g. $\mathbb{C}[x^2] \hookrightarrow \mathbb{C}[x]$ and the ideal (x^2).)

The notion of a prime sequence naturally leads to a way to homogenize non-standard graded prime ideals, which we call *Prime Standardization*. One merely replaces each variable of degree d by a very general form of degree d in many new variables such that the chosen forms define a prime sequence. The following choice of forms works in general. Let $Y = \mathbb{C}[y_1, \ldots, y_n]$ be a positively graded polynomial ring with integer $\deg(y_i) \geq 1$. Consider the standard graded polynomial ring

$$Y' := \mathbb{C}[t_{i,j,\ell} \,|\, 1 \leq i \leq n, 0 \leq j \leq n, 1 \leq \ell \leq \deg(y_i)]$$

and let

$$F_i = \sum_{j=0}^{n} \prod_{\ell=1}^{\deg(y_i)} t_{i,j,\ell} \in Y'.$$

Define the graded map of rings $\zeta : \ Y \longrightarrow Y'$ by setting $\zeta(y_i) = F_i$.

Theorem 6.6. (Mantero-McCullough-Miller [MMM, Proposition 3.3]) *The elements $F_1, \ldots, F_n$ form a prime sequence. For any homogeneous prime ideal $M \subseteq Y$, the ideal $\zeta(M)$ is prime, standard graded, and has the same graded Betti numbers as M.*

This method has the advantage of preserving the codimension of the singular locus of the ideal; of course, just like with Step-by-step homogenization, the many new variables mean the dimension of the associated variety grows considerably, but one may take hyperplane sections and appeal to Bertini's theorem [Fl] to compensate for this. An example where the singular locus of the Step-by-step homogenization differs from that of Prime Standardization is provided in [MMM, Example 3.2].

Example 6.7. We will illustrate how Step-by-step homogenization and Prime standardization work and make a comparison to traditional homogenization (taking projective closure).

We consider the defining ideal of the affine monomial curve parametrized by (t, t^2, t^3). It is the prime ideal

$$E = (x^2 - y, \ xy - z)$$

which is the kernel of the map

$$\begin{aligned} Y := \mathbb{C}[x,y,z] &\longrightarrow \mathbb{C}[t] \\ x &\longmapsto t \\ y &\longmapsto t^2 \\ z &\longmapsto t^3\,. \end{aligned}$$

It is homogeneous with respect to the non-standard grading defined by

$$\deg(x) = 1,\ \deg(y) = 2,\ \deg(z) = 3\,.$$

The non-zero graded Betti numbers of E over Y are

$$\beta_{0,2} = 1,\ \beta_{0,3} = 1,\ \beta_{1,5} = 1$$

and thus $\operatorname{reg}(E) = 4$.

The traditional homogenization (that is, taking projective closure) of E is obtained by homogenizing a Gröbner basis. The generators $x^2 - y,\ xy - z$ and the element $xz - y^2$ form a minimal Gröbner basis with respect to the degree-lex order. Homogenizing them with a new variable w we obtain the homogeneous prime ideal

$$E' = (x^2 - yw,\ xy - zw,\ xz - y^2)$$

in the ring $Y' = \mathbb{C}[x,y,z,w]$, which is standard graded. The ideal E' defines the projective closure of the affine variety $V(E)$ in $\mathbb{P}^3$. Note that:

(1) A Gröbner basis computation is needed in order to obtain the generators of E'.
(2) The non-zero Betti numbers of E' over Y' are $\beta'_{0,2} = 3,\ \beta'_{1,3} = 2$, and so they are different than those of E over Y. Moreover

$$\operatorname{reg}(E') = 2 < \operatorname{reg}(E) = 4.$$

The Step-by-step homogenization works by applying Theorem 6.3 to E. We replace the variable y by yu and replace the variable z by zv^2. Thus, we obtain the homogeneous prime ideal

$$E' = (x^2 - yu,\ xyu - zv^2)$$

in the ring $Y' = \mathbb{C}[x,y,z,u,v]$ which is standard graded (all variables have degree one). The graded Betti numbers of E' over Y' (and thus also the regularity) are the same as the graded Betti numbers of E over Y.

The Prime standardization works by applying Theorem 6.6 to E. The homogenized ideal E' is defined by replacing y by $\sum_{j=0}^{3}\prod_{\ell=1}^{2} t_{2,j,\ell}$ and replacing z by $\sum_{j=0}^{3}\prod_{\ell=1}^{3} t_{3,j,\ell}$. We need not replace x since it has degree 1 already. Thus, E' is generated by the following two elements:

$$x^2 - (t_{2,0,1}t_{2,0,2} + t_{2,1,1}t_{2,1,2} + t_{2,2,1}t_{2,2,2} + t_{2,3,1}t_{2,3,2}),$$
$$x(t_{2,0,1}t_{2,0,2} + t_{2,1,1}t_{2,1,2} + t_{2,2,1}t_{2,2,2} + t_{2,3,1}t_{2,3,2})$$
$$- (t_{3,0,1}t_{3,0,2}t_{3,0,3} + t_{3,1,1}t_{3,1,2}t_{3,1,3} + t_{3,2,1}t_{3,2,2}t_{3,2,3} + t_{3,3,1}t_{3,3,2}t_{3,3,3}).$$

See [MMM, Example 3.8].

7. **Rees-like algebras**

In Section 3 we presented several examples of homogeneous ideals with high regularity. Our goal is to produce similar examples with prime ideals. For this, we need a method which, starting from a homogeneous ideal I, produces a prime ideal P whose regularity and multiplicity can be estimated. One way to produce such ideals is to consider Rees algebras, which have been well-studied in Algebraic Geometry and Commutative Algebra. However, their defining equations (let alone free resolutions) are difficult to find in general (see for example [Hu], [KPU]). Thus, the best we can hope to obtain for Rees algebras are bounds on these invariants. Section 9 is devoted to Rees algebras. We introduced in [MP] another concept, Rees-like algebras, which has the advantage that we can provide simple explicit formulas for the defining equations, projective dimension, regularity, gensdeg, multiplicity, dimension, depth, and codimension of P in terms of numerical invariants of I. The construction of Rees-like algebras was inspired by Hochster's example in [Bec] which, starting with a family of three-generated ideals in a regular local ring, produces prime ideals with fixed embedding dimension and Hilbert-Samuel multiplicity but arbitrarily many minimal generators.

Fix a polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$ with a standard grading defined by $\deg(x_i) = 1$ for every i. Let I be a homogeneous ideal minimally generated by forms $f_1, \dots, f_m$, where $m \geq 2$. We consider the prime ideal Q of defining equations of the *Rees-like algebra* $S[It, t^2]$. For this purpose, introduce a new polynomial ring $Y = S[y_1, \dots, y_m, z]$ graded by $\deg(z) = 2$ and $\deg(y_i) = \deg(f_i) + 1$ for every i. The ideal Q is the homogeneous prime ideal that is the kernel of the graded homomorphism

$$\begin{aligned} \varphi:\ Y &\longrightarrow S[It, t^2] \subset S[t] \\ y_i &\longmapsto f_i t \\ z &\longmapsto t^2\,, \end{aligned}$$

where t is a new variable of $\deg(t) = 1$. In contrast to the defining ideal of a Rees algebra, we can describe a set of generators of Q as follows. If $p \in \mathbb{Z}$, denote by $S(-p)$ the shifted free module for which $S(-p)_i = S_{i-p}$ for all i. The minimal graded presentation $(\mathbf{G}, d)$ of I has the form

$$\mathbf{G}: \quad G_1 \xrightarrow{d_1=(c_{ij})} G_0 \coloneqq S(-a_1) \oplus \cdots \oplus S(-a_m) \xrightarrow{d_0=(f_1 \ \cdots \ f_m)} I \,.$$

Denote by $\xi_1, \ldots, \xi_m$ a homogeneous basis of G_0 such that $d(\xi_i) = f_i$ for every i. Let $r = \operatorname{rank}(G_1)$, and fix a homogeneous basis $\mu_1, \ldots, \mu_r$ of G_1 that is mapped by the differential to a homogeneous minimal system of generators of $\operatorname{Ker}(d_0)$. Let $C = (c_{ij})$ be the matrix of the differential d_1 in these fixed homogeneous bases. Thus $\operatorname{Syz}_1(I) = \operatorname{Ker}(d_0)$ is the module generated by the elements

$$\left\{ \sum_{i=1}^{m} c_{ij}\xi_i \,\middle|\, 1 \le j \le r \right\} .$$

It is clear that the corresponding elements

$$\mathcal{G} \coloneqq \left\{ \sum_{i=1}^{m} c_{ij}y_i \,\middle|\, 1 \le j \le r \right\} \tag{7.1}$$

are in Q. We prove in [MP, Proposition 3.2 and Corollary 3.6] that the ideal Q is minimally generated by the elements

$$Q = \big(\mathcal{G} \cup \{ \, y_iy_j - zf_if_j \,|\, 1 \le i, j \le m \} \big) \,. \tag{7.2}$$

The prime ideal Q is non-degenerate, and so z is a non-zerodivisor on Y/Q. Set $\bar{Y} := Y/z$, and let $\bar{Q} \subset \bar{Y}$ be the homogeneous ideal (which is the image of Q) generated by

$$\bar{Q} = \big(\mathcal{G} \,, \{ \, y_iy_j \,|\, 1 \le i, j \le m \} \big) \,. \tag{7.3}$$

It follows that the graded Betti numbers of Q over Y are equal to those of $\bar{Q}$ over $\bar{Y}$. The minimal free resolution of $\bar{Q}$ is obtained in [MP, Theorem 3.10] using a mapping cone resolution, and we get formulas for its numerical invariants. We describe this resolution in the next section.

The ideal Q is homogeneous in the polynomial ring Y, which is not standard graded. Our goal is to construct a prime ideal in a standard graded ring. We change the degrees of the variables to 1 and homogenize the ideal Q by applying the Step-by-step homogenization technique described in Section 6. This yields a homogeneous, non-degenerate prime ideal P in a standard graded polynomial ring. Since Step-by-step homogenization (or Prime Homogenization) preserves graded Betti numbers, the formulas for the numerical invariants of Q yield formulas for the numerical invariants of P.

Theorem 7.4. [MP, Theorem 1.6] *Let I be an ideal generated minimally by homogeneous elements $f_1, \ldots, f_m$ (with $m \ge 2$) in the standard graded polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$. Consider the standard graded polynomial ring*

$$R = S[y_1, \ldots, y_m, u_1, \ldots, u_m, z, v]$$

with $n+2m+2$ variables. Let P be the ideal in R generated by the Step-by-step homogenizations of the minimal generators of Q listed in (7.1) and (7.2), namely P is generated by:

$$\left\{ y_i y_j u_i^{\deg(f_i)} u_j^{\deg(f_j)} - zv f_i f_j \,\middle|\, 1 \le i, j \le m \right\} \tag{7.5}$$

and

$$\left\{ \sum_{i=1}^{m} c_{ij} y_i u_i^{\deg(f_i)} \,\middle|\, 1 \le j \le r \right\}. \tag{7.6}$$

The ideal P is homogeneous, prime, and non-degenerate. Furthermore:

(1) *The above system of generators is minimal.*

(2) *The maximal degree of a minimal generator of P is:*

$$\operatorname{gensdeg}(P) = \max\left\{ 1 + \operatorname{syzdeg}(I),\ 2\operatorname{gensdeg}(I) + 2 \right\}.$$

Note that $\operatorname{gensdeg}(I) = \max\{\deg(f_i) | 1 \le i \le m\}$.

(3) *The multiplicity of R/P is*

$$\deg(P) = 2\prod_{i=1}^{m}\left(\deg(f_i) + 1\right).$$

(4) *The Castelnuovo-Mumford regularity and the projective dimension of P are:*

$$\operatorname{reg}(P) = \operatorname{reg}(I) + 2 + \sum_{i=1}^{m}\deg(f_i)$$
$$\operatorname{pd}(P) = \operatorname{pd}(I) + m - 1.$$

(5) *The depth, the codimension, and the dimension of R/P are:*

$$\operatorname{depth}(R/P) = \operatorname{depth}(S/I) + m + 3$$
$$\operatorname{codim}(P) = m$$
$$\dim(R/P) = m + n + 2.$$

The key and striking property of the construction of the ideal P is that it has a nicely structured minimal free resolution, which makes it possible to express its regularity, multiplicity, and other invariants in terms of invariants of I.

Example 7.7. We illustrate the constructions and results above. Consider the ideal $I = (x_1, x_2) \subset S = \mathbb{C}[x_1, x_2]$. Let $Y = S[y_1, y_2, z]$. Then the Rees-like algebra $S[It, t^2]$ of I is isomorphic to Y/Q where

$$Q = (y_1x_2 - y_2x_1,\ y_1^2 - x_1^2 z,\ y_1y_2 - x_1x_2z,\ y_2^2 - x_2^2 z).$$

The Step-by-step homogenization $P \subseteq R$ of Q is

$$P = (y_1u_1x_2 - y_2u_2x_1,\ y_1^2u_1^2 - x_1^2zv,\ y_1u_1y_2u_2 - x_1x_2zv,\ y_2^2u_2^2 - x_2^2zv).$$

Then P is a homogeneous prime ideal generated by 1 cubic and 3 quartics in a standard graded polynomial ring over $\mathbb{C}$ with

$$\begin{aligned}
\deg(P) &= 2\prod_{i=1}^{m}(\deg(x_i)+1) = 2^3 = 8\\
\operatorname{reg}(P) &= \operatorname{reg}(I) + 2 + \sum_{i=1}^{2}\deg(x_i) = 1+2+1+1 = 5\\
\operatorname{pd}(P) &= \operatorname{pd}(I) + m - 1 = 1+2-1 = 2\\
\operatorname{depth}(R/P) &= \operatorname{depth}(S/I) + m + 3 = 0+2+3 = 5\\
\operatorname{codim}(P) &= m = 2\\
\dim(R/P) &= m+n+2 = 2+2+2 = 6.
\end{aligned}$$

In particular, R/P is not Cohen-Macaulay, as is the case with any Rees-like algebra when $m \geq 2$. It is easy to check that R/P is not normal either. The projective variety $V(P) \subseteq \mathbb{P}^7$ is 5-dimensional.

8. The minimal free resolution for a Rees-like algebra modulo a non-zerodivisor

Notation 8.1. We adopt the following conventions for shifting: If U is a graded module, denote by $U(-1)$ the shifted module for which $U(-1)_i = U_{i-1}$ for all i; thus, we consider the shift that increases the internal degree by 1. If $(\mathbf{V}, d)$ is a complex, we write $\mathbf{V}[1]$ for the shifted complex with $\mathbf{V}[1]_i = \mathbf{V}_{i-1}$ and differential $(-1)^p d$; thus, we shift the complex one step higher in homological degree.

In this section we describe the construction of the minimal graded free resolution of $\bar{Y}/\bar{Q}$ over $\bar{Y}$, in the notation of the previous section. We will follow [MP, Construction 3.8] which uses a mapping cone. In view of (7.3) we consider the ideals

$$\begin{aligned}
M &:= (\mathcal{G})\\
N &:= (y_1, \dots, y_m)^2\,,
\end{aligned}$$

so $\bar{Q} = M + N$. There is a short exact sequence

$$0 \longrightarrow M/(M\cap N) \xrightarrow{\gamma} \bar{Y}/N \longrightarrow \bar{Y}/(M+N) = \bar{Y}/\bar{Q} \longrightarrow 0\,,$$

where γ is the homogeneous map (of degree 0) induced by $M \subset \bar{Y}$. Let $(\mathbf{B}, d^B)$ and $(\mathbf{G}, d^G)$ be the graded minimal free resolutions of $M/(M\cap N)$ and $\bar{Y}/N$, respectively. Let $\zeta : \mathbf{B} \longrightarrow \mathbf{G}$ be a homogeneous lifting of γ. Its mapping cone $\mathbf{D}$ is a graded free resolution of $\bar{Y}/\bar{Q}$ over $\bar{Y}$. It is a complex with modules

$D_q = G_q \oplus B_{q-1}$. Thus, as a bigraded (graded by homological degree and by internal degree) module

$$\mathbf{D} = \mathbf{G} \oplus \mathbf{B}[1]\,.$$

The resolution $\mathbf{G}$ may be expressed as $\bar{Y} \otimes \mathbf{G}'$, where $\mathbf{G}'$ is the Eliahou-Kervaire resolution (or the Eagon-Northcott resolution) that resolves minimally the module $\mathbb{C}[y_1, \ldots, y_m]/(y_1, \ldots, y_m)^2$ over the polynomial ring $\mathbb{C}[y_1, \ldots, y_m]$. Furthermore,

$$\mathbf{B} = \mathbf{K}_{\bar{Y}}(y_1, \ldots, y_m) \otimes_{\bar{Y}} \big(\mathbf{F}(-1) \otimes_S \bar{Y}\,\big)\,,$$

where

- $\mathbf{K}_{\bar{Y}}(y_1, \ldots, y_m)$ is the Koszul complex on $y_1, \ldots, y_m$ over $\bar{Y}$.
- $\mathbf{F}$ is the minimal S-free resolution of $\mathrm{Syz}_1^S(I)$.

Formulas for the differentials in this construction are given in [MP, Section 3]. The following theorem shows that the construction provides the desired minimal free resolution:

Theorem 8.3. (McCullough-Peeva [MP, Theorem 3.10]) *Use the notation above. The graded minimal $\bar{Y}$-free resolution of $\bar{Y}/\bar{Q}$ can be described as a bigraded (graded by homological degree and by internal degree) module by*

$$\mathbf{D} = \big(\bar{Y} \otimes \mathbf{G}'\big) \;\oplus\; \big(\mathbf{K}_{\bar{Y}}(y_1, \ldots, y_m) \otimes \mathbf{F}(-1)\big)[1]\,,$$

where $[1]$ *stands for shifting one step higher in homological degree, and* (-1) *stands for the shift that increases the internal degree by* 1*.*

9. Rees algebras

Rees algebras are of high interest in Commutative Algebra and Algebraic Geometry because of their geometric properties, see for example [Cu, Theorem 6.4] for the relation to blow-ups. Fix a polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$ with a standard grading defined by $\deg(x_i) = 1$ for every i. Let I be a homogeneous ideal minimally generated by forms $f_1, \ldots, f_m$, where $m \geq 2$. We consider the prime ideal W of defining equations of the *Rees algebra* $S[It]$. For this purpose, introduce a new polynomial ring $V = S[y_1, \ldots, y_m]$ graded by $\deg(y_i) = \deg(f_i) + 1$ for every i. The ideal W is the homogeneous prime ideal that is the kernel of the graded homomorphism

$$\begin{aligned} \varphi:\; V = S[y_1, \ldots, y_m] \;&\longrightarrow S[It] \subset S[t] \\ y_i &\longmapsto f_i t\,, \end{aligned}$$

where t is a new variable and $\deg(t) = 1$.

The following result provides bounds for the regularity and multiplicity of the defining ideal of a Rees algebra by comparing it to the corresponding Rees-like algebra:

Theorem 9.1. *Consider the standard graded polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$. Let I be a homogeneous ideal minimally generated by forms $f_1, \ldots, f_m$, where $m \geq 2$. Consider the Rees algebra $S[It]$ and the Rees-like algebra $S[It, t^2]$. Denote by Q and W the defining ideals of $S[It, t^2]$ and $S[It]$ respectively. Let U and P be the respective Step-by-step homogenizations of W and Q.*

(1) [CCMPV, Theorem 3.2] *The degree of the Rees algebra satisfies*

$$\deg(U) \leq \frac{1}{2}\deg(P) = \prod_{i=1}^{m} \left(\deg(f_i) + 1\right).$$

(2) *The regularity of the Rees algebra satisfies*

$$\operatorname{reg}(U) \geq \operatorname{gensdeg}(U) \geq \operatorname{syzdeg}(I).$$

Theorem 9.1(2) follows from Theorem 7.4(2) and the fact that the generators $\mathcal{G}$ in (7.1) are always minimal generators of W. We remark that it is usually very difficult to determine what other elements are needed to generate W.

10. Rees algebras of ideals generated in one degree

In this section we outline a different approach which has been used in the study of Rees algebras. If M is an ideal generated by $m \geq 1$ forms of the same degree $d \geq 2$ in S, then the Rees algebra $S[Mt]$ can be considered as a standard graded quotient of the polynomial ring $V = S[y_1, \ldots, y_m]$. In this case, we have the following bounds on degree and regularity:

Theorem 10.1. *Let M be an ideal generated by $m \geq 1$ forms of the same degree $d \geq 2$ in $S = \mathbb{C}[x_1, \ldots, x_n]$, and W be the defining ideal of the Rees algebra $S[Mt]$, which is considered as a standard graded quotient of the polynomial ring $S[y_1, \ldots, y_m]$.*

(1) [CCMPV, Theorem 4.3] *The degree of the Rees algebra satisfies*

$$\deg(W) \leq \frac{d^{\min\{m,n\}} - 1}{d-1}.$$

(2) *The regularity of the Rees algebra satisfies*

$$\operatorname{reg}(W) \geq \operatorname{gensdeg}(W) \geq \operatorname{syzdeg}(M) - (d-1).$$

Theorem 10.1(2) follows from the fact that the elements in (7.1) are contained in a minimal system of generators of the ideal W and that we have $\deg(y_i) = 1 = \operatorname{gensdeg}(I) - (d-1)$ for every i.

In order to apply the above theorem, we use the following construction which replaces a homogeneous ideal I in S by an ideal generated in one degree:

Theorem 10.2. [CCMPV, Construction 4.1] *Let I be a homogeneous ideal in S minimally generated by forms $f_1, \ldots, f_m$, where $m \geq 2$. Set*

$$d = \operatorname{gensdeg}(I) = \max\{\deg(f_i) \mid 1 \leq i \leq m\}\,.$$

Consider a new ideal M generated by the forms $\{x^{d-a_i} f_i\}$ of degree d in the polynomial ring $S[x]$. For every i,

$$\operatorname{gensdeg}(\operatorname{Syz}_i^{S[x]}(M)) \geq \operatorname{gensdeg}(\operatorname{Syz}_i^{S}(I))\,.$$

11. **A zoo of counterexamples to the Regularity Conjecture**

The first counterexample that we present in this paper is a threefold computed by Macaulay2 [M2]. Another such example is given in [MP, Example 4.6]. In these examples $X \subset \mathbb{P}^5$ is 3-dimensional. Note that Kwak [Kw2] proved the inequality $\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim} X + 1$ if $X \subset \mathbb{P}^5$ is 3-dimensional, non-degenerate, irreducible, and smooth.

Example 11.1. [MP, Example 4.7] Consider the ideal

$$I = (u^6,\ v^6,\ u^2w^4 + v^2x^4 + uvwy^3 + uvxz^3)$$

constructed in [BMN] (where it is denoted by $I_{2,(2,1,2)}$) in the standard graded polynomial ring

$$S = \mathbb{C}[u, v, w, x, y, z]\,.$$

We consider the defining prime ideal $M \subset W = S[w_1, w_2, w_3]$ of the Rees algebra $S[It]$, with $\deg(w_i) = 1$ for $i = 1, 2, 3$. Computation with Macaulay2 [M2] shows that $\operatorname{gensdeg}(M) = 38$, $\deg(M) = 31$, and $\operatorname{pd}(W/M) = 5$. As $\dim(W) = 9$, we may apply Bertini's Theorem to obtain a singular projective 3-fold X in $\mathbb{P}^5$ whose degree and regularity are

$$\deg(X) = 31$$
$$\operatorname{reg}(X) \geq \operatorname{gensdeg}(X) = 38.$$

10.2. Super-polynomial growth of regularity.

In this subsection, we provide families of counterexamples to the Regularity Conjecture, which lead to our main result in [MP]:

Theorem 11.3. [MP, Theorem 1.9] *The regularity of non-degenerate homogeneous prime ideals is not bounded by any polynomial function of the multiplicity, i.e., for any polynomial $\Theta(x)$ there exists a non-degenerate homogeneous prime ideal L in a standard graded polynomial ring (over $\mathbb{C}$) such that $\operatorname{reg}(L) > \Theta\big(\deg(L)\big)$.*

In the next three examples we will show how each of the three methods in Sections 7, 9, 10 can be applied to Koh's example in Theorem 3.5 in order to produce families of counterexamples to the Regularity Conjecture. Recall that for $r \geq 1$ the example provides an ideal I_r generated by $22r-2$ quadrics in a polynomial ring with $22r$ variables, so that

$$\operatorname{syzdeg}(I_r) \geq 2^{2^{r-1}}.$$

Example 11.4. [MP, Counterexamples 1.8] First, we will apply the Rees-like algebra construction, from Section 7, to Koh's example. By Theorem 7.4, the ideal I_r leads to a homogeneous prime ideal P_r (in a standard graded polynomial ring R_r over $\mathbb{C}$) whose multiplicity and generating degree are:

$$\deg(P_r) = 2 \cdot 3^{22r-2}$$
$$\operatorname{reg}(P_r) \geq \operatorname{gensdeg}(P_r) \geq 2^{2^{r-1}}.$$

Therefore, Conjecture 4.4 predicts

$$2^{2^{r-1}} \leq 2 \cdot 3^{22r-2},$$

which fails for $r \geq 10$. Moreover, the difference

$$\operatorname{reg}(P_r) - \deg(P_r) \geq \operatorname{gensdeg}(P_r) - \deg(P_r) > 2^{2^{r-1}} - 2^{50r}$$

can be made arbitrarily large by choosing a large r.

Example 11.5. [CCMPV, Theorem 3.3] We will apply the Rees algebra construction, from Section 9, to Koh's example. For $r \in \mathbb{N}$ we consider the Step-by-step homogenization U_r of the defining ideal W_r of the Rees algebra $S[I_r t]$. By Theorem 9.1, the multiplicity and generating degree of the prime ideal U_r satisfy:

$$\deg(U_r) \leq 3^{22r-2}$$
$$\operatorname{reg}(U_r) \geq \operatorname{gensdeg}(U_r) \geq 2^{2^{r-1}}.$$

Thus it is a counterexample to the Regularity Conjecture for $r \geq 10$.

Example 11.6. [CCMPV, Example 4.2] We will apply the Rees algebra construction, from Section 10, to Koh's example. Following the setting in Theorem 10.1, let W_r be the defining ideal of the Rees algebra $S[I_r t]$, which is considered as a standard graded quotient of a polynomial ring. By Theorem 10.1, the multiplicity and generating degree of the prime ideal W_r satisfy:

$$\deg(W_r) \leq 2^{\min\{22r-2,\, 22r\}} - 1 \leq 2^{22r-2}$$
$$\operatorname{reg}(W_r) \geq \operatorname{gensdeg}(W_r) \geq 2^{2^{r-1}} - 1.$$

Thus it is a counterexample to the Regularity Conjecture for $r \geq 10$.

Example 11.7. Alternatively, we can produce families of counterexamples by applying the three methods in Sections 7, 9, and 10 to the non-prime examples in Theorems 3.3 and 3.4.

10.8. Counterexamples not based on the Mayr-Meyer construction. In order to construct counterexamples to the Regularity Conjecture using Rees or Rees-like algebras, one needs families of ideals whose regularity grows faster than the product of the degrees of the generators. The families based on the Mayr-Meyer construction certainly suffice. Here is the only other known example, which is sufficient to give counterexamples to the Regularity Conjecture without relying on the Mayr-Meyer ideals.

Example 11.9. [CCMPV, Theorem 7.2] In [Ca, Example 4.2.1], Caviglia showed that if $T = \mathbb{C}[z_1, z_2, z_3, z_4]$ and

$$J = (z_1^d,\, z_2^d,\, z_1 z_3^{d-1} - z_2 z_4^{d-1})$$

with $d \geq 2$, then $\operatorname{reg}(T/J) = d^2 - 2$. We set $S = T[x, y]$ and

$$I = \left(x^3,\, y^3,\, x^2 z_1^d + xy(z_1 z_3^{d-1} - z_2 z_4^{d-1}) + y^2 z_2^d\right).$$

Let P be the Step-by-step homogenization of the defining ideal of the Rees-like algebra $S[It, t^2]$ of the ideal I (defined above), as in Theorem 7.4. Then

$$\deg(P) = 32(d+3)$$
$$\operatorname{reg}(P) > d^2 + d + 12\,.$$

In particular, the Regularity Conjecture fails when $d \geq 34$.

Had we instead computed the defining ideal P' of $T[Jt, t^2]$, we would obtain $\deg(P') = 2d^3$, which is not sufficient for a counterexample.

Note that all counterexamples in this section are counterexamples to the weaker Conjecture 4.4 as well.

12. High regularity of syzygies relative to degrees of generators

In the counterexamples in the previous section, the multiplicity is smaller than the maximal degree of a minimal generator of a prime ideal. One may wonder whether high regularity is mainly caused by high degrees of the minimal generators, or whether even low degree generators can lead to high regularity exhibited later in the resolution.

Question 12.1. *Are there homogeneous non-degenerate prime ideals for which the difference between the maximal degree of a minimal generator and the maximal degree of a minimal first syzygy can be made arbitrarily large?*

In the notation introduced in Section 3, this question asks if we can make the difference

$$\mathrm{syzdeg}(L) - \mathrm{gensdeg}(L)$$

arbitrarily large.

The answer to Question 12.1 is not easy even if we don't require the ideal L to be prime. For non-prime ideals, a positive answer was provided by Theorems 3.3, 3.4, 3.5.

Applying our Rees-like algebra construction to Ullery's designer ideals in [Ul, Theorem 1.3] we prove:

Theorem 12.2. [CCMPV, Theorem 6.2] *Let $s \geq 9$ be a positive integer. There exists a non-degenerate prime ideal L in a standard graded polynomial ring (over $\mathbb{C}$) with*

$$\mathrm{gensdeg}(L) = 6$$
$$\mathrm{syzdeg}(L) = s\,.$$

Note that Rees algebras are not currently useful for tackling Question 12.1 since we don't have a suitable upper bound on the degrees of the elements in a minimal generating set of the defining ideal.

13. Open Problems on Regularity in terms of Multiplicity

12.1. A bound on regularity in terms of multiplicity.

The following example shows that a bound on regularity of primary ideals in terms of the multiplicity alone does not exist.

Example 13.2. [CCMPV, Examples 5.4] For $n \geq 1$, consider the ideal

$$J_n = (x^2,\ y^2,\ a^n x + b^n y)$$

in $S = \mathbb{C}[x, y, a, b]$. Then $\mathrm{Ass}(S/J_n) = \{(x, y)\}$. Therefore the ideal J_n is (x, y)-primary. Since the length of $(S/J_n)_{(x,y)}$ is 2, it follows from the associativity formula [HS, Theorem 11.2.4] that $\deg(J_n) = 2$ for all n. Furthermore,

$$\mathrm{reg}(J_n) \geq \mathrm{gensdeg}(J_n) = n + 1\,.$$

Theorem 11.3 shows that the regularity of prime ideals is not bounded by any polynomial function of the multiplicity. In view of Example 13.2, it is natural to wonder if there exists a bound on regularity of prime ideals in terms of the multiplicity alone. Using the recent work of Ananyan-Hochster [AH1] we prove the existence of such a bound:

Theorem 13.3. [CCMPV, Theorem 5.2] *Let L be homogeneous non-degenerate prime ideal in a polynomial ring over $\mathbb{C}$. There exist constants, depending only on* $\deg(L)$*, bounding the projective dimension, regularity and graded Betti numbers of the ideal L.*

The bound obtained in the proof of Theorem 13.3 relies on a bound in [AH1] (alternatively, one can use bounds from [ESS1, DLL]), which is very large. One may wonder how to improve the bound:

Question 13.4. *What is an optimal function $\Phi(x)$ such that* $\operatorname{reg}(L) \leq \Phi(\deg(L))$ *for any non-degenerate homogeneous prime ideal L in a standard graded polynomial ring over $\mathbb{C}$?*

Since we have doubly exponential bounds for all homogeneous ideals (not only the prime ideals), the following question is of interest:

Question 13.5. *Does there exist a singly exponential bound for regularity of homogeneous non-degenerate prime ideals (in a standard graded polynomial ring over* $\mathbb{C}$*) in terms of the multiplicity alone?*

Finding small bounds is motivated by two different topics: Stillman's conjecture and Computational Algebra. We describe these motivations in Sections 14 and 15.

12.6. The Regularity Conjecture for ideals satisfying additional conditions.
The bound in the Regularity Conjecture is very elegant, so it is of interest to study if it holds when we impose extra conditions on the prime ideal. At this point some possible interesting cases seem to be:

(1) smooth varieties;
(2) projectively normal varieties;
(3) toric ideals (in the sense of the definition in [Pe, Section 65]);
(4) projective singular surfaces. (Recall Theorems 5.5 and 5.6. A 3-dimensional counterexample is given in Example 11.1.)

It seems to us that it is currently unpredictable if the Regularity Conjecture holds in any of these cases. The reason why it is reasonable to consider them is that in these cases there are extra tools available, and the examples in Section 11 do not satisfy these properties.

12.7. Avoiding the Mayr-Meyer construction.
It would be desirable to have families of examples of high regularity that are not based on the Mayr-Meyer construction. The family of examples in 11.9 gives

counterexamples to the Regularity Conjecture, but is not sufficient to prove Theorem 11.3 about super-polynomial growth. The following generalization of 11.9 might be sufficient:

Construction 13.8. Define a family of three-generated ideals in the polynomial ring

$$\mathbb{C}[x_0, x_1, \ldots, y_0, y_1, \ldots]$$

recursively as follows. First set

$$f_1 = x_1^2, \qquad g_1 = y_1^2, \qquad h_1 = x_1 x_0 + y_1 y_0.$$

For $r \geq 2$ set

$$f_r = x_r^{2r}, \qquad g_r = y_r^{2r}, \qquad h_r = x_r^2 f_{r-1} + y_r^2 g_{r-1} + x_r y_r h_{r-1}.$$

Consider the ideal

$$I_r = (f_r,\, g_r,\, h_r)\,,$$

generated by three forms of degree $2r$. With this notation we raise the following question based on computational evidence by Macaulay2:

Conjecture 13.9. (McCullough) *The regularity* $\operatorname{reg}(I_r)$ *has super-polynomial growth (that is, for any polynomial* $\Phi(x) \in \mathbb{R}[x]$*, there exists a positive integer* r *such that* $\operatorname{reg}(I_r) > \Phi(r)$*).*

If the conjecture holds, then both the Rees algebra and the Rees-like algebra of the ideal I_r in Construction 13.8 also have regularity exhibiting super-polynomial growth while the degree grows asymptotically as Cr^3, where the constant C is independent of r.

14. More Open Problems

13.1. Bounds in terms of other invariants.

As always, it is very interesting to find bounds on regularity in terms of other invariants, and/or for classes of interesting ideals. As mentioned in the introduction, Bayer-Mumford (1993) [BM] wrote:

Problem 14.2. (Bayer-Mumford (1993) [BM]) *The biggest missing link between the general case and the smooth case is to obtain a "decent bound on the regularity of all reduced equidimensional ideals".*

In particular, we are interested in a decent bound on the regularity of prime ideals, possibly in terms of other invariants than multiplicity. At this point, the problem is widely open.

13.3. Open Problems inspired by Computational Algebra.
Question 13.4 is of interest in Computational Algebra, as indicated in [BS]. Many computer computations in Commutative Algebra and Algebraic Geometry (for example, when using the computer algebra systems Cocoa [Co], Macaulay2 [M2], Singular [DGPS]) use Gröbner bases. Bayer-Stillman [BS2] proved that in generic coordinates and with respect to reverse lexicographic (revlex) order, one has to compute up to degree $\mathrm{reg}(I)$ in order to compute a Gröbner basis of I. Revlex is usually the most efficient monomial order according to [BM]. Thus the regularity of a homogeneous ideal I is the degree-complexity of the Gröbner basis computation of I. In [BM, Comments after Theorem 3.12] Bayer and Mumford wrote: "*We would conjecture that if a linear bound doesn't hold, at the least a single exponential bound, i.e.* $\mathrm{reg}(L) \le \mathrm{gensdeg}(L)^{O(n)}$, *ought to hold for any reduced equidimensional ideal. This is an essential ingredient in analyzing the worst-case behavior of all algorithms based on Gröbner bases.*" Since our paper is focused on prime ideals, we state their conjecture in this case:

Conjecture 14.4. (Bayer-Mumford, 1993, [BM, Comments after Theorem 3.12]) *If L is a homogeneous non-degenerate prime ideal in the standard graded ring $S = \mathbb{C}[x_1, \ldots, x_n]$, then*

$$\mathrm{reg}(L) \le \mathrm{gensdeg}(L)^{O(n)} .$$

Of course, one would also like to have a relation between the prime case and the general case. In some cases Ravi [Ra] has proven that the regularity of the radical of an ideal is no greater than the regularity of the ideal itself. For a long time, there was a folklore conjecture that this would hold for every homogeneous ideal. It was disproved by Chardin-D'Cruz [CD], who provided examples of ideals related to monomial curves in $\mathbf{P}^3$ (respectively, in $\mathbf{P}^4$) such that the regularity of the radical is essentially the square (respectively, the cube) of that of the ideal. They constructed the following family of examples:

Example 14.5. (Chardin-D'Cruz [CD, Example 2.5]) For $m \ge 1$ and $r \ge 3$, consider the ideal

$$J_{m,r} = (y^m u^2 - x^m z v,\ z^{r+1} - x u^r,\ u^{r+1} - x v^r,\ y^m v^r - x^{m-1} z u^{r-1} v)$$

in the polynomial ring $\mathbb{C}[x,y,z,u,v]$. The regularities of $J_{m,r}$ and its radical are:

$$\begin{aligned} \operatorname{reg}(J_{m,r}) &= m + 2r + 1 \\ \operatorname{reg}\left(\sqrt{J_{m,r}}\right) &= m(r^2 - 2r - 1) + 1\,. \end{aligned}$$

The next best result one might hope for, is described in the following folklore question which is currently open:

Question 14.6. *Is there a singly exponential bound on* $\operatorname{reg}(\sqrt{I})$ *in terms of* $\operatorname{reg}(I)$ *(and possibly* $\operatorname{codim}(I)$ *or n) for every homogeneous ideal I in a standard graded polynomial ring over* $\mathbb{C}$*?*

Conjecture 14.4 and Question 14.6 not only have applications in Computational Algebra, but are very interesting on their own. In order to conjecture reasonable bounds, it would be very helpful to have a method for producing interesting examples. In [La2, Remark 1.8.33] Lazarsfeld wrote: "*the absence of systematic techniques for constructing examples is one of the biggest lacunae in the current state of the theory.*"

13.7. Generating Degree.
It is a very basic and natural problem to find a nice bound on the degrees of the defining equations. The following folklore problem is widely open:

Problem 14.8. (Folklore Problem) *Find a decent bound on the generating degree* $\operatorname{gensdeg}(L)$ *for a homogeneous non-degenerate prime ideal L in a standard graded polynomial ring over* $\mathbb{C}$.

Since $\operatorname{gensdeg}(L) \leq \operatorname{reg}(L)$ for every graded ideal L in S, the following weaker form of the Regularity Conjecture provides an elegant bound: if L is a homogeneous non-degenerate prime ideal, then $\operatorname{gensdeg}(L) \leq \deg(L)$. Unfortunately, the counterexamples in Section 11 refute that weaker conjecture as well. On the other hand, it was shown by Mumford that this property is true up to radical, meaning:

Theorem 14.9. (Mumford [Mu2]; see also [EHV, Proposition 3.5]) *Every homogeneous prime ideal L in S is generated up to radical by forms of degree at most* $\deg(L)$.

In the smooth case, he obtained:

Theorem 14.10. (Mumford [Mu2]; see also [La2, Example 1.8.38.] *A smooth projective variety $X \subset \mathbb{P}^{n-1}$ is defined scheme-theoretically by forms of degree at most* $\deg(X)$.

15. **Applications to Stillman's Conjecture**

In a different direction, Questions 13.4 and 13.5 are also motivated by Stillman's Conjecture. A classical construction of Burch [Bu] and Kohn [Ko] shows that there exist three-generated ideals in polynomial rings whose projective dimension is arbitrarily large. In particular, this means it is not possible to bound the projective dimension of an ideal purely in terms of the number of generators. Later Bruns showed in a very precise sense that all the pathology of minimal free resolutions of modules is exhibited by the resolutions of ideals with three generators [Br]. Yet, when applying his argument to create three-generated ideals with arbitrarily large projective dimension, the degrees of the generators are forced to grow linearly with the length of the resolution. Motivated by this phenomenon and by computational complexity issues, Stillman posed the following conjecture:

Stillman's Conjecture 15.1. [PS1, Problem 3.14] *Fix $m \geq 1$ and a sequence of natural numbers $d_1, \ldots, d_m$. There exist a number p such that $\operatorname{reg}(I) \leq p$ for every homogeneous ideal I in a polynomial ring with a minimal system of generators of degrees $d_1, \ldots, d_m$.*

Note that the number of variables in the polynomial ring, where I lives, is not fixed. The original version of Stillman's Conjecture replaces regularity by projective dimension; the equivalence of the two conjectures was proved by Caviglia [Pe, Theorem 29.5]. The conjecture was first proved by Ananyan-Hochster in [AH1]. Other proofs of Stillman's Conjecture were given by Erman-Sam-Snowden [ESS1] and Draisma-Lasoń-Leykin [DLL].

Next we will explain how Question 13.4 is related to Stillman's Conjecture. Let I be an ideal in a standard graded polynomial ring S over $\mathbb{C}$ minimally generated by homogeneous forms of degrees $d_1, \ldots, d_m$. Let $\Phi(x)$ be a function such that $\operatorname{reg}(L) \leq \Phi(\deg(L))$ for any non-degenerate homogeneous prime ideal L in a standard graded polynomial ring over $\mathbb{C}$. Let P be the prime ideal associated to I according to Theorem 7.4. Then

$$\operatorname{reg}(I) \leq \operatorname{reg}(P) \leq \Phi(\deg(P)) = \Phi\left(2\prod_{i=1}^{m}(d_i+1)\right),$$

where the first inequality holds by Theorem 7.4(4). Thus, $\Phi\big(2\prod_{i=1}^{m}(d_i+1)\big)$ provides a bound on the regularity in terms of the degrees of the generators.

Stillman's Conjecture is usually studied in terms of projective dimension (instead of regularity). There has been substantial interest in finding tight upper bounds for more specific cases. Families of ideals with large projective dimension constructed by McCullough in [Mc] and by Beder, McCullough, Núñez-Betancourt, Seceleanu, Snapp, Stone in [BMN], show that any upper bound on projective dimension must be large. McCullough constructed a family of ideals Q_g generated by $2g$ quadrics with $\mathrm{pd}_S(S/Q_g) = g^2 + g$. Beder et. al. constructed a family of ideals T_g generated by three homogeneous elements of degree g^2 with $\mathrm{pd}_S(S/T_g) = g^{g-1}$. The following example gives the Betti table of one of these ideals.

Example 15.2. Let $S = \mathbb{C}[x, y, a, b, c, d]$ and

$$I = (x^2,\ y^2,\ ax + by,\ cx + dy)\,.$$

Then the Betti table for S/I is

	0	1	2	3	4	5	6
0:	1	-	-	-	-	-	-
1:	-	4	-	-	-	-	-
2:	-	-	13	20	15	6	1

In particular, $\mathrm{pd}_S(S/I) = 6$, showing that the bounds in [HMMS1] and [HMMS2] are optimal.

Currently, the best known explicit bounds on projective dimension are:

(i) If I is generated by 3 quadrics, then [MS, Theorem 3.1] provides the optimal upper bound $\mathrm{pd}(S/I) \leq 4$.
(ii) If I is generated by 4 quadrics, then [HMMS2, Theorem 1.3] provides the optimal upper bound $\mathrm{pd}(S/I) \leq 6$.
(iii) If I is generated by g quadrics and $\mathrm{ht}(I) = 2$, then [HMMS1, Main Theorem] provides the optimal upper bound $\mathrm{pd}(S/I) \leq 2g - 2$.
(iv) If I is generated by g quadrics, then $\mathrm{pd}(S/I) \leq 2^{g+1}(g - 2) + 4$ by [AH2, Theorem 1.11].
(v) If I is generated by 3 cubics, then [MM1, Theorem 1] provides the optimal upper bound $\mathrm{pd}(S/I) \leq 5$.

In the case of quadrics, the first author has asked if (and Ananyan and Hochster have conjectured that [AH2, Conjecture 11.4])

$$\mathrm{pd}(S/I) \leq h(g - h + 1)$$

for an ideal I generated by g quadrics of height h. More generally, Ananyan and Hochster conjecture that the optimal bound for the projective dimension of an ideal generated by g forms of degree at most d is $C_d g^d$ for some positive constant C_d depending only on d, where d is fixed and g varies. Examples in [Mc] show that such a bound would be asymptotically optimal. Ananyan and Hochster also give alternate arguments in [AH2] yielding very large bounds for ideals generated by cubics and quartics.

For more details about Stillman's Conjecture, we refer the reader to the expository papers [MS, ESS2].

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