RESOLUTIONS BY PERMUTATION MODULES

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ABSTRACT. We prove that, up to adding a complement, every modular representation of a finite group admits a finite resolution by permutation modules.

Let G be a finite group and $\Bbbk$ be a field of characteristic $p > 0$ dividing the order of G. It is well-known that if G has non-cyclic Sylow p-subgroups, the $\Bbbk$-linear representation theory of G is complicated. In particular, the Krull–Schmidt abelian category, $\Bbbk G$-mod, of finite-dimensional $\Bbbk G$-modules admits *infinitely many* isomorphism classes of indecomposable objects. On the other hand, there is a much simpler class of $\Bbbk G$-modules, the *permutation modules*, *i.e.,* those isomorphic to $\Bbbk X$ for X a finite G-set. The *finite* collection $\{\Bbbk(G/H)\}_{H \leqslant G}$ additively generates all such modules.

For a $\Bbbk G$-module $M \in \Bbbk G$-mod, we want to analyze the existence of what we'll call a *permutation resolution* for short, *i.e.,* an exact sequence

$$(1) \qquad\qquad 0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

where all P_i are permutation modules. Up to direct summands, it is always possible:

2. **Theorem.** *Let G be a finite group and $M \in \Bbbk G$-mod. Then there exists a $\Bbbk G$-module N such that $M \oplus N$ admits a finite resolution* (1) *by permutation modules.*

The related problem of resolutions (1) that are not only exact but remain exact under all fixed-point functors has been recently discussed in [BSW17]. Allowing p-permutation modules P_i (that is, direct summands of permutation modules), Bouc–Stancu–Webb prove that such resolutions exist for all M if and only if G has a Sylow subgroup that is either cyclic or dihedral (for $p = 2$).

Unsurprisingly, Theorem 2 reduces to a Sylow subgroup S of G, since every M is a direct summand of $\mathrm{Ind}_S^G \mathrm{Res}_S^G(M)$ and since the functor Ind_S^G is exact and preserves permutation modules. So we focus on the case where G is a p-group.

For the proof, we shall consider a stronger property:

3. *Definition.* We say that a resolution (1) is *free up to degree $m \geqslant 0$* if P_i is a free module for $i = 0, \ldots, m$. We say that M admits *good permutation resolutions* if for every integer $m \geqslant 0$, there exists a finite resolution (1) by permutation modules that is free up to degree m.

4. *Remark.* Let G be a p-group. A $\Bbbk G$-module M admits good permutation resolutions if and only if for all $m \geqslant 1$ the mth Heller loop $\Omega^m M$ admits a finite permutation resolution. Also, if Q is free and $M \oplus Q$ admits a permutation resolution as in (1) then the epimorphism $P_0 \twoheadrightarrow M \oplus Q \twoheadrightarrow Q$ forces Q to be a direct summand of P_0 and one can remove $0 \to Q \xrightarrow{=} Q \to 0$ from the resolution. So if $M \oplus Q$ has a permutation resolution that is free up to degree m then so does M.

An advantage of good permutation resolutions is the *two out of three property*:

5. **Proposition.** *Let G be a p-group. Let $0 \to L \to M \to N \to 0$ be an exact sequence of $\Bbbk G$-modules. If two out of L, M and N have good permutation resolutions then so does the third.*

Proof. If $P \twoheadrightarrow N$ is a projective cover, we obtain by 'rotation' an exact sequence $0 \to \Omega^1 N \to L \oplus P \to M \to 0$. In view of Remark 4, we can rotate in this way and reduce to the case where L and M admit good permutation resolutions and then prove that N does. Let $m \geqslant 0$. Choose $P_\bullet \to M$ a permutation resolution of M that is free up to degree m. Let $\ell \geqslant m$ be such that $P_i = 0$ for all $i > \ell$. Now choose $Q_\bullet \to L$ a permutation resolution of L that is free up to degree ℓ. We have the following picture (plain part) with exact rows:

(6)
$$
\begin{array}{ccccccccccc}
0 \to & Q_n & \to \cdots \to & Q_{\ell+1} & \to & Q_\ell & \to \cdots \to & Q_0 & \to & L & \to 0 \\
& \vdots & & \vdots & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & 0 & \to \cdots \to & 0 & \to & P_\ell & \to \cdots \to & P_0 & \to & M & \to 0
\end{array}
$$

The standard lifting argument, using that Q_j is projective for $j = 0, \ldots, \ell$ shows that there exists a lift $f_\bullet \colon Q_\bullet \to P_\bullet$ of the morphism $L \to M$. Then the mapping cone complex $\mathrm{cone}(f_\bullet)$ yields a resolution of $\mathrm{coker}(L \to M) = N$ and this complex $\mathrm{cone}(f_\bullet)$ has free objects in degree $0, \ldots, m$ since $P_\bullet$ and $Q_\bullet$ do. $\qquad \square$

Let us discuss an example of Theorem 2, where we can even take $N = 0$.

7. **Proposition.** *Let $E = (C_p)^{\times r} = C_p \times \cdots \times C_p$ be an elementary abelian group of rank r. Then every $\Bbbk E$-module admits good permutation resolutions.*

Proof. Consider for each $1 \leqslant i \leqslant r$ the ('coordinate-wise') subgroup

$$
H_i = C_p \times \cdots \times C_p \times 1 \times C_p \times \cdots \times C_p
$$

of rank $r-1$. Let $m \geqslant 0$. Inflating from $E/H_i \simeq C_p$ the usual 2-periodic resolutions $0 \to \Bbbk \to \Bbbk C_p \to \cdots \to \Bbbk C_p \to \Bbbk \to 0$ of length at least m, we obtain quasi-isomorphisms of $\Bbbk E$-modules $s_i \colon Q(i) \to \Bbbk[0]$ where the $Q(i)$ are defined as follows:

$$
\begin{array}{ccccccccc}
Q(i) := & 0 \to \Bbbk \to \Bbbk(E/H_i) \to & \cdots & \to \Bbbk(E/H_i) \to \Bbbk(E/H_i) \to 0 \\
s_i \downarrow & \downarrow \qquad\qquad \downarrow & & \downarrow \qquad\qquad \downarrow \\
\Bbbk[0] = & 0 \to 0 \longrightarrow 0 \longrightarrow & \cdots & \longrightarrow 0 \longrightarrow \Bbbk \longrightarrow 0
\end{array}
$$

Tensoring all the above, we obtain a quasi-isomorphism

$$
s_1 \otimes \cdots \otimes s_r \colon P_\bullet := Q(1) \otimes \cdots \otimes Q(r) \to (\Bbbk[0])^{\otimes r} \cong \Bbbk[0],
$$

i.e., a permutation resolution $P_\bullet$ of $\Bbbk$. In other words, we performed an 'external tensor' of all the periodic resolutions over each copy of C_p in E. Since the Mackey formula gives by induction $\Bbbk(E/H_{i_1}) \otimes \cdots \otimes \Bbbk(E/H_{i_n}) \cong \Bbbk(E/(H_{i_1} \cap \cdots \cap H_{i_n}))$, we have produced a permutation resolution $P_\bullet$ of $\Bbbk$ that is easily seen to be free up to degree m. As $m \geqslant 0$ was arbitrary, we proved that the trivial module $\Bbbk$ admits good permutation resolutions. A general module $M \in \Bbbk E$-mod admits a filtration whose successive quotients are trivial. We therefore conclude by induction, via Proposition 5. $\qquad \square$

8. *Remark.* The proof of Proposition 7 shows that the stabilisers in the permutation resolution may be taken to be products of subsets with respect to the given decomposition of E. Applying the proposition to a module and its dual shows that given

a module M we may form a finite exact complex of permutation modules with these stabilisers in such a way that the image of one of the maps is M. This should be compared with the main theorem of [BC] which shows that a finite exact sequence of permutation E-modules in which the set of stabilisers has no containment of index p necessarily splits, so that the image of every map is again a permutation module.

Proof of Theorem 2. As already mentioned, we can reduce to the case where G is a p-group. By [Car00], we know that for every $\Bbbk G$-module M, there exists a $\Bbbk G$-module N and a finite filtration $0 = L_0 \subset L_1 \subset \cdots \subset L_s = M \oplus N$ such that every L_i/L_{i-1} is induced from some elementary abelian subgroup $E_i \leqslant G$. Since the result holds for elementary abelian groups (Proposition 7) and is stable by induction, we see that all L_i/L_{i-1} admit good permutation resolutions. By Proposition 5, we conclude that so does $M \oplus N$. In particular, $M \oplus N$ has a permutation resolution. $\qquad\square$

9. *Remark.* Our arguments go through for infinite-dimensional modules, with minor modifications. We leave this variation on the theme to the interested reader.

References

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