STATIONARY CAHN-HILLIARD-NAVIER-STOKES EQUATIONS FOR THE DIFFUSE INTERFACE MODEL OF COMPRESSIBLE FLOWS

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Abstract. A system of partial differential equations for a diffusion interface model is considered for the stationary motion of two macroscopically immiscible, viscous Newtonian fluids in a three-dimensional bounded domain. The governing equations consist of the stationary Navier-Stokes equations for compressible fluids and a stationary Cahn-Hilliard type equation for the mass concentration difference. Approximate solutions are constructed through a two-level approximation procedure, and the limit of the sequence of approximate solutions is obtained by a weak convergence method. New ideas and estimates are developed to establish the existence of weak solutions with a wide range of adiabatic exponent.

1. Introduction

We are concerned with a diffuse interface model for a mixture of two viscous fluids. The interface is usually caused by continuous but steep change of flow properties of immiscible or partially miscible fluids, which has been studied largely in literature (see [9, 34]). An important analytical and numerical method to model such two-phase flows is the diffuse interface modeling (see [45]). The hydrodynamical system of the mixture of two fluids is naturally the Navier-Stokes equations in each fluid domain with the kinematic and other conditions on the interface. On the other hand, the Allen-Cahn type or Cahn-Hilliard type of mixing models is commonly used based on the choice of the flux and production rate, see [19, 33] and the reference cited therein. In this paper, we study the following stationary Cahn-Hilliard-Navier-Stokes equations of the compressible mixture of fluid flows in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) = \operatorname{div}\left(\mathbb{S}_{ns} + \mathbb{S}_c - p\mathbb{I}\right) + \rho g_1 + g_2, \\ \operatorname{div}(\rho u c) = \operatorname{div}(m\nabla\mu), \\ \rho\mu = \rho\dfrac{\partial f(\rho, c)}{\partial c} - \triangle c, \end{cases} \tag{1.1}$$

where ρ, u, c, μ denote the total density, the mean velocity field, the mass concentration difference of the two components, and the chemical potential, respectively; m is the mobility that is assumed to be one for simplicity, and g_1 and g_2 are given force terms. We denote the Navier-Stokes stress tensor by

$$\mathbb{S}_{ns} = \lambda_1 \left(\nabla u + (\nabla u)^\top\right) + \lambda_2 \operatorname{div} u \mathbb{I}, \tag{1.2}$$

where $(\nabla u)^\top$ denotes the transpose of ∇u, $\mathbb{I}$ is the identity matrix, and λ_1, λ_2 are constants satisfying $\lambda_1 > 0$, $2\lambda_1 + 3\lambda_2 \geq 0$. In comparison with a single fluid, there is an additional

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capillary stress tensor

$$\mathbb{S}_c = -\nabla c \otimes \nabla c + \frac{1}{2}|\nabla c|^2 \mathbb{I}, \tag{1.3}$$

which describes the capillary effect related to the surface energy. In this paper we assume the following form of pressure

$$p = \rho^2 \frac{\partial f(\rho, c)}{\partial \rho}, \tag{1.4}$$

and the free energy density

$$f(\rho, c) = \rho^{\gamma-1} + f_{mix}(\rho, c) = \rho^{\gamma-1} + H_1(c)\ln\rho + H_2(c), \tag{1.5}$$

with the adiabatic exponent $\gamma > 1$ and two given functions $H_i\,(i = 1, 2)$ of one variable. We remark that the mixed free energy density $f_{mix}(\rho, c)$ is mainly motivated by the well-known logarithmic form (cf. [5, 34]). We shall study the stationary equations (1.1) with the following boundary conditions:

$$u = 0, \ \ \frac{\partial c}{\partial n} = 0, \ \ \frac{\partial \mu}{\partial n} = 0, \quad \text{on } \partial\Omega \tag{1.6}$$

and the additional conditions:

$$\int \rho(x)dx = m_1 > 0 \quad \text{and} \quad \int \rho(x)c(x)dx = m_2, \tag{1.7}$$

with two given constants $m_1 > 0$ and m_2.

System (1.1) describes the equilibrium state for the compressible mixture of two macroscopically immiscible, viscous Newtonian fluids (cf. [19, 34, 44]). The goal of this paper is to investigate the existence of solutions to the problem (1.1)-(1.7), and give a rigorous mathematical justification of the existence of an equilibrium state for the mixture of fluids. We recall that the corresponding evolutionary system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = \operatorname{div}\left(\mathbb{S}_{ns} + \mathbb{S}_c - \rho^2 \dfrac{\partial f(\rho, c)}{\partial \rho}\mathbb{I}\right), \\ \partial_t(\rho c) + \operatorname{div}(\rho u c) = \triangle \mu, \\ \rho\mu = \rho \dfrac{\partial f(\rho, c)}{\partial c} - \triangle c, \end{cases} \tag{1.8}$$

was derived in Abels-Feireisl [5, Section 2.2], and can be regarded as a variant of the model suggested by Lowengrub-Truskinovsky [34]. For the evolutionary system (1.8), the existence of multi-dimensional renormalized weak solution of finite energy was obtained in [5] for $\gamma > \frac{3}{2}$, and the one-dimensional weak and strong solutions were studied in [15, 18]. For the results on the stationary compressible Navier-Stokes equations, we refer the readers to the books [32, 37], the papers [27, 35, 36, 38] and references therein. For the models of the Cahn-Hilliard-Navier-Stokes type of multi-component viscous fluids with phase transitions, in the case of incompressible fluids with matched or non-matched densities, there are several different approaches to describe the evolutionary diffusion processes (cf. [6, 10, 20, 21, 30, 43]), and there is a large literature on the existence and long-time behavior of solutions; see, e.g., [1–4, 12, 13, 17, 25, 26, 29, 33, 43] and the references therein; and we also mention the existence results in [11, 40, 41] for weak solutions to the stationary non-Newtonian flows in two and three dimensions. As far as the compressible flow is concerned, Lowengrub-Truskinovsky [34] developed a thermodynamically and mechanically consistent model that extends the Euler and Navier-Stokes models to the case of compressible binary Cahn-Hilliard mixtures; see also

[8, 9, 16] for other different approaches. For general interface dynamics of the mixture of different fluids, solids or gas, including the Allen-Cahn type and Cahn-Hilliard type, see [5, 8–10, 14, 16, 17, 19, 20, 33, 34, 39, 43, 44] and the references therein for the discussions and mathematical results.

We now introduce the notation and state our main result. For two given matrices $\mathbb{A} = (a_{ij})_{3\times 3}$ and $\mathbb{B} = (b_{ij})_{3\times 3}$, we denote their scalar product by $\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^{3} a_{ij}b_{ij}$. For two vectors a, $b \in \mathbb{R}^3$, $a \otimes b = (a_i b_j)_{3\times 3}$. The characteristic function of a set A is denoted by $\mathbf{1}_A$. Let $C_0^\infty(\Omega, \mathbb{R}^3)$ be the set of all smooth and compactly supported functions $f : \Omega \mapsto \mathbb{R}^3$, and $C_0^\infty(\Omega) = C_0^\infty(\Omega, \mathbb{R})$. Similarly we denote by $C^\infty(\overline{\Omega}) = C^\infty(\overline{\Omega}, \mathbb{R})$ the set of uniformly smooth functions on Ω. We use $\int f = \int_\Omega f(x)dx$ to denote the integral of f on Ω. For any $p \in [1, \infty]$ and integer $k \geq 0$, $W^{k,p}(\Omega, \mathbb{R}^3)$ and $W^{k,p}(\Omega)$ are the standard Sobolev spaces (cf. [7]) valued in $\mathbb{R}^3$ or $\mathbb{R}$, and $L^p = W^{0,p}$ and $H^k = W^{k,2}$. We denote by $\overline{f}$ the weak limit of function f.

The definition of weak solutions is as follows.

Definition 1.1. The function (ρ, u, μ, c) is a weak solution to the problem (1.1)-(1.7) if for some $p > \frac{6}{5}$ and $\theta > 0$ with $\theta + \gamma > \frac{3}{2}$,

$$\begin{aligned} &\rho \in L^{\gamma+\theta}(\Omega), \ \ \rho \geq 0 \ a.e. \text{ in } \Omega, \\ &u \in H_0^1(\Omega, \mathbb{R}^3), \quad \mu \in H_n^1(\Omega), \quad c \in W_n^{2,p}(\Omega), \end{aligned}$$

where $W_n^{k,p}(\Omega) = \{f \in W^{k,p}(\Omega) : \ \frac{\partial f}{\partial n}|_{\partial\Omega} = 0\}$ for any positive integer k and $H_n^1 = W_n^{1,2}$, such that,

(i) The system (1.1) is satisfied in the distribution sense in Ω, i.e., for any $\Phi \in C_0^\infty(\Omega, \mathbb{R}^3)$,

$$\int \left(\rho u \otimes u + \rho^2 \frac{\partial f(\rho, c)}{\partial \rho}\mathbb{I} - \mathbb{S}_{ns} - \mathbb{S}_c\right) : \nabla\Phi = \int (\rho g_1 + g_2) \cdot \Phi,$$

and for any $\phi \in C^\infty(\overline{\Omega})$,

$$\int \rho u \cdot \nabla\phi = 0, \quad \int \rho c u \cdot \nabla\phi = \int \nabla\mu \cdot \nabla\phi, \quad \int \rho\mu\phi - \rho\frac{\partial f(\rho, c)}{\partial c}\phi = \int \nabla c \cdot \nabla\phi;$$

and (1.7) holds for some $m_1 > 0$ and $m_2 \in \mathbb{R}$.

(ii) If (ρ, u) is prolonged by zero outside Ω, then both the equation $(1.1)_1$ and

$$\operatorname{div}(b(\rho)u) + \big(b'(\rho)\rho - b(\rho)\big)\operatorname{div} u = 0$$

are satisfied in the distribution sense in $\mathbb{R}^3$, where $b \in C^1([0, \infty))$ with $b'(z) = 0$ if z is large enough.

(iii) The energy inequality is valid

$$\int \left(\lambda_1|\nabla u|^2 + (\lambda_1 + \lambda_2)(\operatorname{div} u)^2 + |\nabla\mu|^2\right) dx \leq \int (\rho g_1 + g_2) \cdot u.$$

We are ready to state our main result.

Theorem 1.1. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded and simply connected domain with C^2 smooth boundary,*

$$g_1, \ g_2 \in L^\infty(\Omega, \mathbb{R}^3), \tag{1.9}$$

the functions in (1.5) *satisfy*

$$|H_i(c)| + |H_i'(c)| \leq \overline{H} < \infty, \quad \forall\, c \in \mathbb{R}, \quad i = 1, 2, \tag{1.10}$$

for some constant $\overline{H}$*, and in addition,*

$$\begin{cases} \gamma > \dfrac{5}{3}, & \text{if } \nabla \times g_1 \equiv 0 \text{ in } \Omega, \\ \gamma > 2, & \text{otherwise.} \end{cases} \tag{1.11}$$

Then, for any given constants $m_1 > 0$ *and* $m_2 \in \mathbb{R}$*, the problem* (1.1)-(1.7) *admits a weak solution* (ρ, u, μ, c) *in the sense of Definition 1.1.*

We shall prove Theorem 1.1 via two levels of approximations and weak convergence methods, which rely on the heuristic approaches in [5, 24, 27, 31, 32, 35–38]. We remark that our stationary problem seems worse than the time-evolutionary one because the energy inequality by itself gives less useful information about the sequence of approximate solutions, and it is much more complicated than the Navier-Stokes equations of the single fluid due to the coupling with the Cahn-Hilliard equations. Our construction of weak solution in Theorem 1.1 of this paper follows the spirit of [27, 32, 35–38] for the stationary compressible Navier-Stokes equations, but we also need to overcome extra barriers from the coupled Cahn-Hilliard equations. The main difficulties and our strategies are described below.

We first construct the approximate system (2.2), which is inspired by the time-discretization of equations (1.8). The main ideas for this approximation are the following: (1) To guarantee the sufficient regularity on density ρ, we add the diffusion term $\varepsilon^4\triangle\rho$ in the transport equation and an artificial pressure in the momentum equation. Our choice of ε^4 as the diffusion coefficient makes it possible to avoid the appearance of new parameters and thus simplify the approximation procedures. (2) In the proof, the total mass and difference of volume fraction should be preserved, namely, both $\int_\Omega \rho(x)dx$ and $\int_\Omega \rho(x)c(x)dx$ are constant. This is necessary from both the physical and mathematical point of view, and can be derived by the Hardy-Poincaré type inequality as well as the well-posedness of solutions. For this purpose, we use $\varepsilon^2(\rho - \rho_0)$ and $\varepsilon(\rho c - \rho_0 c_0)$ in the approximation, which can be regarded as time discretization of $\partial_t\rho$ and $\partial_t(\rho c)$ respectively. (3) For fixed ε and δ, we solve (2.2) by the Schaefer fixed point theorem. Some new ideas are needed in the proof. Firstly, the solution is not self-contained due to the Neumann boundary conditions imposed on μ and c. To fix the constants, we add compatible integral conditions in the system (2.11). Secondly, we use the conservative quantities (1.7) and interpolation techniques to obtain the required estimates so that the uniform *a priori* bounds can be closed. Next, notice that the pressure p relies not only on ρ but also on c, and hence is not monotone in ρ for all range of c. In this connection, we adopt some idea in [5] and decompose

$$p = \rho^2\frac{\partial f}{\partial \rho} = \rho^2\frac{\partial \tilde{f}}{\partial \rho} - 2\overline{H}\rho\mathbf{1}_{\{\rho\le k\}} = \tilde{p} - 2\overline{H}\rho\mathbf{1}_{\{\rho\le k\}}, \tag{1.12}$$

where $\overline{H}\rho\mathbf{1}_{\{\rho\le k\}}$ is bounded for some large but finite constant k. See Remark 2.3 for the detail. Finally, to avoid the appearance of higher order derivatives of c, we replace the capillary stress $\operatorname{div}\left(-\nabla c\otimes\nabla c + \frac{1}{2}|\nabla c|^2\mathbb{I}\right)$ in (2.2) by the equivalent expression $\left(\rho\mu - \rho\frac{\partial f(\rho,c)}{\partial c}\right)\nabla c$.

Then we shall establish the *a priori* estimates uniform in ε to guarantee the ε-limit procedure to obtain the approximation sequence (3.1) by using the compactness theories developed in [24, 31, 32]. In the proof, we need strong convergence of ∇c for taking limit in the momentum equation. For this purpose, we shall make full use of the properties obtained from the higher order diffusion in the Cahn-Hilliard equation; see for example the proof of (3.19). Another difficulty is the non-monotonicity of the pressure with respect to ρ. Thanks to the

decomposition technique (see Remark 2.3) $H_1(c)$ is always positive, which leads to our desired estimates.

Finally we need to show the δ-limit in the vanishing artificial pressure term. The proof shall be based on the compactness theories in [24, 31]. The difficulty is that the approximation sequence does not provide any good estimate on the density but $\|\rho\|_{L^1}$, which is different from the evolutionary equations for which the density ρ is bounded in L^γ with $\gamma > 1$. To overcome the difficulty, we borrow some ideas developed in [27, 36] and derive the higher regularities by means of weighted pressure estimates. However, we need to handle the difficulties caused by the Newmann boundary conditions and the appearance of the strongly nonlinear stress tensor $\mathrm{div}\left(-\nabla c \otimes \nabla c + \frac{1}{2}|\nabla c|^2 \mathbb{I}\right)$. See Lemma 4.1 for the detail.

The rest of the paper is organized as follows. In Section 2, we construct the two-level approximation system, find the solution by the fixed point theorem, and derive some energy estimates. In Section 3, we derive the uniform estimates in ε and pass the limit as ε goes to zero. In Section 4, we derive the uniform estimates in δ and pass the limit as δ goes to zero to finally obtain the weak solution in Theorem 1.1.

2. Construction of approximation solutions

We first set the following fixed constants:

$$\varepsilon \in (0,1),\ \delta \in (0,1); \quad \rho_0 = \frac{m_1}{|\Omega|},\ c_0 = \frac{m_2}{m_1}, \tag{2.1}$$

where $m_1,\ m_2$ are taken from (1.7), and $|\Omega|$ denotes the Lebesgue measure of Ω. Then we consider the following approximate system:

$$\begin{cases} \varepsilon^2\rho + \mathrm{div}(\rho u) = \varepsilon^4 \triangle \rho + \varepsilon^2 \rho_0, \\ \varepsilon^2 \rho u + \mathrm{div}(\rho u \otimes u) + \nabla\left(\delta\rho^4 + \rho^2 \dfrac{\partial f(\rho,c)}{\partial \rho}\right) + \varepsilon^4 \nabla\rho \cdot \nabla u \\ \qquad = \mathrm{div}\mathbb{S}_{ns} + \rho\mu\nabla c - \rho \dfrac{\partial f(\rho,c)}{\partial c}\nabla c + \rho g_1 + g_2, \\ \varepsilon\rho c + \rho u \cdot \nabla c = \triangle \mu + \varepsilon \rho_0 c_0, \\ \rho\mu = \rho \dfrac{\partial f(\rho,c)}{\partial c} - \triangle c, \end{cases} \tag{2.2}$$

with the boundary conditions

$$u = 0, \quad \frac{\partial \rho}{\partial n} = 0,\ \frac{\partial c}{\partial n} = 0,\ \frac{\partial \mu}{\partial n} = 0, \quad \text{on } \partial\Omega. \tag{2.3}$$

Remark 2.1. A direct computation shows, at least formally,

$$\rho\mu\nabla c - \rho\frac{\partial f(\rho,c)}{\partial c}\nabla c = -\triangle c \nabla c = \mathrm{div}\left(-\nabla c \otimes \nabla c + \frac{1}{2}|\nabla c|^2 \mathbb{I}\right) = \mathrm{div}\mathbb{S}_c.$$

The following lemma is concerned with the solvability of $(2.2)_1$, and its proof can be found in [37, Prop. 4.29].

Lemma 2.1 ([37], Proposition 4.29)**.** *Suppose*

$$v \in W_0^{1,\infty}(\Omega, \mathbb{R}^3) := \{v \in W^{1,\infty}(\Omega, \mathbb{R}^3),\ \ v|_{\partial\Omega} = 0\}. \tag{2.4}$$

Then there exists a function $\rho = \rho(v) \in W^{2,p}(\Omega)$ $(1 < p < \infty)$ such that for any $\eta \in C^\infty(\overline{\Omega})$,

$$\varepsilon^4 \int \nabla\rho \cdot \nabla\eta - \int \rho v \cdot \nabla\eta + \varepsilon^2 \int (\rho - \rho_0)\eta = 0, \tag{2.5}$$

where $\varepsilon > 0$ is a fixed constant. Moreover,

$$\rho \geq 0 \text{ a.e. in } \Omega, \quad \|\rho\|_{L^1} = m_1, \quad \|\rho\|_{W^{2,p}} \leq C(\varepsilon, p, \|v\|_{W^{1,\infty}}). \tag{2.6}$$

Next, we consider the Neumann boundary problem

$$\triangle\rho = \mathrm{div} b \quad \text{with} \quad \frac{\partial\rho}{\partial n}\Big|_{\partial\Omega} = 0. \tag{2.7}$$

Lemma 2.2 ([37], Lemma 4.27)**.** *Let $p \in (1, \infty)$ and $b \in L^p(\Omega, \mathbb{R}^3)$ be given. Then the problem* (2.7) *admits a solution $\rho \in W^{1,p}(\Omega)$, satisfying*

$$\int \nabla\rho \cdot \nabla\phi = \int b \cdot \nabla\phi, \quad \forall\ \phi \in C^\infty(\Omega),$$

and the estimates

$$\|\nabla\rho\|_{L^p} \leq C(p, \Omega)\|b\|_{L^p} \quad \text{and} \quad \|\nabla\rho\|_{W^{1,p}} \leq C(p, \Omega)(\|b\|_{L^p} + \|\mathrm{div} b\|_{L^p}).$$

Our main task in this section is to prove the following theorem.

Theorem 2.1. *Under the conditions* (1.9), (1.10) *and* (2.1)*, for any fixed $\varepsilon > 0$ the problem* (2.2)-(2.3) *has a solution $(\rho_\varepsilon, u_\varepsilon, \mu_\varepsilon, c_\varepsilon)$, such that for all $p \in (1, \infty)$,*

$$0 \leq \rho_\varepsilon \in W^{2,p}(\Omega), \quad \|\rho_\varepsilon\|_{L^1(\Omega)} = m_1, \tag{2.8}$$

$$u_\varepsilon \in W_0^{1,p}(\Omega, \mathbb{R}^3) \cap W^{2,p}(\Omega, \mathbb{R}^3), \quad (\mu_\varepsilon,\, c_\varepsilon) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega). \tag{2.9}$$

Proof. We will prove Theorem 2.1 by the fixed point theorem. Setting

$$(v, \tilde{\mu}, \tilde{c}) \in \mathcal{W} := W_0^{1,\infty}(\Omega, \mathbb{R}^3) \times W_n^{1,p}(\Omega) \times W_n^{1,p}(\Omega), \tag{2.10}$$

where $W_n^{1,p}(\Omega) = \{f \in W^{1,p}(\Omega) : \ \frac{\partial f}{\partial n}|_{\partial\Omega} = 0\}$ and $W_0^{1,\infty}$ is from (2.4). Let us consider the elliptic system of (u, μ, c):

$$\begin{cases} \mathrm{div}\mathbb{S}_{ns} = F^1(v, \tilde{\mu}, \tilde{c}) \\ \qquad := \varepsilon^2\rho v + \mathrm{div}(\rho v \otimes v) + \nabla(\delta\rho^4 + \rho^2\dfrac{\partial f(\rho, \tilde{c})}{\partial\rho}) + \varepsilon^4\nabla\rho \cdot \nabla v \\ \qquad\qquad + \rho\dfrac{\partial f(\rho, \tilde{c})}{\partial\tilde{c}}\nabla\tilde{c} - \rho\tilde{\mu}\nabla\tilde{c} - \rho g_1 - g_2, \\ \triangle\mu = F^2(v, \tilde{\mu}, \tilde{c}) := \varepsilon\rho\tilde{c} + \rho v \cdot \nabla\tilde{c} - \varepsilon\rho_0 c_0, \\ \triangle c = F^3(v, \tilde{\mu}, \tilde{c}) := \rho\dfrac{\partial f(\rho, \tilde{c})}{\partial\tilde{c}} - \rho\tilde{\mu}, \\ \displaystyle\int \rho\tilde{c} = m_2 + \varepsilon\int(\rho_0 - \rho)\tilde{c} - \varepsilon^3\int \nabla\rho \cdot \nabla\tilde{c}, \quad \int \rho\tilde{\mu} = \int \rho\frac{\partial f(\rho, \tilde{c})}{\partial\tilde{c}}, \\ \displaystyle\int \rho c = m_2 + \varepsilon\int(\rho_0 - \rho)c - \varepsilon^3\int \nabla\rho \cdot \nabla c, \quad \int \rho\mu = \int \rho\frac{\partial f(\rho, c)}{\partial c}, \\ u = 0, \ \dfrac{\partial\mu}{\partial n} = 0, \ \dfrac{\partial c}{\partial n} = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{2.11}$$

where $\rho = \rho(v)$ is determined in Lemma 2.1. For any given $(v, \tilde{\mu}, \tilde{c})$ satisfying $(2.11)_4$, the system (2.11) has a solution

$$(u, \mu, c) := A[(v, \tilde{\mu}, \tilde{c})]. \tag{2.12}$$

Applying the L^p regularity estimates (cf.[28]), we have

$$\|(u, \mu, c)\|_{W^{2,p}} \le C\|(F^1, F^2, F^3)\|_{L^p} < \infty.$$

Remark 2.2. The condition $(2.11)_4$ guarantees $\int F^2 = \int F^3 = 0$ which is compatible with the Neumann boundary conditions in (2.11). In fact, by this condition together with (2.1) and Lemma 2.1, one has

$$\begin{aligned}\int F^2 &= \int (\varepsilon\rho\tilde{c} + \rho v \cdot \nabla\tilde{c} - \varepsilon\rho_0 c_0) = \int (\varepsilon\rho\tilde{c} + \rho v \cdot \nabla\tilde{c}) - \varepsilon m_2 \\ &= \varepsilon \int \rho\tilde{c} + \varepsilon^4 \int \nabla\rho \cdot \nabla\tilde{c} + \varepsilon^2 \int (\rho - \rho_0)\tilde{c} - \varepsilon m_2 = 0.\end{aligned}$$

The second equality of the condition $(2.11)_4$ yields $\int F^3 = 0$ immediately. Finally, we note that the condition $(2.11)_5$ is for the uniqueness of μ and c. The two conditions $(2.11)_4$ and $(2.11)_5$ coincide after the fixed point argument.

Proposition 2.1. *Suppose that (u, μ, c) is a solution to* (2.11) *and the operator $A : \mathcal{W} \mapsto \mathcal{W}$ is defined in* (2.12)*. Then, the set of possible fixed points*

$$\left\{ (u, \mu, c) \in \mathcal{W} \,\middle|\, \begin{array}{l} (u, \mu, c) := \sigma A[(u, \mu, c)] \\ \text{for some } \ \sigma \in (0, 1] \text{ and } \quad \rho = \rho(u) \end{array} \right\} \tag{2.13}$$

is bounded, where $\mathcal{W}$ is defined in (2.10)*.*

A standard argument shows that A is compact and continuous in $\mathcal{W}$. Therefore, using Proposition 2.1, we conclude from the Schaefer Fixed Point Theorem (Chap. 9, Th. 4 in [22]) that $(u, \mu, c) := A[(u, \mu, c)]$ with $\rho = \rho(u)$. This and Lemma 2.1 guarantee the existence of solution $(\rho_\varepsilon, u_\varepsilon, \mu_\varepsilon, c_\varepsilon)$ to (2.2)-(2.3). Consequently, (2.8) follows directly from (2.6).

It remains to prove Proposition 2.1 as well as (2.9).

Proof of Proposition 2.1. It suffices to show that there is a constant $M < \infty$ independent of σ such that

$$\|(u, \mu, c)\|_{\mathcal{W}} < M, \tag{2.14}$$

where (ρ, u, μ, c) solves

$$\begin{cases} \varepsilon^2\rho + \operatorname{div}(\rho u) = \varepsilon^4 \triangle\rho + \varepsilon^2\rho_0, \\ \operatorname{div}\mathbb{S} = \sigma F^1(u, \mu, c), \\ \triangle\mu = \sigma F^2(u, \mu, c), \\ \triangle c = \sigma F^3(u, \mu, c), \\ \displaystyle\int \rho c = m_2 + \varepsilon \int (\rho_0 - \rho)c - \varepsilon^3 \int \nabla\rho \cdot \nabla c, \quad \int \rho\mu = \int \rho \frac{\partial f(\rho, c)}{\partial c}, \\ \displaystyle u = 0, \quad \frac{\partial\rho}{\partial n} = 0, \ \frac{\partial\mu}{\partial n} = 0, \ \frac{\partial c}{\partial n} = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

We divide the proof into several steps.

Step 1. It follows directly from $(2.15)_1$ that $\|\rho\|_{L^1} = m_1$. Multiplying $(2.15)_1$ by $\frac{1}{2}|u|^2$ and $(2.15)_2$ by u respectively, we get

$$\begin{aligned}&\frac{\varepsilon^2\sigma}{2}\int(\rho+\rho_0)|u|^2+\sigma\int u\cdot\nabla\left(\delta\rho^4+\rho^2\frac{\partial f}{\partial\rho}\right)\\&\quad+\int\mathbb{S}_{ns}:\nabla u+\sigma\int\rho\frac{\partial f}{\partial c}(u\cdot\nabla)c-\sigma\int\rho\mu(u\cdot\nabla)c=\sigma\int(\rho g_1+g_2)\cdot u.\end{aligned}\tag{2.16}$$

Using $(2.15)_1$, one has

$$\begin{aligned}&\int u\cdot\nabla\left(\delta\rho^4+\rho^2\frac{\partial f}{\partial\rho}\right)\\&=\int\rho u\cdot\nabla\left(\frac{4\delta}{3}\rho^3+\frac{\partial(\rho f)}{\partial\rho}\right)+\int u\cdot\left(\nabla\rho\frac{\partial(\rho f)}{\partial\rho}-\nabla(\rho f)\right)\\&=-\int\operatorname{div}(\rho u)\left(\frac{4\delta}{3}\rho^3+\frac{\partial(\rho f)}{\partial\rho}\right)-\int\rho\frac{\partial f}{\partial c}(u\cdot\nabla)c\\&=\varepsilon^2\int\left(\frac{4\delta}{3}\rho^3+\frac{\partial(\rho f)}{\partial\rho}\right)(\rho-\rho_0)+\varepsilon^4\int\left(4\delta\rho^2+\frac{\partial^2(\rho f)}{\partial\rho^2}\right)|\nabla\rho|^2\\&\quad+\varepsilon^4\int\frac{\partial^2(\rho f)}{\partial\rho\partial c}\nabla\rho\cdot\nabla c-\int\rho\frac{\partial f}{\partial c}(u\cdot\nabla)c.\end{aligned}$$

Substitute the above into (2.16) to obtain

$$\begin{aligned}&\frac{\varepsilon^2\sigma}{2}\int(\rho+\rho_0)|u|^2+\varepsilon^2\sigma\int\left(\frac{4\delta}{3}\rho^3+\frac{\partial(\rho f)}{\partial\rho}\right)(\rho-\rho_0)\\&\quad+\int\mathbb{S}_{ns}:\nabla u-\sigma\int\rho\mu(u\cdot\nabla)c+\varepsilon^4\sigma\int\left(4\delta\rho^2+\frac{\partial^2(\rho f)}{\partial\rho^2}\right)|\nabla\rho|^2\\&=\sigma\int(\rho g_1+g_2)\cdot u-\varepsilon^4\sigma\int\frac{\partial^2(\rho f)}{\partial\rho\partial c}\nabla\rho\cdot\nabla c.\end{aligned}\tag{2.17}$$

Next, multiplying $(2.15)_3$ by μ and $(2.15)_4$ by c gives rise to

$$\int|\nabla\mu|^2+\sigma\int\rho\mu(u\cdot\nabla)c+\varepsilon\int|\nabla c|^2=\varepsilon\sigma\int\rho_0c_0\mu-\varepsilon\sigma\int\rho\frac{\partial f}{\partial c}c.\tag{2.18}$$

Combining (2.17) with (2.18) leads to

$$\begin{aligned}&\frac{\varepsilon^2\sigma}{2}\int(\rho+\rho_0)|u|^2+\varepsilon^2\sigma\int\left(\frac{4\delta}{3}\rho^3+\frac{\partial(\rho f)}{\partial\rho}\right)(\rho-\rho_0)\\&\quad+\varepsilon\int|\nabla c|^2+\int|\nabla\mu|^2+\int\mathbb{S}_{ns}:\nabla u+\varepsilon^4\sigma\int\left(4\delta\rho^2+\frac{\partial^2(\rho f)}{\partial\rho^2}\right)|\nabla\rho|^2\\&=\sigma\int(\rho g_1+g_2)\cdot u-\varepsilon^4\sigma\int\frac{\partial^2(\rho f)}{\partial\rho\partial c}\nabla\rho\cdot\nabla c+\varepsilon\sigma\int\rho_0c_0\mu-\varepsilon\sigma\int\rho\frac{\partial f}{\partial c}c.\end{aligned}\tag{2.19}$$

We first assume that

$$H_1(c)\geq 1\quad\text{for all}\quad c\in\mathbb{R}.\tag{2.20}$$

(See Remark 2.3 for the opposite case). Then, from (1.5) we compute

$$4\delta\rho^2+\frac{\partial^2(\rho f)}{\partial\rho^2}\geq 4\delta\rho^2+\gamma(\gamma-1)\rho^{\gamma-2}+\rho^{-1}\geq 0.\tag{2.21}$$

Therefore,

$$\int \left(4\delta\rho^2 + \frac{\partial^2(\rho f)}{\partial \rho^2}\right)|\nabla\rho|^2 \geq \int \left(\delta|\nabla\rho^2|^2 + 4(\gamma-1)\gamma^{-1}|\nabla\rho^{\frac{\gamma}{2}}|^2 + 4|\nabla\sqrt{\rho}|^2\right)$$

and

$$\int \left(\frac{4\delta}{3}\rho^3 + \frac{\partial(\rho f)}{\partial\rho}\right)(\rho-\rho_0) \geq \int \left(\frac{\delta}{3}\rho^4 + \rho f(\rho,c)\right) - \int \left(\frac{\delta}{3}\rho_0^4 + \rho_0 f(\rho_0,c)\right).$$

Taking the last two inequalities into accounts, we estimate (2.19) as

$$\begin{aligned}
&\varepsilon^2\int(\rho+\rho_0)|u|^2 + \varepsilon^2\sigma\int\frac{\delta}{3}\rho^4 + \varepsilon^2\sigma\int\rho f(\rho,c)\\
&\quad + \varepsilon\int|\nabla c|^2 + \int|\nabla\mu|^2 + \int\mathbb{S}:\nabla u\\
&\quad + \varepsilon^4\sigma\int\left(\delta\left|\nabla\rho^2\right|^2 + 4(\gamma-1)\gamma^{-1}|\nabla\rho^{\frac{\gamma}{2}}|^2 + 4|\nabla\sqrt{\rho}|^2\right) \qquad (2.22)\\
&\leq \sigma\int(\rho g_1 + g_2)\cdot u + \varepsilon^2\sigma\int\frac{\delta}{3}\rho_0^4 - \varepsilon^2\sigma\int\rho_0 f(\rho_0,c)\\
&\quad - \varepsilon^4\sigma\int\frac{\partial^2(\rho f)}{\partial\rho\partial c}\nabla\rho\cdot\nabla c + \varepsilon\sigma\int\rho_0 c_0\mu - \varepsilon\sigma\int\rho\frac{\partial f}{\partial c}c.
\end{aligned}$$

Remark 2.3. If (2.20) fails, we follow the idea in [5] and express the $f(\rho,c)$ in (1.5) as

$$f(\rho,c) = \underbrace{\rho^{\gamma-1} + \left(H_1(c) + 2\overline{H}\mathbf{1}_{\{\rho\leq k\}}\right)\ln\rho + H_2(c)}_{\tilde{f}(\rho,c)} - 2\ln\rho\overline{H}\mathbf{1}_{\{\rho\leq k\}},$$

where $\overline{H}$ is taken from (1.10), and the constant k is large but fixed. Let us decompose the pressure function as

$$p = \rho^2\frac{\partial f}{\partial\rho} = \rho^2\frac{\partial\tilde{f}}{\partial\rho} - 2\overline{H}\rho\mathbf{1}_{\{\rho\leq k\}} = \tilde{p} - 2\overline{H}\rho\mathbf{1}_{\{\rho\leq k\}} \qquad (2.23)$$

and replace $p = \rho^2\frac{\partial f}{\partial\rho}$ with $\tilde{p} = \rho^2\frac{\partial\tilde{f}}{\partial\rho}$ in (2.16). We claim that (2.21) is also valid. To see this, if $\rho\leq k$, we have

$$\begin{aligned}
4\delta\rho^2 + \frac{\partial^2(\rho\tilde{f})}{\partial\rho^2} &= 4\delta\rho^2 + \gamma(\gamma-1)\rho^{\gamma-2} + (H_1+2\overline{H})\rho^{-1}\\
&\geq 4\delta\rho^2 + \gamma(\gamma-1)\rho^{\gamma-2} + \overline{H}\rho^{-1} > 0,
\end{aligned}$$

owing to $H_1(c) + 2\overline{H} > \overline{H}$; while if $\rho > k$,

$$\begin{aligned}
4\delta\rho^2 + \frac{\partial^2(\rho\tilde{f})}{\partial\rho^2} &= 4\delta\rho^2 + \frac{\partial^2(\rho f)}{\partial\rho^2}\\
&\geq 4\delta\rho^2 + \gamma(\gamma-1)\rho^{\gamma-2} + H_1(c)\rho^{-1} \geq 4\delta\rho^2 - \overline{H}\rho^{-1} > 2\delta\rho^2 > 0,
\end{aligned}$$

as long as $k = k(\delta,\overline{H})$ is taken to be large enough. However, the following extra term will be induced by the decomposition (2.23),

$$\int 2\overline{H}\rho\mathbf{1}_{\{\rho\leq k\}}\mathrm{div}u.$$

Fortunately, it can be bounded by $\|\nabla u\|_{L^2}$ because $\rho \mathbf{1}_{\{\rho \le k\}}$ is bounded. Without loss of generality, in what follows, we always assume that, for all $c \in \mathbb{R}$, $H_1(c)$ is positive and bounded from below.

Step 2. Let us deal with the terms on the right-hand side of (2.22). Thanks to (1.5), (1.10), (1.9), (2.1), and the Hölder inequality, the first three terms satisfy

$$\begin{aligned} &\sigma \int (\rho g_1 + g_2) \cdot u + \varepsilon^2 \sigma \int \frac{\delta}{3} \rho_0^4 + \varepsilon^2 \sigma \int \rho_0 f(\rho_0, c) \\ &\le C \|u\|_{L^6} \|\rho\|_{L^{\frac{6}{5}}} \|g_1\|_{L^\infty} + C \|u\|_{L^6} \|g_2\|_{L^\infty} + C \\ &\le C \left(1 + \|\rho\|^2_{L^{\frac{6}{5}}}\right) + \frac{\lambda_1}{2} \|\nabla u\|^2_{L^2}. \end{aligned} \tag{2.24}$$

Throughout this section, the positive constants $C, C_i\,(i = 1, 2, \cdots)$ may depend on g_1, g_2, λ_1, m_1, m_2, δ, γ, $\overline{H}$, $|\Omega|$, but not on ε or σ.

Using (1.5) and (1.10) again, one has

$$\begin{aligned} \varepsilon^4 \sigma \int \frac{\partial^2 (\rho f)}{\partial \rho \partial c} \nabla \rho \cdot \nabla c &\le C \varepsilon^4 \sigma \|\nabla c\|_{L^2} \|(1 + \ln \rho) \nabla \rho\|_{L^2} \\ &\le \frac{\varepsilon}{4} \|\nabla c\|^2_{L^2} + C_1 \varepsilon^7 \sigma \int \left(\delta \left|\nabla \rho^2\right|^2 + 4 |\nabla \sqrt{\rho}|^2\right). \end{aligned} \tag{2.25}$$

It follows from $(2.15)_4$ that

$$\int \rho \mu = \int \left(\rho \frac{\partial f}{\partial c} + \sigma^{-1} \triangle c\right) = \int \rho \frac{\partial f}{\partial c} \le C \left(\|\rho \ln \rho\|_{L^1} + 1\right). \tag{2.26}$$

Then we have, from (2.26) together with (2.1) and the Poincaré inequality,

$$\begin{aligned} \int \mu &= \frac{1}{\rho_0} \int \rho \left(\frac{1}{|\Omega|} \int \mu\right) \\ &= \frac{1}{\rho_0} \int \rho \mu - \frac{1}{\rho_0} \int \rho \left(\mu - \frac{1}{|\Omega|} \int \mu\right) \\ &\le C \left(\|\rho \ln \rho\|_{L^1} + 1\right) + C \|\rho\|_{L^{\frac{6}{5}}} \|\nabla \mu\|_{L^2}, \end{aligned} \tag{2.27}$$

which implies

$$\begin{aligned} \|\mu\|_{L^1} &\le C \|\nabla \mu\|_{L^2} + C \left(\|\rho \ln \rho\|_{L^1} + 1\right) + C \|\rho\|_{L^{\frac{6}{5}}} \|\nabla \mu\|_{L^2} \\ &\le C (1 + \|\nabla \mu\|_{L^2})(1 + \|\rho\|_{L^{\frac{6}{5}}}), \end{aligned}$$

where we have used $\|\rho \ln \rho\|_{L^1} \le C + \|\rho\|_{L^{\frac{6}{5}}}$, owing to the interpolation and $\|\rho\|_{L^1} = m_1$. Thus,

$$\|\mu\|_{L^p} \le C \left(1 + \|\nabla \mu\|_{L^2})(1 + \|\rho\|_{L^{\frac{6}{5}}}\right), \quad \forall\ p \in [1, 6]. \tag{2.28}$$

Thanks to (2.1) and (2.15), one has

$$\begin{aligned} \int \rho c &= m_2 + \varepsilon \int (\rho_0 - \rho) c + \varepsilon^3 \int c \triangle \rho \\ &= m_2 + \varepsilon \int (\rho_0 - \rho) c + \varepsilon^3 \int \rho \triangle c \\ &\le C + C \varepsilon \|\rho\|_{L^{\frac{6}{5}}} \|\nabla c\|_{L^2} + C \varepsilon^3 (\|\rho\|^2_{L^{\frac{12}{5}}} \|\mu\|_{L^6} + \|\rho^2 \ln \rho\|_{L^1}), \end{aligned} \tag{2.29}$$

where we have used the estimate:

$$\int(\rho-\rho_0)c=\int\rho c-\frac{1}{|\Omega|}\int m_1 c=\int\rho\left(c-\frac{1}{|\Omega|}\int c\right)\le C\|\rho\|_{L^{\frac{6}{5}}}\|\nabla c\|_{L^2}.$$

With the aid of (2.29), the same method as (2.27) yields,

$$\begin{aligned}\int c&=\frac{1}{\rho_0}\int\rho c-\frac{1}{\rho_0}\int\rho\left(c-\frac{1}{|\Omega|}\int c\right)\\&\le C+C\|\rho\|_{L^{\frac{6}{5}}}\|\nabla c\|_{L^2}+C\varepsilon^3(\|\rho\|^2_{L^{\frac{12}{5}}}\|\mu\|_{L^6}+\|\rho^2\ln\rho\|_{L^1}).\end{aligned}\tag{2.30}$$

Thus, for $p\in[1,6]$,

$$\|c\|_{L^p}\le C(1+\|\rho\|_{L^{\frac{6}{5}}})(1+\|\nabla c\|_{L^2})+C\varepsilon^3(\|\rho\|^2_{L^{\frac{12}{5}}}\|\mu\|_{L^6}+\|\rho^2\ln\rho\|_{L^1}).\tag{2.31}$$

Having (2.28) and (2.31) in hand, we can make the following computation and estimate,

$$\begin{aligned}&\varepsilon\sigma\int\rho_0 c_0\mu-\varepsilon\sigma\int\rho\frac{\partial f}{\partial c}c\\&\le C\sigma\varepsilon\left(\|\mu\|_{L^1}+\|\rho(\ln\rho+1)\|_{L^{\frac{6}{5}}}\|c\|_{L^6}\right)\\&\le C\sigma\varepsilon\left(1+\|\nabla\mu\|_{L^2}\right)\left(1+\|\rho\|_{L^{\frac{6}{5}}}\right)+C\sigma\varepsilon\left(1+\|\rho\ln\rho\|^2_{L^{\frac{6}{5}}}\right)(1+\|\nabla c\|_{L^2})\\&\quad+C\sigma\varepsilon^4\left(1+\|\rho\ln\rho\|_{L^{\frac{6}{5}}}\right)(\|\rho\|^2_{L^{\frac{12}{5}}}\|\mu\|_{L^6}+\|\rho^2\ln\rho\|_{L^1})\\&\le C+\frac{1}{2}\|\nabla\mu\|^2_{L^2}+\frac{\varepsilon}{4}\|\nabla c\|^2_{L^2}\\&\quad+C\sigma\left(\varepsilon\|\rho\ln\rho\|^4_{L^{\frac{6}{5}}}+\varepsilon^4\|\rho\ln\rho\|_{L^{\frac{6}{5}}}\|\rho^2\ln\rho\|_{L^1}+\varepsilon^8\|\rho\ln\rho\|^4_{L^{\frac{6}{5}}}\|\rho\|^4_{L^{\frac{12}{5}}}\right).\end{aligned}\tag{2.32}$$

Then, we compute

$$\begin{aligned}&\sigma\left(\varepsilon\|\rho\ln\rho\|^4_{L^{\frac{6}{5}}}+\varepsilon^4\|\rho\ln\rho\|_{L^{\frac{6}{5}}}\|\rho^2\ln\rho\|_{L^1}+\varepsilon^8\|\rho\ln\rho\|^4_{L^{\frac{6}{5}}}\|\rho\|^4_{L^{\frac{12}{5}}}\right)\\&\le C+\frac{\varepsilon^2\sigma}{8}\|\rho\ln\rho\|^8_{L^{\frac{6}{5}}}+\frac{\varepsilon^2\sigma}{8}\|\rho\|^4_{L^4}+\sigma\varepsilon^8\|\rho\ln\rho\|^4_{L^{\frac{6}{5}}}\|\rho\|^4_{L^{\frac{12}{5}}}\\&\le C+\frac{\varepsilon^2\sigma}{4}\|\rho\|^4_{L^4}+\sigma\varepsilon^8\|\rho\ln\rho\|^4_{L^{\frac{6}{5}}}\|\rho\|^4_{L^{\frac{12}{5}}}\\&\le C+\frac{\varepsilon^2\sigma}{4}\|\rho\|^4_{L^4}+\sigma\varepsilon^8\|\rho^2\|^{\frac{20}{11}}_{L^6}\\&\le C(\delta)+\frac{\varepsilon^2\sigma}{2}\|\rho\|^4_{L^4}+\frac{\sigma\varepsilon^4\delta}{2}\|\nabla(\rho^2)\|^2_{L^2},\end{aligned}$$

where in the third inequality sign we have used

$$\|\rho\ln\rho\|^4_{L^{\frac{6}{5}}}\|\rho\|^4_{L^{\frac{12}{5}}}\le C\|\rho\|^{\frac{38}{11}}_{L^{12}}\le C+\|\rho^2\|^{\frac{20}{11}}_{L^6},$$

owing to interpolation and the fact $\|\rho\|_{L^1}=m_1$. By the above estimates, substituting (2.24)-(2.25) and (2.32) into (2.22) concludes

$$\begin{aligned}&\varepsilon^2\sigma\|\rho\|^4_{L^4}+\|\nabla u\|^2_{L^2}+\|\nabla\mu\|^2_{L^2}+\varepsilon\|\nabla c\|^2_{L^2}+\varepsilon^4\sigma\int\left(\delta\left|\nabla\rho^2\right|^2+4|\nabla\sqrt{\rho}|^2\right)\\&\le C+C\|\rho\|^2_{L^{\frac{6}{5}}},\end{aligned}$$

which, along with (2.28) and (2.31), implies

$$\varepsilon^2\|\rho\|_{L^4}^4+\|u\|_{H_0^1}^2+\varepsilon\|c\|_{H^1}^2+\|\mu\|_{H^1}^2+\varepsilon^4\|\left|\nabla\rho^2\right|+|\nabla\sqrt{\rho}|\|_{L^2}^2\le C+C\|\rho\|_{L^{\frac{6}{5}}}^2. \tag{2.33}$$

Step 3. By (2.33), it is clear that

$$\|\rho\|_{L^4}^4+\|u\|_{H_0^1}^2+\|\mu\|_{H^1}^2+\|c\|_{H^1}^2+\|\left|\nabla\rho^2\right|+|\nabla\sqrt{\rho}|\|_{L^2}^2\le C(\varepsilon). \tag{2.34}$$

From [37, Lemma 3.17] we take the Bogovskii operator

$$\mathcal{B}=[\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3]:\ \left\{f\in L^p\mid\int f=0\right\}\mapsto W_0^{1,p}(\Omega),\quad p\in(1,\infty). \tag{2.35}$$

Then, $\mathrm{div}\mathcal{B}(f)=f$ a.e. in Ω, and moreover,

$$\|\nabla\mathcal{B}(f)\|_{L^p}\le C\|f\|_{L^p},\quad \|\mathcal{B}(f)\|_{L^p}\le C\|g\|_{L^p}, \tag{2.36}$$

where $f=\mathrm{div}g$ and $g\in L^p$ with $g\cdot n|_{\partial\Omega}=0$. Furthermore, we write $(2.15)_1$ as the equivalent form

$$\varepsilon^4\triangle\rho=\mathrm{div}(\rho u+\varepsilon^2\mathcal{B}(\rho-\rho_0)). \tag{2.37}$$

Applying Lemma 2.2 to (2.37), using (2.34) and (2.36), we find

$$\begin{aligned}\|\nabla\rho\|_{L^4}&\le\|\rho u+\varepsilon^2\mathcal{B}(\rho-\rho_0)\|_{L^4}\\&\le\|\rho u\|_{L^4}+\|\nabla\mathcal{B}(\rho-\rho_0)\|_{L^4}\\&\le\|u\|_{L^6}\|\rho^2\|_{L^6}^{\frac{1}{2}}+\|\rho-\rho_0\|_{L^4}\le C(\varepsilon),\end{aligned} \tag{2.38}$$

and hence,

$$\begin{aligned}\|\rho\|_{H^2}&\le C\|\mathrm{div}(\rho u+\varepsilon^2\mathcal{B}(\rho-\rho_0))\|_{L^2}\\&\le\|u\cdot\nabla\rho+\rho\mathrm{div}u\|_{L^2}+\|\mathrm{div}\mathcal{B}(\rho-\rho_0)\|_{L^2}\le C(\varepsilon).\end{aligned} \tag{2.39}$$

Combining (2.34) with (2.39) gives

$$\|\sigma F^1(u,\mu,c)\|_{L^{\frac{3}{2}}}+\|\sigma F^2(u,\mu,c)\|_{L^6}+\|\sigma F^3(u,\mu,c)\|_{L^6}\le C(\varepsilon).$$

By L^p regularity estimates, we obtain

$$\|u\|_{W^{2,\frac{3}{2}}}+\|\mu\|_{W^{2,6}}+\|c\|_{W^{2,6}}\le C(\varepsilon). \tag{2.40}$$

From (2.40) we have $\|\sigma F^1(u,\mu,c)\|_{L^6}\le C(\varepsilon)$, and thus $\|u\|_{W^{2,6}}\le C(\varepsilon)$. By a bootstrap procedure,

$$\|(u,\mu,c)\|_{W^{2,p}}\le C(\varepsilon),\quad\forall\ p\in(1,\infty).$$

This completes the proof of Proposition 2.1 and (2.9). □

3. ε-Limit procedure for the approximation solutions

In this section, we shall take the ε-limit procedure and prove the following result.

Theorem 3.1. *Under the same assumptions as in Theorem 2.1, the system*

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) + \nabla\left(\delta\rho^4 + \rho^2\dfrac{\partial f}{\partial \rho}\right) = \operatorname{div}\mathbb{S}_{ns} + \rho\mu\nabla c - \rho\dfrac{\partial f}{\partial c}\nabla c + \rho g_1 + g_2, \\ \rho u\cdot\nabla c = \triangle\mu, \\ \rho\mu = \rho\dfrac{\partial f}{\partial c} - \triangle c, \end{cases} \tag{3.1}$$

with the boundary conditions (1.6) *admits a weak solution* (ρ, u, μ, c) *such that*

$$\int \rho = m_1, \quad \int \rho c = m_2, \tag{3.2}$$

$$0 \le \rho \in L^5(\Omega), \ \ u \in H_0^1(\Omega, \mathbb{R}^3), \ \ (\mu,\, c) \in H^1(\Omega)\times H^1(\Omega). \tag{3.3}$$

Moreover, $\forall\ \Phi \in C_0^\infty(\Omega, \mathbb{R}^3)$ *and* $\phi \in C^\infty(\overline{\Omega})$,

$$\begin{aligned} &\int \left(\delta\rho^4 + \rho^2\frac{\partial f}{\partial\rho}\right)\operatorname{div}\Phi \\ &= \int (\mathbb{S}_{ns} - \rho u\otimes u) : \nabla\Phi + \int\left(\rho\frac{\partial f}{\partial c}\nabla c - \rho\mu\nabla c - \rho g_1 - g_2\right)\cdot\Phi, \end{aligned} \tag{3.4}$$

and

$$\int \rho u\cdot\nabla c\phi + \int \nabla\mu\cdot\nabla\phi = 0, \quad \int \rho\mu\phi - \rho\frac{\partial f}{\partial c}\phi = \int \nabla c\cdot\nabla\phi; \tag{3.5}$$

when (ρ, u) *is prolonged by zero outside* Ω,

$$\int_{\mathbb{R}^3} b(\rho)u\cdot\nabla\phi = \int_{\mathbb{R}^3}\phi\left(b'(\rho)\rho - b(\rho)\right)\operatorname{div}u, \tag{3.6}$$

where $b(z) = z$, *or* $b(z) \in C^1([0,\infty))$ *with* $b'(z) = 0$ *if* z *is large; and the following energy inequality holds:*

$$\int \left(\lambda_1|\nabla u|^2 + (\lambda_1+\lambda_2)(\operatorname{div}u)^2 + |\nabla\mu|^2\right) \le \int (\rho g_1 + g_2)\cdot u. \tag{3.7}$$

Theorem 3.1 is indeed a result of ε-limit of the solutions $(\rho_\varepsilon, u_\varepsilon, \mu_\varepsilon, c_\varepsilon)$ obtained in Theorem 2.1, as shown below. First the following lemma derives some uniform in ε estimates on $(\rho_\varepsilon, u_\varepsilon, \mu_\varepsilon, c_\varepsilon)$.

Lemma 3.1. *Let* $(\rho_\varepsilon, u_\varepsilon, \mu_\varepsilon, c_\varepsilon)$ *be a solution in Theorem 2.1. Then there exists a constant* C *which is independent of* ε, *such that*

$$\|\rho_\varepsilon^5\|_{L^1} + \|\rho_\varepsilon^3\frac{\partial f}{\partial\rho_\varepsilon}\|_{L^1} + \|c_\varepsilon\|_{H^1} \le C. \tag{3.8}$$

Proof. Let $\mathcal{B}$ be the Bogovskii operator as defined in (2.35). If we test $(2.2)_2$ by $\mathcal{B}(\rho_\varepsilon-\rho_0)$, we infer

$$\begin{aligned}
&\int\left(\delta\rho_\varepsilon^4+\rho_\varepsilon^2\frac{\partial f}{\partial\rho_\varepsilon}\right)\rho_\varepsilon\\
&=\int\left(\delta\rho_\varepsilon^4+\rho_\varepsilon^2\frac{\partial f}{\partial\rho_\varepsilon}\right)\rho_0-\int(\rho_\varepsilon g_1+g_2)\cdot\mathcal{B}(\rho_\varepsilon-\rho_0)\\
&\quad+\varepsilon^2\int\rho_\varepsilon u_\varepsilon\cdot\mathcal{B}(\rho_\varepsilon-\rho_0)+\varepsilon^4\int\nabla\rho_\varepsilon\cdot\nabla u_\varepsilon\mathcal{B}(\rho_\varepsilon-\rho_0)\\
&\quad-\int\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon:\nabla\mathcal{B}(\rho_\varepsilon-\rho_0)\\
&\quad+\int\lambda_1(\nabla u_\varepsilon+(\nabla u_\varepsilon)^\top):\nabla\mathcal{B}(\rho_\varepsilon-\rho_0)+\lambda_2\mathrm{div}u_\varepsilon\mathrm{div}\mathcal{B}(\rho_\varepsilon-\rho_0)\\
&\quad+\int\left(\rho_\varepsilon\frac{\partial f}{\partial c_\varepsilon}-\rho_\varepsilon\mu_\varepsilon\right)\mathcal{B}(\rho_\varepsilon-\rho_0)\cdot\nabla c_\varepsilon\\
&=\sum_{i=1}^7 I_i.
\end{aligned}\tag{3.9}$$

Owing to (1.5), (1.10), (2.8), (2.36), and the simple fact $\rho^2\frac{\partial f}{\partial\rho}=(\gamma-1)\rho^\gamma+H_1\rho$, we get

$$\begin{aligned}
I_1+I_2&\le\left|\int\left(\delta\rho_\varepsilon^4+\rho_\varepsilon^2\frac{\partial f}{\partial\rho_\varepsilon}\right)\rho_0\right|+\left|\int(\rho_\varepsilon g_1+g_2)\mathcal{B}(\rho_\varepsilon-\rho_0)\right|\\
&\le C\int\left(\delta\rho_\varepsilon^4+\rho_\varepsilon^2\frac{\partial f}{\partial\rho_\varepsilon}\right)+C(1+\|\rho_\varepsilon\|_{L^{\frac{6}{5}}})\|\nabla\mathcal{B}(\rho_\varepsilon-\rho_0)\|_{L^2}\\
&\le\frac{1}{8}\int\left(\delta\rho_\varepsilon^5+\rho_\varepsilon^3\frac{\partial f}{\partial\rho_\varepsilon}\right)+C.
\end{aligned}$$

Next, by (2.33) we have

$$\begin{aligned}
&I_3+I_4+I_5\\
&\le\left(\varepsilon^2\|\rho_\varepsilon\|_{L^2}+\varepsilon^4\|\nabla\rho_\varepsilon\|_{L^2}\right)\|u_\varepsilon\|_{H^1}\|\mathcal{B}(\rho_\varepsilon-\rho_0)\|_{L^\infty}\\
&\quad+\|\rho_\varepsilon\|_{L^{\frac{12}{5}}}\|u_\varepsilon\|_{L^6}^2\|\nabla\mathcal{B}(\rho_\varepsilon-\rho_0)\|_{L^4}\\
&\le C\left(1+\|\rho_\varepsilon\|_{L^{\frac{6}{5}}}^2\right)\|\rho_\varepsilon\|_{L^{\frac{12}{5}}}\|\mathcal{B}(\rho_\varepsilon-\rho_0)\|_{W^{1,4}}\\
&\le\frac{\delta}{8}\|\rho_\varepsilon\|_{L^5}^5+C.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_6&\le\left|\int\lambda_1(\nabla u_\varepsilon+(\nabla u_\varepsilon)^\top):\nabla\mathcal{B}(\rho_\varepsilon-\rho_0)+\lambda_2\mathrm{div}u_\varepsilon\mathrm{div}\mathcal{B}(\rho_\varepsilon-\rho_0)\right|\\
&\le C\|\nabla u_\varepsilon\|_{L^2}\|\rho_\varepsilon-\rho_0\|_{L^2}\le\frac{\delta}{8}\|\rho_\varepsilon\|_{L^5}^5+C.
\end{aligned}$$

To deal with the last term, we multiply $(2.2)_4$ by c_ε, then use (1.5), (1.10), (2.31), (2.33) and the interpolation inequality to deduce

$$\begin{aligned}\|\nabla c_\varepsilon\|_{L^2}^2 &= \int \left(\rho_\varepsilon\mu_\varepsilon - \rho_\varepsilon\frac{\partial f}{\partial c_\varepsilon}\right) c_\varepsilon \\ &\le \|c_\varepsilon\|_{L^2}\left(\int |\rho_\varepsilon\mu_\varepsilon - \rho_\varepsilon\frac{\partial f}{\partial c_\varepsilon}|^2\right)^{\frac{1}{2}} \\ &\le C\|c_\varepsilon\|_{L^2}\left(\|\rho_\varepsilon\|_{L^3}\|\mu_\varepsilon\|_{L^6} + \|\rho_\varepsilon\ln\rho_\varepsilon\|_{L^2} + 1\right) \\ &\le \frac{1}{2}\|\nabla c_\varepsilon\|_{L^2}^2 + C\left(\|\rho_\varepsilon\|_{L^5}^{\frac{5}{2}} + 1\right),\end{aligned} \tag{3.10}$$

whence,

$$\begin{aligned}I_7 &\le \left|\int \left(\rho_\varepsilon\frac{\partial f}{\partial c_\varepsilon} - \rho_\varepsilon\mu_\varepsilon\right)\mathcal{B}(\rho_\varepsilon - \rho_0)\cdot\nabla c_\varepsilon\right| \\ &\le C\|\mathcal{B}(\rho_\varepsilon - \rho_0)\|_{L^\infty}\|\nabla c_\varepsilon\|_{L^2}\left(\int |\rho_\varepsilon\mu_\varepsilon - \rho_\varepsilon\frac{\partial f}{\partial c_\varepsilon}|^2\right)^{\frac{1}{2}} \\ &\le C\|\rho_\varepsilon\|_{L^5}^4 + C.\end{aligned}$$

In summary, substituting the estimates above back into (3.9), using (3.10), we conclude

$$\|\nabla c_\varepsilon\|_{L^2} + \int\left(\delta\rho_\varepsilon^5 + \rho_\varepsilon^3\frac{\partial f}{\partial\rho_\varepsilon}\right) \le C.$$

This, along with (2.33) and (2.31), gives rise to (3.8). The proof of Lemma 3.1 is completed.

Having (1.5), (1.10), (2.33) and (3.8) in hand, we can take the limit as $\varepsilon\to 0$ of $(\rho_\varepsilon, u_\varepsilon, \mu_\varepsilon, c_\varepsilon)$, subject to some subsequence, so that,

$$\rho_\varepsilon \rightharpoonup \rho \text{ in } L^5\cap L^{\gamma+1}, \quad \rho_\varepsilon^4 \rightharpoonup \overline{\rho^4} \quad \text{in} \quad L^{\frac{5}{4}}, \tag{3.11}$$

$$(\nabla u_\varepsilon,\, \nabla\mu_\varepsilon, \nabla c_\varepsilon) \rightharpoonup (\nabla u,\, \nabla\mu,\, \nabla c_\varepsilon) \text{ in } L^2, \tag{3.12}$$

$$(u_\varepsilon,\, \mu_\varepsilon,\, c_\varepsilon) \to (u,\, \mu,\, c) \text{ in } L^{p_1} \ (1\le p_1<6), \tag{3.13}$$

$$\varepsilon^4\nabla\rho_\varepsilon \to 0 \text{ in } L^2, \tag{3.14}$$

$$\varepsilon^2\rho_\varepsilon\to 0, \quad \varepsilon^2\rho_\varepsilon u_\varepsilon\to 0, \quad \varepsilon\rho_\varepsilon c_\varepsilon\to 0, \ \varepsilon^4\nabla\rho_\varepsilon\nabla u_\varepsilon\to 0 \text{ in } L^1. \tag{3.15}$$

Moreover, it follows from (3.11) and (3.13) that

$$(\rho_\varepsilon u_\varepsilon,\, \rho_\varepsilon\mu_\varepsilon) \rightharpoonup (\rho u,\, \rho\mu) \text{ in } L^2, \qquad \rho_\varepsilon u_\varepsilon\otimes u_\varepsilon \rightharpoonup \rho u\otimes u \text{ in } L^p \text{ (for some } p>1), \tag{3.16}$$

and

$$\rho_\varepsilon\frac{\partial f}{\partial c_\varepsilon} = \rho_\varepsilon\ln\rho_\varepsilon H_1'(c_\varepsilon) + \rho_\varepsilon H_2'(c_\varepsilon) \rightharpoonup \overline{\rho\ln\rho}H_1'(c) + \rho H_2'(c) = \overline{\rho\frac{\partial f}{\partial c}} \text{ in } L^2, \tag{3.17}$$

$$\rho_\varepsilon^2\frac{\partial f}{\partial\rho_\varepsilon} = (\gamma-1)\rho_\varepsilon^\gamma + \rho_\varepsilon H_1(c_\varepsilon) \rightharpoonup (\gamma-1)\overline{\rho^\gamma} + \rho H_1(c) = \overline{\rho^2\frac{\partial f}{\partial\rho}} \text{ in } L^{\frac{\gamma+1}{\gamma}}. \tag{3.18}$$

It remains to verify the strong convergence of ∇c_ε, i.e.,

$$\nabla c_\varepsilon\to\nabla c \quad \text{in} \quad L^2. \tag{3.19}$$

In fact, as in [5], we use equality $(2.2)_4$ to obtain

$$\int\nabla c_\varepsilon\nabla\phi = \int\rho_\varepsilon\mu_\varepsilon\phi - \int\rho_\varepsilon\frac{\partial f}{\partial c_\varepsilon}\phi, \tag{3.20}$$

which, together with (3.13), (3.16)-(3.17), provides us

$$\lim_{\varepsilon\to 0}\int |\nabla c_\varepsilon|^2 = \lim_{\varepsilon\to 0}\int \rho_\varepsilon\mu_\varepsilon c_\varepsilon - \lim_{\varepsilon\to 0}\int \rho_\varepsilon\frac{\partial f}{\partial c_\varepsilon}c_\varepsilon = \int \rho\mu c - \int \overline{\rho\frac{\partial f}{\partial c}}c.$$

On the other hand, if we select $\phi = c$ in (3.20), we obtain

$$\int |\nabla c|^2 = \lim_{\varepsilon\to 0}\int \nabla c_\varepsilon\cdot\nabla c = \lim_{\varepsilon\to 0}\int \rho_\varepsilon\mu_\varepsilon c - \lim_{\varepsilon\to 0}\int \rho_\varepsilon\frac{\partial f}{\partial c_\varepsilon}c = \int \rho\mu c - \int \overline{\rho\frac{\partial f}{\partial c}}c.$$

Thus

$$\lim_{\varepsilon\to 0}\int |\nabla c_\varepsilon|^2 = \int |\nabla c|^2,$$

which, together with (3.12), guarantees (3.19).

From (3.11)-(3.19), we are able to pass limit and get the integral equalities (3.4)-(3.5) with ρ^4, $\rho\frac{\partial f}{\partial c}$, $\rho^2\frac{\partial f}{\partial \rho}$ replaced by $\overline{\rho^4}$, $\overline{\rho\frac{\partial f}{\partial c}}$, $\overline{\rho^2\frac{\partial f}{\partial \rho}}$, respectively. In addition, we obtain (3.2) and (3.7) from (2.1), $(2.2)_1$, (2.29), and (3.8).

Finally, (3.6) is guaranteed by the following lemma, whose proof is available in [38, Lemma 2.1] and [37, Lemma 3.3].

Lemma 3.2. *Let (ρ, u) be a solution to $(3.1)_1$. Assume that $\rho \in L^2(\Omega)$ and $u \in H_0^1(\Omega, \mathbb{R}^3)$. If we extend (ρ, u) by zero outside Ω, we have*

$$\mathrm{div}(b(\rho)u) + (b'(\rho)\rho - b(\rho))\mathrm{div}u = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3), \tag{3.21}$$

where $b(z) = z$, or $b \in C^1([0,\infty))$ with $b'(z) = 0$ for large z.

In order to complete the proof of Theorem 3.1, we need to verify

$$\overline{\rho^4} = \rho^4, \quad \overline{\rho\frac{\partial f}{\partial c}} = \rho\frac{\partial f}{\partial c}, \quad \overline{\rho^2\frac{\partial f}{\partial \rho}} = \rho^2\frac{\partial f}{\partial \rho}. \tag{3.22}$$

For that purpose, let us define

$$C^2([0,\infty)) \ni b_n(\rho) = \begin{cases} \rho\ln(\rho + \dfrac{1}{n}), & \rho \le n; \\ (n+1)\ln(n+1+\dfrac{1}{n}), & \rho \ge n+1. \end{cases}$$

First we see that $b_n(\rho) \to \rho\ln\rho$ a.e. because of the fact: $\rho \in L^1$. Select b_n in (3.21) and send $n \to \infty$ to obtain

$$\mathrm{div}(u\rho\ln\rho) + \rho\mathrm{div}u = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3).$$

This implies

$$\int \rho\mathrm{div}u = 0. \tag{3.23}$$

On the other hand, multiplying $(2.2)_1$ by $b_n'(\rho_\varepsilon)$ gives

$$\begin{aligned} &\int (b_n'(\rho_\varepsilon)\rho_\varepsilon - b_n(\rho_\varepsilon))\mathrm{div}u_\varepsilon \\ &= \varepsilon^2\int \rho_0 b_n'(\rho_\varepsilon) - \varepsilon^2\int \rho_\varepsilon b_n'(\rho_\varepsilon) - \varepsilon^4\int b_n''(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2 \\ &\le \varepsilon^2\int \rho_0 b_n'(\rho_\varepsilon) - \varepsilon^2\int \rho_\varepsilon b_n'(\rho_\varepsilon) - \varepsilon^4\int_{\{x:\, b_n''(\rho_\varepsilon)\le 0\}} b_n''(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2. \end{aligned} \tag{3.24}$$

It follows from (2.34) that $\|\nabla\rho_\varepsilon\|_{L^2} \le C\left(\|\nabla\rho_\varepsilon^2\|_{L^2} + \|\nabla\sqrt{\rho_\varepsilon}\|_{L^2}\right) \le C(\varepsilon)$. Then, for fixed $\varepsilon > 0$,

$$\begin{aligned}\left|-\varepsilon^4 \int_{\{x:\, b_n''(\rho_\varepsilon)\le 0\}} b_n''(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2\right| &\le C(\varepsilon)\int_{\{x:\, b_n''(\rho_\varepsilon)\le 0\}} |\nabla\rho_\varepsilon|^2 \\ &\le C(\varepsilon)\int_{\{x:\, n\le\rho_\varepsilon\le n+1\}} |\nabla\rho_\varepsilon|^2 \to 0 \quad (n\to\infty),\end{aligned}$$

where in the second inequality we have used the fact $b_n''(\rho_\varepsilon) \ge 0$ if $\rho_\varepsilon \le n$ or $\rho_\varepsilon \ge n+1$.

Recalling (3.8) and the definition of b_n, one deduces

$$\begin{aligned}&\lim_{n\to\infty}\int \rho_0 b_n'(\rho_\varepsilon) \\ &= \lim_{n\to\infty}\left(\int_{\{\rho_\varepsilon\le n\}} \rho_0 b_n'(\rho_\varepsilon) + \int_{\{\rho_\varepsilon> n\}} \rho_0 b_n'(\rho_\varepsilon)\right) \\ &\le \lim_{n\to\infty}\int_{\{\rho_\varepsilon\le n\}} \rho_0\left(\ln(\rho_\varepsilon+\frac{1}{n}) + \frac{\rho_\varepsilon}{\rho_\varepsilon+\frac{1}{n}}\right) + C\lim_{n\to\infty}\text{meas}\,|\{x;\,\rho_\varepsilon\ge n\}| \\ &\le \lim_{n\to\infty}\int_{\{1/2\le\rho_\varepsilon\le n\}} \rho_0\ln(\rho_\varepsilon+\frac{1}{n}) + \lim_{n\to\infty}\int\frac{\rho_0\rho_\varepsilon}{\rho_\varepsilon+\frac{1}{n}} \\ &\le C.\end{aligned}$$

Similarly,

$$\lim_{n\to\infty}\int \rho_\varepsilon b_n'(\rho_\varepsilon) \le C.$$

Therefore, taking sequentially $n\to\infty$ and $\varepsilon\to 0$ in (3.24), using (3.23), one has

$$\int\overline{\rho\text{div}u} = \lim_{\varepsilon\to 0}\int\rho_\varepsilon\text{div}u_\varepsilon \le 0 = \int\rho\text{div}u. \tag{3.25}$$

To proceed, define the following effective viscous flux:

$$\mathbb{F}_\varepsilon = \delta\rho_\varepsilon^4 + \rho_\varepsilon^2\frac{\partial f}{\partial\rho_\varepsilon} - (2\lambda_1+\lambda_2)\text{div}u_\varepsilon \quad\text{and}\quad \overline{\mathbb{F}} = \delta\overline{\rho^4} + \overline{\rho^2\frac{\partial f}{\partial\rho}} - (2\lambda_1+\lambda_2)\text{div}u.$$

We have the following lemma.

Lemma 3.3. *Under the assumptions in Theorem 3.1, the following property holds:*

$$\lim_{\varepsilon\to 0}\int\phi\rho_\varepsilon\mathbb{F}_\varepsilon = \int\phi\rho\overline{\mathbb{F}}, \quad \forall\ \phi\in C_0^\infty(\Omega). \tag{3.26}$$

Let us continue to prove (3.22) with the aid of (3.26). The proof of Lemma 3.3 will be postponed to the end of this section.

In view of (3.25), $\mathbb{F}_\varepsilon$ and $\overline{\mathbb{F}}$, we take $\phi\to 1$ in (3.26) and deduce

$$\lim_{\varepsilon\to 0}\int\rho_\varepsilon\left(\delta\rho_\varepsilon^4 + \rho_\varepsilon^2\frac{\partial f}{\partial\rho_\varepsilon}\right) \le \int\rho\left(\delta\overline{\rho^4} + \overline{\rho^2\frac{\partial f}{\partial\rho}}\right). \tag{3.27}$$

According to (3.18) and (3.27), we have

$$\begin{aligned}
&\int \left(\delta\overline{\rho^5} + (\gamma-1)\overline{\rho^{\gamma+1}} + \overline{\rho^2}H_1(c)\right) \\
&= \lim_{\varepsilon\to 0}\int \rho_\varepsilon\left(\delta\rho_\varepsilon^4 + (\gamma-1)\rho_\varepsilon^\gamma + \rho_\varepsilon H_1(c_\varepsilon)\right) \\
&= \lim_{\varepsilon\to 0}\int \rho_\varepsilon\left(\delta\rho_\varepsilon^4 + \rho_\varepsilon^2\frac{\partial f}{\partial\rho_\varepsilon}\right) \\
&\le \int \rho\left(\delta\overline{\rho^4} + \overline{\rho^2\frac{\partial f}{\partial\rho}}\right) = \int \rho\left(\delta\overline{\rho^4} + (\gamma-1)\overline{\rho^\gamma} + \rho H_1(c)\right),
\end{aligned}$$

which implies

$$\int \delta\left(\rho\overline{\rho^4} - \overline{\rho^5}\right) \ge (\gamma-1)\int\left(\overline{\rho^{\gamma+1}} - \rho\overline{\rho^\gamma}\right) + \int\left(\overline{\rho^2} - \rho^2\right)H_1(c) \ge 0, \tag{3.28}$$

where the last inequality is due to convexity and $H_1(c) \ge 0$. Next, for the given constant $\beta > 0$ and any $\eta \in C^\infty(\Omega)$,

$$\begin{aligned}
0 &\le \int \left(\rho_\varepsilon^4 - (\rho+\beta\eta)^4\right)\left(\rho_\varepsilon - (\rho+\beta\eta)\right) \\
&= \int \left(\rho_\varepsilon^5 - \rho_\varepsilon^4\rho - \rho_\varepsilon^4\beta\eta - (\rho+\beta\eta)^4\rho_\varepsilon + (\rho+\beta\eta)^5\right).
\end{aligned}$$

By (3.28), as $\varepsilon \to 0$,

$$0 \le \int\left(\overline{\rho^5} - \rho\overline{\rho^4} - \overline{\rho^4}\beta\eta + (\rho+\beta\eta)^4\beta\eta\right) \le \int\left(-\overline{\rho^4} + (\rho+\beta\eta)^4\right)\beta\eta.$$

Replacing $-\beta$ with β in the argument above, and then taking $\beta \to 0$, we get

$$\int\left(\rho^4 - \overline{\rho^4}\right)\eta = 0.$$

This implies $\overline{\rho^4} = \rho^4$, and thus $\rho_\varepsilon \to \rho$ a.e. in Ω since η is arbitrary. Moreover, (3.11) implies that, for all $s \in [1,5)$,

$$\rho_\varepsilon \to \rho \quad \text{in} \quad L^s. \tag{3.29}$$

As a result of (3.29), (3.11), (3.17)-(3.18), we obtain (3.22). The proof of Theorem 3.1 is completed. □

Proof of Lemma 3.3. We will prove Lemma 3.3 by the results developed in [32]. Let $\triangle^{-1}(h) = K * h$ be the convolution of h with the fundamental solution K of the Laplacian in $\mathbb{R}^3$. For $\partial_i\triangle^{-1}$ $(i = 1,2,3)$, by the Mikhlin multiplier theory (cf. [42]),

$$\begin{cases}
\|\partial_i\triangle^{-1}(h)\|_{W^{1,p}(\Omega)} \le C(\Omega,p)\|h\|_{L^p(\mathbb{R}^3)}, \quad p \in (1,\infty), \\
\|\partial_i\triangle^{-1}(h)\|_{L^{p^*}(\Omega)} \le C(\Omega,p)\|\partial_i\triangle^{-1}(h)\|_{W^{1,p}(\mathbb{R}^3)}, \quad p^* = \dfrac{3p}{3-p},\ p<3, \\
\|\partial_i\triangle^{-1}(h)\|_{L^\infty(\Omega)} \le C(\Omega,p)\|h\|_{L^p(\mathbb{R}^3)}, \quad p > 3.
\end{cases} \tag{3.30}$$

If $h_n \rightharpoonup h$ in $L^p(\mathbb{R}^3)$, we have

$$\partial_j\partial_i\triangle^{-1}(h_n) \rightharpoonup \partial_j\partial_i\triangle^{-1}(h) \quad \text{in} \quad L^p, \tag{3.31}$$

and additionally, by the Rellich-Kondrachov compactness theorem,

$$\partial_i\triangle^{-1}(h_n)\to\partial_i\triangle^{-1}(h)\quad\text{in } L^q, \tag{3.32}$$

where $q<p^*$ if $p<3$ and $q\le\infty$ if $p>3$.

Prolonging ρ_ε to the whole space $\mathbb{R}^3$ by zero, multiplying $(2.2)_2^i$ by $\phi\partial_i\triangle^{-1}(\rho_\varepsilon)$ with $\phi\in C_0^\infty(\Omega)$, we obtain

$$\begin{aligned}
&\int\phi\rho_\varepsilon\mathbb{F}_\varepsilon\\
&=-\int\partial_i\triangle^{-1}(\rho_\varepsilon)\partial_i\phi\left(\delta\rho_\varepsilon^4+\rho_\varepsilon^2\frac{\partial f}{\partial\rho_\varepsilon}-(\lambda_1+\lambda_2)\mathrm{div}u_\varepsilon\right)\\
&\quad+\lambda_1\int\left(\partial_ju_\varepsilon^i\partial_i\triangle^{-1}(\rho_\varepsilon)\partial_j\phi-u_\varepsilon^i\partial_j\partial_i\triangle^{-1}(\rho_\varepsilon)\partial_j\phi+\rho_\varepsilon u_\varepsilon\cdot\nabla\phi\right)\\
&\quad-\int\left(\rho_\varepsilon\mu_\varepsilon\partial_ic_\varepsilon-\rho_\varepsilon\frac{\partial f}{\partial c_\varepsilon}\partial_ic_\varepsilon\right)\phi\partial_i\triangle^{-1}(\rho_\varepsilon)-\int(\rho_\varepsilon g_1+g_2)\,\phi\partial_i\triangle^{-1}(\rho_\varepsilon)\\
&\quad-\int\rho_\varepsilon u_\varepsilon^ju_\varepsilon^i\partial_j\phi\partial_i\triangle^{-1}(\rho_\varepsilon)-\int\rho_\varepsilon u_\varepsilon^ju_\varepsilon^i\phi\partial_j\partial_i\triangle^{-1}(\rho_\varepsilon)\\
&\quad+\varepsilon^2\int\rho_\varepsilon u_\varepsilon^i\phi\partial_i\triangle^{-1}(\rho_\varepsilon)+\varepsilon^4\int\nabla\rho_\varepsilon\cdot\nabla u_\varepsilon^i\phi\partial_i\triangle^{-1}(\rho_\varepsilon),
\end{aligned}\tag{3.33}$$

where the second line on the right-hand side comes from

$$\begin{aligned}
&\lambda_1\int\partial_ju_\varepsilon^i\left(\partial_i\triangle^{-1}(\rho_\varepsilon)\partial_j\phi+\partial_j\partial_i\triangle^{-1}(\rho_\varepsilon)\phi\right)\\
&=\lambda_1\int\left(\partial_ju_\varepsilon^i\partial_i\triangle^{-1}(\rho_\varepsilon)\partial_j\phi-u_\varepsilon^i\partial_j\partial_i\triangle^{-1}(\rho_\varepsilon)\partial_j\phi-u_\varepsilon^i\partial_i\rho_\varepsilon\phi\right)\\
&=\lambda_1\int\left(\partial_ju_\varepsilon^i\partial_i\triangle^{-1}(\rho_\varepsilon)\partial_j\phi-u_\varepsilon^i\partial_j\partial_i\triangle^{-1}(\rho_\varepsilon)\partial_j\phi+\rho_\varepsilon u_\varepsilon\cdot\nabla\phi\right)+\lambda_1\int\rho_\varepsilon\mathrm{div}u_\varepsilon\phi.
\end{aligned}$$

Next, since $(\rho_\varepsilon,u_\varepsilon)\in(H^1,H_0^1)$, then $\mathrm{div}(\rho_\varepsilon u_\varepsilon)\in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $\mathrm{div}(\rho_\varepsilon u_\varepsilon)=0$ in $\mathbb{R}^3\backslash\Omega$. In addition, $\rho_\varepsilon\in H^2$ and $\frac{\partial\rho_\varepsilon}{\partial n}|_{\partial\Omega}=0$ imply

$$\mathrm{div}(\mathbf{1}_\Omega\nabla\rho_\varepsilon)=\begin{cases}\triangle\rho_\varepsilon, & \text{in }\Omega,\\ 0, & \mathbb{R}^3\setminus\Omega.\end{cases}$$

Thus, it makes sense to extend $(2.2)_1$ to the whole space by zero,

$$\varepsilon^2(\rho_\varepsilon-\rho_0)+\mathrm{div}(\rho_\varepsilon u_\varepsilon)=\varepsilon^4\mathrm{div}(\mathbf{1}_\Omega\nabla\rho_\varepsilon)\quad\text{in}\quad\mathbb{R}^3,$$

which yields by straight forward computations,

$$\begin{aligned}
&-\int\rho_\varepsilon u_\varepsilon^i\phi\partial_i\partial_j\triangle^{-1}(\rho_\varepsilon u_\varepsilon^j)\\
&=-\int\rho_\varepsilon u_\varepsilon^i\phi\partial_i\triangle^{-1}(\mathrm{div}(\rho_\varepsilon u_\varepsilon))\\
&=-\varepsilon^4\int\rho_\varepsilon u_\varepsilon^i\phi\partial_i\triangle^{-1}(\mathrm{div}(\mathbf{1}_\Omega\nabla\rho_\varepsilon))+\varepsilon^2\int\rho_\varepsilon u_\varepsilon^i\phi\partial_i\triangle^{-1}(\rho_\varepsilon-\rho_0),
\end{aligned}$$

and

$$
\begin{aligned}
&-\int \rho_\varepsilon u_\varepsilon^j u_\varepsilon^i \partial_j\phi\partial_i\triangle^{-1}(\rho_\varepsilon) - \int \rho_\varepsilon u_\varepsilon^j u_\varepsilon^i \phi\partial_j\partial_i\triangle^{-1}(\rho_\varepsilon)\\
&= -\int \rho_\varepsilon u_\varepsilon^j u_\varepsilon^i \partial_j\phi\partial_i\triangle^{-1}(\rho_\varepsilon) + \int u_\varepsilon^i\phi\left[\rho_\varepsilon\partial_i\partial_j\triangle^{-1}(\rho_\varepsilon u_\varepsilon^j) - \rho_\varepsilon u_\varepsilon^j\partial_j\partial_i\triangle^{-1}(\rho_\varepsilon)\right]\\
&\quad -\int \rho_\varepsilon u_\varepsilon^i\phi\partial_i\partial_j\triangle^{-1}(\rho_\varepsilon u_\varepsilon^j)\\
&= -\int \rho_\varepsilon u_\varepsilon^j u_\varepsilon^i \partial_j\phi\partial_i\triangle^{-1}(\rho_\varepsilon) + \int u_\varepsilon^i\phi\left[\rho_\varepsilon\partial_i\partial_j\triangle^{-1}(\rho_\varepsilon u_\varepsilon^j) - \rho_\varepsilon u_\varepsilon^j\partial_j\partial_i\triangle^{-1}(\rho_\varepsilon)\right]\\
&\quad -\varepsilon^4\int \rho_\varepsilon u_\varepsilon^i\phi\partial_i\triangle^{-1}(\mathrm{div}(\mathbf{1}_\Omega\nabla\rho_\varepsilon)) + \varepsilon^2\int \rho_\varepsilon u_\varepsilon^i\phi\partial_i\triangle^{-1}(\rho_\varepsilon-\rho_0).
\end{aligned}
\tag{3.34}
$$

Now, replace the second line from the bottom in (3.33) by (3.34) to obtain

$$
\begin{aligned}
&\int \phi\rho_\varepsilon\mathbb{F}_\varepsilon\\
&= -\int \partial_i\triangle^{-1}(\rho_\varepsilon)\partial_i\phi\left(\delta\rho_\varepsilon^4 + \rho_\varepsilon^2\frac{\partial f}{\partial\rho_\varepsilon} - (\lambda_1+\lambda_2)\mathrm{div}u_\varepsilon\right)\\
&\quad +\lambda_1\int\left(\partial_j u_\varepsilon^i\partial_i\triangle^{-1}(\rho_\varepsilon)\partial_j\phi - u_\varepsilon^i\partial_j\partial_i\triangle^{-1}(\rho_\varepsilon)\partial_j\phi + \rho_\varepsilon u_\varepsilon\cdot\nabla\phi\right)\\
&\quad -\int\left((\rho_\varepsilon\mu_\varepsilon\partial_i c_\varepsilon + \rho_\varepsilon\frac{\partial f}{\partial c_\varepsilon}\partial_i c_\varepsilon)\phi\partial_i\triangle^{-1}(\rho_\varepsilon) - (\rho g_1+g_2)\phi\partial_i\triangle^{-1}(\rho_\varepsilon)\right)\\
&\quad -\int \rho_\varepsilon u_\varepsilon^j u_\varepsilon^i\partial_j\phi\partial_i\triangle^{-1}(\rho_\varepsilon) + \int u_\varepsilon^i\phi\left[\rho_\varepsilon\partial_i\partial_j\triangle^{-1}(\rho_\varepsilon u_\varepsilon^j) - \rho_\varepsilon u_\varepsilon^j\partial_j\partial_i\triangle^{-1}(\rho_\varepsilon)\right]\\
&\quad -\varepsilon^4\int \rho_\varepsilon u_\varepsilon^i\phi\partial_i\triangle^{-1}(\mathrm{div}(\mathbf{1}_\Omega\nabla\rho_\varepsilon)) - \nabla\rho_\varepsilon\cdot\nabla u_\varepsilon^i\phi\partial_i\triangle^{-1}(\rho_\varepsilon)\\
&\quad +\varepsilon^2\int \rho_\varepsilon u_\varepsilon^i\phi\partial_i\triangle^{-1}(2\rho_\varepsilon-\rho_0)\\
&= \sum_{i=1}^7 J_i^\varepsilon,
\end{aligned}
\tag{3.35}
$$

where J_i^ε denotes the i^{th} integral quantity on the right hand side of (3.35).

On the other hand, if we take ε-limit in $(2.2)_2$ first and then multiply the resulting equation by $\phi\partial_i\triangle^{-1}(\rho)$, we obtain

$$\begin{aligned}
&\int \phi\rho\overline{\mathbb{F}}\\
&= -\int \partial_i\triangle^{-1}(\rho)\partial_i\phi\left(\delta\overline{\rho^4}+\overline{\rho^2\frac{\partial f}{\partial\rho}}-(\lambda_1+\lambda_2)\mathrm{div}u\right)\\
&\quad+\lambda_1\int\left(\partial_j u^i\partial_i\triangle^{-1}(\rho)\partial_j\phi-u^i\partial_j\partial_i\triangle^{-1}(\rho)\partial_j\phi+\rho u\cdot\nabla\phi\right)\\
&\quad-\int\left((\rho\mu\partial_i c+\overline{\rho\frac{\partial f}{\partial c}}\partial_i c)\phi\partial_i\triangle^{-1}(\rho)-(\rho g_1+g_2)\phi\partial_i\triangle^{-1}(\rho)\right)\\
&\quad-\int\rho u^j u^i\partial_j\phi\partial_i\triangle^{-1}(\rho)+\int u^i\phi\left[\rho\partial_i\partial_j\triangle^{-1}(\rho u^j)-\rho u^j\partial_j\partial_i\triangle^{-1}(\rho)\right]\\
&=\sum_{i=1}^{5}J_i.
\end{aligned}\tag{3.36}$$

In terms of (3.35) and (3.36), to prove (3.26) it suffices to check

$$\lim_{\varepsilon\to0}J_i^\varepsilon=J_i\ (i=1,2,\cdots,5)\quad\text{and}\quad\lim_{\varepsilon\to0}J_i^\varepsilon=0\ (i=6,7).$$

In fact, owing to (3.32), (3.11)-(3.12), (3.18), we have $\lim_{\varepsilon\to0}J_1^\varepsilon=J_1$. In a similar way, for $i=2,3,4$, we obtain $\lim_{\varepsilon\to0}J_i^\varepsilon=J_i$ from (3.31)-(3.32), (3.11)-(3.13), (3.16)-(3.17), and (3.19). Next, by (3.30), (2.33), and (3.8), we estimate

$$\begin{aligned}
&|J_6^\varepsilon+J_7^\varepsilon|\\
&\le\varepsilon^4\|\nabla\rho_\varepsilon\|_{L^2}\|\rho_\varepsilon\|_{L^3}\|u_\varepsilon\|_{L^6}+\varepsilon^2\|\rho_\varepsilon\|_{L^2}\|u_\varepsilon\|_{L^2}\|\partial_i\triangle^{-1}(2\rho_\varepsilon-\rho_0)\|_{L^\infty}\\
&\quad+\varepsilon^4\|\nabla\rho_\varepsilon\|_{L^2}\|\nabla u_\varepsilon\|_{L^2}\|\partial_i\triangle^{-1}(\rho_\varepsilon)\|_{L^\infty}\\
&\le\varepsilon^2\left(\varepsilon^2\|\nabla\rho_\varepsilon\|_{L^2}\right)\|u_\varepsilon\|_{H_0^1}\left(\|\rho_\varepsilon\|_{L^3}+\|\partial_i\triangle^{-1}(\rho_\varepsilon)\|_{L^\infty}\right)\\
&\quad+\varepsilon^2\|\rho_\varepsilon\|_{L^2}\|u_\varepsilon\|_{L^2}\|\partial_i\triangle^{-1}(2\rho_\varepsilon-\rho_0)\|_{L^\infty}\\
&\le C\varepsilon\to0\quad\text{as}\quad\varepsilon\to0.
\end{aligned}$$

Finally, in order to check J_5, we present the following div-curl Lemma.

Lemma 3.4 ([23]). *Let $\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r}$ and $1\le r,r_1,r_2<\infty$. Suppose that*

$$v_\varepsilon\rightharpoonup v\text{ in }L^{r_1}\quad\text{and}\quad w_\varepsilon\rightharpoonup w\text{ in }L^{r_2}.$$

Then, for $i,j=1,2,3$,

$$v_\varepsilon\partial_i\partial_j\triangle^{-1}(w_\varepsilon)-w_\varepsilon\partial_i\partial_j\triangle^{-1}(v_\varepsilon)\rightharpoonup v\partial_i\partial_j\triangle^{-1}(w)-w\partial_i\partial_j\triangle^{-1}(v)\quad\text{in }L^r.$$

Taking $v_\varepsilon=\rho_\varepsilon u_\varepsilon^j$ and $w_\varepsilon=\rho_\varepsilon$ in Lemma 3.4, and using (3.13), (3.16), we get $\lim_{\varepsilon\to0}J_5^\varepsilon=J_5$. This completes the proof of Lemma 3.3.

4. Vanishing artificial pressure

In this section, we take the δ-limit in the artificial pressure and prove the main result in Theorem 1.1.

By $(3.1)_1$, it follows from $(3.5)_1$ that

$$\int \nabla\mu\cdot\nabla\phi = -\int \rho u\cdot\nabla c\phi = -\int \rho u\cdot\nabla(c\phi) + \int \rho c u\cdot\nabla\phi = \int \rho c u\cdot\nabla\phi. \tag{4.1}$$

Next, from (1.5), (2.28), and (3.3), one can easily check that $c\in W^{2,p}(\Omega)$ for some $p>1$. Let $\phi = \nabla c\cdot\Phi$ in $(3.5)_2$ with $\Phi\in C_0^\infty(\Omega;\mathbb{R}^3)$. We have, by approximation if necessary,

$$\int(\rho\mu - \rho\frac{\partial f}{\partial c})\nabla c\cdot\Phi = \int \nabla c\cdot\nabla(\nabla c\cdot\Phi) = -\int \mathbb{S}_c : \nabla\Phi,$$

which, along with (3.4), leads to

$$\int\left(\delta\rho^4 + \rho^2\frac{\partial f}{\partial\rho}\right)\mathrm{div}\Phi = \int(\mathbb{S}_{ns}+\mathbb{S}_c-\rho u\otimes u):\nabla\Phi - \int(\rho g_1+g_2)\cdot\Phi. \tag{4.2}$$

As a result of (4.1), (4.2), and Theorem 3.1, we have the following theorem.

Theorem 4.1. *Under the same conditions in Theorem 3.1, for any fixed $\delta>0$, the following system*

$$\begin{cases} \mathrm{div}(\rho u) = 0, \\ \mathrm{div}(\rho u\otimes u) + \nabla\left(\delta\rho^4+\rho^2\dfrac{\partial f}{\partial\rho}\right) = \mathrm{div}\,(\mathbb{S}_{ns}+\mathbb{S}_c) + \rho g_1 + g_2, \\ \mathrm{div}(\rho u c) = \triangle\mu, \\ \rho\mu = \rho\dfrac{\partial f}{\partial c} - \triangle c, \end{cases} \tag{4.3}$$

with the boundary conditions (1.6) *admits a weak solution $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ which satisfies* (3.2) *and* (3.3).

We will prove Theorem 1.1 by taking $\delta\to 0$ in the solutions $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ obtained in Theorem 4.1. Firstly, we derive some refined estimates on $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ which are uniform in δ.

Lemma 4.1. *Let $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ be a solution obtained in Theorem 4.1. Assume that* (1.11) *is satisfied. Then there is some $p>\frac{6}{5}$ and $\theta>0$ with $\gamma+\theta>2$ such that*

$$\delta\|\rho_\delta^{4+\theta}\|_{L^1} + \|\rho_\delta^{2+\theta}\frac{\partial f}{\partial\rho_\delta}\|_{L^1} + \|u_\delta\|_{H_0^1} + \|\mu_\delta\|_{H^1} + \|c_\delta\|_{W^{2,p}} \le C, \tag{4.4}$$

where, and in what follows, the constant C is independent of δ.

Proof. We shall borrow some ideas from [27, 36] to give a weighted estimate on pressure. Owing to (1.9) and (1.11), it follows from (3.7) that

$$\begin{aligned} \lambda_1\int|\nabla u_\delta|^2 + \int|\nabla\mu_\delta|^2 &\le \int(\rho_\delta g_1+g_2)\cdot u_\delta \\ &\le \mathbf{1}_{g_1}\|\rho_\delta u_\delta\|_{L^1}\|g_1\|_{L^\infty} + \|u_\delta\|_{H_0^1}\|g_2\|_{L^{\frac{6}{5}}}, \end{aligned} \tag{4.5}$$

where $\mathbf{1}_{g_1}=1$ if $\nabla\times g_1\neq 0$ and $\mathbf{1}_{g_1}=0$ if $\nabla\times g_1=0$. Taking

$$b\ge\frac{3s-2}{s} \ \text{ and } \ s\in\left[1,\frac{6(\gamma+\theta)}{5\gamma+2\theta}\right], \tag{4.6}$$

we have

$$\begin{aligned}\|\rho_\delta u_\delta\|_{L^s}^s &= \int \left(\rho_\delta^b |u_\delta|^2\right)^{\frac{7s-6}{6b-4}} \left(|u_\delta|^6\right)^{\frac{2+(b-3)s}{6b-4}} (\rho_\delta)^{\frac{(6-s)b-4s}{6b-4}} \\ &\le \|\rho_\delta^b |u_\delta|^2\|_{L^1}^{\frac{7s-6}{6b-4}} \|u_\delta\|_{L^6}^{\frac{6(2+(b-3)s)}{6b-4}} \|\rho_\delta\|_{L^1}^{\frac{(6-s)b-4s}{6b-4}}.\end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.5) gives rise to

$$\|u_\delta\|_{H_0^1} + \|\nabla \mu_\delta\|_{L^2} \le C\left(\|\rho_\delta^b |u_\delta|^2\|_{L^1}^{\frac{\mathbf{1}_{g_1}}{6b-2}} + 1\right). \tag{4.8}$$

The case of $\nabla \times g_1 \neq 0$. The estimate is divided into several steps.

Step 1. Let $\mathcal{B}$ be the Bogovskii operator defined in (2.35). Choosing $\Phi = \mathcal{B}(\rho_\delta^\theta - |\Omega|^{-1}\int_\Omega \rho_\delta^\theta)$ in (4.2) with $\theta = \theta(\gamma)$ small and to be determined, we get

$$\begin{aligned}&\int \left(\delta\rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta}\right)\rho_\delta^\theta \\ &= \left(|\Omega|^{-1}\int \rho_\delta^\theta\right)\int \left(\delta\rho_\delta^4 + \rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right) + \int \mathbb{S}_{ns} : \nabla\Phi - (\rho_\delta g_1 + g_2)\cdot\Phi \\ &\quad - \int \rho_\delta u_\delta \otimes u_\delta : \nabla\Phi + \int \mathbb{S}_c : \nabla\Phi \\ &= \sum_{i=1}^4 K_i.\end{aligned} \tag{4.9}$$

Firstly, by (1.5) and (1.10), we have

$$\begin{aligned}K_1 &\le C\int \left(\delta\rho_\delta^4 + \rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right) \\ &\le \frac{1}{8}\int \left(\delta\rho_\delta^{4+\theta} + (\gamma-1)\rho_\delta^{\gamma+\theta} + \rho_\delta^{1+\theta}H_1(c_\delta)\right) + C \\ &= \frac{1}{8}\int \left(\delta\rho_\delta^4 + \rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right)\rho_\delta^\theta + C.\end{aligned} \tag{4.10}$$

Secondly, thanks to (4.8) and (1.11),

$$\begin{aligned}K_2 &= \int \mathbb{S}_{ns} : \nabla\Phi - (\rho_\delta g_1 + g_2)\cdot\Phi \\ &\le \left(\|\nabla u_\delta\|_{L^2} + \|\rho_\delta\|_{L^{\frac{6}{5}}}\|g_1\|_{L^\infty} + \|g_2\|_{L^{\frac{6}{5}}}\right)\|\nabla\Phi\|_{L^2} \\ &\le C\left(\|\rho_\delta^b|u_\delta|^2\|_{L^1}^{\frac{\mathbf{1}_{g_1}}{6b-2}} + \|\rho_\delta\|_{L^{\frac{6}{5}}} + 1\right)\|\rho_\delta^\theta\|_{L^2} \\ &\le \frac{1}{8}\int_\Omega \rho_\delta^{2+\theta}\frac{\partial f}{\partial\rho_\delta} + C\|\rho_\delta^b|u_\delta|^2\|_{L^1}^{\frac{1}{6b-2}} + C.\end{aligned} \tag{4.11}$$

Next, by (4.8), one has

$$\begin{aligned}\|\rho_\delta|u_\delta|^2\|_{L^t}^t &= \int \left(\rho_\delta^b|u_\delta|^2\right)^{\frac{4t-3}{(3b-2)}} \left(|u_\delta|^6\right)^{\frac{1-t(2-b)}{(3b-2)}} \rho_\delta^{\frac{3b-t(2+b)}{(3b-2)}} \\ &\le \|\rho_\delta^b|u_\delta|^2\|_{L^1}^{\frac{4t-3}{(3b-2)}} \|u_\delta\|_{H_0^1}^{\frac{6(1-t(2-b))}{(3b-2)}} \|\rho_\delta\|_{L^1}^{\frac{3b-t(2+b)}{(3b-2)}} \\ &\le C\left(1+\|\rho_\delta^b|u_\delta|^2\|_{L^1}^{\frac{5t-3}{(3b-1)}}\right).\end{aligned} \tag{4.12}$$

Let $t=\frac{\gamma+\theta}{\gamma}$ in (4.12), then we have

$$\begin{aligned}K_3 &= -\int \rho_\delta u_\delta\otimes u_\delta : \nabla\Phi \\ &\le \|\nabla\Phi\|_{L^{\frac{\gamma+\theta}{\theta}}} \||\rho_\delta|u_\delta|^2\|_{L^{\frac{\gamma+\theta}{\gamma}}} \\ &\le C\|\rho_\delta\|_{L^{\gamma+\theta}}^{\theta} \||\rho_\delta|u_\delta|^2\|_{L^{\frac{\gamma+\theta}{\gamma}}} \\ &\le \frac{1}{8}\int_\Omega \rho_\delta^{2+\theta}\frac{\partial f}{\partial\rho_\delta} + C\left(1+\|\rho_\delta^b|u_\delta|^2\|_{L^1}^{\frac{2\gamma+5\theta}{\gamma(3b-1)}}\right).\end{aligned} \tag{4.13}$$

Finally, if we replace u_δ with μ_δ and take $\bar{b}=3-\frac{2}{s}$ in (4.7), we find

$$\|\rho_\delta\mu_\delta\|_{L^s}^s \le C\|\rho_\delta^{\bar{b}}\mu_\delta^2\|_{L^1}^{\frac{7s-6}{6b-4}}. \tag{4.14}$$

Taking $s=\frac{6(\gamma+\theta)}{5\gamma+2\theta}$ in (4.14), we deduce

$$\begin{aligned}\|\nabla^2 c_\delta\|_{L^{\frac{6(\gamma+\theta)}{5\gamma+2\theta}}}^{\frac{6(\gamma+\theta)}{5\gamma+2\theta}} &\le C\|\triangle c_\delta\|_{L^{\frac{6(\gamma+\theta)}{5\gamma+2\theta}}}^{\frac{6(\gamma+\theta)}{5\gamma+2\theta}} \\ &\le C\|\rho_\delta\mu_\delta+\rho_\delta\frac{\partial f}{\partial c_\delta}\|_{L^{\frac{6(\gamma+\theta)}{5\gamma+2\theta}}}^{\frac{6(\gamma+\theta)}{5\gamma+2\theta}} \\ &\le C+C\|\rho_\delta^{\bar{b}}\mu_\delta^2\|_{L^1}^{\frac{3(\gamma+\theta)}{5\gamma+2\theta}} + C\|\rho_\delta\ln\rho_\delta\|_{L^{\frac{6(\gamma+\theta)}{5\gamma+2\theta}}}^{\frac{6(\gamma+\theta)}{5\gamma+2\theta}},\end{aligned} \tag{4.15}$$

where the exponents in the last inequality are due to

$$s=\frac{6(\gamma+\theta)}{5\gamma+2\theta} \quad\text{and}\quad \bar{b}=\frac{4\gamma+7\theta}{3(\gamma+\theta)}. \tag{4.16}$$

With the help of (4.15) and $\|\rho_\delta\|_{L^1}=m_1$, we have the following estimate,

$$\begin{aligned}K_4 &\le C\|\nabla\Phi\|_{L^{\frac{\gamma+\theta}{\theta}}} \|\nabla c_\delta\|_{L^{\frac{2(\gamma+\theta)}{\gamma}}}^2 \\ &\le C\|\rho_\delta\|_{L^{(\gamma+\theta)}}^{\theta}\left(\|\nabla^2 c_\delta\|_{L^{\frac{6(\gamma+\theta)}{5\gamma+2\theta}}}^{\frac{6(\gamma+\theta)}{5\gamma+2\theta}}\right)^{\frac{5\gamma+2\theta}{3(\gamma+\theta)}} \\ &\le C+\frac{1}{8}\int\rho_\delta^{2+\theta}\frac{\partial f}{\partial\rho_\delta} + C\|\rho_\delta^{\bar{b}}\mu_\delta^2\|_{L^1}^{\frac{\gamma+\theta}{\gamma}}.\end{aligned} \tag{4.17}$$

In conclusion, substituting (4.10)-(4.13), (4.17) back into (4.9) gives rise to

$$\begin{aligned}\int \left(\delta\rho_\delta^4+\rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right)\rho_\delta^\theta &\le C+C\|\rho_\delta^{\bar b}|u_\delta|^2\|_{L^1}^{\frac{2\gamma+5\theta}{\gamma(3\bar b-1)}}+C\|\rho_\delta^{\bar b}\mu_\delta^2\|_{L^1}^{\frac{\gamma+\theta}{\gamma}}\\ &\le C+C\|\rho_\delta^{\bar b}(|u_\delta|^2+\mu_\delta^2)\|_{L^1}^{\frac{\gamma+\theta}{\gamma}},\end{aligned}\tag{4.18}$$

from (4.16) and $\frac{2\gamma+5\theta}{\gamma(3\bar b-1)}\le\frac{2\gamma+5\theta}{\gamma(3\bar b-2)}=\frac{\gamma+\theta}{\gamma}$.

Step 2. We show the following estimate:

Proposition 4.1. *For any fixed $\alpha_0\in(0,1)$ and $x^*\in\overline{\Omega}$, there is some constant C independent of δ or x^*, such that*

$$\int\frac{\rho_\delta^\gamma(x)}{|x-x^*|^{\alpha_0}}dx\le C\left(1+\|\rho_\delta^{\bar b}(|u_\delta|^2+\mu_\delta^2)\|_{L^1}\right),\tag{4.19}$$

with $\bar b$ being defined in (4.16).

Proof. We consider two cases.

Case 1: boundary point $x^*\in\partial\Omega$. As in [27], we introduce

$$\xi^i(x)=\phi(x)\partial_i\phi(x)\left(\phi(x)+|x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{-\alpha_0},\quad i=1,2,3,\tag{4.20}$$

where the function $\phi(x)\in C^2(\overline{\Omega})$ satisfies the following properties:

$$\begin{cases}\phi(x)>0 \text{ in } \Omega \text{ and } \phi(x)=0 \text{ on } \partial\Omega,\\ |\phi(x)|\ge k_1 \text{ if } x\in\Omega \text{ and } dist(x,\partial\Omega)\ge k_2,\\ |\nabla\phi(x)|\ge k_1 \text{ if } x\in\Omega \text{ and } dist(x,\partial\Omega)\le k_2,\end{cases}\tag{4.21}$$

and the constants $k_i>0$ are given.

Remark 4.1. The function $\phi(x)$ satisfying (4.21) is in fact the distance function near the boundary with C^2 extension to the whole Ω. Moreover, for every point $x\in\Omega$ near the boundary, there is a unique $\tilde x\in\partial\Omega$ such that

$$\nabla\phi=\frac{x-\tilde x}{\phi(x)}\quad\text{and}\quad\phi(x)=|x-\tilde x|.\tag{4.22}$$

See, e.g., [46, Exercise 1.15] for the detail.

It is clear that $\xi\in L^\infty(\Omega)$ and $\xi=0$ on $\partial\Omega$. In addition, a direct computation yields

$$\begin{aligned}\partial_j\xi^i=&\frac{\phi\partial_j\partial_i\phi}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0}}+\frac{\partial_j\phi\partial_i\phi}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0}}\\ &-\alpha_0\frac{\phi\partial_i\phi\partial_j\phi}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0+1}}-\alpha_0\frac{\phi\partial_i\phi\partial_j|x-x^*|^{\frac{2}{2-\alpha_0}}}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0+1}}.\end{aligned}\tag{4.23}$$

Thus, $|\nabla\xi|\in L^q$ for all $q\in[2,\frac{3}{\alpha_0})$ because $|\partial_j\xi^i|\le C+C|x-x^*|^{-\alpha_0}$. Due to (4.21) and $\frac{2}{2-\alpha_0}>1$, the following inequalities hold true:

$$\phi<\phi+|x-x^*|^{\frac{2}{2-\alpha_0}}\le C|x-x^*|.\tag{4.24}$$

With (4.21)-(4.24), one deduces that, for $dist(x,\,\partial\Omega)\leq k_2$,

$$\mathrm{div}\xi\geq -C+\frac{1}{2(1-\alpha_0)}\frac{|\nabla\phi|^2}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0}}\geq -C+\frac{C}{|x-x^*|^{\alpha_0}}. \tag{4.25}$$

Take $\Phi=\xi$ in (4.2) to obtain

$$\begin{aligned}&\int\left(\delta\rho_\delta^4+\rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right)\mathrm{div}\xi+\int\rho_\delta u_\delta\otimes u_\delta:\nabla\xi\\&=\int\mathbb{S}_{ns}:\nabla\xi-\int(\rho_\delta g_1+g_2)\cdot\xi+\int\mathbb{S}_c:\nabla\xi.\end{aligned} \tag{4.26}$$

The first two terms on the right-hand side of (4.26) satisfy

$$\left|\int\mathbb{S}_{ns}:\nabla\xi-\int(\rho_\delta g_1+g_2)\cdot\xi\right|\leq C(\alpha_0)\left(\|\nabla u_\delta\|_{L^2}+1\right). \tag{4.27}$$

Next, let

$$\frac{\gamma+\theta}{\theta}<\frac{3}{\alpha_0}, \tag{4.28}$$

where θ and α_0 will be determined in (4.46). One deduces

$$\begin{aligned}\left|\int\mathbb{S}_c:\nabla\xi\right|&\leq C\|\nabla\xi\|_{L^{\frac{\gamma+\theta}{\theta}}}\|\nabla c_\delta\|^2_{L^{\frac{2(\gamma+\theta)}{\gamma}}}\leq C\|\nabla^2c_\delta\|^2_{L^{\frac{6(\gamma+\theta)}{5\gamma+2\theta}}}\\&\leq C+C\|\rho_\delta^{\bar{b}}\mu_\delta^2\|_{L^1}+C\int\rho_\delta^2\frac{\partial f}{\partial\rho_\delta},\end{aligned} \tag{4.29}$$

where the last inequality is from (4.15).

Now let us focus on the left-hand side of (4.26). Owing to (4.25), one has

$$\int\left(\delta\rho_\delta^4+\rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right)\mathrm{div}\xi\geq -C\int\left(\delta\rho_\delta^4+\rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right)+C\int_{\Omega\cap B_{k_2}(x^*)}\frac{\left(\delta\rho_\delta^4+\rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right)}{|x-x^*|^{\alpha_0}}. \tag{4.30}$$

By (4.22), one has $\partial_j\partial_i\phi=\frac{\partial_i(x-\tilde{x})^j}{\phi}-\frac{\partial_j\phi\partial_i\phi}{\phi}$. Then,

$$\int\frac{\phi\rho_\delta u_\delta\otimes u_\delta:(\partial_j\partial_i\phi)_{3\times3}}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0}}=\int\frac{\rho_\delta|u_\delta|^2}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0}}-\int\frac{\rho_\delta|u_\delta\cdot\nabla\phi|^2}{\left(\phi+|x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0}},$$

and hence, by (4.23) and (4.24), we have

$$
\begin{aligned}
&\int \rho_\delta u_\delta \otimes u_\delta : \nabla \xi \\
&= \int \frac{\rho_\delta |u_\delta|^2}{\left(\phi + |x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0}} - \alpha_0 \int \frac{\phi \rho_\delta (u_\delta \cdot \nabla \phi)^2}{\left(\phi + |x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0+1}} \\
&\quad - \alpha_0 \int \frac{\phi \rho_\delta (u_\delta \cdot \nabla |x-x^*|^{\frac{2}{2-\alpha_0}})(u_\delta \cdot \nabla \phi)}{\left(\phi + |x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0+1}} \\
&\geq (1-\alpha_0) \int \frac{\rho_\delta |u_\delta|^2}{\left(\phi + |x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0}} - \alpha_0 \int \frac{\phi \rho_\delta (u_\delta \cdot \nabla |x-x^*|^{\frac{2}{2-\alpha_0}})(u_\delta \cdot \nabla \phi)}{\left(\phi + |x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0+1}} \\
&\geq \frac{(1-\alpha_0)}{2} \int \frac{\rho_\delta |u_\delta|^2}{\left(\phi + |x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0}} - C(\alpha_0) \int \frac{\phi^2 \rho_\delta |u_\delta|^2 |x-x^*|^{\frac{2\alpha_0}{2-\alpha_0}}}{\left(\phi + |x-x^*|^{\frac{2}{2-\alpha_0}}\right)^{\alpha_0+2}} \\
&\geq C \int_{\Omega \cap B_{k_2}(x^*)} \frac{\rho_\delta |u_\delta|^2}{|x-x^*|^{\alpha_0}} - C \|\rho_\delta |u_\delta|^2\|_{L^1}.
\end{aligned}
\tag{4.31}
$$

Therefore, (4.26) together with (4.27) and (4.29)-(4.31) yield

$$
\begin{aligned}
&\int_{\Omega \cap B_{k_2}(x^*)} \left(\frac{\left(\delta \rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta}\right)}{|x-x^*|^{\alpha_0}} + \frac{\rho_\delta |u_\delta|^2}{|x-x^*|^{\alpha_0}} \right) \\
&\leq C \int \left(\delta \rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta} \right) + C \left(\|u_\delta\|_{H_0^1} + \|\rho_\delta^{\bar{b}} \mu_\delta^2\|_{L^1} + \|\rho_\delta |u_\delta|^2\|_{L^1} + 1 \right) \\
&\leq C(\gamma, \theta, \overline{H}) \left(\int \left(\delta \rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta} \right) \rho_\delta^\theta \right)^{\frac{\gamma}{\gamma+\theta}} \\
&\quad + C \left(\|u_\delta\|_{H_0^1} + \|\rho_\delta^{\bar{b}} \mu_\delta^2\|_{L^1} + \|\rho_\delta |u_\delta|^2\|_{L^1} + 1 \right) \\
&\leq C + C \|\rho_\delta^{\bar{b}} (|u_\delta|^2 + \mu_\delta^2)\|_{L^1},
\end{aligned}
\tag{4.32}
$$

where, for the last two inequalities we have used the Hölder inequality, (4.8), (4.18) as well as

$$
\|\rho_\delta |u_\delta|^2\|_{L^1} \leq C + C \|\rho_\delta^{\bar{b}} |u_\delta|^2\|_{L^1}^{\frac{2}{3b-1}} \leq C + C \|\rho_\delta^{\bar{b}} |u_\delta|^2\|_{L^1},
$$

which comes from (4.12).

Case 2: interior point $x^* \in \Omega$. There is a constant $r > 0$ such that $dist(x^*, \partial\Omega) = 3r$. Let χ be a smooth cut-off function satisfying $\chi = 1$ in $B_r(x^*)$ and $\chi = 0$ outside $B_{2r}(x^*)$, as well

as $|\nabla\chi| \le 2r^{-1}$. Choosing $\Phi(x) = \frac{x-x^*}{|x-x^*|^{\alpha_0}}\chi^2$ in (4.2), we find

$$\begin{aligned}
&\int \left(\delta\rho_\delta^4 + \rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right)\frac{3-\alpha_0}{|x-x^*|^{\alpha_0}}\chi^2 + \int \rho_\delta u_\delta\otimes u_\delta : \nabla\left(\frac{x-x^*}{|x-x^*|^{\alpha_0}}\chi^2\right)\\
&= -\int (\rho_\delta g_1 + g_2)\cdot\frac{x-x^*}{|x-x^*|^{\alpha_0}}\chi^2 + \int \mathbb{S}_{ns} : \nabla\left(\frac{x-x^*}{|x-x^*|^{\alpha_0}}\chi^2\right)\\
&\quad - 2\int \left(\delta\rho_\delta^4 + \rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right)\chi\frac{\nabla\chi\cdot(x-x^*)}{|x-x^*|^{\alpha_0}} + \int \mathbb{S}_c : \nabla\left(\frac{x-x^*}{|x-x^*|^{\alpha_0}}\chi^2\right).
\end{aligned} \tag{4.33}$$

By a direct computation, one has

$$\begin{aligned}
&\partial_i\left(\frac{x^j-(x^*)^j}{|x-x^*|^{\alpha_0}}\chi^2\right)\\
&= \frac{\partial_i(x^j-(x^*)^j)}{|x-x^*|^{\alpha_0}}\chi^2 - \alpha_0\frac{(x^j-(x^*)^j)(x^i-(x^*)^i)}{|x-x^*|^{\alpha_0+2}}\chi^2 + 2\chi\frac{x^j-(x^*)^j}{|x-x^*|^{\alpha_0}}\partial_i\chi\\
&\in L^q, \quad q\in[2,\frac{3}{\alpha_0}),
\end{aligned}$$

and hence, the second term on the left-hand side of (4.33) satisfies

$$\begin{aligned}
&\int \rho_\delta u_\delta\otimes u_\delta : \nabla\left(\frac{x-x^*}{|x-x^*|^{\alpha_0}}\chi^2\right)\\
&\ge (1-\alpha_0)\int\frac{\rho_\delta|u_\delta|^2}{|x-x^*|^{\alpha_0}}\chi^2 + 2\int\frac{\chi\rho_\delta(u_\delta\cdot\nabla\chi)(u_\delta\cdot(x-x^*))}{|x-x^*|^{\alpha_0}}\\
&\ge \frac{1-\alpha_0}{2}\int\frac{\rho_\delta|u_\delta|^2}{|x-x^*|^{\alpha_0}}\chi^2 - C\int_{r<|x-x^*|<2r}\frac{\rho_\delta|u_\delta|^2}{|x-x^*|^{\alpha_0}},
\end{aligned} \tag{4.34}$$

where C is independent of r.

For the terms on the right-hand side of (4.33), we have

$$\begin{aligned}
&\left|-\int(\rho_\delta g_1+g_2)\cdot\frac{x-x}{|x-x^*|^{\alpha_0}}\chi^2 + \int\mathbb{S}_{ns}:\nabla\left(\frac{x-x^*}{|x-x^*|^{\alpha_0}}\chi^2\right)\right|\\
&\le C + C\|u_\delta\|_{H_0^1}
\end{aligned}$$

and

$$-2\int\left(\delta\rho_\delta^4+\rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right)\chi\frac{\nabla\chi\cdot(x-x^*)}{|x-x^*|^{\alpha_0}} \le C\int_{r<|x-x^*|<2r}\frac{\left(\delta\rho_\delta^4+\rho_\delta^2\frac{\partial f}{\partial\rho_\delta}\right)}{|x-x^*|^{\alpha_0}},$$

where C is independent of r. Similarly to (4.29), we deduce

$$\int\mathbb{S}_c:\nabla\left(\frac{x-x^*}{|x-x^*|^{\alpha_0}}\chi^2\right) \le C + C\|\rho_\delta^{\bar b}\mu_\delta^2\|_{L^1} + C\int\rho_\delta^2\frac{\partial f}{\partial\rho_\delta}.$$

From the above estimates, we obtain

$$\begin{aligned}
&\int_{B_r(x^*)} \left(\frac{\left(\delta\rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta}\right)}{|x-x^*|^{\alpha_0}} + \frac{\rho_\delta |u_\delta|^2}{|x-x^*|^{\alpha_0}} \right) \\
&\le C \int \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta} + C \left(\|u_\delta\|_{H_0^1} + \|\rho_\delta^{\bar{b}} \mu_\delta^2\|_{L^1} + \|\rho_\delta |u_\delta|^2\|_{L^1} + 1 \right) \\
&\quad + C \int_{r<|x-x^*|<2r} \left(\frac{\left(\delta\rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta}\right)}{|x-x^*|^{\alpha_0}} + \frac{\rho_\delta |u_\delta|^2}{|x-x^*|^{\alpha_0}} \right) \\
&\le C + C\|\rho_\delta^{\bar{b}} (|u_\delta|^2 + \mu_\delta^2)\|_{L^1} \\
&\quad + C \int_{r<|x-x^*|<2r} \left(\frac{\left(\delta\rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta}\right)}{|x-x^*|^{\alpha_0}} + \frac{\rho_\delta |u_\delta|^2}{|x-x^*|^{\alpha_0}} \right),
\end{aligned} \tag{4.35}$$

where the last inequality follows from (4.32).

We need to discuss two situations: (i) $x^* \in \Omega$ is far from the boundary. (ii) $x^* \in \Omega$ is close to the boundary.

(i) The case of $dist(x^*, \partial\Omega) = 3r \ge \frac{k_2}{2} > 0$, where k_2 is the same as in (4.21). From (4.35), one has

$$\begin{aligned}
&\int_{B_r(x^*)} \frac{\left(\delta\rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta}\right)}{|x-x^*|^{\alpha_0}} \\
&\le C + C\|\rho_\delta^{\bar{b}} (|u_\delta|^2 + \mu_\delta^2)\|_{L^1} + C(k_2) \int \left(\delta\rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta} + \rho_\delta |u_\delta|^2 \right) \\
&\le C \left(1 + \|\rho_\delta^{\bar{b}} (|u_\delta|^2 + \mu_\delta^2)\|_{L^1} \right),
\end{aligned} \tag{4.36}$$

where in the last inequality we have also used (4.32).

(ii) The case of $dist(x^*, \partial\Omega) = 3r < \frac{k_2}{2}$. Let $|x^* - \tilde{x}^*| = dist(x^*, \partial\Omega)$. Then,

$$4|x - x^*| \ge |x - \tilde{x}^*|, \quad \forall\ x \notin B_r(x^*). \tag{4.37}$$

In view of (4.37), we infer from (4.35) that

$$\begin{aligned}
&\int_{B_r(x^*)} \left(\frac{\left(\delta\rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta}\right)}{|x-x^*|^{\alpha_0}} + \frac{\rho_\delta |u_\delta|^2}{|x-x^*|^{\alpha_0}} \right) \\
&\le C + C\|\rho_\delta^{\bar{b}} (|u_\delta|^2 + \mu_\delta^2)\|_{L^1} + C \int_{r<|x-x^*|<2r} \left(\frac{\left(\delta\rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta}\right)}{|x-x^*|^{\alpha_0}} + \frac{\rho_\delta |u_\delta|^2}{|x-x^*|^{\alpha_0}} \right) \\
&\le C + C\|\rho_\delta^{\bar{b}} (|u_\delta|^2 + \mu_\delta^2)\|_{L^1} + C \int_{\Omega \cap B_{k_2}(\tilde{x}^*)} \left(\frac{\left(\delta\rho_\delta^4 + \rho_\delta^2 \frac{\partial f}{\partial \rho_\delta}\right)}{|x-\tilde{x}^*|^{\alpha_0}} + \frac{\rho_\delta |u_\delta|^2}{|x-\tilde{x}^*|^{\alpha_0}} \right) \\
&\le C \left(1 + \|\rho_\delta^{\bar{b}} (|u_\delta|^2 + \mu_\delta^2)\|_{L^1} \right),
\end{aligned} \tag{4.38}$$

where for the last inequality we have also used (4.32).

In summary, we obtain (4.19) from (4.32), (4.36) and (4.38). □

Remark 4.2. The case that $x^* \in \Omega$ is close to the boundary was first treated by Mucha-Pokorný-Zatorska [36], where they combined the test functions for both the interior and boundary cases.

Step 3. By (4.19) and the Hölder inequality, we have

$$\begin{aligned}\int \frac{\rho_\delta^{\bar{b}}}{|x-x^*|} &\le \left(\int \frac{\rho_\delta^{\gamma}}{|x-x^*|^{\alpha_0}}\right)^{\frac{\bar{b}}{\gamma}} \left(\int \frac{1}{|x-x^*|^{\frac{\gamma-\bar{b}\alpha_0}{\gamma-\bar{b}}}}\right)^{\frac{\gamma-\bar{b}}{\gamma}} \\ &\le C\left(\int \frac{\rho_\delta^{\gamma}}{|x-x^*|^{\alpha_0}}\right)^{\frac{\bar{b}}{\gamma}} \\ &\le C\left(1+\|\rho_\delta^{\bar{b}}(|u_\delta|^2+\mu_\delta^2)\|_{L^1}\right)^{\frac{\bar{b}}{\gamma}},\end{aligned} \tag{4.39}$$

if

$$\frac{\bar{b}(3-\alpha_0)}{2} < \gamma. \tag{4.40}$$

We note that (4.40) implies $\frac{\gamma-\bar{b}\alpha_0}{\gamma-\bar{b}} < 3$.

Consider the Neumann boundary value problem:

$$\begin{cases} \triangle h(x^*) = \rho_\delta^{\bar{b}} - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \rho_\delta^{\bar{b}} = \rho_\delta^{\bar{b}} - \mathfrak{m} \quad \text{in } \Omega, \\ \dfrac{\partial h(x^*)}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.41}$$

where $\mathfrak{m} = \frac{1}{|\Omega|}\int_\Omega \rho_\delta^{\bar{b}}$. Recalling the Green's function representation

$$h(x^*) = \int_\Omega G(x^*,x)\left(\rho_\delta^{\bar{b}}(x) - \mathfrak{m}\right)dx,$$

and using (4.39), we have

$$\begin{aligned}\|h\|_{L^\infty} &\le \sup_{x^*\in\Omega}\int_\Omega \frac{\left(\rho_\delta^{\bar{b}}(x)-\mathfrak{m}\right)}{|x-x^*|}dx \\ &\le \sup_{x^*\in\Omega}\int_\Omega \frac{\rho_\delta^{\bar{b}}(x)}{|x-x^*|}dx + C\mathfrak{m} \\ &\le C\left(1+\|\rho_\delta^{\bar{b}}(\mu_\delta^2+|u_\delta|^2)\|_{L^1}^{\frac{\bar{b}}{\gamma}}\right).\end{aligned} \tag{4.42}$$

From (4.41) one has

$$\begin{aligned}\|\rho_\delta^{\bar{b}}\mu_\delta^2\|_{L^1} &= \int \mu_\delta^2(\mathfrak{m}+\triangle h) \\ &= \mathfrak{m}\int \mu_\delta^2 - 2\int \mu_\delta \nabla\mu_\delta\cdot\nabla h \\ &\le \mathfrak{m}\int \mu_\delta^2 + 2\|\nabla\mu_\delta\|_{L^2}\left(\int \mu_\delta^2|\nabla h|^2\right)^{\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}\int \mu_\delta^2|\nabla h|^2 &= -\int \left(\mu_\delta^2 h\triangle h + 2\mu_\delta\nabla\mu_\delta h\nabla h\right)\\ &\le \|h\|_{L^\infty}\left(\mathfrak{m}\int\mu_\delta^2 + \int\rho_\delta^{\bar b}\mu_\delta^2 + 2\|\nabla\mu_\delta\|_{L^2}\left(\int\mu_\delta^2|\nabla h|^2\right)^{\frac12}\right),\end{aligned}$$

thus,

$$\|\rho_\delta^{\bar b}\mu_\delta^2\|_{L^1} \le C\left(\|h\|_{L^\infty}\|\nabla\mu_\delta\|_{L^2}^2 + \mathfrak{m}\|\mu_\delta\|_{L^2}^2\right). \tag{4.43}$$

Thanks to the interpolation inequality and $\|\rho_\delta\|_{L^1} = m_1$, we have

$$\mathfrak{m} \le C\|\rho_\delta^{\bar b}\|_{L^1} \le C\|\rho_\delta^\gamma\|_{L^1}^{\frac{\bar b-1}{\gamma-1}} \quad\text{and}\quad \|\rho_\delta\|_{L^{\frac65}}^2 \le C\|\rho_\delta^\gamma\|_{L^1}^{\frac{1}{3(\gamma-1)}}.$$

Then, by (2.28) one has

$$\mathfrak{m}\|\mu_\delta\|_{L^2}^2 \le C\|\rho_\delta^\gamma\|_{L^1}^{\frac{3\bar b-2}{3(\gamma-1)}}(1+\|\nabla\mu_\delta\|_{L^2}^2).$$

Substituting it back into (4.43) yields

$$\|\rho_\delta^{\bar b}\mu_\delta^2\|_{L^1} \le C\left(\|h\|_{L^\infty} + \|\rho_\delta^\gamma\|_{L^1}^{\frac{3\bar b-2}{3(\gamma-1)}}\right)(1+\|\nabla\mu_\delta\|_{L^2}^2).$$

Similarly, we have

$$\|\rho_\delta^{\bar b}|u_\delta|^2\|_{L^1} \le C\|h\|_{L^\infty}\|\nabla u_\delta\|_{L^2}^2.$$

Thus, from (4.42), (4.19) and (4.8), we obtain

$$\begin{aligned}\|\rho_\delta^{\bar b}(|u_\delta|^2+\mu_\delta^2)\|_{L^1} &\le C\left(\|h\|_{L^\infty} + \|\rho_\delta^\gamma\|_{L^1}^{\frac{3\bar b-2}{3(\gamma-1)}}\right)\left(1+\|\nabla\mu_\delta\|_{L^2}+\|\nabla u_\delta\|_{L^2}\right)^2\\ &\le C + C\|\rho_\delta^{\bar b}(|u_\delta|^2+\mu_\delta^2)\|_{L^1}^\beta,\end{aligned} \tag{4.44}$$

with

$$\beta = \max\left\{\frac{\bar b}{\gamma},\ \frac{3\bar b-2}{3(\gamma-1)}\right\} + \frac{1}{3\bar b-1}.$$

Step 4. If we can prove

$$\|\rho_\delta^{\bar b}(|u_\delta|^2+\mu_\delta^2)\|_{L^1} \le C, \tag{4.45}$$

then, we conclude (4.4) from (4.8), (4.15), and (4.18), and thus complete the proof of Lemma 4.1.

To prove (4.45), it suffices to show $\beta < 1$ in view of (4.44). By (4.28), we may take θ close to zero as $\alpha_0 \to 0$. From (4.40) and (4.16) we see that

$$\frac{\bar b}{\gamma} < \frac{2}{3-\alpha_0} \to \frac23 \quad(\text{as } \alpha_0\to 0) \quad\text{and}\quad \frac{1}{3\bar b-1} = \frac{\gamma+\theta}{3\gamma+6\theta} \to \frac13 \ (\text{as } \theta\to 0). \tag{4.46}$$

Hence, $\beta = \frac{\bar b}{\gamma} + \frac{1}{3\bar b-1} < 1$ if both α_0 and θ are chosen small enough. Besides, to guarantee (4.40), from (4.16) we have

$$\gamma > \frac{3-\alpha_0}{2}\bar b = \frac{3-\alpha_0}{2}\cdot\frac{4\gamma+7\theta}{3(\gamma+\theta)} \to 2 \quad(\text{as } \alpha_0,\ \theta\to 0).$$

If $\beta = \frac{3\bar{b}-2}{3(\gamma-1)} + \frac{1}{3\bar{b}-1}$ (we have no need checking (4.40) any more), we see that $\frac{3\bar{b}-2}{3(\gamma-1)} < \frac{2}{3}$ is equivalent to $\gamma > \frac{3}{2}\bar{b}$. By (4.16),

$$\gamma > \frac{3}{2}\bar{b} = \frac{3}{2}\cdot\frac{4\gamma+7\theta}{3(\gamma+\theta)} \to 2 \quad (\text{as } \theta \to 0). \tag{4.47}$$

This and (4.46) guarantee that $\beta < 1$ as long as θ is small.

The case of $\nabla \times g_1 = 0$. In this case, from (4.8) we have

$$\|u_\delta\|_{H_0^1} + \|\nabla \mu_\delta\|_{L^2} \le C.$$

Then, the same deduction as (4.44) yields

$$\begin{aligned} \|\rho_\delta^{\bar{b}}(|u_\delta|^2 + \mu_\delta^2)\|_{L^1} &\le C\left(\|h\|_{L^\infty} + \|\rho_\delta^\gamma\|_{L^1}^{\frac{3\bar{b}-2}{3(\gamma-1)}}\right) \\ &\le C + C\|\rho_\delta^{\bar{b}}(|u_\delta|^2 + \mu_\delta^2)\|_{L^1}^{\beta}, \end{aligned} \tag{4.48}$$

with

$$\beta = \max\left\{\frac{\bar{b}}{\gamma},\ \frac{3\bar{b}-2}{3(\gamma-1)}\right\}.$$

By (4.40), we see that $\beta = \frac{\bar{b}}{\gamma} < \frac{2}{3-\alpha_0} < 1$ is always valid for all $\alpha_0 \in (0,1)$. In order for (4.40) and (4.28) to be satisfied, from (4.16) we have

$$\gamma > \frac{3-\alpha_0}{2}\bar{b} > \bar{b} = \frac{4\gamma+7\theta}{3(\gamma+\theta)} > \frac{4}{3} + \frac{\alpha_0}{3} \to \frac{5}{3} \quad (\text{as } \alpha_0 \to 1).$$

If $\beta = \frac{3\bar{b}-2}{3(\gamma-1)}$ (we have no need checking (4.40) any more), to guarantee $\beta < 1$, it suffices to require

$$\gamma > \frac{1}{3} + \bar{b} = \frac{1}{3} + \frac{4\gamma+7\theta}{3(\gamma+\theta)} > \frac{5}{3} + \frac{\alpha_0}{3} \to \frac{5}{3} \quad (\text{as } \alpha_0 \to 0).$$

The proof of Lemma 4.1 is completed. □

By Lemma 4.1, we can take the following limits, subject to some subsequence,

$$(\nabla u_\delta,\ \nabla \mu_\delta) \rightharpoonup (\nabla u,\ \nabla \mu) \text{ in } L^2, \tag{4.49}$$

$$(u_\delta,\ \mu_\delta) \to (u,\ \mu) \ \text{ in } \ L^{p_1} \ (1 \le p_1 < 6), \tag{4.50}$$

$$c_\delta \to c \text{ in } \ W^{1,p_2} \ \ (\text{for some } p_2 > 2), \tag{4.51}$$

$$\delta\rho_\delta^{4+\theta} \to 0 \ \text{ in } \ L^1, \quad \text{and} \quad \rho_\delta \rightharpoonup \rho \text{ in } L^{\gamma+\theta}, \tag{4.52}$$

where (4.52) is due to $\rho_\delta^{\gamma+\theta} \le (\gamma-1)\rho_\delta^{2+\theta}\frac{\partial f}{\partial \rho_\delta}$. As a result of (4.50)-(4.52),

$$(\rho_\delta u_\delta,\ \rho_\delta \mu_\delta) \rightharpoonup (\rho u,\ \rho\mu) \ \text{ in } \ L^{p_3} \ \ (\text{for some } p_3 > 6/5), \tag{4.53}$$

$$(\rho_\delta u_\delta \otimes u_\delta,\ \rho_\delta u_\delta c_\delta) \rightharpoonup (\rho u \otimes u,\ \rho u c) \ \text{ in } \ L^p \ (\text{for some } p > 1); \tag{4.54}$$

and furthermore,

$$\rho_\delta^2\frac{\partial f}{\partial \rho_\delta} = (\gamma-1)\rho_\delta^\gamma + \rho_\delta H_1(c_\delta) \rightharpoonup (\gamma-1)\overline{\rho^\gamma} + \rho H_1(c) := \overline{\rho^2\frac{\partial f}{\partial \rho}} \ \text{ in } \ L^{\frac{\gamma+\theta}{\gamma}}, \tag{4.55}$$

$$\rho_\delta\frac{\partial f}{\partial c_\delta} = \rho_\delta \ln \rho_\delta H_1'(c_\delta) + \rho_\delta H_2'(c_\delta) \rightharpoonup \overline{\rho\ln\rho}H_1'(c) + \rho H_2'(c) := \overline{\rho\frac{\partial f}{\partial c}} \text{ in } \ L^{p_3}. \tag{4.56}$$

With (4.49)-(4.56) in hand, we are able to take δ-limit in (4.3) and obtain the following equations in the distribution sense:

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) + \nabla \left(\overline{\rho^2 \frac{\partial f}{\partial \rho}} \right) = \operatorname{div} \left(\mathbb{S}_{ns} + \mathbb{S}_c \right) + \rho g_1 + g_2, \\ \operatorname{div}(\rho u c) = \triangle \mu, \\ \rho \mu = \overline{\rho \frac{\partial f}{\partial c}} - \triangle c. \end{cases} \tag{4.57}$$

In order to complete the proof of Theorem 1.1, it remains to verify

$$\overline{\rho^2 \frac{\partial f}{\partial \rho}} = \rho^2 \frac{\partial f}{\partial \rho} \quad \text{and} \quad \overline{\rho \frac{\partial f}{\partial c}} = \rho \frac{\partial f}{\partial c}.$$

To this end it suffices to prove $\rho_\delta \to \rho$ in L^1, which is our task in the rest of the paper.

Let $T_k(z)$ be an increasing and concave function, in particular,

$$C^1([0,\infty)) \ni T_k(z) = \begin{cases} z, & z \le k \in \mathbb{N}, \\ k+1, & z \ge k+1. \end{cases} \tag{4.58}$$

Clearly,

$$T_k(\rho_\delta) \rightharpoonup \overline{T_k(\rho)} \quad \text{in} \quad L^p(\Omega), \ \ \forall \ p \in [1,\infty]. \tag{4.59}$$

Lemma 4.2. *Let $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ be a solution obtained in Theorem 4.1. Then, for the effective viscous flux the following holds,*

$$\lim_{\delta \to 0} \int T_k(\rho_\delta) \left(\rho_\delta^2 \frac{\partial f}{\partial \rho_\delta} - (2\lambda_1 + \lambda_2) \operatorname{div} u_\delta \right) = \int \overline{T_k(\rho)} \left(\overline{\rho^2 \frac{\partial f}{\partial \rho}} - (2\lambda_1 + \lambda_2) \operatorname{div} u \right), \tag{4.60}$$

where T_k is defined in (4.58).

Proof. The argument is similar to that in Lemma 3.3. Choose $\Phi = \phi\nabla\triangle^{-1}(T_k(\rho_\delta))$ in (4.2) to get

$$
\begin{aligned}
&\int \phi T_k(\rho_\delta)\left(\rho_\delta^2\frac{\partial f}{\partial \rho_\delta} - (2\lambda_1+\lambda_2)\mathrm{div}u_\delta\right)\\
&= -\int \delta\phi T_k(\rho_\delta)\rho_\delta^4 + \partial_i\triangle^{-1}(T_k(\rho_\delta))\partial_i\phi\left(\delta\rho_\delta^4 + \rho_\delta^2\frac{\partial f}{\partial\rho_\delta} - (\lambda_1+\lambda_2)\mathrm{div}u_\delta\right)\\
&\quad + \lambda_1\int \partial_j u_\delta^i\partial_i\triangle^{-1}(T_k(\rho_\delta))\partial_j\phi - u_\delta^i\partial_j\partial_i\triangle^{-1}(T_k(\rho_\delta))\partial_j\phi + T_k(\rho_\delta)u_\delta\cdot\nabla\phi\\
&\quad - \int(\rho_\delta g_1 + g_2)\phi\partial_i\triangle^{-1}(T_k(\rho_\delta))\\
&\quad + \frac{1}{2}\int|\nabla c_\delta|^2\left(\phi T_k(\rho_\delta) + \partial_i\phi\partial_i\triangle^{-1}(T_k(\rho_\delta))\right)\\
&\quad - \int\nabla c_\delta\otimes\nabla c_\delta\left(\phi\partial_j\partial_i\triangle^{-1}(T_k(\rho_\delta)) + \partial_j\phi\partial_i\triangle^{-1}(T_k(\rho_\delta))\right)\\
&\quad - \int\rho_\delta u_\delta^j u_\delta^i\partial_j\phi\partial_i\triangle^{-1}(T_k(\rho_\delta))\\
&\quad - \int u_\delta^i\phi\left[\rho_\delta u_\delta^j\phi\partial_j\partial_i\triangle^{-1}(T_k(\rho_\delta)) - T_k(\rho_\delta)\partial_i\partial_j\triangle^{-1}(\rho_\delta u_\delta^j)\right]\\
&= \sum_{i=1}^7 R_i^\delta.
\end{aligned}
$$

On the other hand, if we use $\phi\nabla\triangle^{-1}\left(\overline{T_k(\rho)}\right)$ as a test function in $(4.57)_2$, we infer

$$
\begin{aligned}
&\int \phi\overline{T_k(\rho)}\left(\overline{\rho^2\frac{\partial f}{\partial\rho}} - (2\lambda_1+\lambda_2)\mathrm{div}u\right)\\
&= -\int\partial_i\triangle^{-1}\left(\overline{T_k(\rho)}\right)\partial_i\phi\left(\overline{\rho^2\frac{\partial f}{\partial\rho}} - (\lambda_1+\lambda_2)\mathrm{div}u\right)\\
&\quad + \lambda_1\int\partial_j u^i\partial_i\triangle^{-1}\left(\overline{T_k(\rho)}\right)\partial_j\phi - u^i\partial_j\partial_i\triangle^{-1}\left(\overline{T_k(\rho)}\right)\partial_j\phi + \overline{T_k(\rho)}u\cdot\nabla\phi\\
&\quad - \int(\rho g_1+g_2)\phi\partial_i\triangle^{-1}\left(\overline{T_k(\rho)}\right)\\
&\quad + \frac{1}{2}\int|\nabla c|^2\left(\phi\overline{T_k(\rho)} + \partial_i\phi\partial_i\triangle^{-1}\left(\overline{T_k(\rho)}\right)\right)\\
&\quad - \int\nabla c\otimes\nabla c\left(\phi\partial_j\partial_i\triangle^{-1}\left(\overline{T_k(\rho)}\right) + \partial_j\phi\partial_i\triangle^{-1}\left(\overline{T_k(\rho)}\right)\right)\\
&\quad - \int\rho u^j u^i\partial_j\phi\partial_i\triangle^{-1}\left(\overline{T_k(\rho)}\right)\\
&\quad - \int u^i\phi\left[\rho u^j\partial_j\partial_i\triangle^{-1}\left(\overline{T_k(\rho)}\right) - \overline{T_k(\rho)}\partial_i\partial_j\triangle^{-1}(\rho u^j)\right]\\
&= \sum_{i=1}^7 R_i.
\end{aligned}
$$

Therefore, we obtain (4.60) provided

$$\lim_{\delta\to 0} R_i^\delta = R_i \quad (i = 1, 2, \cdots, 7). \tag{4.61}$$

In fact, (4.61) can be verified by modifying slightly the argument in Lemma 3.3. The detail is omitted here. □

Finally, let us prove the strong convergence of density. By (1.5) and the simple fact

$$(\rho_\delta^\gamma - \rho^\gamma)\left(T_k(\rho_\delta) - T_k(\rho)\right) \geq \left(T_k(\rho_\delta) - T_k(\rho)\right)^{\gamma+1},$$

one has

$$\begin{aligned}
&\int (\rho_\delta^2 \frac{\partial f(\rho_\delta, c_\delta)}{\partial \rho_\delta} - \rho^2 \frac{\partial f}{\partial \rho}) \left(T_k(\rho_\delta) - T_k(\rho)\right) \\
&= \int (\gamma - 1)(\rho_\delta^\gamma - \rho^\gamma)\left(T_k(\rho_\delta) - T_k(\rho)\right) \\
&\quad + \int (\rho_\delta H_1(c_\delta) - \rho H_1(c))\left(T_k(\rho_\delta) - T_k(\rho)\right) \\
&\geq \int (\gamma - 1)\left(T_k(\rho_\delta) - T_k(\rho)\right)^{\gamma+1} \\
&\quad + \int \rho(H_1(c_\delta) - H_1(c))\left(T_k(\rho_\delta) - T_k(\rho)\right) + (\rho_\delta - \rho)H_1(c_\delta)\left(T_k(\rho_\delta) - T_k(\rho)\right) \\
&\geq \int (\gamma - 1)\left(T_k(\rho_\delta) - T_k(\rho)\right)^{\gamma+1} + \int \rho(H_1(c_\delta) - H_1(c))\left(T_k(\rho_\delta) - T_k(\rho)\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\lim_{\delta\to 0} \int (\rho_\delta^2 \frac{\partial f(\rho_\delta, c_\delta)}{\partial \rho_\delta} - \rho^2 \frac{\partial f}{\partial \rho}) \left(T_k(\rho_\delta) - T_k(\rho)\right) \\
&\geq (\gamma - 1) \lim_{\delta\to 0} \int \left(T_k(\rho_\delta) - T_k(\rho)\right)^{\gamma+1}.
\end{aligned} \tag{4.62}$$

By virtue of (4.62) and (4.60),

$$\begin{aligned}
&(2\lambda_1 + \lambda_2) \lim_{\delta\to 0} \int \left(T_k(\rho_\delta)\mathrm{div} u_\delta - \overline{T_k(\rho)}\mathrm{div} u\right) \\
&= \lim_{\delta\to 0} \int \left(T_k(\rho_\delta)\rho_\delta^2 \frac{\partial f(\rho_\delta, c_\delta)}{\partial \rho_\delta} - \overline{T_k(\rho)}\ \overline{\rho^2 \frac{\partial f}{\partial \rho}}\right) \\
&= \lim_{\delta\to 0} \int (\rho_\delta^2 \frac{\partial f(\rho_\delta, c_\delta)}{\partial \rho_\delta} - \rho^2 \frac{\partial f}{\partial \rho}) \left(T_k(\rho_\delta) - T_k(\rho)\right) \\
&\quad + \int (\overline{\rho^2 \frac{\partial f}{\partial \rho}} - \rho^2 \frac{\partial f}{\partial \rho}) \left(T_k(\rho) - \overline{T_k(\rho)}\right) \\
&\geq \lim_{\delta\to 0} \int (\rho_\delta^2 \frac{\partial f(\rho_\delta, c_\delta)}{\partial \rho_\delta} - \rho^2 \frac{\partial f}{\partial \rho}) \left(T_k(\rho_\delta) - T_k(\rho)\right),
\end{aligned} \tag{4.63}$$

where the last inequality is due to the concavity of T_k and

$$\overline{\rho^2 \frac{\partial f}{\partial \rho}} = (\gamma - 1)\overline{\rho^\gamma} + \overline{\rho \ln \rho} H_1(c) \geq (\gamma - 1)\rho^\gamma + \rho \ln \rho H_1(c) = \rho^2 \frac{\partial f}{\partial \rho}.$$

Following [24], we define

$$L_k = \begin{cases} z\ln z, & z \le k, \\ z\ln k + z\int_k^z \dfrac{T_k(s)}{s^2}ds, & z \ge k. \end{cases}$$

A direct computation shows that

$$b_k(z) = L_k(z) - \left(\ln k + \int_k^{k+1} \frac{T_k(s)}{s^2} + 1\right) z$$

belongs to $C([0,\infty))\cap C^1((0,\infty))$, $b_k'(z) = 0$ if $z \ge k+1$, and $b_k'(z)z - b_k(z) = T_k(z)$. Choosing $b = b_k$ in (3.21), we infer (approximating $b_k(z)$ near $z = 0$)

$$\mathrm{div}(b_k(\rho)u) + T_k(\rho)\mathrm{div}u = 0 \quad \text{in } \mathcal{D}(\mathbb{R}^3),$$

which implies

$$\int T_k(\rho)\mathrm{div}u = 0. \tag{4.64}$$

Also, one has

$$\int \overline{T_k(\rho)\mathrm{div}u} = \lim_{\delta\to 0}\int T_k(\rho_\delta)\mathrm{div}u_\delta = 0. \tag{4.65}$$

From (4.64)-(4.65) we obtain

$$\begin{aligned} &C\|T_k(\rho) - \overline{T_k(\rho)}\|_{L^2} \\ &\ge (2\lambda_1+\lambda_2)\int \left(T_k(\rho) - \overline{T_k(\rho)}\right)\mathrm{div}u \\ &= (2\lambda_1+\lambda_2)\int \overline{T_k(\rho)\mathrm{div}u} - \overline{T_k(\rho)}\mathrm{div}u \\ &= (2\lambda_1+\lambda_2)\lim_{\delta\to 0}\int \left(T_k(\rho_\delta)\mathrm{div}u_\delta - \overline{T_k(\rho)}\mathrm{div}u\right) \\ &\ge \lim_{\delta\to 0}\int (\rho_\delta^2\frac{\partial f}{\partial \rho_\delta} - \rho^2\frac{\partial f}{\partial \rho})\left(T_k(\rho_\delta) - T_k(\rho)\right) \\ &\ge (\gamma-1)\lim_{\delta\to 0}\int \left(T_k(\rho_\delta) - T_k(\rho)\right)^{\gamma+1}, \end{aligned} \tag{4.66}$$

where the inequalities are due to (4.62)-(4.63). Therefore, (4.66) gives

$$\begin{aligned} &\lim_{k\to\infty}\lim_{\delta\to 0}\|T_k(\rho) - T_k(\rho_\delta)\|_{L^{\gamma+1}}^{\gamma+1} \\ &\le C\lim_{k\to\infty}\|T_k(\rho) - \overline{T_k(\rho)}\|_{L^2} \\ &\le C\lim_{k\to\infty}\lim_{\delta\to 0}\left(\|T_k(\rho)-\rho\|_{L^2} + \|T_k(\rho_\delta)-\rho_\delta\|_{L^2}\right). \end{aligned} \tag{4.67}$$

However, by Lemma 4.1,

$$\|\rho_\delta\|_{L^2}^2 \le C\|\rho_\delta\|_{L^{\gamma+\theta}}^{\gamma+\theta} \le \|\rho_\delta^{2+\theta}\frac{\partial f}{\partial \rho_\delta}\|_{L^1} \le C.$$

Then, the following estimate holds true

$$\begin{aligned} \|T_k(\rho_\delta) - \rho_\delta\|_{L^p} &= \|T_k(\rho_\delta) - \rho_\delta\|_{L^2(\{\rho_\delta \ge k\})} \\ &\le 2\|\rho_\delta\|_{L^2(\{\rho_\delta\ge k\})} \le Ck^{\frac{p-\gamma}{2}} \to 0 \quad \text{as } k\to\infty, \end{aligned}$$

which is uniform in δ. Consequently,

$$\lim_{k\to\infty} \|\overline{T_k(\rho)} - \rho\|_{L^2} \le \lim_{k\to\infty}\lim_{\delta\to 0} \|T_k(\rho_\delta) - \rho_\delta\|_{L^2} = 0. \tag{4.68}$$

The same argument yields

$$\lim_{k\to\infty} \|T_k(\rho) - \rho\|_{L^2} = 0. \tag{4.69}$$

In terms of (4.67)-(4.69), one has

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{\delta\to 0} \|\rho_\delta - \rho\|_{L^1} \\
&\le \lim_{k\to\infty}\lim_{\delta\to 0} \left(\|\rho_\delta - T_k(\rho_\delta)\|_{L^1} + \|T_k(\rho_\delta) - T_k(\rho)\|_{L^1} + \|T_k(\rho) - \rho\|_{L^1}\right) \\
&= 0.
\end{aligned}$$

The proof of Theorem 1.1 is completed.

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