Twistors, Hyper-Kähler Manifolds, and Complex Moduli

Claude LeBrun

For my good friend and admired colleague Simon Salamon, on the occasion of his sixtieth birthday.

Abstract A theorem of Kuranishi (Ann Math 75(2):536–577, 1962) tells us that the moduli space of complex structures on any smooth compact manifold is always *locally* a finite-dimensional space. *Globally*, however, this is simply not true; we display examples in which the moduli space contains a sequence of regions for which the local dimension tends to infinity. These examples naturally arise from the twistor theory of hyper-Kähler manifolds.

Keywords Complex structure • Hyper-Kähler • Kodaira-Spencer map • Moduli • Twistor space

If Y is a smooth compact manifold, the moduli space $\mathfrak{M}(Y)$ of complex structures on Y is defined to be the quotient of the set of all smooth integrable almost-complex structure J on Y, equipped with the topology it inherits from the space of almost-complex structures, modulo the action of the group of self-diffeomorphisms of Y. When we focus only on complex structures near some given J_0, an elaboration of Kodaira-Spencer theory [3] due to Kuranishi [4] shows that the moduli space is locally finite dimensional. Indeed, if Θ denotes the sheaf of holomorphic vector fields on (Y, J_0), Kuranishi shows that there is a family of complex structures parameterized by an analytic subvariety of the unit ball in $H^1(Y, \Theta)$ which, up to biholomorphism, sweeps out every complex structure near J_0. This subvariety of $H^1(Y, \Theta)$ is defined by equations taking values in $H^2(Y, \Theta)$, and one must then also divide by the group of complex automorphisms of (Y, J), which is a Lie group with Lie algebra $H^0(Y, \Theta)$. But, in any case, near a given complex structure, this says

C. LeBrun (✉)
Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651, USA
e-mail: claude@math.stonybrook.edu

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that the moduli space is a finite-dimensional object, with dimension bounded above by $h^1(Y, \Theta)$.

What we will observe here, however, is that this local finite-dimensionality can completely break down in the large:

Theorem A *Let X^{4k} be a smooth simply connected compact manifold that admits a hyper-Kähler metric. Then the moduli space $\mathfrak{M}$ of complex structures on $S^2 \times X$ is infinite dimensional, in the following sense: for every $N \in \mathbb{Z}^+$, there are holomorphic embeddings $D^N \hookrightarrow \mathfrak{M}$ of the N-complex-dimensional unit polydisk $D^N := D \times \cdots \times D \subset \mathbb{C}^N$ into the moduli space.*

In fact, for every natural number N, we will construct proper holomorphic submersions $\mathfrak{Y} \to D^N$ with fibers diffeomorphic to $X \times S^2$ such that no two fibers are biholomorphically equivalent. Focusing on this concrete assertion should help avoid confusing the phenomenon under study with other possible structural pathologies of the moduli space $\mathfrak{M}$.

Before proceeding further, it might help to clarify how our construction differs from various off-the-shelf examples where Kodaira-Spencer theory produces mirages of moduli that should not be mistaken for the real thing. Consider the Hirzebruch surfaces $F_\ell = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(\ell)) \to \mathbb{CP}_1$. These are all diffeomorphic to $S^2 \times S^2$ or $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$, depending on whether ℓ is even or odd. For $\ell > 0$, $h^1(F_\ell, \Theta_\ell) = (\ell - 1) \to \infty$ and $h^2(F_\ell, \Theta_\ell) = 0$, so it might appear that the dimension of the moduli space is growing without bound. However, when these infinitesimal deformations are realized by a versal family, most of the fibers always turn out to be mutually biholomorphic, because $h^0(F_\ell, \Theta_\ell) = (\ell + 5) \to \infty$, too, and a cancellation arises from the action of the automorphisms of the central fiber on the versal deformation. In fact, the F_ℓ represent *all* the complex structures on $S^2 \times S^2$ and $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$; thus, while the corresponding moduli spaces are highly non-Hausdorff, they are in fact just 0-dimensional. Similar phenomena also arise from projectivizations of higher-rank vector bundles over $\mathbb{CP}_1$; even though it is easy to construct examples with $h^1(\Theta) \to \infty$ in this context, the piece of the moduli space one constructs in this way is once again non-Hausdorff and 0-dimensional.

Let us now recall that a smooth compact Riemannian manifold (X^{4k}, g) is said to be *hyper-Kähler* if its holonomy is a subgroup of $\mathbf{Sp}(k)$. One then says that a hyper-Kähler manifold is *irreducible* if its holonomy is exactly $\mathbf{Sp}(k)$. This in particular implies [1] that X is simply connected. Conversely, any simply connected compact hyper-Kähler manifold is a Cartesian product of irreducible ones, since its deRham decomposition [2] cannot involve any flat factors. In order to prove Theorem A, one therefore might as well assume that (X, g) is irreducible, since any hyper-Kähler manifold admits complex structures, and $S^2 \times (X \times \tilde{X}) = (S^2 \times X) \times \tilde{X}$. Note that examples of irreducible hyper-Kähler $(4k)$-manifolds are in fact known [1, 6] for every $k \geq 1$. When $k = 1$, the unique choice for X is $K3$. For $k \geq 2$, the smooth manifold X is no longer uniquely determined by k, but the the Hilbert scheme of k points on a $K3$ surface always provides one simple and elegant example.

The construction we will use to prove Theorem A crucially involves the use of twistor spaces [2, 7]. Recall that the standard representation of $\mathbf{Sp}(k)$ on $\mathbb{R}^{4k} = \mathbb{H}^k$ commutes with every almost-complex structure arising from a quaternionic scalar in

$S^2 \subset \Im m\,\mathbb{H}$, and that every hyper-Kähler manifold is therefore Kähler with respect to a 2-sphere's worth of parallel almost-complex structures. Concretely, if we let J_1, J_2, and J_3 denote the complex structures corresponding to the quaternions $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, then the integrable complex structures in question are those given by $aJ_1 + bJ_2 + cJ_3$ for any $(a, b, c) \in \mathbb{R}^3$ with $a^2 + b^2 + c^2 = 1$. We can then assemble these to form an integrable almost-complex structure on $X \times S^2$ by using the round metric and standard orientation on S^2 to make it into a $\mathbb{CP}_1$, and then giving the X the integrable complex structure $aJ_1 + bJ_2 + cJ_3$ determined by $(a, b, c) \in S^2$. For each $x \in X$, the stereographic coordinate $\zeta = (b + ic)/(a + 1)$ on $\{x\} \times S^2$ is thus a compatible complex coordinate system on the so-called *real twistor line* $\mathbb{CP}_1 \subset Z$ near the point $(1, 0, 0)$ representing $J_1|_x$. We will make considerable use of the fact that the factor projection $X \times S^2 \to S^2$ now becomes a holomorphic submersion $\varpi : Z \to \mathbb{CP}_1$ with respect to the twistor complex structure, so that ϖ can therefore be thought of as a family of complex structures on X.

Lemma 1 *Let (X^{4k}, g), $k \geq 1$, be a hyper-Kähler manifold, and let Z be its twistor space. Consider the holomorphic submersion $\varpi : Z \to \mathbb{CP}_1$ as a family of compact complex manifolds, and set $X_\zeta := \varpi^{-1}(\zeta)$ for any $\zeta \in \mathbb{CP}_1$. Then the Kodaira-Spencer map $T^{1,0}_{\zeta_0}\mathbb{CP}_1 \to H^1(X_{\zeta_0}, \mathcal{O}(T^{1,0}X_\zeta))$ is non-zero at every $\zeta_0 \in \mathbb{CP}_1$.*

Proof Since we can always change our basis for the parallel complex structures on (X, g) by the action of $\mathbf{SO(3)}$, we may assume that the value ζ_0 of $\zeta \in \mathbb{CP}_1$ at which we wish to check the claim represents the complex structure on X we have temporarily chosen to call J_1. Observe that the 2-forms $\omega_\alpha = g(J_\alpha \cdot, \cdot)$, $\alpha = 1, 2, 3$, are all parallel. Moreover, notice that, with respect to J_1, the 2-form ω_1 is just the Kähler form of g, while $\omega_2 + i\omega_3$ is a non-degenerate holomorphic $(2, 0)$-form.

By abuse of notation, we will now also use ζ to denote a local complex coordinate on $\mathbb{CP}_1$, with $\zeta = 0$ representing the complex structure J_1 of interest. Now recall that the Kodaira-Spencer map sends $d/d\zeta$ to an element of $H^1(X, \mathcal{O}_{J_1}(T^{1,0}_{J_1}X))$ that literally encodes the derivative of the complex structure J_ζ with respect to ζ. Indeed, since we already have chosen a differentiable trivialization of our family, this element is represented in Dolbeault cohomology by the $(0, 1)$-form φ with values in $T^{1,0}$ given by

$$\varphi(v) := \left[\frac{d}{d\zeta} J_\zeta(v^{0,1})\right]^{1,0}\Bigg|_{\zeta=0}$$

where the decomposition $T_{\mathbb{C}}X = T^{1,0} \oplus T^{0,1}$ used here is understood to be the one determined by J_1. Now taking ζ to specifically be the stereographic coordinate $\zeta = \xi + i\eta$, where $\xi = b/(1 + a)$ and $\eta = c/(1 + a)$, we then have

$$\frac{d}{d\xi} J_\zeta\Big|_{\zeta=0} = J_2 \quad \text{and} \quad \frac{d}{d\eta} J_\zeta\Big|_{\zeta=0} = J_3,$$

and hence

$$\frac{d}{d\zeta}J_\zeta\Big|_{\zeta=0} = \frac{1}{2}(J_2 - iJ_3).$$

Since $T^{0,1}$ is the $(-i)$-eigenspace of J_1, we therefore have

$$\begin{aligned}\varphi(v) &= \frac{1}{2}\left[(J_2 - iJ_3)v^{0,1}\right]^{1,0}\\ &= \frac{1}{2}\left[(J_2 + iJ_2J_1)v^{0,1}\right]^{1,0}\\ &= \left[J_2(v^{0,1})\right]^{1,0}\\ &= J_2(v^{0,1})\end{aligned}$$

where the last step uses the fact that J_2 anti-commutes with J_1, and therefore interchanges the $(\pm i)$-eigenspaces $T^{1,0}$ and $T^{0,1}$ of J_1.

On the other hand, since $\omega_2 + i\omega_3$ is a non-degenerate holomorphic 2-form on (X, J_1), contraction with this form induces a holomorphic isomorphism $T^{1,0} \cong \Lambda^{1,0}$, and hence an isomorphism $H^1(X, \mathcal{O}(T^{1,0})) \cong H^1(X, \Omega^1)$. In Dolbeault terms, the Kodaira-Spencer class $[\varphi]$ is thus mapped by this isomorphism to the element of $H^{1,1}_{\bar\partial_{J_1}}(X) = H^1(X, \Omega^1)$ represented by the contraction $\varphi \lrcorner (\omega_2 + i\omega_3)$. Since

$$\begin{aligned}[\varphi(v^{0,1})]\lrcorner(\omega_2 + i\omega_3) &= g([J_2 + iJ_3]\varphi(v^{0,1}), \cdot)\\ &= g([J_2 + iJ_1J_2]J_2(v^{0,1}), \cdot)\\ &= g(-[I + iJ_1]v^{0,1}, \cdot)\\ &= -2i\,\omega_1(v^{0,1}, \cdot)\\ &= 2i\,\omega_1(\cdot, v^{0,1}),\end{aligned}$$

the Kodaira-Spencer class is therefore mapped to $2i[\omega_1] \in H^{1,1}_{\bar\partial_{J_1}}(X)$. However, since $[\omega_1]^{2k}$ pairs with fundamental cycle $[X]$ to yield $(2k)!$ times the total volume of (X, g), $2i\,[\omega_1]$ is certainly non-zero in deRham cohomology, and is therefore non-zero in Dolbeault cohomology, too. The Kodaira-Spencer map of such a twistor family is thus everywhere non-zero, as claimed. ∎

We next define many new complex structures on $X \times S^2$ by generalizing a construction [5] originally introduced in the $k = 1$ case to solve a different problem. Let $f : \mathbb{CP}_1 \to \mathbb{CP}_1$ be a holomorphic map of arbitrary degree ℓ. We then define a holomorphic family $f^*\varpi$ over $\mathbb{CP}_1$ by pulling ϖ back via f:

$$\begin{array}{ccc} f^*Z & \stackrel{\hat f}{\longrightarrow} & Z\\ {\scriptstyle f^*\varpi}\downarrow & & \downarrow{\scriptstyle \varpi}\\ \mathbb{CP}_1 & \stackrel{f}{\longrightarrow} & \mathbb{CP}_1.\end{array}$$

In other words, if $\Gamma \subset \mathbb{CP}_1 \times \mathbb{CP}_1$ is the graph of f, then f^*Z is the inverse image of Γ under $Z \times \mathbb{CP}_1 \stackrel{\varpi \times 1}{\longrightarrow} \mathbb{CP}_1 \times \mathbb{CP}_1$. Since ϖ is differentiably trivial, so is $\hat{\varpi} := f^*\varpi$, and $\hat{Z} := f^*Z$ may therefore be viewed as $X \times S^2$ equipped with some new complex structure J_f.

Lemma 2 *Let $\hat{Z} = f^*Z$ be the complex $(2k+1)$-manifold associated with a holomorphic map $f : \mathbb{CP}_1 \to \mathbb{CP}_1$ of degree ℓ, and let $\hat{\varpi} = f^*\varpi$ be the associated holomorphic submersion $\hat{\varpi} = f^*\varpi$. Then the canonical line bundle $K_{\hat{Z}}$ is isomorphic to $\hat{\varpi}^*\mathcal{O}(-2k\ell - 2)$ as a holomorphic line bundle.*

Proof The twistor space of any hyper-Kähler manifold (X^{4k}, g) satisfies $K_Z = \varpi^*\mathcal{O}(-2k-2)$. On the other hand, the branch locus B of $\hat{f} : \hat{Z} \to Z$ is the inverse image via $\hat{\varpi}$ of $2\ell - 2$ points in $\mathbb{CP}_1$. Thus

$$K_{\hat{Z}} = [B] \otimes \hat{f}^* K_Z \cong \hat{\varpi}^*[\mathcal{O}(2\ell - 2) \otimes \mathcal{O}(\ell(-2k-2))] = \hat{\varpi}^*\mathcal{O}(-2k\ell - 2),$$

as claimed. ∎

This now provides one cornerstone of our argument:

Proposition 1 *If $\hat{Z} = f^*Z$ is the complex $(2k+1)$-manifold arising from a simply connected hyper-Kähler manifold (X^{4k}, g) and a holomorphic map $f : \mathbb{CP}_1 \to \mathbb{CP}_1$ of degree ℓ, then there is a unique holomorphic line bundle $K^{-1/(2k\ell+2)}$ whose $(2+2k\ell)$th tensor power is isomorphic to the anti-canonical line bundle. Moreover, $h^0(Z, \mathcal{O}(K^{-1/(2k\ell+2)})) = 2$, and the pencil of sections of this line bundle exactly reproduces the holomorphic map $\hat{\varpi} : \hat{Z} \to \mathbb{CP}_1$. Thus the holomorphic submersion $\hat{\varpi}$ is an intrinsic property of the compact complex manifold $\hat{Z} = (X \times S^2, J_f)$, and is uniquely determined, up to Möbius transformation, by the complex structure J_f.*

Proof Because $\hat{Z} \approx X \times S^2$ is simply connected, $H^1(\hat{Z}, \mathbb{Z}_{2k\ell+2}) = 0$, and the long exact sequence induced by the short exact sequence of sheaves

$$0 \to \mathbb{Z}_{2k\ell+2} \to \mathcal{O}^\times \to \mathcal{O}^\times \to 0$$

therefore guarantees that there can be at most one holomorphic line bundle $K^{-1/(2k\ell+2)}$ whose $(2+2k\ell)$th tensor power is the anti-canonical line bundle K^*. Since Lemma 2 guarantees that $\hat{\varpi}^*\mathcal{O}(1)$ is one candidate for this root of K^*, it is therefore the unique such root. On the other hand, since $\hat{\varpi}^*\mathcal{O}(1)$ is trivial on the compact fibers of $\hat{\varpi}$, any holomorphic section of this line bundle on $\hat{Z}$ is fiber-wise constant, and is therefore the pull-back of a section of $\mathcal{O}(1)$ on $\mathbb{CP}_1$. Thus $h^0(Z, \mathcal{O}(K^{-1/(2k\ell+2)})) = h^0(\mathbb{CP}_1, \mathcal{O}(1)) = 2$, and the pencil of sections of $K^{-1/(2k\ell+2)}$ thus exactly reproduces $\hat{\varpi} : \hat{Z} \to \mathbb{CP}_1$. ∎

Here, the role of the Möbius transformations is of course unavoidable. After all, preceding f by a Möbius transformation will certainly result in a biholomorphic manifold!

Since $\hat{\varpi}$ is intrinsically determined by the complex structure of $\hat{Z}$, its complex structure also completely determines those elements of $\mathbb{CP}_1$ at which the Kodaira-Spencer map of the family $\hat{\varpi} : \hat{Z} \to \mathbb{CP}_1$ vanishes; this is the same as asking for fibers for which there is a transverse holomorphic foliation of the first formal neighborhood. Similarly, one can ask whether there are elements of $\mathbb{CP}_1$ at which the Kodaira-Spencer map vanishes to order m; this is the same as asking for fibers for which there is a transverse holomorphic foliation of the $(m+1)^{\text{st}}$ formal neighborhood.

Proposition 2 *The critical points of $f : \mathbb{CP}_1 \to \mathbb{CP}_1$, along with their multiplicities, can be reconstructed from the submersion $f^*\varpi : f^*Z \to \mathbb{CP}_1$.*

Proof The Kodaira-Spencer map is functorial, and transforms with respect to pull-backs like a bundle-valued 1-form. Since the Kodaira-Spencer map of ϖ is everywhere non-zero by Lemma 1, the points at which the Kodaira-Spencer map of $\hat{\varpi} = f^*\varpi$ vanishes to order m are exactly those points at which the derivative of $f : \mathbb{CP}_1 \to \mathbb{CP}_1$ has a critical point of order m. ∎

Taken together, Propositions 1 and 2 thus imply the following:

Theorem B *Modulo Möbius transformations, the configuration of critical points of $f : \mathbb{CP}_1 \to \mathbb{CP}_1$, along with their multiplicities, is an intrinsic invariant of the compact complex manifold $\hat{Z} = f^*Z$.*

By displaying suitable families of holomorphic maps $\mathbb{CP}_1 \to \mathbb{CP}_1$, we will now use Theorem B prove Theorem A. Indeed, for any $(a_1, \ldots, a_N) \in \mathbb{C}^N$ with $|a_j - 2j| < 1$, let $P_{a_1,\ldots,a_N}(\zeta)$ be the polynomial of degree $N+6$ in the complex variable ζ defined by

$$P_{a_1,\ldots,a_N}(\zeta) = \int_0^\zeta t^2(t-1)^3(t-a_1)\cdots(t-a_N)dt,$$

and let $f_{a_1,\ldots,a_N} : \mathbb{CP}_1 \to \mathbb{CP}_1$ be the self-map of $\mathbb{CP}_1 = \mathbb{C} \cup \{\infty\}$ obtained by extending $P_{a_1,\ldots,a_N} : \mathbb{C} \to \mathbb{C}$ via $\infty \mapsto \infty$; in other words,

$$f_{a_1,\ldots,a_N}([\zeta_1, \zeta_2]) = [P_{a_1,\ldots,a_N}(\zeta_1, \zeta_2), \zeta_2^{N+6}],$$

where $P_{a_1,\ldots,a_N}(\zeta_1, \zeta_2)$ is the homogeneous polynomial formally defined by

$$P_{a_1,\ldots,a_N}(\zeta_1, \zeta_2) = \zeta_2^{N+6} P_{a_1,\ldots,a_N}(\tfrac{\zeta_1}{\zeta_2}).$$

Since the constraints we have imposed on our auxiliary parameters force the complex numbers $0, 1, a_1, \ldots, a_N$ to all be distinct, the critical points of $f_{a_1,\ldots,a_N} : \mathbb{CP}_1 \to \mathbb{CP}_1$ are just the $a_1, \ldots, a_N$, each with multiplicity 1, along with 0, 1, and ∞, which are individually distinguishable by their respective multiplicities of 2, 3, and $N+5$. Since any Möbius transformation that fixes 0, 1, and ∞ must be the identity, Theorem B implies that different values of the parameters $(a_1, \ldots, a_N)$, subject the constraints $|a_j - 2j| < 1$, will always result in non-biholomorphic complex manifolds

$\hat{Z}_{a_1,\ldots,a_N} := f^*_{a_1,\ldots,a_N} Z$. Thus, pulling back $\varpi : Z \to \mathbb{CP}_1$ via the holomorphic map

$$\Phi : D^N \times \mathbb{CP}_1 \longrightarrow \mathbb{CP}_1$$
$$(u_1, \ldots, u_n, [\zeta_1, \zeta_2]) \longmapsto f_{u_1+2,\ldots,u_N+2N}([\zeta_1, \zeta_2])$$

now produces a family $\Phi^*\varpi : \Phi^*Z \to D^N$ of mutually non-biholomorphic complex manifolds over the unit polydisk $D^N \subset \mathbb{C}^N$. Since these manifolds are all diffeomorphic to $X \times S^2$, and since this works for any positive integer N, Theorem A is therefore an immediate consequence.

Of course, the above proof is set in the world of general compact complex manifolds, and so has little to say about conditions prevailing in the tidier realm of, say, complex algebraic varieties. In fact, one should probably expect the examples described in this article to never be of Kähler type, since there are results in this direction [5] when $k = 1$. It would certainly be interesting to see this definitively established for general k.

On the other hand, the feature of the $k = 1$ case highlighted in [5] readily generalizes to higher dimensions; namely, the Chern numbers of the complex structures J_f change as we vary the degree of f. Indeed, notice the tangent bundle of $X \times S^2$ is stably isomorphic to the pull-back of the tangent bundle of X, and that TX has some non-trivial Pontrjagin numbers; for example, if we assume for simplicity that X is irreducible, we then have $\hat{A}(X) = k + 1$. Since the fibers of $f^*\varpi$ are Poincaré dual to $c_1(f^*Z)/(2k\ell + 2)$, we have $(\mathbf{c_1}\hat{\mathbf{A}})(f^*Z) = 2(k\ell + 1)(k + 1)$, and a certain combination of the Chern numbers of $f^*(Z)$ therefore grows linearly in $\ell = \deg f$. Consequently, as $N \to \infty$, the families of complex structures we have constructed skip through infinitely many connected components of the moduli space $\mathfrak{M}(X \times S^2)$. Is this necessary for a complex moduli space to fail to be finite-dimensional?

Finally, notice that the dimension of each exhibited component of the moduli space $\mathfrak{M}(X \times S^2)$ is higher than what might be inferred from our construction. Indeed, we have only made use of a single hyper-Kähler metric g on X, whereas these in practice always come in large families. Hyper-Kähler twistor spaces also carry a tautological anti-holomorphic involution, whereas their generic small deformations generally will not. In short, these moduli spaces are still largely *terra incognita*. Perhaps some interested reader will take up the challenge, and tell us much more about them!

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