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Einstein Metrics, Harmonic Forms and Conformally Kähler Geometry

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Abstract

The author has elsewhere given a complete classification of the compact oriented Einstein 4-manifolds that satisfy $W^+(\omega, \omega) > 0$ for some self-dual harmonic 2-form ω , where W^+ denotes the self-dual Weyl curvature. In this chapter, similar results are obtained when $W^+(\omega, \omega) \geq 0$, provided the self-dual harmonic 2-form ω is transverse to the zero section of $\Lambda^+ \rightarrow M$. However, this transversality condition plays an essential role in the story; dropping it leads one into wildly different territory where entirely different phenomena predominate.

9.1 Introduction

Recall that a Riemannian metric h is said to be *Einstein* [3] if it has constant Ricci curvature, or in other words if it solves the Einstein equation

$$r = \lambda h \tag{9.1}$$

for some real number λ , where r is the Ricci tensor of h . When this happens, λ is called the *Einstein constant* of h , and of course has the same sign as the Einstein metric's scalar curvature.

Dimension four seems to represent a sort of “Goldilocks zone” for the Einstein equation. In lower dimensions, Einstein metrics are extremely rigid, in the sense that they necessarily have constant sectional curvature, and so do not really exhibit any interesting local differential geometry. In higher dimensions, on the other hand, they are extremely flexible, existing in such profusion on familiar manifolds [6, 7, 32] that their local geometry seems to offer little clue as to the identity of the manifold where

they reside. By contrast, dimension four seems “just right” for (9.1), as four-dimensional Einstein metrics exhibit a well-tempered combination of local flexibility and global rigidity that often makes their geometry perfectly reflect the manifold on which they live. For example, if M is a compact real or complex-hyperbolic 4-manifold, a 4-torus, or $K3$, the moduli space of Einstein metrics on M is known explicitly, and moreover turns out to be connected [3, 4, 16].

Unfortunately, however, we do not have a similarly complete understanding of the moduli space of Einstein metrics on most of the 4-manifolds where this moduli space is non-empty. An important family of test-cases is provided by the *del Pezzo surfaces*, here understood to mean the smooth compact oriented 4-manifolds that support complex structures with ample anti-canonical line bundle. Up to diffeomorphism, there are exactly ten such manifolds, namely $S^2 \times S^2$ and the nine connected sums $\mathbb{CP}_2 \# m \overline{\mathbb{CP}}_2$, $m = 0, 1, \dots, 8$. These 4-manifolds are completely characterized [8] by two properties: they admit Einstein metrics with $\lambda > 0$, and they also admit symplectic structures. However, it is currently unclear whether the known Einstein metrics on these spaces sweep out the entire Einstein moduli space. One of our main objectives here will be to generalize and strengthen a characterization of the known Einstein metrics on del Pezzo surfaces previously proved by the author in [20].

In order to formulate our results, first recall that the bundle of 2-forms $\Lambda^2 \rightarrow M$ over an oriented Riemannian 4-manifold (M, h) decomposes invariantly as the Whitney sum

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- \quad (9.2)$$

of the eigenspaces of the Hodge star operator $\star : \Lambda^2 \rightarrow \Lambda^2$. Sections of the $(+1)$ -eigenbundle Λ^+ are called self-dual 2-forms, while the sections of the (-1) -eigenbundle Λ^- are called anti-self-dual 2-forms. The decomposition (9.2) is moreover *conformally invariant*, meaning that it is unchanged by multiplying the metric by an arbitrary positive function.

One important consequence of the decomposition (9.2) is that it induces an invariant decomposition of the Riemann curvature tensor $\mathcal{R}$ into simpler pieces. Indeed, if we identify the Riemann curvature tensor with the self-adjoint endomorphism $\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$ of the 2-forms defined by

$$\varphi_{ab} \mapsto \frac{1}{2} \mathcal{R}^{cd}{}_{ab} \varphi_{cd}$$

and known as the *curvature operator*, then (9.2) allows us to decompose

$\mathcal{R}$ into irreducible pieces:

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12} & \overset{\circ}{r} \\ \hline \overset{\circ}{r} & W^- + \frac{s}{12} \end{array} \right), \quad (9.3)$$

where s denotes the scalar curvature, $\overset{\circ}{r} = r - \frac{s}{4}g$ is the trace-free Ricci curvature, and where the remaining pieces $W^\pm$, known as the *self-dual* and *anti-self-dual* Weyl tensors, are the trace-free parts of the endomorphisms of $\Lambda^\pm$ induced by $\mathcal{R}$. Remarkably, the corresponding pieces $(W^\pm)^a_{bcd}$ of the Riemann curvature tensor are both *conformally invariant*, in the sense that they remain unaltered if the metric is multiplied by an arbitrary smooth positive function.

Let us now assume that (M, h) is a *compact* oriented Riemannian 4-manifold. The Hodge theorem then tells us that every de-Rham class on M has a unique harmonic representative. In particular, there is a canonical isomorphism

$$H^2(M, \mathbb{R}) \cong \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, d\star\varphi = 0\}.$$

However, since the Hodge star operator $\star$ defines an involution of the right-hand side, we obtain a direct-sum decomposition

$$H^2(M, \mathbb{R}) = \mathcal{H}_h^+ \oplus \mathcal{H}_h^-, \quad (9.4)$$

where

$$\mathcal{H}_h^\pm = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

are the spaces of self-dual and anti-self-dual harmonic forms. Since the conditions of being closed and belonging to $\Lambda^\pm$ are both conformally invariant, it follows that the spaces $\mathcal{H}^\pm$ are both conformally invariant, too. Moreover, the dimensions $b_\pm = \dim \mathcal{H}^\pm$ of these spaces are completely metric independent, and can easily be shown to be oriented homotopy invariants of the 4-manifold M .

Now, if (M, h) is a compact oriented Riemannian 4-manifold, and if $\omega \in \mathcal{H}^+$ is a fixed self-dual harmonic 2-form, the quantity

$$W^+(\omega, \omega) := \langle W^+(\omega), \omega \rangle = \frac{1}{4}(W^+)^{abcd}\omega_{ab}\omega_{cd}$$

transforms in an extremely simple manner under conformal rescaling;

namely, if we change our metric by

$$h \rightsquigarrow u^2 h$$

for some positive function u , then the quantity in question changes by

$$W^+(\omega, \omega) \rightsquigarrow u^{-6} W^+(\omega, \omega).$$

In particular, the *sign* of this quantity at a given point is unchanged by conformal rescalings. This makes this hybrid measure of curvature particularly compelling when $b_+(M) = 1$, because in this case there is, up to a non-zero constant factor, only one non-trivial choice of ω , and the sign of $W^+(\omega, \omega)$ at each point then becomes a natural global conformal invariant of (M, h) .

The main result of [20] was that if a compact 4-dimensional Einstein manifold satisfies

$$W^+(\omega, \omega) > 0 \tag{9.5}$$

for some self-dual harmonic 2-form ω , then (M, h) is one of the known Einstein metrics on some del Pezzo surface. Conversely, the known Einstein metrics on del Pezzo surfaces all have this property. Combining these two observations then shows, as a corollary, that the known Einstein metrics on these spaces exactly sweep out one connected component of the Einstein moduli space. Here it is worth noting that every del Pezzo surface has $b_+ = 1$, so that condition (9.5) represents a rather natural characterization of the known Einstein metrics on these 4-manifolds.

On the other hand, since condition (9.5) trivially implies that both W^+ and ω are nowhere zero, it might seem desirable to relax this overly stringent condition by merely requiring that $W^+(\omega, \omega)$ be non-negative. What we will show here is that this can indeed be done, provided one imposes an interesting and natural condition on the 2-form. Namely, if ω is a harmonic self-dual 2-form on a compact oriented Riemannian 4-manifold (M, h) , one says that ω is *near-symplectic* if its graph is transverse to the zero section of the rank-3 vector bundle $\Lambda^+ \rightarrow M$. This is a *generic* condition, as has come to be understood through the work of Taubes [27, 29] and others [14, 18, 24]; indeed, on any smooth compact oriented 4-manifold with $b_+ \neq 0$, the set of metrics admitting a near-symplectic self-dual harmonic 2-form is open and dense. Of course, a dimension count immediately reveals that the zero locus of a near-symplectic self-dual harmonic 2-form ω on (M, h) is automatically a (possibly empty) finite disjoint union Z of circles:

$$Z \approx \sqcup_{j=1}^n S^1. \tag{9.6}$$

Imposing this reasonable assumption on the behavior of ω will actually allow us to prove some natural generalizations of the main result of [20]. Namely, the main results of the present chapter are the following:

Theorem 9.1.1 *Let (M, h) be a compact oriented Einstein 4-manifold that carries a near-symplectic self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) \geq 0, \quad W^+(\omega, \omega) \not\equiv 0. \quad (9.7)$$

Then $W^+(\omega, \omega) > 0$ everywhere, M is diffeomorphic to a del Pezzo surface, and h is conformally related to a positive-scalar-curvature extremal Kähler metric g on M with Kähler form ω . Conversely, every del Pezzo surface admits an Einstein metric h satisfying (9.7) for a self-dual harmonic 2-form ω that is nowhere zero (and hence near-symplectic).

Theorem 9.1.2 *Let (M, h) be a compact oriented $\lambda \geq 0$ Einstein 4-manifold that carries a near-symplectic self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) \geq 0 \quad (9.8)$$

everywhere. Then ω is nowhere zero, and h is conformally related to an extremal Kähler metric g on M with Kähler form ω . Moreover, M is diffeomorphic to a del Pezzo surface, a K3 surface, an Enriques surface, an Abelian surface, or a hyper-elliptic surface. Conversely, each of these complex surfaces admits a $\lambda \geq 0$ Einstein metric h satisfying (9.8) for a self-dual harmonic 2-form ω that is nowhere zero (and hence near-symplectic).

Theorem 9.1.3 *The near-symplectic hypothesis in Theorem 9.1.1 is essential: counter-examples show that the result fails without this assumption.*

The proofs of these main results can be found Section 9.4 below, following the proofs, in Sections 9.2–9.3, of the technical results that underpin these theorems.

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9.2 An Integral Weitzenböck Formula

Let (M, h) be a compact oriented Riemannian 4-manifold with *harmonic self-dual Weyl curvature*, in the sense that $\delta W^+ := -\nabla \cdot W^+ = 0$. When h is Einstein, this property automatically holds, by virtue of the second Bianchi identity. We will further assume throughout that h is at least C^4 . The latter assumption is of course innocuous in the Einstein case, as elliptic regularity for (9.1) implies that Einstein metrics are always [10] real-analytic in harmonic coordinates.

We will henceforth also assume that $b_+(M) \neq 0$. This is equivalent to saying that (M, h) admits a self-dual harmonic 2-form $\omega \neq 0$. We now choose some such form, and regard it as fixed for the remainder of the discussion. Let $Z \subset M$ denote the zero set of ω . Since ω is self-dual by assumption,

$$\omega \wedge \omega = \omega \wedge \star \omega = |\omega|_h^2 d\mu_h,$$

and it therefore follows that ω is actually a symplectic form on the open set $X := M - Z$, where ω is non-zero. Moreover, the Riemannian metric g on X defined by $g = 2^{-1/2}|\omega|_h h$ is then an *almost-Kähler metric*, in the sense that g is related to the symplectic form ω by $g = \omega(\cdot, J\cdot)$ for a unique almost-complex structure J on X .

Let us now re-express the conformal relationship between our two metrics as

$$h = f^2 g,$$

where $f = 2^{1/4}|\omega|_h^{-1/2}$. The fact that h satisfies $\delta W^+ = 0$ then implies [23] that g satisfies $\delta(fW^+) = 0$. Since our assumptions imply that g is also at least C^4 , we therefore have [9, 12, 20, 23] the Weitzenböck formula

$$0 = \nabla^* \nabla(fW^+) + \frac{s}{2}fW^+ - 6fW^+ \circ W^+ + 2f|W^+|^2 I \quad (9.9)$$

for fW^+ , which for notational simplicity has been represented here as a trace-free section of $\text{End}(\Lambda^+)$, while s and ∇ respectively denote the

scalar curvature and Levi-Civita connection of our almost-Kähler metric g on X .

Our strategy is now to contract (9.9) with $\omega \otimes \omega$, integrate on $X = M - Z$, and then try to integrate by parts in order to throw the Bochner Laplacian $\nabla^* \nabla$ onto $\omega \otimes \omega$. In order to accomplish this, we first exhaust X by domains X_ε with smooth boundary, where X_ε is the region where $|\omega|_h \geq \varepsilon$, where $\varepsilon > 0$ is any regular value of the smooth non-negative function $|\omega|_h : X \rightarrow \mathbb{R}$. Integrating by parts on X_ε then has the following effect:

Lemma 9.2.1 *There is a constant C , independent of $\varepsilon \in (0, 1)$, but depending on (M, h, ω) , such that*

$$\left| \int_{X_\varepsilon} [\langle \nabla^* \nabla(fW^+), \omega \otimes \omega \rangle - \langle fW^+, \nabla^* \nabla(\omega \otimes \omega) \rangle] d\mu_g \right| \leq C\varepsilon^{-3/2} \text{Vol}(\partial X_\varepsilon, h),$$

where all terms in the integral on the left are computed with respect to g , but where the 3-dimensional boundary volume on the right is computed with respect to h .

Proof By the divergence version of Stokes' theorem, we have

$$\begin{aligned} \int_{X_\varepsilon} \langle \nabla^* \nabla(fW^+), \omega \otimes \omega \rangle d\mu_g &= \int_{X_\varepsilon} \langle -\nabla \cdot \nabla fW^+, \omega \otimes \omega \rangle d\mu_g \\ &= - \int_{X_\varepsilon} \nabla \cdot \langle \nabla fW^+, \omega \otimes \omega \rangle d\mu_g \\ &\quad + \int_{X_\varepsilon} \langle \nabla fW^+, \nabla(\omega \otimes \omega) \rangle d\mu_g \\ &= - \int_{\partial X_\varepsilon} \langle \nabla_\nu fW^+, \omega \otimes \omega \rangle da_g \\ &\quad + \int_{X_\varepsilon} \langle \nabla fW^+, \nabla(\omega \otimes \omega) \rangle d\mu_g \\ &= - \int_{\partial X_\varepsilon} \nabla_\nu \langle fW^+, \omega \otimes \omega \rangle da_g \\ &\quad + \int_{\partial X_\varepsilon} \langle fW^+, \nabla_\nu(\omega \otimes \omega) \rangle da_g \\ &\quad + \int_{X_\varepsilon} \nabla \cdot \langle fW^+, \nabla(\omega \otimes \omega) \rangle d\mu_g \\ &\quad + \int_{X_\varepsilon} \langle fW^+, -\nabla \cdot \nabla(\omega \otimes \omega) \rangle d\mu_g \end{aligned}$$

$$\begin{aligned}
&= - \int_{\partial X_\varepsilon} \nabla_\nu \langle fW^+, \omega \otimes \omega \rangle da_g \\
&\quad + 2 \int_{\partial X_\varepsilon} \langle fW^+, \nabla_\nu(\omega \otimes \omega) \rangle da_g \\
&\quad + \int_{X_\varepsilon} \langle fW^+, \nabla^* \nabla(\omega \otimes \omega) \rangle d\mu_g \\
&= - \int_{\partial X_\varepsilon} \nabla_\nu [fW^+(\omega, \omega)] da_g \\
&\quad + 4 \int_{\partial X_\varepsilon} fW^+(\omega, \nabla_\nu \omega) da_g \\
&\quad + \int_{X_\varepsilon} \langle fW^+, \nabla^* \nabla(\omega \otimes \omega) \rangle d\mu_g,
\end{aligned}$$

where ν is the outward-pointing unit normal of ∂X_ε with respect to g , and where da_g is the g -induced volume 3-form on the boundary. Here, every term is thus understood to be computed with respect to g .

We now estimate the boundary integrals by first re-expressing them in terms of the original metric $h = f^2 g$. For emphasis and clarity, we will temporarily use $\hat{\nu} = f^{-1} \nu$ to denote the unit normal of ∂X_ε with respect to h , and $\hat{\nabla}$ to denote the Levi-Civita connection of h , which differs from the Levi-Civita connection of ∇ of g by

$$\delta_b^a \beta_c + \delta_c^a \beta_b - \beta_d h^{da} h_{bc},$$

where $\beta = d \log f = -\frac{1}{2} d \log |\omega|_h$. In other cases where the meaning of a term depends on a choice of metric, we will indicate the metric used by means of a subscript; for example, since index-raising is needed to define $W^+(\omega, \omega)$, one has

$$[W^+(\omega, \omega)]_g = f^6 [W^+(\omega, \omega)]_h.$$

With these conventions in hand, we thus have

$$\begin{aligned}
\left| \int_{\partial X_\varepsilon} \nabla_\nu [fW^+(\omega, \omega)]_g da_g \right| &= \left| \int_{\partial X_\varepsilon} f \nabla_{\hat{\nu}} [f^7 W^+(\omega, \omega)]_h f^{-3} da_h \right| \\
&\leq 7 \left| \int_{\partial X_\varepsilon} f^4 (\nabla_{\hat{\nu}} f) [W^+(\omega, \omega)]_h da_h \right| \\
&\quad + \left| \int_{\partial X_\varepsilon} f^5 \nabla_{\hat{\nu}} [W^+(\omega, \omega)]_h da_h \right|
\end{aligned}$$

$$\begin{aligned}
&\leq 7 \left| \int_{\partial X_\varepsilon} f^5 |f^{-1} df|_h |W^+|_h |\omega|_h^2 da_h \right| \\
&\quad + \left| \int_{\partial X_\varepsilon} f^5 |\hat{\nabla} W^+|_h |\omega|_h^2 da_h \right| \\
&\quad + 2 \left| \int_{\partial X_\varepsilon} f^5 |W^+|_h |\omega|_h |\hat{\nabla} \omega|_h da_h \right| \\
&= 7 \left| \int_{\partial X_\varepsilon} 2^{1/4} |\omega|_h^{-3/2} |d|\omega|_h| |W^+|_h da_h \right| \\
&\quad + \left| \int_{\partial X_\varepsilon} 2^{5/4} |\omega|_h^{-1/2} |\hat{\nabla} W^+|_h da_h \right| \\
&\quad + 2 \left| \int_{\partial X_\varepsilon} 2^{5/4} |\omega|_h^{-3/2} |W^+|_h |\hat{\nabla} \omega|_h da_h \right| \\
&\leq C_1 \varepsilon^{-3/2} \text{Vol}^{(3)}(\partial X_\varepsilon, h),
\end{aligned}$$

where $C_1 = \sqrt[4]{2} \left[11(\max_M |W^+|_h)(\max_M |\hat{\nabla} \omega|_h) + 2 \max_M |\hat{\nabla} W^+|_h \right]$. (In the last step, we have used the Kato inequality $|d|\omega|| \leq |\hat{\nabla} \omega|$, and have remembered that $\varepsilon < 1$ by hypothesis.) Similarly,

$$\begin{aligned}
\left| \int_{\partial X_\varepsilon} f W^+(\omega, \nabla_\nu \omega)_g da_g \right| &= \left| \int_{\partial X_\varepsilon} f \cdot f^6 W^+(\omega, \nabla_{f\hat{\nu}} \omega)_h f^{-3} da_h \right| \\
&= \left| \int_{\partial X_\varepsilon} f^5 W^+(\omega, \nabla_{\hat{\nu}} \omega)_h da_h \right| \\
&\leq 2 \left| \int_{\partial X_\varepsilon} f^5 |W^+|_h |\omega|_h |\nabla \omega|_h da_h \right| \\
&\leq 2 \left| \int_{\partial X_\varepsilon} f^5 |W^+|_h |\omega|_h |\hat{\nabla} \omega|_h da_h \right| \\
&\quad + 6 \left| \int_{\partial X_\varepsilon} f^5 |W^+|_h |\omega|_h^2 |\beta|_h da_h \right| \\
&= 2 \left| \int_{\partial X_\varepsilon} f^5 |W^+|_h |\omega|_h |\hat{\nabla} \omega|_h da_h \right| \\
&\quad + 3 \left| \int_{\partial X_\varepsilon} f^5 |W^+|_h |\omega|_h |d|\omega|_h| da_h \right| \\
&\leq 5 \left| \int_{\partial X_\varepsilon} 2^{5/4} |\omega|_h^{-3/2} |W^+|_h |\hat{\nabla} \omega|_h da_h \right| \\
&\leq C_2 \varepsilon^{-3/2} \text{Vol}^{(3)}(\partial X_\varepsilon, h),
\end{aligned}$$

where $C_2 = 10\sqrt[4]{2}(\max_M |W^+|_h)(\max_M |\hat{\nabla} \omega|_h)$. Setting $C = C_1 + 4C_2$,

and referring back to our integration-by-parts calculation, we thus see that the claim now follows immediately from the triangle inequality. $\square$

So far, we have only assumed that ω is a non-trivial self-dual harmonic form on (M, h) . However, the information we have just gleaned becomes much more useful when ω happens to be near-symplectic:

Lemma 9.2.2 *Let ω be a near-symplectic self-dual harmonic 2-form on a compact oriented Riemannian 4-manifold. Let $X = M - Z$ be the complement of the zero set Z of ω , set $f = 2^{1/4}|\omega|_h^{-1/2}$ on X , and let $g = f^{-2}h$ be the almost-Kähler metric on (X, ω) obtained by conformally rescaling h to make $|\omega|_g \equiv \sqrt{2}$. Then*

$$\int_X \langle \nabla^* \nabla (fW^+), \omega \otimes \omega \rangle d\mu_g = \int_X \langle fW^+, \nabla^* \nabla (\omega \otimes \omega) \rangle d\mu_g, \quad (9.10)$$

where the integrands on both sides are defined with respect to g , and where both moreover belong to L^1 . In particular, both integrals are finite, and may be treated either as improper Riemann integrals or as Lebesgue integrals.

Proof To say that ω is near-symplectic means, by definition, that the section ω of $\Lambda^+ \rightarrow M$ is transverse to the zero section along its zero locus $Z \approx \sqcup_{j=1}^n S^1$. In particular, the derivative of ω along Z induces an isomorphism between the normal bundle of $Z \subset M$ and the vector bundle $\Lambda^+|_Z \rightarrow Z$. This moreover allows us to construct a diffeomorphism between a sufficiently small tubular neighborhood $\mathcal{U}$ of Z and $Z \times B_\varepsilon^3$, where $B_\varepsilon^3 \subset \mathbb{R}^3$ is the standard 3-ball of some small radius ε , by combining the nearest-point projection $\mathcal{U} \rightarrow Z$ with the components of ω relative to some orthonormal framing of the vector bundle $\Lambda^+ \rightarrow \mathcal{U}$. (Here, we are using the fact that $\Lambda^+|_{\mathcal{U}}$ is necessarily trivial because Λ^+ is oriented, $\mathbf{SO}(3)$ is connected, and $\mathcal{U}$ deformation retracts to a union of circles.) Via this diffeomorphism, the function $|\omega|_h$ on $\mathcal{U}$ then just becomes the standard radius function on B_ε^3 . Moreover, after reducing the size of ε if necessary, the Riemannian metric h on $\mathcal{U}$ becomes quasi-isometric to the standard flat product metric h_0 on $Z \times B_\varepsilon^3$, in the sense that $h_0/\kappa < h < \kappa h_0$ for some constant $\kappa > 1$, and where we have $|\omega|_h \geq \varepsilon$ on the complement $M - \mathcal{U}$ of $\mathcal{U}$. It then follows that the hypersurfaces $(\partial X_\varepsilon, h)$ are uniformly quasi-isometric to $(Z \times S_\varepsilon^2, h_0)$, so there consequently exists a positive constant $L = 4\pi|Z|\kappa^{3/2}$ such that

$$\text{Vol}^{(3)}(\partial X_\varepsilon, h) < L\varepsilon^2$$

for all $\varepsilon \in (0, \varepsilon)$. Combining this with Lemma 9.2.1 then tells us that

$$\left| \int_{X_\varepsilon} [\langle \nabla^* \nabla(fW^+), \omega \otimes \omega \rangle - \langle fW^+, \nabla^* \nabla(\omega \otimes \omega) \rangle] d\mu_g \right| \leq CL\sqrt{\varepsilon}$$

for all $\varepsilon \in (0, \varepsilon)$. In particular, this implies that

$$\lim_{\varepsilon \searrow 0} \int_{X_\varepsilon} [\langle \nabla^* \nabla(fW^+), \omega \otimes \omega \rangle - \langle fW^+, \nabla^* \nabla(\omega \otimes \omega) \rangle] d\mu_g = 0. \quad (9.11)$$

To prove the claim, it therefore suffices to show that both integrands in (9.10) are absolutely integrable, and so belong to L^1 . To see this, first notice that

$$\begin{aligned} \int_X |\langle \nabla^* \nabla(fW^+), \omega \otimes \omega \rangle_g| d\mu_g &\leq 2 \int_X |\nabla^* \nabla(fW^+)|_g d\mu_g \\ &= 2 \int_X f^2 |[\nabla \cdot \nabla(fW^+)]_g|_h f^{-4} d\mu_h \\ &\leq 2 \int_X |\nabla^* \nabla(fW^+)|_h d\mu_h \\ &\quad + 8 \int_X |\nabla(\beta \otimes fW^+)|_h d\mu_h \\ &\quad + 10 \int_X |\beta \otimes \nabla(fW^+)|_h d\mu_h \\ &\quad + 40 \int_X |\beta \otimes \beta \otimes fW^+|_h d\mu_h \\ &\leq 2 \int_X f |\nabla^* \nabla W^+|_h d\mu_h \\ &\quad + 22 \int_X |\nabla f|_h |\nabla W^+|_h d\mu_h \\ &\quad + 10 \int_X |\nabla \nabla f|_h |W^+|_h d\mu_h \\ &\quad + 50 \int_X f^{-1} |\nabla f|_h^2 |W^+|_h d\mu_h \\ &\leq C_3 \int_M \left[|\omega|_h^{-1/2} + |\nabla |\omega|_h^{-1/2}|_h \right. \\ &\quad \left. + |\omega|_h^{1/2} |\nabla |\omega|_h^{-1/2}|_h^2 \right. \\ &\quad \left. + |\nabla \nabla |\omega|_h^{-1/2}|_h \right] d\mu_h \end{aligned}$$

$$\begin{aligned}
&\leq C_3 \int_M \left[|\omega|_h^{-1/2} + \frac{1}{2} |\omega|_h^{-3/2} |\nabla \omega|_h \right. \\
&\quad \left. + \frac{23}{4} |\omega|_h^{-5/2} |\nabla \omega|_h^2 + 2 |\omega|_h^{-3/2} |\nabla \nabla \omega|_h \right] d\mu_h \\
&\leq C_4 \int_M |\omega|_h^{-5/2} d\mu_h \\
&< \infty,
\end{aligned}$$

where C_3 is a positive constant depending on (M, h) , C_4 is a positive constant depending on (M, h, ω) , and where, as in the remainder of the chapter, ∇ denotes the Levi-Civita connection $\hat{\nabla}$ of h when its relation to h is clearly indicated by a subscript. Here, in the last step, we have used the fact that $|\omega|^{-5/2}$ is comparable, near $Z = M - X$, to $r^{-5/2}$ on $B_\varepsilon^3 \times S^1$, where $r = |\vec{x}|$ is the distance from the origin in the ε -ball $B_\varepsilon^3 \subset \mathbb{R}^3$, and therefore has finite integral because

$$\int_{B_\varepsilon^3} |\vec{x}|^{-5/2} dx^1 \wedge dx^2 \wedge dx^3 = 4\pi \int_0^\varepsilon r^{-5/2} r^2 dr = 8\pi\sqrt{\varepsilon} < \infty.$$

In much the same way,

$$\begin{aligned}
&\int_X |\langle fW^+, \nabla^* \nabla (\omega \otimes \omega) \rangle_g| d\mu_g \\
&\leq 2\sqrt{2} \int_X f |W^+|_g |\nabla^* \nabla \omega|_g d\mu_g + 2 \int_X f |W^+|_g |\nabla \omega|_g^2 d\mu_g \\
&\leq 2^{3/2} \int_X f^3 |W^+|_h f^4 \left[|\nabla^* \nabla \omega|_h + 2 |\nabla \beta|_h |\omega|_h \right. \\
&\quad \left. + 4 |\beta|_h |\nabla \omega|_h + |\beta|^2 |\omega|_h \right] f^{-4} d\mu_h \\
&\quad + 2 \int_X f^3 |W^+|_h f^6 \left[|\nabla \omega|_h^2 + 4 |\beta|_h |\omega|_h |\nabla \omega|_h \right. \\
&\quad \left. + 4 |\beta|_h^2 |\omega|_h^2 \right] f^{-4} d\mu_h \\
&\leq 2^{3/2} \int_X |W^+|_h \left[f^3 |\nabla^* \nabla \omega|_h + 2 f^2 |\nabla (f^{-1} \nabla f)|_h |\omega|_h \right. \\
&\quad \left. + 4 f^2 |\nabla f|_h |\nabla \omega|_h + f |\nabla f|^2 |\omega|_h \right] d\mu_h \\
&\quad + 2 \int_X |W^+|_h \left[f^5 |\nabla \omega|_h^2 + 4 f^4 |\nabla f|_h |\omega|_h |\nabla \omega|_h \right. \\
&\quad \left. + 4 f^3 |\nabla f|_h^2 |\omega|_h^2 \right] d\mu_h
\end{aligned}$$

$$\begin{aligned}
&\leq C_5 \int_X \left[|\omega|_h^{-3/2} |\nabla^* \nabla \omega|_h + |\omega|_h^{-2} |\nabla \omega|_h^2 \right. \\
&\quad \left. + |\omega|_h^{-2} |\nabla \nabla \omega|_h^2 + |\omega|_h^{-5/2} |\nabla \omega|_h^2 \right] d\mu_h \\
&\leq C_6 \int_M |\omega|_h^{-5/2} d\mu_h < \infty,
\end{aligned}$$

where C_5 and C_6 are positive constants depending, respectively, on (M, h) and (M, h, ω) . Thus, the integrands in (9.10) both belong to L^1 , and (9.11) therefore implies that the two integrals in (9.10) are equal. $\square$

Since we are thus entitled to carry out the desired integration by parts in the near-symplectic case, (9.9) therefore implies an interesting integral Weitzenböck formula when h also satisfies $\delta W^+ = 0$.

Proposition 9.2.3 *Let ω be a near-symplectic self-dual harmonic 2-form on a compact oriented Riemannian 4-manifold (M, h) with $\delta W^+ = 0$. Let $X = M - Z$ be the complement of the zero set Z of ω , set $f = 2^{1/4} |\omega|_h^{-1/2}$ on X , and let $g = f^{-2} h$ be the almost-Kähler metric on (X, ω) obtained by conformally rescaling h to make $|\omega|_g \equiv \sqrt{2}$. Then g satisfies*

$$\int_X \left[\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle + \frac{s}{2} W^+(\omega, \omega) - 6 |W^+(\omega)|^2 + 2 |W^+|^2 |\omega|^2 \right] f d\mu_g = 0,$$

both as a Lebesgue integral and as an improper Riemann integral.

Proof Contraction of (9.9) with $\omega \otimes \omega$ tells us that

$$\langle \nabla^* \nabla (f W^+), \omega \otimes \omega \rangle + \frac{s}{2} f W^+(\omega, \omega) - 6 f |W^+(\omega)|^2 + 2 f |W^+|^2 |\omega|^2 = 0$$

on (X, g) , so integration certainly tells us that

$$\int_X \left[\langle \nabla^* \nabla (f W^+), \omega \otimes \omega \rangle + \frac{s}{2} f W^+(\omega, \omega) - 6 f |W^+(\omega)|^2 + 2 f |W^+|^2 |\omega|^2 \right] d\mu_g = 0.$$

However, because the first term is L^1 , equation (9.9) tells us that the same is also true of the sum of the remaining terms, and Lemma 9.2.2 therefore allows us to rewrite the above expression as

$$\int_X \left[(f W^+, \nabla^* \nabla (\omega \otimes \omega)) + f \frac{s}{2} W^+(\omega, \omega) - 6 f |W^+(\omega)|^2 + 2 f |W^+|^2 |\omega|^2 \right] d\mu_g = 0.$$

Collecting the common of factor of f now yields the desired result. $\square$

9.3 Some Almost-Kähler Geometry

When an oriented Riemannian manifold (M, h) with $\delta W^+ = 0$ carries a near-symplectic self-dual harmonic 2-form ω , we saw in Proposition 9.2.3 that, if we set $f = 2^{1/4}|\omega|_h^{-1/2}$ on the open set X where $\omega \neq 0$, the conformally related almost-Kähler metric $g = f^{-2}h$ then satisfies an integral Weitzenböck formula on X . In order to exploit this effectively, we will need a universal identity previously pointed out in [20]:

Lemma 9.3.1 *Any 4-dimensional almost-Kähler manifold satisfies*

$$\langle W^+, \nabla^* \nabla(\omega \otimes \omega) \rangle = [W^+(\omega, \omega)]^2 + 4|W^+(\omega)|^2 - sW^+(\omega, \omega)$$

at every point.

Proof First notice that the oriented Riemannian 4-manifold (X, g) satisfies

$$\Lambda^+ \otimes \mathbb{C} = \mathbb{C}\omega \oplus K \oplus \overline{K},$$

where $K = \Lambda_J^{2,0}$ is the canonical line bundle of the almost-complex structure J defined by $\omega = g(J\cdot, \cdot)$. Locally choosing a unit section φ of K , we thus have

$$\nabla\omega = \alpha \otimes \varphi + \bar{\alpha} \otimes \bar{\varphi}$$

for a unique 1-form $\alpha \in \Lambda_J^{1,0}$, since $\nabla_{[a}\omega_{bc]} = 0$ and $\omega^{bc}\nabla_a\omega_{bc} = 0$. If

$$\circledast : \Lambda^+ \times \Lambda^+ \rightarrow \odot_0^2 \Lambda^+$$

denotes the symmetric trace-free product, we therefore have

$$(\nabla_e \omega) \circledast (\nabla^e \omega) = 2|\alpha|^2 \varphi \circledast \bar{\varphi} = -\frac{1}{4}|\nabla\omega|^2 \omega \circledast \omega$$

and we thus deduce that

$$\begin{aligned} \langle W^+, \nabla^* \nabla(\omega \otimes \omega) \rangle &= 2W^+(\omega, \nabla^* \nabla\omega) - 2W^+(\nabla_e \omega, \nabla^e \omega) \\ &= 2W^+(\omega, \nabla^* \nabla\omega) + \frac{1}{2}|\nabla\omega|^2 W^+(\omega, \omega) \\ &= 2W^+(\omega, 2W^+(\omega) - \frac{s}{3}\omega) \\ &\quad + \left[W^+(\omega, \omega) - \frac{s}{3}\right] W^+(\omega, \omega) \\ &= -\frac{2}{3}sW^+(\omega, \omega) + 4|W^+(\omega)|^2 \\ &\quad + \left[W^+(\omega, \omega) - \frac{s}{3}\right] W^+(\omega, \omega) \\ &= [W^+(\omega, \omega)]^2 + 4|W^+(\omega)|^2 - sW^+(\omega, \omega), \end{aligned}$$

where we have used the Weitzenböck formula

$$0 = \nabla^* \nabla \omega - 2W^+(\omega) + \frac{s}{3}\omega$$

for the harmonic self-dual 2-form ω , as well as the associated key identity

$$\frac{1}{2}|\nabla \omega|^2 = W^+(\omega, \omega) - \frac{s}{3} \quad (9.12)$$

resulting from the fact that $|\omega|^2 \equiv 2$. $\square$

In conjunction with Proposition 9.2.3, this now yields the following:

Theorem 9.3.2 *Let ω be a near-symplectic self-dual harmonic 2-form on a compact oriented Riemannian 4-manifold (M, h) with $\delta W^+ = 0$. Let $X = M - Z$ be the complement of the zero set Z of ω , set $f = 2^{1/4}|\omega|_h^{-1/2}$ on X , and let $g = f^{-2}h$ be the almost-Kähler metric on (X, ω) obtained by conformally rescaling h to make $|\omega|_g \equiv \sqrt{2}$. Then the almost-Kähler metric g satisfies*

$$\int_X \left[8 \left(|W^+|^2 - \frac{1}{2} |W^+(\omega)^\perp|^2 \right) - sW^+(\omega, \omega) \right] f \, d\mu_g = 0, \quad (9.13)$$

where s is the scalar curvature of g , and where $W^+(\omega)^\perp$ denotes the orthogonal projection of $W^+(\omega)$ to the orthogonal complement of $\omega \in \Lambda^+$. Moreover, the integrand belongs to L^1 , so the statement holds whether the left-hand-side is construed as a Lebesgue integral or as an improper Riemann integral.

Proof Combining Proposition 9.2.3 with Lemma 9.3.1, we have

$$\begin{aligned} 0 &= \int_X \left[\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle + \frac{s}{2} W^+(\omega, \omega) \right. \\ &\quad \left. - 6|W^+(\omega)|^2 + 2|W^+|^2 |\omega|^2 \right] f \, d\mu \\ &= \int_X \left[([W^+(\omega, \omega)]^2 + 4|W^+(\omega)|^2 - sW^+(\omega, \omega)) \right. \\ &\quad \left. + \frac{s}{2} W^+(\omega, \omega) - 6|W^+(\omega)|^2 + 4|W^+|^2 \right] f \, d\mu \\ &= \int_X \left[[W^+(\omega, \omega)]^2 - \frac{s}{2} W^+(\omega, \omega) - 2|W^+(\omega)|^2 + 4|W^+|^2 \right] f \, d\mu. \end{aligned}$$

Since $|W^+(\omega)^\perp|^2 = |W^+(\omega)|^2 - \frac{1}{2}[W^+(\omega, \omega)]^2$, multiplication by 2 thus yields the desired formula (9.13). Moreover, this calculation shows that the integrand is the sum of two L^1 functions, and is therefore itself L^1 by the triangle inequality. $\square$

Next, we prove a refinement of a point-wise inequality used in [20]:

Lemma 9.3.3 *Any 4-dimensional almost-Kähler manifold satisfies*

$$|W^+|^2 - \frac{1}{2}|W^+(\omega)^\perp|^2 \geq \frac{3}{8} [W^+(\omega, \omega)]^2 + \frac{1}{2}|W^+(\omega)^\perp|^2$$

at every point.

Proof If $A = [A_{jk}]$ is any symmetric trace-free 3×3 matrix, the fact that $A_{33} = -(A_{11} + A_{22})$ implies that

$$\sum_{jk} A_{jk}^2 \geq 2A_{21}^2 + A_{11}^2 + A_{22}^2 + (A_{11} + A_{22})^2 = 2A_{21}^2 + \frac{3}{2}A_{11}^2 + 2\left(\frac{A_{11}}{2} + A_{22}\right)^2$$

and we therefore conclude that

$$|A|^2 \geq 2A_{21}^2 + \frac{3}{2}A_{11}^2.$$

If we now let A represent $W^+ : \Lambda^+ \rightarrow \Lambda^+$ with respect to an orthogonal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for Λ^+ such that $\omega = \sqrt{2}\mathbf{e}_1$ and $W^+(\omega)^\perp \propto \mathbf{e}_2$, this inequality becomes

$$|W^+|^2 \geq |W^+(\omega)^\perp|^2 + \frac{3}{8} [W^+(\omega, \omega)]^2,$$

and subtracting $\frac{1}{2}|W^+(\omega)^\perp|^2$ from both sides thus proves the claim. $\square$

This now yields a key inequality:

Lemma 9.3.4 *Let (M, h) , ω , X , g , and f be as in Theorem 9.3.2. Then the almost-Kähler metric $g = f^{-2}h$ satisfies*

$$0 \geq \int_X \left[W^+(\omega, \omega) |\nabla \omega|^2 + \frac{8}{3} |W^+(\omega)^\perp|^2 \right] f \, d\mu_g, \quad (9.14)$$

in the sense that the Lebesgue integral on the right is well defined and belongs to $[-\infty, 0]$.

Proof Theorem 9.3.2 tells us that

$$0 = \int_X \left[8 \left(|W^+|^2 - \frac{1}{2} |W^+(\omega)^\perp|^2 \right) - sW^+(\omega, \omega) \right] f \, d\mu_g$$

and that the positive and negative parts of the integrand are both L^1 functions. The point-wise inequality of integrands provided by Lemma 9.3.3 therefore implies that

$$0 \geq \int_X \left[3 [W^+(\omega, \omega)]^2 - sW^+(\omega, \omega) + 4 |W^+(\omega)^\perp|^2 \right] f \, d\mu_g$$

in the Lebesgue sense. After dividing by 3, we can then re-express this as

$$0 \geq \int_X \left[W^+(\omega, \omega) \left(W^+(\omega, \omega) - \frac{s}{3} \right) + \frac{4}{3} |W^+(\omega)^\perp|^2 \right] f \, d\mu_g. \quad (9.15)$$

However, (9.12) tells us that $W^+(\omega, \omega) - \frac{s}{3} = \frac{1}{2} |\nabla \omega|^2$ for any almost-Kähler 4-manifold. Making this substitution in (9.15) and then multiplying by 2 thus yields the desired inequality (9.14). $\square$

In the special case where (M, h, ω) satisfies the conformally invariant condition $W^+(\omega, \omega) \geq 0$, we thus obtain the following:

Proposition 9.3.5 *Let (M, h) be a compact oriented Riemannian 4-manifold that satisfies $\delta W^+ = 0$, and suppose that ω is a near-symplectic self-dual harmonic 2-form on (M, h) that satisfies $W^+(\omega, \omega) \geq 0$. Let X , g , and f be as in Theorem 9.3.2. Then the almost-Kähler manifold (X, g, ω) satisfies*

$$\int_X \left[W^+(\omega, \omega) |\nabla \omega|^2 + \frac{8}{3} |W^+(\omega)^\perp|^2 \right] f \, d\mu_g = 0, \quad (9.16)$$

both as a Lebesgue and as an improper Riemann integral.

Proof The added assumption that $W^+(\omega, \omega) \geq 0$ obviously implies

$$\int_X \left[W^+(\omega, \omega) |\nabla \omega|^2 + \frac{8}{3} |W^+(\omega)^\perp|^2 \right] f \, d\mu_g \geq 0$$

as an extended real number, because the integrand is now non-negative. But, in conjunction with (9.14), this immediately implies that

$$\int_X \left[W^+(\omega, \omega) |\nabla \omega|^2 + \frac{8}{3} |W^+(\omega)^\perp|^2 \right] f \, d\mu_g = 0$$

as a Lebesgue integral. Moreover, since the integrand is also L^1 , the integral also necessarily vanishes as an improper Riemann integral. $\square$

This very strong statement now has even stronger consequences:

Proposition 9.3.6 *Let M , h , ω , X , g , and f be as in Proposition 9.3.5. Then either g is a Kähler metric on X whose scalar curvature is given by $s = \mathbf{c}/f$ for some constant $\mathbf{c} > 0$, or else g satisfies $W^+ \equiv 0$, and so is an anti-self-dual metric.*

Proof Since $f > 0$ by construction, and since $W^+(\omega, \omega) \geq 0$ by assumption, both terms in the integrand of (9.16) must vanish identically.

We thus have

$$W^+(\omega, \omega)|\nabla\omega|^2 = 0 \quad \text{and} \quad W^+(\omega)^\perp = 0 \quad (9.17)$$

at every point of X . In particular, $\nabla\omega = 0$ wherever $W^+(\omega, \omega) \neq 0$. If $\mathcal{V} \subset X$ is the open subset where $W^+(\omega, \omega) \neq 0$, the restriction of g to $\mathcal{V}$ is therefore Kähler. On the other hand, since $h = f^2g$ satisfies $\delta W^+ = 0$, conformal invariance of this equation tells us that g satisfies $\delta(fW^+) = 0$, as previously noted. On $(\mathcal{V}, g)$ we therefore have

$$\begin{aligned} 0 &= \omega^{ab}\omega^{cd}\nabla^e(fW_{ebcd}^+) = \nabla^e(fW_{ebcd}^+\omega^{ab}\omega^{cd}) \\ &= \nabla^e(f\frac{s}{3}\omega_{eb}\omega^{ab}) = \frac{1}{3}\nabla^e(fs\delta_e^a) = \frac{1}{3}\nabla^a(fs) = \nabla^a[fW^+(\omega, \omega)], \end{aligned}$$

since at each point of any Kähler manifold of real dimension 4, the Kähler form ω is an eigenvector of $W^+ : \Lambda^+ \rightarrow \Lambda^+$, with eigenvalue one-sixth of the scalar curvature s . This shows that $d[fW^+(\omega, \omega)] = 0$ on $\mathcal{V}$, and therefore, by continuity, on the closure $\overline{\mathcal{V}}$ of $\mathcal{V}$, too. On the other hand, since our definition of $\mathcal{V}$ guarantees that $fW^+(\omega, \omega) \equiv 0$ on the open set $X - \overline{\mathcal{V}}$, we also have $d[fW^+(\omega, \omega)] = 0$ on $X - \overline{\mathcal{V}}$. It follows that $d[fW^+(\omega, \omega)] = 0$ on all of X . Since X is connected, and since $fW^+(\omega, \omega) \geq 0$, we therefore conclude that $fW^+(\omega, \omega) = \mathbf{c}/3$ for some non-negative constant $\mathbf{c} \geq 0$.

If $\mathbf{c} > 0$, $\mathcal{V} = X$, and it follows that (X, g, ω) is a Kähler manifold, with

$$s = 3W^+(\omega, \omega) = \frac{\mathbf{c}}{f}.$$

Otherwise, $\mathbf{c} = 0$, and we have $W^+(\omega, \omega) \equiv 0$. On the other hand, (9.17) also tells us that $W^+(\omega)^\perp \equiv 0$ on X . Substituting these two facts into (9.13) then yields

$$\int_X |W^+|^2 f \, d\mu_g = 0.$$

Thus, when $\mathbf{c} = 0$, we conclude that $W^+ \equiv 0$, and g is therefore anti-self-dual in this remaining case, exactly as claimed. $\square$

Sharpening these conclusions now supplies our mainspring result:

Theorem 9.3.7 *Let (M, h) be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$ that admits a near-symplectic self-dual harmonic 2-form ω such that*

$$W^+(\omega, \omega) \geq 0.$$

Then either h satisfies $W^+ \equiv 0$, and so is anti-self-dual, or else $W^+(\omega, \omega)$

is everywhere positive, in which case M admits a global Kähler metric g with scalar curvature $s > 0$ such that $h = s^{-2}g$.

Proof If (X, g) satisfies $W^+ \equiv 0$, the conformal invariance of this condition implies that (X, h) satisfies $W^+ \equiv 0$, too. But since $X \subset M$ is dense, it then follows by continuity that h satisfies $W^+ \equiv 0$ on all of M . Thus, (M, h) must be a compact anti-self-dual manifold in this case.

Otherwise, $W^+ \not\equiv 0$, and Proposition 9.3.6 then guarantees that $g = f^{-2}h$ must be a Kähler metric on $X = M - Z$, with Kähler form ω and

$$3W^+(\omega, \omega) = s = \mathbf{c}f^{-1}$$

for some positive constant $\mathbf{c}$. However, since $h = f^2g$, we also have

$$[W^+(\omega, \omega)]_h = f^{-6}[W^+(\omega, \omega)]_g,$$

and it therefore follows that

$$[W^+(\omega, \omega)]_h = \frac{\mathbf{c}}{3} f^{-7}.$$

But since $f = 2^{1/4}|\omega|_h^{-1/2}$ by construction, this means that

$$[W^+(\omega, \omega)]_h = \mathbf{b} |\omega|_h^{7/2} \quad (9.18)$$

on X , where $\mathbf{b} = \sqrt[4]{2}\mathbf{c}/12$ is a positive constant. However, since g is Kähler, with positive scalar curvature and Kähler form ω , it follows that W^+ has a repeated negative eigenvalue at every point of X , and that ω everywhere belongs to the positive eigenspace. This implies that

$$W^+(\omega, \omega) = \sqrt{\frac{2}{3}}|W^+||\omega|^2$$

at every point of X , both for g and for h . Thus (9.18) implies that

$$|W^+|_h = \mathbf{a}|\omega|_h^{3/2} \quad (9.19)$$

everywhere on X , where $\mathbf{a} = \sqrt{\frac{3}{2}}\mathbf{b}$ is another positive constant. However, since $X \subset M$ is dense, and because the two sides are both continuous functions, it then follows that (9.19) actually holds on all of M . Now notice that this implies that $|W^+|$ is everywhere differentiable, and that W^+ must vanish to first order along Z ; thus, $\nabla W^+ = 0$ at every point of Z , where ∇ denotes the Levi-Civita connection of h . Next, notice that (9.19) also implies that

$$|d|W^+|_h|_h = \frac{3}{2}\mathbf{a}|\omega|_h^{1/2}|d|\omega|_h|_h$$

on $X = M - Z$. Since the near-symplectic nature of ω moreover guarantees that $|d|\omega|_h|$ is bounded away from zero near Z , we therefore have

$$|d|W^+|_h|_h \geq \mathbf{A}|\omega|_h^{1/2}$$

on some neighborhood $\mathcal{U}$ of Z , where $\mathbf{A} := \frac{3}{2}\mathbf{a} \inf_{\mathcal{U}-Z} |d|\omega|_h|_h$ is another positive constant. By the Kato inequality, we therefore have

$$|\nabla W^+|_h \geq \mathbf{A}|\omega|_h^{1/2}$$

on $\mathcal{U}$. But since h has been assumed throughout to be a C^4 metric, ∇W^+ is a differentiable tensor field, and we have moreover previously observed that this field vanishes along Z . It thus follows that $|\nabla W^+|_h$ is a Lipschitz function that vanishes along Z . But since ω is near-symplectic, $|\omega|_h$ is commensurate with the distance from Z in a small enough neighborhood $\mathcal{U} \supset Z$, and we must therefore have $\mathbf{B}|\omega|_h > |\nabla W^+|_h$ on a sufficiently small neighborhood $\mathcal{U}$ of Z , for some positive constant $\mathbf{B}$. But this then says that

$$\mathbf{B}|\omega|_h > \mathbf{A}|\omega|_h^{1/2}$$

on $\mathcal{U}$, and so implies that

$$|\omega|_h > \frac{\mathbf{A}^2}{\mathbf{B}^2} > 0$$

on $\mathcal{U} - Z$. But since $X - (\mathcal{U} - Z) = M - \mathcal{U}$ is compact, and since $\omega \neq 0$ on X , this implies that $|\omega|_h$ is uniformly bounded away from zero on all of X . But since X is dense in M , it therefore follows by continuity that $|\omega|_h$ is bounded away from zero on all of M . Since Z is by definition the zero set of ω , we are therefore forced to conclude that $Z = \emptyset$.

Thus, g is a globally defined Kähler metric with scalar curvature $s > 0$ such that $h = f^2 g = \mathbf{c}^2 s^{-2} g$ on all of M . By now replacing ω with $\mathbf{c}^{-2/3} \omega$ and thus replacing g with $\mathbf{c}^{-2/3} g$, we can now arrange for h to simply be given by $s^{-2} g$, as promised. $\square$

This tells us quite a bit about the 4-manifolds that carry metrics h of the type covered by Theorem 9.3.7. Indeed [3, 9], if (M, J, g) is a compact Kähler surface of scalar curvature $s > 0$, then $h = s^{-2} g$ is a metric on M with $\delta W^+ = 0$, and with $W^+(\omega, \omega) > 0$ for the Kähler form ω of g . On the other hand, if a compact complex surface (M, J) admits Kähler metrics g with $s > 0$, it is necessarily rational or ruled [33]. Conversely, any rational or ruled surface has arbitrarily small deformations that admit such metrics [13, 26]. Up to oriented

diffeomorphism, we can therefore give a complete list of the 4-manifolds that admit solutions of this first type: they are $\mathbb{CP}_2$, $(\Sigma^2 \times S^2) \# k \overline{\mathbb{CP}}_2$, and $\Sigma^2 \rtimes S^2$, where Σ is any compact orientable surface, k is any non-negative integer, and $\Sigma^2 \rtimes S^2$ is the non-trivial oriented 2-sphere bundle over Σ . The moduli space of solutions on any of these manifolds is, moreover, infinite dimensional.

The other class of solutions allowed by Theorem 9.3.7 is rather different, both because the moduli spaces of solutions are always finite dimensional, and because the near-symplectic self-dual harmonic 2-form ω is allowed to have non-empty zero set. Of course, a vast menagerie of smooth compact oriented 4-manifolds with $b_+ \neq 0$ is known to admit anti-self-dual metrics [21, 28], but little is known about when their self-dual harmonic 2-forms ω are near-symplectic. There certainly are many examples with nowhere-zero ω that are not conformally Kähler [15], but there are also related explicit families [5] with $b_+ = 1$ where the self-dual harmonic 2-form ω transmutes from being nowhere zero to having non-empty zero locus. For the latter explicit anti-self-dual manifolds, it seems likely that the self-dual harmonic 2-form ω is usually near-symplectic, but this is equivalent to the non-degeneracy of all critical points for a preferred harmonic function on a quasi-Fuchsian hyperbolic 3-manifold associated with the solution. Perhaps some interested reader will decide that this tractable-looking open problem merits careful investigation!

9.4 The Main Theorems

With the results of Section 9.3 in hand, we are now ready to prove our main theorems, starting with Theorem 9.1.1.

Proof of Theorem 9.1.1 If (M, h) is an oriented 4-dimensional Einstein manifold, the second Bianchi identity implies that $\delta W^+ = 0$. If (M, h) is moreover compact, connected, and admits a near-symplectic self-dual harmonic 2-form ω such that $W^+(\omega, \omega) \geq 0$, the conclusions of Theorem 9.3.7 then apply. Thus, if $W^+(\omega, \omega) > 0$ at some point, we know that $W^+ \not\equiv 0$, and Theorem 9.3.7 then tells us that $W^+(\omega, \omega) > 0$ everywhere, and $h = s^{-2}g$ for some globally defined Kähler metric g on M with scalar curvature $s > 0$. However, any 4-dimensional Einstein metric is Bach-flat, and, because this is a conformally invariant condition, the Kähler metric g must therefore be Bach-flat, too. In particular, this implies [8, 9] that g is an extremal Kähler metric. Moreover, one can

also show [17] that the complex structure associated with any such g has $c_1 > 0$, and it therefore follows that M is necessarily diffeomorphic to a del Pezzo surface. Conversely, each del Pezzo diffeotype carries [8, 22, 25, 30, 31] an Einstein metric h which can be written as $s^{-2}g$ for a suitable extremal Kähler metric g with scalar curvature $s > 0$. In fact, h is actually Kähler–Einstein in most cases, the only exceptions being when M is diffeomorphic to $\mathbb{CP}_2 \# \mathbb{CP}_2$ or $\mathbb{CP}_2 \# 2\mathbb{CP}_2$. $\square$

For each del Pezzo diffeotype, the moduli space of all Einstein metrics h with $W^+(\omega, \omega) > 0$ is actually connected [20]. Moreover, it follows from [19, Thm. A] and a modicum of elementary Seiberg–Witten theory [16, Thm. 3] that, for each del Pezzo M , this moduli space exactly coincides with the moduli space of all conformally Kähler, Einstein metrics. We now prove Theorem 9.1.2.

Proof of Theorem 9.1.2 If (M^4, h) is a compact oriented $\lambda \geq 0$ Einstein manifold that carries a near-symplectic self-dual harmonic ω with $W^+(\omega, \omega) \geq 0$, then Theorem 9.3.7 tells us that either $W^+(\omega, \omega) > 0$ everywhere, or else $W^+ \equiv 0$. Since the former case is covered by Theorem 9.1.1, we may therefore assume that $W^+ \equiv 0$. However, by the Weitzenböck formula for the Hodge Laplacian, the non-trivial self-dual harmonic 2-form ω satisfies

$$0 = \nabla^* \nabla \omega - 2W^+(\omega) + \frac{s}{3}\omega,$$

and, since $W^+ = 0$ and $s = 4\lambda \geq 0$ in our case, taking the inner product with ω and integrating yields

$$0 = \int_M \left[|\nabla \omega|^2 + \frac{4\lambda}{3} |\omega|^2 \right] d\mu_h.$$

We therefore conclude that $\nabla \omega = 0$ and $\lambda = 0$, so that (M^4, h) is necessarily Ricci-flat and Kähler. Thus, after multiplying ω by a positive constant if necessary in order to give it constant length $|\omega|_h \equiv \sqrt{2}$, we see that (M, h) carries an integrable, metric-compatible almost-complex structure J such that $\omega = h(J\cdot, \cdot)$. Moreover, since the Kähler metric h is Ricci-flat, the canonical line bundle K of (M, J) is flat, and $c_1(M, J)$ must therefore be a torsion class. The Kodaira classification of complex surfaces [2, 11] therefore tells us that (M, J) must be a $K3$ surface, an Enriques surface, an Abelian surface, or a hyper-elliptic surface. Conversely, Yau’s solution of the Calabi conjecture [35] tells us that each complex surface of one of these types carries a unique Ricci-flat Kähler

metric in each Kähler class, and every such Calabi–Yau metric satisfies $W^+ \equiv 0$. $\square$

It is worth pointing out that the moduli space of Ricci-flat Kähler metrics is connected. Indeed, since the Kähler cone is contractible for each complex structure, Yau’s theorem reduces this statement to the known fact [2] that all the $c_1^{\mathbb{R}} = 0$ complex structures on these 4-manifolds are swept out by a single connected family.

Finally we prove Theorem 9.1.3, by showing that the near-symplectic hypothesis is absolutely essential for Theorem 9.1.1:

Proof of Theorem 9.1.3 Let (M, J, h) be a Kähler–Einstein metric with $\lambda < 0$ on a compact complex surface (M, J) with $p_g(M) := h^{2,0}(M) \neq 0$. (For example, we could take (M, J) to be a smooth quintic hypersurface in $\mathbb{CP}_3$, so that $c_1(M) < 0$ and $p_g(M) = 4$, and let h be the Kähler–Einstein metric whose existence is guaranteed by the Aubin–Yau theorem [1, 34].) Now recall that the self-dual Weyl curvature $W^+ : \Lambda^+ \rightarrow \Lambda^+$ of any Kähler surface (M^4, J, g) takes the form

$$\begin{bmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{bmatrix}$$

in any orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for Λ^+ in which $\mathbf{e}_3$ is a multiple of the Kähler form, where s is the scalar curvature. Rather than taking ω to be the Kähler form, we now instead take $\omega = \operatorname{Re}(\varphi)$ for some holomorphic 2-form $\varphi \neq 0$, on (M, J) . Of course, the existence of such a φ is guaranteed by our assumption that $h^{2,0} \neq 0$. Notice that φ is automatically self-dual and harmonic as a consequence of standard Kähler identities, and that the same is therefore automatically true of its real part ω .

However, since $\omega \in \operatorname{Re} \Lambda^{2,0}$ is everywhere point-wise orthogonal to the Kähler form, we now see that

$$W^+(\omega, \omega) = -\frac{s}{12}|\omega|^2 = \frac{|\lambda|}{3}|\omega|^2 \geq 0,$$

since the Einstein constant λ of h is assumed to be negative. Moreover, since $\omega \neq 0$, this non-negative expression is somewhere positive. On the other hand, the canonical line bundle of (M, J) is non-trivial, because $c_1(K) = -c_1 > 0$, so φ , and therefore ω , must vanish along some non-empty holomorphic curve $\Sigma \subset M$. Thus, $W^+(\omega, \omega)$ vanishes

somewhere, and the conclusion of Theorem 9.1.1 therefore fails for this class of examples. $\square$

Of course, in light of counter-examples like those detailed in the proof of Theorem 9.1.3, it is important to explain exactly where the proof of Theorem 9.1.1 breaks down when ω is not near-symplectic. In fact, the key failure occurs at the very beginning of our chain of reasoning, when Lemma 9.2.2 is deduced from Lemma 9.2.1. Recall that Lemma 9.2.1 tells us that the boundary terms arising from integration by parts have size $\sim \varepsilon^{-3/2} \text{Vol}^{(3)}(\partial X_\varepsilon, h)$, where ∂X_ε is the hypersurface where $|\omega|_h = \varepsilon$. In the near-symplectic case, $\text{Vol}^{(3)}(\partial X_\varepsilon, h) \sim \varepsilon^2$, so the boundary terms are no worse than $\varepsilon^{1/2}$, and so vanish in the limit as $\varepsilon \rightarrow 0$. By contrast, in the above examples, the zero locus $Z = \Sigma$ of ω has real codimension 2, and we instead have $\text{Vol}^{(3)}(\partial X_\varepsilon, h) \sim \varepsilon$. This means that the boundary terms could in principle blow up as fast as $\varepsilon^{-1/2}$, and so, in particular, can then no longer be expected to become negligible as ε tends to zero.

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