The Gauss Image Problem

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Dedicated to the memory of Professor Delin Ren

1 Introduction

The Brunn-Minkowski theory and the dual Brunn-Minkowski theory are two core theories in convex geometric analysis that center on the investigation of global geometric invariants and geometric measures associated with convex bodies. The two theories display an amazing conceptual duality that involves many dual concepts in both geometry and analysis such as dual spaces in functional analysis, polarity in convex geometry, and projection and intersection in geometric tomography; see Schneider [49, p. 507] for a lucid explanation.

In the conceptual duality, a central role is assumed by the *radial Gauss image* $\boldsymbol{\alpha}_K$ (defined immediately below) of a convex body K in euclidean n-space, $\mathbb{R}^n$. The radial Gauss image is a map on the unit sphere, S^{n-1}, of $\mathbb{R}^n$ whose values are subsets of the unit sphere. It is known that *Aleksandrov's integral curvature* on S^{n-1} and spherical Lebesgue measure are "linked" via the radial Gauss image, and so are the classical *surface area measure* of Aleksandrov-Fenchel-Jessen and Federer's $(n-1)^{\text{th}}$ *curvature measure* (see Schneider [49, theorem 4.2.3] and [27]). The importance of the radial Gauss image was made more evident in the recent work [27], in which the long-sought *dual curvature measures* (the dual counterparts of Federer's curvature measures) were unveiled. In [27] new links were established between the Brunn-Minkowski theory and the dual Brunn-Minkowski theory by making critical use of the radial Gauss image. Motivated by the manner in which these new geometric measures are defined via the radial Gauss image, it becomes

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natural to introduce a general new concept—the *Gauss image measure* associated with a convex body. Among other things, this concept bridges the classical and the recently discovered geometric measures of convex bodies.

In light of the role that the radial Gauss image plays in connecting various spherical Borel measures, a central question regarding Gauss image measures is: Given two spherical Borel measures, under what conditions does there exist a convex body so that one measure is the Gauss image measure of the other? We call this the *Gauss image problem* and state it more precisely immediately below.

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interior) in n-dimensional euclidean space, $\mathbb{R}^n$, with $\mathcal{K}^n_o$ denoting the bodies that contain the origin in their interiors.

If $K \in \mathcal{K}^n_o$ and $x \in \partial K$ is a boundary point, then the *normal cone* of K at x is defined by

$$N(K,x) = \{v \in S^{n-1} : (y-x)\cdot v \leq 0 \text{ for all } y \in K\},$$

where $(y-x)\cdot v$ denotes the standard inner product of $y-x$ and v in $\mathbb{R}^n$. The *radial map* $r_K : S^{n-1} \to \partial K$ of K is defined for $u \in S^{n-1}$ by $r_K(u) = ru \in \partial K$, where $r > 0$. For $\omega \subset S^{n-1}$, the *radial Gauss image* of ω is defined by

$$\boldsymbol{\alpha}_K(\omega) = \bigcup_{x \in r_K(\omega)} N(K,x) \subset S^{n-1}.$$

The radial Gauss image is the composite of the multivalued Gauss map and the radial map. It is well-known (see Schneider [49]) that for a Borel measurable $\omega \subset S^{n-1}$, the set $\boldsymbol{\alpha}_K(\omega) \subset S^{n-1}$ is spherically Lebesgue measurable but not necessarily Borel measurable.

Recall (see, e.g., [30, p. 1117]) that a submeasure differs from a measure in that the countable additivity in the definition of a measure is replaced by countable subadditivity. (See Section 3 for precise definitions.)

DEFINITION. Suppose λ is a submeasure defined on spherical Lebesgue measurable subsets of S^{n-1}, and $K \in \mathcal{K}^n_o$. Then $\lambda(K,\cdot)$, the *Gauss image measure of* λ *via* K, is the submeasure on S^{n-1} defined by

$$\lambda(K,\omega) = \lambda(\boldsymbol{\alpha}_K(\omega))$$

for each Borel $\omega \subset S^{n-1}$.

When we write that a Borel measure μ on S^{n-1} is *absolutely continuous*, we shall always mean that it is absolutely continuous with respect to spherical Lebesgue measure. Obviously, the completion of an absolutely continuous Borel measure is defined on all spherically Lebesgue measurable subsets of S^{n-1}. When we speak of Borel measures on S^{n-1}, we shall always assume them to be finite, nonnegative, and nonzero.

As will be shown, when λ is an absolutely continuous Borel measure on S^{n-1}, and $K \in \mathcal{K}^n_o$, then $\lambda(K,\cdot)$, is a Borel measure on S^{n-1}. When λ is Lebesgue measure on S^{n-1}, then $\lambda(K,\cdot)$ is simply *Aleksandrov's integral curvature* of the body

K (see, e.g., [2]). Moreover, the classical surface area measures of Aleksandrov-Fenchel-Jessen [1, 49], and the recently discovered, in [27], dual curvature measures are all Gauss image measures. This makes the Gauss image measure an object of significant interest that requires extensive study.

It is the aim of this work to introduce and attack the *Gauss image problem*:

The Gauss image problem. *Suppose λ is a submeasure defined on the Lebesgue measurable subsets of S^{n-1}, and μ is a Borel submeasure on S^{n-1}. What are the necessary and sufficient conditions, on λ and μ, so that there exists a convex body $K \in \mathcal{K}_o^n$ such that*

$$\lambda(K, \cdot) = \mu \tag{1.1}$$

on the Borel subsets of S^{n-1}? And if such a body exists, to what extent is it unique?

When λ is spherical Lebesgue measure, the Gauss image problem is just the classical Aleksandrov problem. Note that since obviously $\boldsymbol{\alpha}_K(S^{n-1}) = S^{n-1}$, a solution to (1.1) is only possible if $|\lambda| = |\mu|$; i.e., $\lambda(S^{n-1}) = \mu(S^{n-1})$.

Purely as an aside, we note that for the special case in which μ is a measure that has a density, say f, and λ is a measure that has a density, say g, the geometric problem (1.1) is the equation of Monge-Ampère type,

$$g\left(\frac{\nabla h + h\iota}{|\nabla h + h\iota|}\right)|\nabla h + h\iota|^{-n} h \det(\nabla^2 h + hI) = f, \tag{1.2}$$

where $h: S^{n-1} \to (0, \infty)$ is the unknown function. In (1.2), I is the standard Riemannian metric on S^{n-1}, the map $\iota: S^{n-1} \to S^{n-1}$ is the identity, while ∇h and $\nabla^2 h$ are, respectively, the gradient and the Hessian of h with respect to I.

The focus of this work will be on solving the general question posed by (1.1). Special cases, such as (1.2), shall be ignored. Our approach in attacking equation (1.1) uses convex geometric methods of a variational nature. What will be needed are delicate estimates for geometric invariants in order to solve an associated maximization problem. The techniques developed in this work in order to obtain these critical estimates are new and different from those developed in [16, 27].

If $K \in \mathcal{K}_o^n$, then its radial function $\rho_K: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is defined, for each $x \neq 0$, by $\rho_K(x) = \max\{r > 0 : rx \in K\}$. If μ is a Borel measure on S^{n-1}, then for a real $q \neq 0$, define the *q^{th} dual volume of K with respect to μ* by

$$\mu_q(K) = \left(\frac{1}{|\mu|}\int_{S^{n-1}} \rho_K^q(u)\, d\mu(u)\right)^{\frac{1}{q}}.$$

Recall that $\mu_q(K)$ is monotone nondecreasing and continuous in q. Define the *log-volume* of K with respect to μ by $\mu_0(K) = \lim_{q\to 0} \mu_q(K)$. If μ is the spherical Lebesgue measure, then the dual volume $\mu_q(K)$ is just the normalized classical q^{th} dual volume. Dual volumes associated with the spherical Lebesgue measure are fundamental geometric invariants. Their connections to dual curvature measures and the dual Minkowski problem were discovered in [27]. Surprisingly, as will be seen, log-volumes are closely related to the Gauss image problem.

For $Q \in \mathcal{K}_o^n$, let $Q^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$ denote the *polar* of Q. As will be shown, the solutions of the Gauss image problem are closely tied to the following:

Maximizing the log-volume-product. *If μ, λ are Borel measures on S^{n-1} of the same total mass, what are the necessary and sufficient conditions on λ and μ so that there exists a convex body $K \in \mathcal{K}_o^n$ such that*

$$\sup_{Q \in \mathcal{K}_o^n} \mu_0(Q)\lambda_0(Q^*) = \mu_0(K)\lambda_0(K^*)?$$

If $\omega \subset S^{n-1}$ is contained in a closed hemisphere, then the *polar set* ω^* is defined by

$$\omega^* = \{v \in S^{n-1} : u \cdot v \leq 0 \text{ for all } u \in \omega\} = \bigcap_{u \in \omega} \{v \in S^{n-1} : u \cdot v \leq 0\}. \tag{1.3}$$

A critical new concept introduced here is that of two Borel measures on S^{n-1} being *Aleksandrov related*.

DEFINITION. Two Borel measures μ and λ on S^{n-1} are called *Aleksandrov related* if

$$\lambda(S^{n-1}) = \mu(S^{n-1}) > \lambda(\omega^*) + \mu(\omega)$$

for each compact, spherically convex set $\omega \subset S^{n-1}$.

This relationship is easily seen to be symmetric since $\omega^{**} = \omega$ for each compact, spherically convex set $\omega \subset S^{n-1}$. If μ is Aleksandrov related to spherical Lebesgue measure, then the measure μ is said to *satisfy the Aleksandrov condition*, which is an important well-known notion.

The following solution to a critical case of the Gauss image problem will be presented:

THEOREM 1.1. *Suppose μ and λ are Borel measures on S^{n-1} and λ is absolutely continuous. If μ and λ are Aleksandrov related, then there exists a body $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$.*

It will be shown that when the measure λ is strictly positive on nonempty open sets, the requirement that the measures be Aleksandrov related is also necessary. Moreover, it will be shown that the convex body in the solution is unique up to dilation.

When the measure λ is spherical Lebesgue measure, Theorem 1.1 is originally due to Aleksandrov. New proofs were presented by Oliker [47] and later by Bertrand [10]. The approach taken below is different from these.

It will be shown that in an important case, the Gauss image problem and the problem of maximizing the log-volume-product are equivalent.

THEOREM 1.2. *Suppose λ and μ are Borel measures on S^{n-1}, and λ is both absolutely continuous and strictly positive on nonempty open sets. If $|\mu| = |\lambda|$, then the following statements are equivalent*:

(1) *There exists a body* $K \in \mathcal{K}_o^n$ *such that* $\lambda(K, \cdot) = \mu$.
(2) *There exists a body* $K \in \mathcal{K}_o^n$ *such that*

$$\sup_{Q \in \mathcal{K}_o^n} \mu_0(Q)\lambda_0(Q^*) = \mu_0(K)\lambda_0(K^*).$$

(3) μ *and* λ *are Aleksandrov related.*

Moreover, if the convex body K *exists, then it is unique up to dilation.*

It can be shown that two even Borel measures with the same total mass are always Aleksandrov related (a Borel measure is even if its value is the same for each Borel set and its antipode).

THEOREM 1.3. *Suppose* μ *is an even Borel measure on* S^{n-1} *that is not concentrated on any great hypersphere, and* λ *is an even Borel measure on* S^{n-1} *that is absolutely continuous and strictly positive on nonempty open sets. If* $|\mu| = |\lambda|$, *then there exists an origin-symmetric convex body* $K \in \mathcal{K}_o^n$, *unique up to dilation, such that*

(1) $\lambda(K, \cdot) = \mu$, *and*
(2) *the maximum of* $\mu_0(Q)\lambda_0(Q^*)$ *over* $Q \in \mathcal{K}_o^n$ *is attained at* K.

It is necessary to contrast the Gauss image problem with the various Minkowski problems and dual Minkowski problems that have been extensively studied (see, e.g., [14, 16–18, 27–29, 37–42, 45–47, 54, 56–59]). A good way to do that is to contrast the Gauss image problem with a specific Minkowski problem, say the log-Minkowski problem. The cone volume measure of a convex body has been of considerable recent interest (see, e.g., [9, 12, 13, 16, 25, 26, 43, 44, 54]). The *cone-volume measure* V_K of a convex body K is a Borel measure on the unit sphere, defined for Borel $\omega \subset S^{n-1}$ as the n-dimensional Lebesgue measure of the cone

$$\{tx : 0 \leq t \leq 1 \text{ and } x \in \partial K \text{ with } N(K, x) \cap \omega \neq \varnothing\}.$$

The *log-Minkowski problem* asks: Given a Borel measure μ, does there exist a convex body K such that $\mu = V_K$? And if the body exists, to what extent is it unique? (For recent work on this, see, e.g., [6–8, 16, 25, 26].) It is precisely here that we can see the difference between Minkowski problems and the Gauss image problem. In the Gauss image problem, a pair of submeasures is given and it is asked if there exists a convex body "linking" them via its radial Gauss image. Thus, we need to construct a convex body whose radial Gauss image "links" the two given submeasures. On the other hand, in a Minkowski problem, only one measure is given, and the question asks if this measure is a specific geometric measure of a convex body, such as the cone-volume measure of a convex body. To solve a Minkowski problem, we are attempting to construct a convex body for a specific geometric measure of convex bodies. However, the Gauss image problem could be a Minkowski problem. For example, if λ is spherical Lebesgue measure, then $\lambda(K, \cdot)$ is just Aleksandrov's integral curvature of K. Here we are dealing with a Minkowski problem, namely, the Minkowski problem for Aleksandrov's integral

curvature: Given a Borel measure μ, does there exist a convex body K such that $\mu = \lambda(K, \cdot)$; i.e., does there exist a convex body K whose integral curvature is the given measure μ? And if the body exists, to what extent is it unique? In this sense, the Gauss image problem broadens the study of Minkowski problems. But the essence of the problem is an attempt at a deeper understanding of the Gauss image map.

2 Preliminaries

For $x \in \mathbb{R}^n$, let $|x| = \sqrt{x \cdot x}$ be the euclidean norm of x. For $x \in \mathbb{R}^n \setminus \{0\}$, define $\bar{x} = x/|x|$. For a subset $E \subset \mathbb{R}^n$, let $\bar{E} = \{\bar{x} : x \in E \setminus \{0\}\}$. The origin-centered unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$ is always denoted by B.

Lebesgue measure in $\mathbb{R}^n$ is denoted by V, which is also called "volume." Write ω_n for the volume of B. We shall write $\mathcal{H}^{n-1}$ for $(n-1)$-dimensional Hausdorff measure.

For the set of continuous functions defined on S^{n-1}, write $C(S^{n-1})$, and for $f \in C(S^{n-1})$, write $\|f\|_\infty = \max_{v \in S^{n-1}} |f(v)|$. We shall view $C(S^{n-1})$ as endowed with the topology induced by this *max-norm*. We write $C^+(S^{n-1})$ for the set of strictly positive functions in $C(S^{n-1})$, and $C_e^+(S^{n-1})$ for the set of even functions in $C^+(S^{n-1})$.

Let $\mathcal{K}^n$ denote the set of compact, convex subsets of $\mathbb{R}^n$. For $K \in \mathcal{K}^n$, the support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of K is defined by $h_K(x) = \max\{x \cdot y : y \in K\}$ for $x \in \mathbb{R}^n$. The support function is convex and homogeneous of degree 1. A compact convex subset of $\mathbb{R}^n$ is uniquely determined by its support function. The set $\mathcal{K}^n$ is viewed as endowed with the *Hausdorff metric*. So, the distance between $K, L \in \mathcal{K}^n$ is simply $d(K, L) = \|h_K - h_L\|_\infty$. If A is a compact subset of $\mathbb{R}^n$, then conv A, the *convex hull* of A, is the smallest convex set that contains A. It is easily seen that its support function is given by

$$h_{\mathrm{conv}A}(x) = \max\{x \cdot y : y \in A\}, \tag{2.1}$$

for $x \in \mathbb{R}^n$.

A *convex body* in $\mathbb{R}^n$ is a compact convex set with nonempty interior. Denote by $\mathrm{int}\, K$ the interior of the convex body K. Denote by $\mathcal{K}_e^n$ the class of origin-symmetric convex bodies in $\mathbb{R}^n$. Obviously, $\mathcal{K}_o^n$ is a subspace of $\mathcal{K}^n$, and $\mathcal{K}_e^n$ is a subspace of $\mathcal{K}_o^n$.

The radial function $\rho_K : S^{n-1} \to \mathbb{R}$ of a compact set K that is star-shaped, with respect to the origin, is defined by $\rho_K(x) = \max\{a : au \in K\}$ for $u \in S^{n-1}$. A compact star-shaped set with respect to the origin is uniquely determined by its radial function. The radial function of a convex body in $\mathcal{K}_o^n$ is continuous and positive. If $K \in \mathcal{K}_o^n$, then obviously

$$\partial K = \{\rho_K(u)u : u \in S^{n-1}\}.$$

The *radial metric* defines the distance between $K, L \in \mathcal{K}_o^n$ as $\|\rho_K - \rho_L\|_\infty$. We shall use the well-known fact that on $\mathcal{K}_o^n$, the Hausdorff metric and radial metric are topologically equivalent.

For a Borel measure μ on S^{n-1}, define

$$\mu_p(f) = \left(\frac{1}{|\mu|}\int_{S^{n-1}} f^p \, d\mu\right)^{1/p}, \quad p \neq 0,$$

and

$$\mu_0(f) = \exp\left(\frac{1}{|\mu|}\int_{S^{n-1}} \log f \, d\mu\right)$$

for each $f \in C^+(S^{n-1})$. When $f = \rho_K$, for some $K \in \mathcal{K}_o^n$, then $\mu_p(f)$ will be written as $\mu_p(K)$. When μ is spherical Lebesgue measure, then the $\mu_p(K)$ are the normalized dual volumes from the dual Brunn-Minkowski theory—a theory that played a critical role in the ultimate solution of the Busemann-Petty problem (see, e.g., [19, 21, 31, 32, 55]).

If $K \in \mathcal{K}_o^n$, then it is easily seen that the radial function and the support function of K are related by

$$h_K(v) = \max_{u \in S^{n-1}} (u \cdot v)\, \rho_K(u), \qquad v \in S^{n-1},$$

and

$$1/\rho_K(u) = \max_{v \in S^{n-1}} (u \cdot v)/h_K(v), \quad u \in S^{n-1}.$$

From the definition of the polar body, we see that

$$\rho_K = 1/h_{K^*} \quad \text{and} \quad h_K = 1/\rho_{K^*} \tag{2.2}$$

on S^{n-1}.

For $K, L \in \mathcal{K}^n$ and real $a, b \geq 0$, the *Minkowski combination*, $aK + bL \in \mathcal{K}^n$, is the compact convex set defined by

$$aK + bL = \{ax + by : x \in K \text{ and } y \in L\},$$

and its support function is given by

$$h_{aK+bL} = ah_K + bh_L.$$

Suppose $\Omega \subset S^{n-1}$ is closed and not contained in any closed hemisphere of S^{n-1}. For a function $f : \Omega \to (0, \infty)$, define $\langle f \rangle$ to be the convex hull in $\mathbb{R}^n$,

$$\langle f \rangle = \text{conv}\{f(u)u : u \in \Omega\}.$$

Since f is strictly positive and Ω is not contained in any closed hemisphere of S^{n-1}, it follows that $\langle f \rangle \in \mathcal{K}_o^n$. Note that $\langle af \rangle = a\langle f \rangle$ for $a > 0$. From (2.1), we see that the support function of $\langle f \rangle$ is given by

$$h_{\langle f \rangle}(x) = \max_{u \in \Omega} (x \cdot u) f(u), \tag{2.3}$$

for $x \in \mathbb{R}^n$. We shall make use of the fact that if $f_0, f_1, \ldots \in C^+(S^{n-1})$, then

$$\lim_{k \to \infty} f_k = f_0 \text{ uniformly on } S^{n-1} \quad \Longrightarrow \quad \langle f_k \rangle \to \langle f_0 \rangle \text{ in } \mathcal{K}_o^n. \tag{2.4}$$

See, e.g., [27, p. 345] for a proof.

If $\omega \subset S^{n-1}$, define $\operatorname{cone}\omega$, the *cone generated by* ω, as

$$\operatorname{cone}\omega := \{tu : t \geq 0 \text{ and } u \in \omega\}$$

and define $\widehat{\omega}$, the *restricted cone generated by* ω, as

$$\widehat{\omega} = \{tu : 0 \leq t \leq 1 \text{ and } u \in \omega\}.$$

A subset $\omega \subset S^{n-1}$ is *spherically convex*, if $\operatorname{cone}\omega$ is a nonempty proper convex subset of $\mathbb{R}^n$.This definition implies that a spherically convex set on S^{n-1} is nonempty and is always contained in a closed hemisphere of S^{n-1}. A spherically convex set $\omega \subset S^{n-1}$ is said to be *strongly spherically convex* if it is contained in an open hemisphere.

If ω is a compact spherically convex set in S^{n-1}, then ω is strongly spherically convex if and only if $\omega \cap (-\omega) = \varnothing$, or equivalently ω does not contain a pair of antipodal points. Indeed, when $\omega \subset S^{n-1}$ is compact spherically convex and $\omega \cap (-\omega) = \varnothing$, then $\operatorname{conv}\omega$ and $\operatorname{conv}(-\omega)$, the convex hulls in $\mathbb{R}^n$, are disjoint. If this were not the case, then this would immediately imply that the origin belongs to $\operatorname{conv}\omega$. But, to see that this is impossible write the origin as a convex combination of $u_1, \ldots, u_r \in \omega$ with strictly positive coefficients. This would imply that the point $-u_1 \in \operatorname{cone}\omega$, and since $-u_1 \in S^{n-1}$, it would follow that $-u_1 \in \omega$, thus contradicting the fact that $u_1 \in \omega$. Since $\operatorname{conv}\omega$ and $\operatorname{conv}(-\omega)$ are disjoint compact convex sets in $\mathbb{R}^n$ that do not contain the origin, the hyperplane separation theorem tells us that $\operatorname{conv}\omega$ and $\operatorname{conv}(-\omega)$ are contained in the opposite open sides of a hyperplane passing through the origin. Thus, ω is contained in an open hemisphere.

For a subset $\omega \subset S^{n-1}$ that is contained in a closed hemisphere, its *polar set* ω^* is defined by

$$\omega^* = \{v \in S^{n-1} : u \cdot v \leq 0 \text{ for all } u \in \omega\}.$$

The *spherical convex hull*, $\langle\omega\rangle$, of ω is defined by

$$\langle\omega\rangle = S^{n-1} \cap \operatorname{conv}(\operatorname{cone}\omega).$$

The polar set ω^* is always convex and

$$\omega^* = \langle\omega\rangle^*. \tag{2.5}$$

For recent work on spherical convex bodies, see Besau and Werner [11].

As is well-known, the Hausdorff metric can be extended to the set of all nonempty compact subsets of $\mathbb{R}^n$. If K and L are nonempty compact subsets of $\mathbb{R}^n$, then the Hausdorff distance between them can be defined by

$$\max\Big\{\sup_{x\in K}\inf_{y\in L}|x-y|,\ \sup_{y\in L}\inf_{x\in K}|x-y|\Big\}.$$

Let $\mathcal{O}^{n-1}$ denote the set of spherically compact convex sets of S^{n-1} endowed with the topology of the Hausdorff metric. It is easily verified that a sequence $\omega_i \in \mathcal{O}^{n-1}$ converges to $\omega \in \mathcal{O}^{n-1}$ if and only if $\widehat{\omega}_i$ converges to $\widehat{\omega}$.

Let $\Omega \subset S^{n-1}$ be a closed set that is not contained in a closed hemisphere of S^{n-1}. Let $f:\Omega \to \mathbb{R}$ be continuous and $\delta > 0$. Let $h_t:\Omega \to (0,\infty)$ be a continuous function defined for each $t \in (-\delta,\delta)$ by

$$\log h_t = \log h + tf + o(t,\cdot),$$

where $o(t,\cdot):\Omega \to \mathbb{R}$ is continuous and $\lim_{t\to 0} o(t,\cdot)/t = 0$ uniformly on Ω. Denote by

$$[h_t] = \{x \in \mathbb{R}^n : x \cdot v \le h_t(v) \text{ for all } v \in \Omega\}$$

the Wulff shape determined by h_t. We shall call $[h_t]$ a *logarithmic family of Wulff shapes formed by* (h, f). On occasion, we shall write $[h_t]$ as $[h, f]$, and, if h happens to be the support function of a convex body K, perhaps as $[K, f]$ or $[K, f, t]$ or $[K, f, o, t]$, if required for clarity. We call $[K, f]$ a logarithmic family of Wulff shapes formed by (K, f).

Let $g:\Omega \to \mathbb{R}$ be continuous and $\delta > 0$. Let $\rho_t:\Omega \to (0,\infty)$ be a continuous function defined for each $t \in (-\delta,\delta)$ by

$$\log \rho_t = \log \rho + tg + o(t,\cdot),$$

where again $o(t,\cdot):\Omega \to \mathbb{R}$ is continuous and $\lim_{t\to 0} o(t,\cdot)/t = 0$ uniformly on Ω. Denote by

$$\langle \rho_t \rangle = \operatorname{conv}\{\rho_t(u)u : u \in S^{n-1}\}$$

the convex hull generated by ρ_t. We will call $\langle \rho_t \rangle$ a *logarithmic family of convex hulls generated by* (ρ, g). On occasion, we shall write $\langle \rho_t \rangle$ as $\langle \rho, g, t \rangle$, and if ρ happens to be the radial function of a convex body $K \in \mathcal{K}_o^n$ as $\langle K, g \rangle$ or $\langle K, g, t \rangle$ or $\langle K, g, o, t \rangle$, if required for clarity. We call $\langle K, g \rangle$ a logarithmic family of convex hulls generated by (K, g).

From [27] we will use the easily established fact that if $K \in \mathcal{K}_o^n$ and $f:\Omega \to \mathbb{R}$ is continuous, where $\Omega \subset S^{n-1}$ is a closed set that is not contained in a closed hemisphere of S^{n-1}, then

$$\langle K, f \rangle^* = [K^*, -f]. \tag{2.6}$$

It will be important to recall the fact that every Borel measure that is absolutely continuous vanishes on the boundaries of spherically convex subsets of the sphere.

Schneider's book [49] is our standard reference for the basics regarding convex bodies. The books [20, 22] are also good references.

3 The Gauss Image Measure

Let K be a convex body in $\mathbb{R}^n$. For each $v \in S^{n-1}$, the hyperplane

$$H_K(v) = \{x \in \mathbb{R}^n : x \cdot v = h_K(v)\}$$

is called the *supporting hyperplane to K with unit normal v*. For $\sigma \subset \partial K$, the *spherical image of σ* is defined by

$$\boldsymbol{\nu}_K(\sigma) = \{v \in S^{n-1} : x \in H_K(v) \text{ for some } x \in \sigma\} \subset S^{n-1}.$$

For $\eta \subset S^{n-1}$, the *reverse spherical image of* η is defined by

$$\boldsymbol{x}_K(\eta) = \{x \in \partial K : x \in H_K(v) \text{ for some } v \in \eta\} \subset \partial K.$$

Let $\sigma_K \subset \partial K$ be the set consisting of all $x \in \partial K$ for which the set $\boldsymbol{\nu}_K(\{x\})$, abbreviated as $\boldsymbol{\nu}_K(x)$, contains more than a single element. The points in $\partial K \setminus \sigma_K$ are called *regular* points of ∂K. It is well-known (Schneider [49, p. 84]) that the $(n-1)$-dimensional Hausdorff measure of the set of *singular* (i.e., nonregular) points of a convex body is 0; i.e., $\mathcal{H}^{n-1}(\sigma_K) = 0$. The function

$$\nu_K \colon \partial K \setminus \sigma_K \to S^{n-1},$$

defined by letting $\nu_K(x)$ be the unique element in $\boldsymbol{\nu}_K(x)$ for each $x \in \partial K \setminus \sigma_K$, is called the *spherical image map* (also known as the Gauss map) of K and is known to be continuous (see lemma 2.2.12 of Schneider [49]).

The set $\eta_K \subset S^{n-1}$ consisting of all $v \in S^{n-1}$ for which the set $\boldsymbol{x}_K(v)$ contains more than a single element is of $\mathcal{H}^{n-1}$-measure 0 (see theorem 2.2.11 of Schneider [49]). The function

$$x_K \colon S^{n-1} \setminus \eta_K \to \partial K,$$

defined, for each $v \in S^{n-1} \setminus \eta_K$, by letting $x_K(v)$ be the unique element in $\boldsymbol{x}_K(v)$, is called the *reverse spherical image map*. The vectors in $S^{n-1} \setminus \eta_K$ are called the *regular normal vectors* of K. Thus, $v \in S^{n-1}$ is a regular normal vector of K if and only if $\partial K \cap H_K(v)$ consists of a single point. The function x_K is well-known to be continuous (see lemma 2.2.12 of Schneider [49]).

For $K \in \mathcal{K}^n_o$, define the *radial map* of K,

$$r_K \colon S^{n-1} \to \partial K \quad \text{by} \quad r_K(u) = \rho_K(u)u \in \partial K$$

for $u \in S^{n-1}$. Note that the mapping $r_K^{-1} \colon \partial K \to S^{n-1}$ is just the restriction of the map $\bar{\cdot} \colon \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ to the set ∂K. The radial map is bi-Lipschitz.

For $\omega \subset S^{n-1}$, define the *radial Gauss image of* ω by

$$\boldsymbol{\alpha}_K(\omega) = \boldsymbol{\nu}_K(r_K(\omega)) \subset S^{n-1}.$$

Thus, for $u \in S^{n-1}$,

$$\boldsymbol{\alpha}_K(\{u\}) = \{v \in S^{n-1} \colon r_K(u) \in H_K(v)\}.$$

We will need the fact that $\boldsymbol{\alpha}_K$ maps closed sets of S^{n-1} into closed sets of S^{n-1}.

LEMMA 3.1. *If $\omega \subset S^{n-1}$ is closed, then $\boldsymbol{\alpha}_K(\omega)$ is also closed.*

PROOF. Suppose the points $v_i \in \boldsymbol{\alpha}_K(\omega)$ are such that $v_i \to v_0$. We will show that $v_0 \in \boldsymbol{\alpha}_K(\omega)$. Now $v_i \in \boldsymbol{\nu}_K(r_K(\omega))$ means that v_i is a unit outer normal to K at $r_K(u_i)$ for some $u_i \in \omega$; i.e.,

$$x \cdot v_i \le r_K(u_i) \cdot v_i \quad \text{for all } x \in K. \tag{3.1}$$

Since $\omega \subset S^{n-1}$ is compact, $u_i \in \omega$ has a convergent subsequence, which we will again denote by u_i, that is, $u_i \to u_0 \in \omega$. Since r_K is a continuous function,

$r_K(u_i) \to r_K(u_0)$, and together with $v_i \to v_0$ and (3.1) gives

$$x \cdot v_0 \leq r_K(u_0) \cdot v_0 \quad \text{for all } x \in K.$$

Hence, $v_0 \in \boldsymbol{\nu}_K(r_K(\omega)) = \boldsymbol{\alpha}_K(\omega)$. □

Define the *radial Gauss map* of the convex body $K \in \mathcal{K}_o^n$

$$\alpha_K : S^{n-1} \setminus \omega_K \to S^{n-1} \quad \text{by } \alpha_K = \nu_K \circ r_K,$$

where $\omega_K = \overline{\sigma_K} = r_K^{-1}(\sigma_K)$. Since $r_K^{-1} = \bar{\cdot}$ is a bi-Lipschitz map between the spaces ∂K and S^{n-1}, it follows that ω_K has spherical Lebesgue measure 0. Observe that if $u \in S^{n-1} \setminus \omega_K$, then $\boldsymbol{\alpha}_K(\{u\})$ contains only the element $\alpha_K(u)$. Note that since both ν_K and r_K are continuous, α_K is continuous.

From [27] Lemma 2.2, if $K_0, K_1, \ldots \in \mathcal{K}_o^n$, then

$$K_i \to K_0 \implies \alpha_{K_i} \to \alpha_{K_0}, \tag{3.2}$$

almost everywhere, with respect to spherical Lebesgue measure.

For $\eta \subset S^{n-1}$, define the *reverse radial Gauss image* of η by

$$\boldsymbol{\alpha}_K^*(\eta) = r_K^{-1}(\boldsymbol{x}_K(\eta)) = \overline{\boldsymbol{x}_K(\eta)}. \tag{3.3}$$

Thus,

$$\boldsymbol{\alpha}_K^*(\eta) = \{\overline{x} : x \in \partial K \text{ where } x \in H_K(v) \text{ for some } v \in \eta\}. \tag{3.4}$$

Define the *reverse radial Gauss map* of the convex body $K \in \mathcal{K}_o^n$,

$$\alpha_K^* \colon S^{n-1} \setminus \eta_K \to S^{n-1} \quad \text{by} \quad \alpha_K^* = r_K^{-1} \circ x_K.$$

Note that since both r_K^{-1} and x_K are continuous, α_K^* is continuous.

If $\eta \subset S^{n-1}$ is a Borel set, then $\boldsymbol{\alpha}_K^*(\eta) = \overline{\boldsymbol{x}_K(\eta)} \subset S^{n-1}$ is spherical Lebesgue measurable. This fact is lemma 2.2.14 of Schneider [49]; an alternate proof was given in [27]. It was shown in [27] that if $v \notin \eta_K$ and $\omega \subset S^{n-1}$, then

$$v \in \boldsymbol{\alpha}_K(\omega) \quad \text{if and only if} \quad \alpha_K^*(v) \in \omega. \tag{3.5}$$

Hence (3.5) holds for almost all $v \in S^{n-1}$, with respect to spherical Lebesgue measure. It was also shown in [27] that if $K \in \mathcal{K}_o^n$, then the reverse radial Gauss image of K and the radial Gauss image of the polar body, K^*, are identical; i.e.,

$$\boldsymbol{\alpha}_K^*(\eta) = \boldsymbol{\alpha}_{K^*}(\eta), \tag{3.6}$$

for each $\eta \subset S^{n-1}$. It follows that for $K \in \mathcal{K}_o^n$, the set $\boldsymbol{\alpha}_{K^*}(\eta)$ is spherical Lebesgue measurable whenever $\eta \subset S^{n-1}$ is a Borel set. Since $K^{**} = K$, this shows that $\boldsymbol{\alpha}_K(\omega)$ is spherically Lebesgue measurable whenever $\omega \subset S^{n-1}$ is a Borel set and $K \in \mathcal{K}_o^n$. From (3.6) we also see that for $K \in \mathcal{K}_o^n$,

$$\alpha_K^* = \alpha_{K^*} \tag{3.7}$$

almost everywhere on S^{n-1}, with respect to spherical Lebesgue measure.

If $K_0, K_1, \ldots \in \mathcal{K}_o^n$ are such that $K_i \to K_0$, then $K_i^* \to K_0^*$. This and (3.2) give us $\alpha_{K_i^*} \to \alpha_{K_0^*}$ almost everywhere with respect to spherical Lebesgue measure. Now (3.7) allows us to conclude that

$$K_i \to K_0 \implies \alpha^*_{K_i} \to \alpha^*_{K_0}, \tag{3.8}$$

almost everywhere, with respect to spherical Lebesgue measure.

For $K \in \mathcal{K}_o^n$, Aleksandrov's *integral curvature*, $C_0(K, \cdot)$, is a Borel measure on S^{n-1} defined, for Borel $\omega \subset S^{n-1}$, by

$$C_0(K, \omega) = \mathcal{H}^{n-1}(\boldsymbol{\alpha}_K(\omega)); \tag{3.9}$$

i.e., $C_0(K, \omega)$ is the spherical Lebesgue measure of $\boldsymbol{\alpha}_K(\omega)$. The total measure $C_0(K, S^{n-1})$ of integral curvature of each convex body K is $n\omega_n$, the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$.

The *solid-angle measure* $\widetilde{C}_0(K, \cdot)$, also known as the 0^{th} dual curvature measure, introduced in [27], can be defined by

$$n\widetilde{C}_0(K, \eta) = \mathcal{H}^{n-1}(\boldsymbol{\alpha}^*_K(\eta)) \tag{3.10}$$

for each Borel $\eta \subset S^{n-1}$. From (3.9), (3.10), and (3.6), we have

$$C_0(K, \cdot) = n\widetilde{C}_0(K^*, \cdot).$$

The $(n-1)^{\text{th}}$ area measure $S_{n-1}(K, \cdot)$ is the classical surface area measure $S(K, \cdot)$, which is defined, for each Borel $\eta \subset S^{n-1}$, by

$$S_{n-1}(K, \eta) = \mathcal{H}^{n-1}(\boldsymbol{x}_K(\eta)). \tag{3.11}$$

Federer's $(n-1)^{\text{th}}$ curvature measure $C_{n-1}(K, \cdot)$ on S^{n-1} can be defined, for each Borel $\omega \subset S^{n-1}$, by

$$C_{n-1}(K, \omega) = \mathcal{H}^{n-1}(r_K(\omega)). \tag{3.12}$$

From (3.11) and (3.12), and the definition (3.3) that $\boldsymbol{\alpha}^*_K = r_K^{-1} \circ \boldsymbol{x}_K$, we see that the $(n-1)^{\text{th}}$ curvature measure $C_{n-1}(K, \cdot)$ on S^{n-1} and the $(n-1)^{\text{th}}$ area measure $S_{n-1}(K, \cdot)$ on S^{n-1} are related by

$$C_{n-1}(K, \boldsymbol{\alpha}^*_K(\eta)) = S_{n-1}(K, \eta) \tag{3.13}$$

for each Borel $\eta \subset S^{n-1}$. See Schneider [49, theorem 4.2.3].

The following lemma establishes a fundamental property of the radial Gauss image.

LEMMA 3.2. *Let $K \in \mathcal{K}_o^n$. If $\omega \subset S^{n-1}$ is a spherically convex set, then*

$$\boldsymbol{\alpha}_K(\omega) \subset S^{n-1} \setminus \omega^*, \tag{3.14}$$

and furthermore the set $(S^{n-1} \setminus \omega^*) \setminus \boldsymbol{\alpha}_K(\omega)$ has interior points.

PROOF. Consider an arbitrary $u \in \omega$ and an arbitrary $v \in \boldsymbol{\alpha}_K(u)$; i.e., v is an outer unit normal of K at $r_K(u)$. From the definition of the support function, we see that

$$\rho_0 \le h_K(v) = \rho_K(u)u \cdot v \le \rho_1 u \cdot v,$$

which implies

$$u \cdot v \ge \rho_0/\rho_1, \tag{3.15}$$

where ρ_0 is the minimum of ρ_K on S^{n-1} and ρ_1 is the maximum of ρ_K on S^{n-1}. The definition of ω^* and the fact that $u \in \omega$ now give us that $v \notin \omega^*$, which yields (3.14).

Now (3.14) is just $\boldsymbol{\alpha}_K(\omega) \cap \omega^* = \varnothing$. When ω is spherically convex, ω^* is nonempty. However, (3.15) implies that if we choose $\delta_0 \in (0, \rho_0/\rho_1)$, then the set

$$\omega'_{\delta_0} = \bigcap_{u \in \omega} \{v \in S^{n-1} : v \cdot u < \delta_0\} \setminus \omega^*$$

is disjoint from $\boldsymbol{\alpha}_K(\omega)$. Note that ω'_{δ_0} has nonempty interior. Therefore, the set

$$(S^{n-1} \setminus \omega^*) \setminus \boldsymbol{\alpha}_K(\omega)$$

has interior points. □

A *spherical submeasure* $\mu\colon \mathcal{B} \to [0, \infty)$, defined on a σ-algebra $\mathcal{B}$ of subsets of S^{n-1}, is a function that satisfies the following:

(1) $\mu(\varnothing) = 0$.
(2) If $A, B \in \mathcal{B}$ are such that $A \subset B$, then $\mu(A) \le \mu(B)$.
(3) If $A_1, A_2, \ldots \in \mathcal{B}$, then $\mu(\bigcup_1^\infty A_i) \le \sum_1^\infty \mu(A_i)$.

Our interest will be limited to spherical Lebesgue submeasures and spherical Borel submeasures, where $\mathcal{B}$ is the collection of spherical Lebesgue measurable subsets of S^{n-1} and spherical Borel subsets of S^{n-1}, respectively.

Suppose λ is a spherical Lebesgue submeasure and $K \in \mathcal{K}_o^n$. The *Gauss image measure* $\lambda(K, \cdot)$ of λ via K is the spherical Borel submeasure defined by

$$\lambda(K, \omega) = \lambda(\boldsymbol{\alpha}_K(\omega)) \tag{3.16}$$

for each Borel set $\omega \subset S^{n-1}$. To see that $\lambda(K, \cdot)$ is indeed a submeasure, we recall the basic properties of the Gauss image $\boldsymbol{\alpha}_K$ of a body $K \in \mathcal{K}_o^n$:

(1) $\boldsymbol{\alpha}_K(\varnothing) = \varnothing$.
(2) If $\omega, \omega' \subset S^{n-1}$ are such that $\omega \subset \omega'$, then $\boldsymbol{\alpha}_K(\omega) \subset \boldsymbol{\alpha}_K(\omega')$.
(3) If $\omega_1, \omega_2, \ldots \subset S^{n-1}$, then $\boldsymbol{\alpha}_K(\bigcup_1^\infty \omega_i) = \bigcup_1^\infty \boldsymbol{\alpha}_K(\omega_i)$.
(4) If $\omega_1, \omega_2, \ldots \subset S^{n-1}$ are pairwise disjoint, then up to a set of spherical Lebesgue measure 0, the sets $\boldsymbol{\alpha}_K(\omega_1), \boldsymbol{\alpha}_K(\omega_2), \ldots$ are pairwise disjoint as well.

Properties (1) and (2) are completely trivial, while Property (3) follows directly from the trivial lemma 2.3 in [27] together with (3.6). Property (4) is lemma 2.4

in [27]. The *reverse Gauss image measure* $\lambda^*(K,\cdot)$ of λ via K is the Borel sub-measure on S^{n-1} defined by

$$\lambda^*(K,\omega) = \lambda(\boldsymbol{\alpha}_K^*(\omega)) = \lambda(\boldsymbol{\alpha}_{K^*}(\omega)) \tag{3.17}$$

for each Borel set $\omega \subset S^{n-1}$. Note that the second identity in (3.17) is from (3.6).

Since for $a > 0$ obviously $\boldsymbol{\alpha}_{aK} = \boldsymbol{\alpha}_K$ and $\boldsymbol{\alpha}_{aK}^* = \boldsymbol{\alpha}_K^*$, it follows, from their definitions, that

$$\lambda(aK,\cdot) = \lambda(K,\cdot) \quad \text{and} \quad \lambda^*(aK,\cdot) = \lambda^*(K,\cdot)$$

for all $a > 0$; i.e., the Gauss image measure and the reverse Gauss image measure of a convex body are invariant under dilations of the convex body. From (3.16), (3.17), and (3.6), we immediately obtain

$$\lambda^*(K,\cdot) = \lambda(K^*,\cdot). \tag{3.18}$$

When λ is spherical Lebesgue measure $\mathcal{H}^{n-1}\big|_{S^{n-1}}$, it follows from (3.9) and (3.10) that the Gauss image measure $\lambda(K,\cdot)$ is integral curvature and the reverse Gauss image measure $\lambda^*(K,\cdot)$ is n times the solid-angle measure, i.e.,

$$\lambda = \mathcal{H}^{n-1}|_{S^{n-1}} \implies \lambda(K,\cdot) = C_0(K,\cdot) \quad \text{and} \quad \lambda^*(K,\cdot) = n\widetilde{C}_0(K,\cdot).$$

If λ is the curvature measure $C_{n-1}(K,\cdot)$ of a convex body K, then, by (3.13), the reverse Gauss image measure $\lambda^*(K,\cdot)$ is the surface measure $S_{n-1}(K,\cdot)$, i.e.,

$$\lambda = C_{n-1}(K,\cdot) \implies \lambda^*(K,\cdot) = S_{n-1}(K,\cdot).$$

When λ is an absolutely continuous Borel measure, the Gauss image measure is a Borel measure, for which we have the following integral representation.

LEMMA 3.3. *If λ is an absolutely continuous Borel measure and $K \in \mathcal{K}_o^n$, then*

$$\int_{S^{n-1}} f(u)\,d\lambda(K,u) = \int_{S^{n-1}} f(\alpha_K^*(v))\,d\lambda(v) \tag{3.19}$$

for each bounded Borel $f\colon S^{n-1} \to \mathbb{R}$.

PROOF. Let ϕ be a simple function on S^{n-1} given by

$$\phi = \sum_i c_i \mathbb{1}_{\omega_i}$$

where $c_i \in \mathbb{R}$, where $\omega_i \subset S^{n-1}$ are Borel sets, and where $\mathbb{1}_{\omega_i}$ is the indicator function of ω_i. Since η_K has spherical Lebesgue measure 0, we can conclude from (3.5) that

$$\mathbb{1}_{\boldsymbol{\alpha}_K(\omega_i)}(v) = \mathbb{1}_{\omega_i}(\alpha_K^*(v)), \tag{3.20}$$

for almost all $v \in S^{n-1}$, with respect to spherical Lebesgue measure. Since λ is absolutely continuous, (3.20) gives

$$\int_{S^{n-1}} \mathbb{1}_{\boldsymbol{\alpha}_K(\omega_i)}(v)\,d\lambda(v) = \int_{S^{n-1}} \mathbb{1}_{\omega_i}(\alpha_K^*(v))\,d\lambda(v). \tag{3.21}$$

We now use (3.16) and (3.21), and get

$$\begin{aligned}\int_{S^{n-1}} \phi(u)d\lambda(K,u) &= \int_{S^{n-1}} \sum_i c_i \mathbb{1}_{\omega_i}(u)d\lambda(K,u)\\ &= \sum_i c_i \lambda(K,\omega_i)\\ &= \sum_i c_i \lambda(\boldsymbol{\alpha}_K(\omega_i))\\ &= \int_{S^{n-1}} \sum_i c_i \mathbb{1}_{\boldsymbol{\alpha}_K(\omega_i)}(v)d\lambda(v)\\ &= \int_{S^{n-1}} \sum_i c_i \mathbb{1}_{\omega_i}(\alpha_K^*(v))d\lambda(v)\\ &= \int_{S^{n-1}} \phi(\alpha_K^*(v))d\lambda(v).\end{aligned}$$

This establishes (3.19) for simple functions. Given a bounded Borel f, we now choose a sequence of simple functions $\phi_k \to f$ uniformly. Then $\phi_k \circ \alpha_K^*$ converges to $f \circ \alpha_K^*$ a.e. with respect to spherical Lebesgue measure, and thus a.e. with respect to λ. Since f is a Borel function on S^{n-1} and the inverse radial Gauss map α_K^* is continuous on $S^{n-1} \setminus \eta_K$, the composite function $f \circ \alpha_K^*$ is a Borel function on $S^{n-1} \setminus \eta_K$. Since $\phi_k \to f$ uniformly and f is bounded, the functions ϕ_k are uniformly bounded. Note that both λ and $\lambda(K,\cdot)$ are finite measures. By the dominated convergence theorem, we take the limit $k \to \infty$ to establish (3.19). □

When the measure λ is an absolutely continuous Borel measure, we can (and will) speak of its Gauss image measure (as opposed to submeasure). The Gauss image measure as a functional from the space $\mathcal{K}_o^n$ to the space of Borel measures on S^{n-1} is weakly convergent with respect to the Hausdorff metric.

LEMMA 3.4. *If λ is an absolutely continuous Borel measure on S^{n-1} and the bodies $K_0, K_1, \ldots \in \mathcal{K}_o^n$ are such that $K_i \to K_0$, then $\lambda(K_i,\cdot) \to \lambda(K,\cdot)$ weakly.*

PROOF. Since $K_i \to K_0$, from (3.8) we see that $\alpha_{K_i}^* \to \alpha_{K_0}^*$ almost everywhere with respect to spherical Lebesgue measure. Then for each continuous function f on S^{n-1}, we have $f \circ \alpha_{K_i}^* \to f \circ \alpha_{K_0}^*$ almost everywhere with respect to spherical Lebesgue measure, and thus almost everywhere with respect to λ. Since $|f \circ \alpha_{K_i}^*|$ is obviously bounded by $\max_{v \in S^{n-1}} |f(v)|$, we have

$$\int_{S^{n-1}} f(\alpha_{K_i}^*(v))d\lambda(v) \;\to\; \int_{S^{n-1}} f(\alpha_{K_0}^*(v))d\lambda(v).$$

This and Lemma 3.3 show that

$$\int_{S^{n-1}} f(u)d\lambda(K_i,u) \;\to\; \int_{S^{n-1}} f(u)d\lambda(K_0,u)$$

for each continuous $f: S^{n-1} \to \mathbb{R}$. Thus, $\lambda(K_i, \cdot) \to \lambda(K_0, \cdot)$ weakly. □

LEMMA 3.5. *If λ is an absolutely continuous Borel measure on S^{n-1}, then for each $K \in \mathcal{K}_o^n$, the Gauss image measure $\lambda(K, \cdot)$ is absolutely continuous with respect to the surface area measure $S(K^*, \cdot)$ of the polar body K^* of K.*

PROOF. Since the polar of the polar is the original body, from (3.18) we see that all we need show is that the reverse Gauss image measure $\lambda^*(K, \cdot)$ is absolutely continuous with respect to the surface area measure $S(K, \cdot)$ of K.

Suppose $\eta \subset S^{n-1}$ is such that $S(K, \eta) = 0$. Then from the definition of $S(K, \cdot)$ we know that $\mathcal{H}^{n-1}(\boldsymbol{x}_K(\eta)) = 0$. But since the map $\bar{\cdot}: \partial K \to S^{n-1}$ is bi-Lipschitz, we have $\mathcal{H}^{n-1}(\overline{\boldsymbol{x}_K(\eta)}) = 0$. This, in turn, can be rewritten using the definition (3.3) of $\boldsymbol{\alpha}_K^*$, as

$$\mathcal{H}^{n-1}(\boldsymbol{\alpha}_K^*(\eta)) = 0.$$

This, (3.17), and the fact that λ is absolutely continuous imply

$$\lambda^*(K, \eta) = \lambda(\boldsymbol{\alpha}_K^*(\eta)) = 0. \qquad \square$$

Taking λ to be spherical Lebesgue measure in Lemma 3.5 and using definition (3.9) give the following:

COROLLARY 3.6. *The integral curvature $C_0(K, \cdot)$ of K is absolutely continuous with respect to the surface area measure $S(K^*, \cdot)$ of the polar body K^* of K.*

The following lemma shows that an absolutely continuous Borel measure λ that is positive on nonempty open subsets of S^{n-1} and its Gauss image measure $\lambda(K, \cdot)$ are always Aleksandrov related. As will be seen, this turns out to be a critical property.

LEMMA 3.7. *Suppose λ is an absolutely continuous Borel measure that is strictly positive on nonempty open subsets of S^{n-1}. If $K \in \mathcal{K}_o^n$, then the Gauss image measure $\lambda(K, \cdot)$ satisfies*

$$\lambda(K, \omega) < \lambda(S^{n-1} \setminus \omega^*) \tag{3.22}$$

for each spherically convex set $\omega \subset S^{n-1}$.

PROOF. Lemma 3.2 tells us that $\boldsymbol{\alpha}_K(\omega) \subset S^{n-1} \setminus \omega^*$ for each convex set $\omega \subset S^{n-1}$, and that $(S^{n-1} \setminus \omega^*) \setminus \boldsymbol{\alpha}_K(\omega)$ has interior points. Thus,

$$\lambda(\boldsymbol{\alpha}_K(\omega)) \leq \lambda(S^{n-1} \setminus \omega^*),$$

and since λ is strictly positive on open sets, we also know that

$$\lambda((S^{n-1} \setminus \omega^*) \setminus \boldsymbol{\alpha}_K(\omega)) > 0.$$

Thus,

$$\lambda(\boldsymbol{\alpha}_K(\omega)) < \lambda(S^{n-1} \setminus \omega^*).$$

This and (3.16), the definition of the Gauss image measure, $\lambda(K, \cdot)$, immediately yield (3.22). □

The following lemma establishes uniqueness, up to dilation, for the Gauss image measure. The proof below is in the spirit of Aleksandrov's proof for the case of integral curvature.

We shall use the fact that if the convex bodies K and L have parallel support hyperplanes at the points $r_K(u)$ and $r_L(u)$ whenever both points are regular, then K and L are dilates (of one another).

LEMMA 3.8. *Suppose λ is an absolutely continuous Borel measure on S^{n-1} that is strictly positive on open sets. If $K, L \in \mathcal{K}_o^n$ are such that $\lambda(K,\cdot) = \lambda(L,\cdot)$, then K and L are dilates* (*of one another*).

PROOF. We will show that K and L have parallel support hyperplanes at points $r_K(u)$ and $r_L(u)$ that are regular. Assume that there exists a $u_0 \in S^{n-1}$ so that $r_K(u_0)$ and $r_L(u_0)$ are regular and the support hyperplane of K at $r_K(u_0)$ and the support hyperplane of L at $r_L(u_0)$ are not parallel; i.e., $\alpha_K(u_0) \neq \alpha_L(u_0)$. Let $c > 0$ be such that $c\, r_K(u_0) = r_L(u_0)$, and let $K' = cK$. Define the regular point $x_0 = r_{K'}(u_0) = r_L(u_0)$.

Define the disjoint decomposition $S^{n-1} = \omega' \cup \omega \cup \omega_0$ by letting

$$\omega' = \{u \in S^{n-1} : \rho_{K'}(u) > \rho_L(u)\},$$
$$\omega = \{u \in S^{n-1} : \rho_{K'}(u) < \rho_L(u)\},$$
$$\omega_0 = \{u \in S^{n-1} : \rho_{K'}(u) = \rho_L(u)\}.$$

Suppose $u \in \omega'$ and ξ_L is a support hyperplane of L at $r_L(u)$. Obviously, $r_{K'}(\omega')$ is not completely contained in the half-space containing L that is generated by ξ_L. Thus, there is a support hyperplane $\xi_{K'}$ of K' at some point of $r_{K'}(\omega')$ that is parallel to ξ_L. This implies that

$$\boldsymbol{\alpha}_L(\omega') \subset \boldsymbol{\alpha}_{K'}(\omega') = \boldsymbol{\alpha}_K(\omega'), \tag{3.23}$$

from which follows

$$\lambda(L, \omega') \leq \lambda(K, \omega'). \tag{3.24}$$

To obtain the contradiction, we shall show that the inequality (3.24) is strict.

The continuity of the radial function and the definitions of ω and ω' show that the sets $\omega \cup \omega_0$ and $\omega' \cup \omega_0$ are closed, and thus by Lemma 3.1 the Gauss images $\boldsymbol{\alpha}_{K'}(\omega \cup \omega_0)$ and $\boldsymbol{\alpha}_L(\omega' \cup \omega_0)$ are closed as well. Thus $S^{n-1} \setminus \boldsymbol{\alpha}_{K'}(\omega \cup \omega_0)$ and $S^{n-1} \setminus \boldsymbol{\alpha}_L(\omega' \cup \omega_0)$ are open. Observe that, from the definitions of ω, ω_0, and ω' and the definition of the Gauss image, we have

$$S^{n-1} \setminus \boldsymbol{\alpha}_{K'}(\omega \cup \omega_0) \subset \boldsymbol{\alpha}_{K'}(\omega') \tag{3.25}$$

and

$$(S^{n-1} \setminus \boldsymbol{\alpha}_L(\omega' \cup \omega_0)) \cap \boldsymbol{\alpha}_L(\omega') = \varnothing. \tag{3.26}$$

Let

$$\beta = (S^{n-1} \setminus \boldsymbol{\alpha}_{K'}(\omega \cup \omega_0)) \cap (S^{n-1} \setminus \boldsymbol{\alpha}_L(\omega' \cup \omega_0)).$$

Then β is an open set, and from (3.26) and (3.25) we obviously have

$$\beta \cap \boldsymbol{\alpha}_L(\omega') = \varnothing \quad \text{and} \quad \beta \subset \boldsymbol{\alpha}_{K'}(\omega'). \tag{3.27}$$

Let ξ_0' be the support hyperplane of K' at the regular point $x_0 = r_{K'}(u_0) \in \partial K'$ with outer unit normal $\alpha_K(u_0)$, and let ξ_0 be the support hyperplane of L at the regular point $x_0 = r_L(u_0) \in \partial L$ with outer unit normal $\alpha_L(u_0)$. Recall that we assumed that the point u_0 is such that $\xi_0 \neq \xi_0'$. Note that $\alpha_K(u_0)$ and $\alpha_L(u_0)$ cannot be opposite of each other, since both K and L contain the origin in the interior.

Consider the hyperplane P that is orthogonal to

$$v_1 = (\alpha_K(u_0) + \alpha_L(u_0))/|\alpha_K(u_0) + \alpha_L(u_0)|$$

and passes through the point x_0. Note that $v_1 \cdot \alpha_K(u_0) > 0$ and $v_1 \cdot \alpha_L(u_0) > 0$.

Let P_+ be the half-space defined by

$$P_+ = \{x \in \mathbb{R}^n : x \cdot v_1 > x_0 \cdot v_1\}.$$

Since x_0 is a regular point for both K' and L, the intersections $P_+ \cap K'$ and $P_+ \cap L$ must be nonempty.

Observe that if $r_{K'}(u) \in H_{K'}(v_1)$ then $u \in \omega'$. To see this, note that

$$r_{K'}(u) = x' + cv_1 \tag{3.28}$$

for some $x' \in P$ and $c > 0$. By definition of support function,

$$\begin{aligned} x_0 \cdot \alpha_{K'}(u_0) = h_{K'}(\alpha_{K'}(u_0)) &\geq r_{K'}(u) \cdot \alpha_{K'}(u_0) \\ &= x' \cdot \alpha_{K'}(u_0) + cv_1 \cdot \alpha_{K'}(u_0). \end{aligned}$$

Since $v_1 \cdot \alpha_{K'}(u_0) > 0$, we have $x' \cdot \alpha_{K'}(u_0) < x_0 \cdot \alpha_{K'}(u_0)$. This, combined with the fact that $x' \cdot v_1 = x_0 \cdot v_1$ (since $x', x_0 \in P$) and the definition of v_1, implies

$$x' \cdot \alpha_L(u_0) > x_0 \cdot \alpha_L(u_0). \tag{3.29}$$

By (3.28), (3.29), and the fact that $v_1 \cdot \alpha_L(u_0) > 0$,

$$r_{K'}(u) \cdot \alpha_L(u_0) = x' \cdot \alpha_L(u_0) + cv_1 \cdot \alpha_L(u_0) > x_0 \cdot \alpha_L(u_0) = h_L(\alpha_L(u_0)).$$

This implies that $r_{K'}(u) \notin L$, which in turn gives $\rho_{K'}(u) > \rho_L(u)$ or $u \in \omega'$. This implies that $v_1 \notin \alpha_{K'}(\omega \cup \omega_0)$.

The same argument gives $v_1 \notin \alpha_L(\omega' \cup \omega_0)$. Hence, $v_1 \in \beta$. Therefore, β is a nonempty open set. Since λ is by hypothesis positive on open sets, $\lambda(\beta) > 0$.

From (3.23) and (3.27),

$$\boldsymbol{\alpha}_L(\omega') = \boldsymbol{\alpha}_L(\omega') \setminus \beta \ \subset \ \boldsymbol{\alpha}_{K'}(\omega') \setminus \beta. \tag{3.30}$$

Thus, (3.30) and $\lambda(\beta) > 0$, give

$$\begin{aligned} \lambda(L, \omega') = \lambda(\boldsymbol{\alpha}_L(\omega')) &\leq \lambda(\boldsymbol{\alpha}_{K'}(\omega') \setminus \beta) \\ &< \lambda(\boldsymbol{\alpha}_{K'}(\omega') \setminus \beta) + \lambda(\beta) = \lambda(\boldsymbol{\alpha}_{K'}(\omega')) = \lambda(K, \omega'), \end{aligned}$$

which contradicts $\lambda(L, \cdot) = \lambda(K, \cdot)$. □

It is easily seen that the integral curvature of a convex body is not concentrated in any closed hemisphere, and the total measure of the integral curvature of a convex body is the surface area of the unit sphere. Then it is natural to find a complete set of properties that characterize the integral curvature. The following result shows that, when λ is an absolutely continuous Borel measure, then the Gauss image measure as a functional from the space $\mathcal{K}_o^n$ of convex bodies to the space of Borel measures is a valuation. The theory of valuations has seen explosive growth in the last quarter century (see, e.g., [3–5, 15, 23, 24, 33–36, 50–53], and the references therein). It would be interesting to characterize this valuation.

PROPOSITION 3.9. *If λ is an absolutely continuous Borel measure on S^{n-1}, then the Gauss image measure of λ is a valuation*; *i.e., for $K, L \in \mathcal{K}_o^n$,*

$$\lambda(K,\cdot) + \lambda(L,\cdot) = \lambda(K \cup L,\cdot) + \lambda(K \cap L,\cdot),$$

whenever $K \cup L \in \mathcal{K}_o^n$.

PROOF. Since r_K and r_L are bijections between S^{n-1} and ∂K and ∂L, respectively, we have the following disjoint partition of $S^{n-1} = \Omega_0 \cup \Omega_L \cup \Omega_K$, where

$$\Omega_0 = r_K^{-1}(\partial K \cap \partial L) = r_L^{-1}(\partial K \cap \partial L) = \{u \in S^{n-1} : \rho_K(u) = \rho_L(u)\},$$
$$\Omega_L = r_K^{-1}(\partial K \cap \mathrm{int} L) = r_L^{-1}((\mathbb{R}^n \setminus K) \cap \partial L) = \{u \in S^{n-1} : \rho_K(u) < \rho_L(u)\},$$
$$\Omega_K = r_K^{-1}(\partial K \cap (\mathbb{R}^n \setminus L)) = r_L^{-1}(\mathrm{int} K \cap \partial L) = \{u \in S^{n-1} : \rho_K(u) > \rho_L(u)\}.$$

Since $K \cup L$ is a convex body, we have, for $\mathcal{H}^{n-1}$-almost all $u \in \Omega_K$,

$$\alpha_K(u) = \alpha_{K\cup L}(u) \quad \text{and} \quad \alpha_L(u) = \alpha_{K\cap L}(u);$$

for $\mathcal{H}^{n-1}$-almost all $u \in \Omega_L$,

$$\alpha_K(u) = \alpha_{K\cap L}(u) \quad \text{and} \quad \alpha_L(u) = \alpha_{K\cup L}(u);$$

and for $\mathcal{H}^{n-1}$-almost all $u \in \Omega_0$,

$$\alpha_K(u) = \alpha_L(u) = \alpha_{K\cap L}(u) = \alpha_{K\cup L}(u).$$

Since λ is absolutely continuous, for a Borel set $\omega \subset S^{n-1}$, we have

$$\begin{aligned}\lambda(K, \omega \cap \Omega_K) = \lambda(\boldsymbol{\alpha}_K(\omega \cap \Omega_K)) &= \lambda(\boldsymbol{\alpha}_{K\cup L}(\omega \cap \Omega_K)) \\ &= \lambda(K \cup L, \omega \cap \Omega_K),\end{aligned}$$

and also

$$\lambda(L, \omega \cap \Omega_K) = \lambda(K \cap L, \omega \cap \Omega_K).$$

Adding the last two, we obtain

$$\lambda(K, \omega \cap \Omega_K) + \lambda(L, \omega \cap \Omega_K) = \lambda(K \cup L, \omega \cap \Omega_K) + \lambda(K \cap L, \omega \cap \Omega_K).$$

Similarly, we have

$$\lambda(K, \omega \cap \Omega_L) + \lambda(L, \omega \cap \Omega_L) = \lambda(K \cup L, \omega \cap \Omega_L) + \lambda(K \cap L, \omega \cap \Omega_L),$$
$$\lambda(K, \omega \cap \Omega_0) + \lambda(L, \omega \cap \Omega_0) = \lambda(K \cup L, \omega \cap \Omega_0) + \lambda(K \cap L, \omega \cap \Omega_0).$$

Summing up the last three gives the desired valuation property. □

4 Variational Formulas for the Log-Volumes of Convex Bodies

Let λ be a Borel measure on S^{n-1}. The *log-volume* $\lambda_0(K)$ *of a convex body* $K \in \mathcal{K}_o^n$ *with respect to* λ is defined by

$$\lambda_0(K) = \exp\left\{\frac{1}{|\lambda|}\int_{S^{n-1}} \log \rho_K(v) d\lambda(v)\right\}. \tag{4.1}$$

We require the following lemma established in [27].

LEMMA 4.1. *Suppose* $\Omega \subset S^{n-1}$ *is a closed set that is not contained in any closed hemisphere of* S^{n-1}. *Suppose* $\rho_0 \colon \Omega \to (0,\infty)$ *and* $g \colon \Omega \to \mathbb{R}$ *are continuous. If* $\langle \rho_t \rangle$ *is a logarithmic family of convex hulls of* (ρ_0, g), *then*

$$\lim_{t\to 0} \frac{\log h_{\langle \rho_t\rangle}(v) - \log h_{\langle \rho_0\rangle}(v)}{t} = g(\alpha^*_{\langle \rho_0\rangle}(v)) \tag{4.2}$$

for all $v \in S^{n-1} \setminus \eta_{\langle \rho_0\rangle}$; *i.e., for all regular normals* v *of* $\langle \rho_0 \rangle$. *Hence* (4.2) *holds a.e. with respect to spherical Lebesgue measure. Moreover, there exists* $\delta_0 > 0$ *and* $M > 0$ *so that*

$$|\log h_{\langle \rho_t\rangle}(v) - \log h_{\langle \rho_0\rangle}(v)| \le M|t|$$

for all $v \in S^{n-1}$ *and all* $t \in (-\delta_0, \delta_0)$.

We require the following lemma. When the measure is spherical Lebesgue measure, it was established in [27].

LEMMA 4.2. *Suppose* λ *is an absolutely continuous Borel measure on* S^{n-1}; *the body* $K \in \mathcal{K}_o^n$ *and* $f, g \colon S^{n-1} \to \mathbb{R}$ *are continuous. If* $\langle K, g\rangle$ *is a logarithmic family of convex hulls generated by* (K, g), *then*

$$\frac{d}{dt}\log \lambda_0(\langle K, g, t\rangle^*)\Big|_{t=0} = -\frac{1}{|\lambda|}\int_{S^{n-1}} g(u) d\lambda(K,u). \tag{4.3}$$

If $[K, f]$ *is a logarithmic family of Wulff shapes formed by* (K, f), *then*

$$\frac{d}{dt}\log \lambda_0([K, f, t])\Big|_{t=0} = \frac{1}{|\lambda|}\int_{S^{n-1}} f(v) d\lambda^*(K,v). \tag{4.4}$$

PROOF. Write $\rho_t = \rho_K + tg + o(t,\cdot)$. Note that $\langle K, g, t\rangle = \langle \rho_t\rangle$. In particular, $\rho_0 = \rho_K$ and $\langle \rho_0\rangle = K$.

From Lemma 4.1, the dominated convergence theorem, and (3.19), we have

$$\begin{aligned}
\frac{d}{dt}\log \lambda_0(\langle K, g, t\rangle^*)\Big|_{t=0} &= -\lim_{t\to 0}\frac{1}{|\lambda|}\int_{S^{n-1}} \frac{\log h_{\langle \rho_t\rangle}(v) - \log h_{\langle \rho_0\rangle}(v)}{t}\, d\lambda(v) \\
&= -\frac{1}{|\lambda|}\int_{S^{n-1}} g(\alpha^*_{\langle \rho_0\rangle}(v)) d\lambda(v) \\
&= -\frac{1}{|\lambda|}\int_{S^{n-1}} g(u) d\lambda(K,u).
\end{aligned}$$

From (2.6) we know that $\langle K^*, -f\rangle^* = [K, f]$, so (4.3) gives

$$\frac{d}{dt}\log\lambda_0([K, f, t])\Big|_{t=0} = \frac{1}{|\lambda|}\int_{S^{n-1}} f(v)d\lambda(K^*, v),$$

which using (3.18) now gives the desired (4.4). □

5 Strengthening the Aleksandrov Relation

Let $O_\alpha(u)$ be the spherical cap on S^{n-1} that is centered at u and is of radius α; i.e.,

$$O_\alpha(u) = \{v \in S^{n-1} : u \cdot v > \cos\alpha\}.$$

For a nonempty compact set $\omega \subset S^{n-1}$ that is contained in some closed hemisphere, the outer parallel set ω_α, where $\alpha \in (0, \frac{\pi}{2}]$, is defined by

$$\omega_\alpha = \bigcup_{u\in\omega}\{v \in S^{n-1} : u \cdot v > \cos\alpha\}. \tag{5.1}$$

(Observe that, as defined, ω_α may not be contained in any closed hemisphere.) Obviously, ω_α is open and increasing (with respect to set inclusion) as a function of α. Also obvious is the fact that, by (1.3),

$$\omega_{\pi/2} = S^{n-1} \setminus \omega^*, \quad \text{or equivalently,} \quad \omega^* = S^{n-1} \setminus \omega_{\pi/2}. \tag{5.2}$$

From (2.5), we see that

$$\omega_{\pi/2} = \langle\omega\rangle_{\pi/2}. \tag{5.3}$$

From definition (5.1), we immediately have the following:

LEMMA 5.1. *Let $\omega_1, \ldots, \omega_k \subset S^{n-1}$ be nonempty compact sets that are contained in some closed hemisphere, and let $\alpha \in (0, \frac{\pi}{2}]$. Then*

$$\left(\bigcup_{j=1}^{k}\omega_j\right)_\alpha = \bigcup_{j=1}^{k}(\omega_j)_\alpha.$$

For a nonempty compact set ω on S^{n-1} that is contained in some closed hemisphere and $\alpha \in [0, \frac{\pi}{2})$, define

$$\omega_\alpha^- = S^{n-1} \setminus \omega_{\frac{\pi}{2}-\alpha} \tag{5.4}$$

or, equivalently,

$$\begin{aligned}\omega_\alpha^- &= \{v \in S^{n-1} : u \cdot v \le \sin\alpha \text{ for all } u \in \omega\}\\ &= \bigcap_{u\in\omega}\{v \in S^{n-1} : u \cdot v \le \sin\alpha\}.\end{aligned}$$

Obviously, the sets ω_α^- are compact and increasing (with respect to set inclusion) as a function of α. Also obvious is the fact that

$$\omega_0^- = \omega^* \quad \text{and thus} \quad \omega^* \subset \omega_\alpha^-. \tag{5.5}$$

LEMMA 5.2. *Suppose $\omega_i \subset S^{n-1}$ is a sequence of nonempty compact sets, each contained in some closed hemisphere such that $\omega_i \to \omega$, and $\alpha_i \in (0, \frac{\pi}{2})$ is a sequence such that $\alpha_i \to 0$. Then, if*

$$\eta_j = \bigcup_{i=j}^{\infty} (\omega_i)^-_{\alpha_i},$$

it follows that

$$\omega^* = \bigcap_{j=1}^{\infty} (\omega^* \cup \eta_j).$$

PROOF. To see that $\bigcap_{j=1}^{\infty}(\omega^* \cup \eta_j) \subset \omega^*$, suppose $v \notin \omega^*$. By definition (of ω^*) there exists a $u_0 \in \omega$ such that $u_0 \cdot v = 2\varepsilon > 0$. Since $\omega_i \to \omega$, we may choose a sequence $u_i \in \omega_i$ such that $u_i \to u_0$. Hence $\lim_{i\to\infty} u_i \cdot v = u_0 \cdot v = 2\varepsilon > 0$. This and the fact that $\alpha_i \to 0$ show that there exists j_0 such that $u_i \cdot v > \varepsilon > \sin\alpha_i$ for all $i \geq j_0$. Therefore, $v \notin (\omega_i)^-_{\alpha_i}$ for all $i \geq j_0$. Therefore, $v \notin \eta_{j_0}$, and thus $v \notin \omega^* \cup \eta_{j_0}$, which show that $v \notin \bigcap_{j=1}^{\infty}(\omega^* \cup \eta_j)$, as desired.

That $\omega^* \subset \bigcap_{j=1}^{\infty}(\omega^* \cup \eta_j)$ is obvious. □

We recall the definition of being Aleksandrov related: If μ and λ are Borel measures on S^{n-1}, then the measures μ and λ are said to be *Aleksandrov related* provided

$$\lambda(S^{n-1}) = \mu(S^{n-1}) > \lambda(\omega^*) + \mu(\omega) \tag{5.6}$$

or, equivalently,

$$\lambda(S^{n-1}) = \mu(S^{n-1}) > \lambda(\omega) + \mu(\omega^*)$$

for each compact, spherically convex $\omega \subset S^{n-1}$. (Recall that each ω is required to be contained in a closed hemisphere.)

If $|\mu| = |\lambda|$, it is easily seen, from (5.2), that condition (5.6) is equivalent to

$$\mu(\omega) < \lambda(S^{n-1} \setminus \omega^*) = \lambda(\omega_{\pi/2}).$$

If the set ω is a closed hemisphere, then $S^{n-1} \setminus \omega$ is an open hemisphere and the set ω^* consists of a single point. Since $\lambda(\omega^*) \geq 0$, condition (5.6) shows that

$$\mu(S^{n-1} \setminus \omega) > 0,$$

which means that μ must be strictly positive on open hemispheres. Thus, condition (5.6) implies that μ (and hence λ) cannot be concentrated on any closed hemisphere. For quick future reference, we state this in the following:

LEMMA 5.3. *If λ and μ are Borel measures on S^{n-1} that are Aleksandrov related, then neither λ nor μ can be concentrated in any closed hemisphere of S^{n-1}.*

For convenience, we restate Lemma 3.7 in terms of being Aleksandrov related.

LEMMA 5.4. *Suppose $K \in \mathcal{K}_o^n$ and λ is an absolutely continuous Borel measure on S^{n-1} that is strictly positive on nonempty open sets. Then λ and the Gauss image measure $\lambda(K, \cdot)$ are Aleksandrov related.*

LEMMA 5.5. *Let λ be a Borel measure on S^{n-1} that vanishes on all great hyperspheres. For nonempty compact $\omega \subset S^{n-1}$ contained in a closed hemisphere,*

$$\omega \cap (-\omega) \neq \varnothing \;\Rightarrow\; \lambda(\omega_{\pi/2}) = \lambda(S^{n-1} \setminus \omega^*) = |\lambda|. \tag{5.7}$$

PROOF. If $\omega \cap (-\omega) \neq \varnothing$, then there exists $u_1 \in S^{n-1}$ so that both $u_1, -u_1 \in \omega$. Thus, for any $v \in \omega^*$, we have $u_1 \cdot v \leq 0$ and $-u_1 \cdot v \leq 0$. Thus, ω^* is contained in the great hypersphere orthogonal to u_1. Since λ vanishes on great hyperspheres, we have $\lambda(\omega^*) = 0$. This and (5.2) give (5.7). □

The next lemma tells us that, under mild assumptions, even measures are Aleksandrov related.

LEMMA 5.6. *Let μ be an even Borel measure on S^{n-1} that is not concentrated on any great hypersphere, and let λ be an even Borel measure that vanishes on great hyperspheres and is strictly positive on nonempty open sets. If $|\mu| = |\lambda|$, then μ and λ are Aleksandrov related.*

PROOF. First, assume that ω is strongly spherically convex on S^{n-1}; that is, ω is contained in an open hemisphere Ω_0. The spherically convex set ω^* is contained in a closed hemisphere

Since we are given that μ and λ are even and that $\lambda(\partial(S^{n-1} \setminus \Omega_0')) = 0$, we have

$$\mu(\omega) \leq \mu(\Omega_0) \leq \frac{1}{2}|\mu| = \frac{1}{2}|\lambda| = \lambda(S^{n-1} \setminus \Omega_0') \leq \lambda(S^{n-1} \setminus \omega^*). \tag{5.8}$$

If ω contains only one point, then $\mu(\omega) < \frac{1}{2}|\mu|$ because μ is not concentrated on a pair of antipodal points. If ω contains at least two points, then ω^* is contained in the intersection of two closed hemispheres, and thus $(S^{n-1} \setminus \omega^*) \setminus (S^{n-1} \setminus \Omega_0')$ contains nonempty open sets. Since λ is strictly positive on open sets,

$$\lambda(S^{n-1} \setminus \Omega_0') < \lambda(S^{n-1} \setminus \omega^*).$$

We have just shown that equality in both of the inequalities $\mu(\Omega_0) \leq \frac{1}{2}|\mu|$ and $\lambda(S^{n-1} \setminus \Omega_0') \leq \lambda(S^{n-1} \setminus \omega^*)$ cannot hold simultaneously in (5.8). Thus, from (5.8) we get

$$\mu(\omega) < \lambda(S^{n-1} \setminus \omega^*),$$

as desired.

Suppose ω is not strongly convex; then $\omega \cap (-\omega) \neq \varnothing$. From Lemma 5.5, we know that

$$\lambda(S^{n-1} \setminus \omega^*) = |\lambda|.$$

Since $|\mu| = |\lambda|$, to show that μ and λ are Aleksandrov related, we need to show that $\mu(\omega) < |\mu|$. Suppose this were not the case; i.e., $\mu(\omega) = |\mu|$. Thus,

$$|\mu| = \mu(\omega \cup (-\omega)) = \mu(\omega) + \mu(-\omega) - \mu(\omega \cap (-\omega)).$$

Since μ is even, it follows that $|\mu| = \mu(\omega \cap (-\omega))$. The fact that ω is spherically convex tells us that $\omega \cap (-\omega)$ is contained in a great hypersphere, and hence μ is concentrated on a great hypersphere. This provides the desired contradiction. □

Let $\omega \subset S^{n-1}$. Obviously,

$$\widehat{\omega} \cap S^{n-1} = \omega, \tag{5.9}$$

and $u \in \omega$ if and only if there exists $t \in (0, 1]$ such that $tu \in \widehat{\omega}$. In particular,

$$\mathbb{1}_{\widehat{\omega}}(u) = \mathbb{1}_{\widehat{\omega}}(\tfrac{1}{2}u), \tag{5.10}$$

for each $u \in S^{n-1}$. We shall make use of the fact that a proper subset $\omega \subset S^{n-1}$ is spherically convex if and only if $\widehat{\omega}$ is convex in $\mathbb{R}^n$. Let $\{\omega_i\}$ be a sequence of spherically convex sets in S^{n-1}. Recall that ω_i converges to a spherically convex set $\omega \subset S^{n-1}$ in the Hausdorff metric if $\widehat{\omega_i}$ converges to $\widehat{\omega}$ in the Hausdorff metric in $\mathbb{R}^n$.

We shall need the following trivial facts.

LEMMA 5.7. *If K_i is a sequence of compact convex sets in $\mathbb{R}^n$ that converges to a compact convex set $K \subset \mathbb{R}^n$ in the Hausdorff metric, then*

$$\lim_{i\to\infty} \mathbb{1}_{K_i}(x) = 0 \quad \text{if } x \notin K. \tag{5.11}$$

Moreover, if int K *is not empty, then*

$$\lim_{i\to\infty} \mathbb{1}_{K_i}(y) = 1 \quad \text{if } y \in \operatorname{int} K. \tag{5.12}$$

PROOF. Consider a fixed $x \notin K$. Since K is compact, we know the Hausdorff distance $d(x, K) > 0$. Since K_i converges to K in the Hausdorff metric, $d(x, K_i) > 0$ for sufficiently large i, and thus $x \notin K_i$ for sufficiently large i. From this (5.11) follows.

Assume int K is not empty. Consider a fixed $y_0 \in \operatorname{int} K$. Let δ be such that $B_\delta(y_0) \subset K$, where $B_\delta(y_0)$ is the closed ball of radius $\delta > 0$ centered at y_0. Then $y_0 + B_\delta(o) \subset K$. Thus,

$$y_0 \cdot v \le h_K(v) - \delta$$

for all $v \in S^{n-1}$. We are given that $h_{K_i} \to h_K$, uniformly on S^{n-1}. Thus, there exists $i_0 > 0$ such that

$$y_0 \cdot v < h_{K_i}(v) - \frac{\delta}{2}$$

for all $v \in S^{n-1}$ and for all $i > i_0$. This shows that $y_0 \in K_i$ for $i > i_0$, which gives (5.12). □

The following lemma concerns the continuity of finite Borel measures on S^{n-1} when regarded as defined on spherical compact convex sets endowed with the Hausdorff metric. For $\omega \subset S^{n-1}$, we write $\widetilde{\partial}\omega$ to denote the boundary of the set ω viewed as a subset of S^{n-1}.

LEMMA 5.8. *Let λ be a Borel measure on S^{n-1}, and ω_i be a sequence of compact, spherically convex sets in S^{n-1} that converges to the compact, spherically convex set ω in the Hausdorff metric. If λ vanishes on the boundary of ω, then*

$$\lim_{i\to\infty} \lambda(\omega_i) = \lambda(\omega).$$

PROOF. From the definition of spherical convex set, the sets $\widehat{\omega}$ and $\widehat{\omega_i}$ are nonempty compact convex sets, and since ω_i converges to ω in the Hausdorff metric, it follows that $\widehat{\omega_i}$ converges to $\widehat{\omega}$ in the Hausdorff metric. We claim that

$$\lim_{i\to\infty} \mathbb{1}_{\omega_i}(u) = \begin{cases} 1, & u \in \omega \setminus \widetilde{\partial}\omega, \\ 0, & u \notin \omega. \end{cases} \tag{5.13}$$

First, assume $u \notin \omega$. Then $u \notin \widehat{\omega}$. Since $\widehat{\omega_i}$ converges to $\widehat{\omega}$, from Lemma 5.7 we have

$$\lim_{i\to\infty} \mathbb{1}_{\widehat{\omega_i}}(u) = 0. \tag{5.14}$$

From (5.9) we know that $\mathbb{1}_{\widehat{\omega_i}}(u) = \mathbb{1}_{\omega_i}(u)$ for all $u \in S^{n-1}$. Hence (5.14) can be rewritten as

$$\lim_{i\to\infty} \mathbb{1}_{\omega_i}(u) = 0 \tag{5.15}$$

when $u \notin \omega$.

Suppose $u \in \omega \setminus \widetilde{\partial}\omega$. (If $\omega \setminus \widetilde{\partial}\omega = \varnothing$, then (5.13) hold by vacuous implication.) From the definition of $\widehat{\omega}$, we can conclude that $\frac{1}{2}u \in \operatorname{int}\widehat{\omega}$. Since $\widehat{\omega_i}$ converges to $\widehat{\omega}$, from Lemma 5.7 we have

$$\lim_{i\to\infty} \mathbb{1}_{\widehat{\omega_i}}\left(\tfrac{1}{2}u\right) = 1. \tag{5.16}$$

Now (5.9), (5.10), and (5.16) imply

$$\lim_{i\to\infty} \mathbb{1}_{\omega_i}(u) = \lim_{i\to\infty} \mathbb{1}_{\widehat{\omega_i}}(u) = \lim_{i\to\infty} \mathbb{1}_{\widehat{\omega_i}}\left(\tfrac{1}{2}u\right) = 1.$$

This and (5.15) yield (5.13).

By assumption, λ vanishes on the boundary of ω; i.e., $\lambda(\widetilde{\partial}\omega) = 0$. This and (5.13) give us

$$\lim_{i\to\infty} \mathbb{1}_{\omega_i}(u) = \mathbb{1}_{\omega}(u) \tag{5.17}$$

almost everywhere with respect to λ. Since λ is finite, it follows by the dominated convergence theorem and (5.17) that

$$\lim_{i\to\infty} \lambda(\omega_i) = \lim_{i\to\infty} \int_{S^{n-1}} \mathbb{1}_{\omega_i}(u) d\lambda(u) = \int_{S^{n-1}} \mathbb{1}_{\omega}(u) d\lambda(u) = \lambda(\omega),$$

which is the desired result. □

The following lemma establishes uniform continuity of $\omega \mapsto \lambda(\omega_\alpha)$ at $\alpha = \frac{\pi}{2}$.

LEMMA 5.9. *Let λ be a Borel measure on S^{n-1} that vanishes on the boundary of all compact, spherically convex sets. Then, given $\varepsilon > 0$, there exists $\alpha \in \left(0, \frac{\pi}{2}\right)$ so that*

$$\lambda(\omega_{\frac{\pi}{2}}) - \lambda(\omega_{\frac{\pi}{2}-\alpha}) < \varepsilon, \tag{5.18}$$

for each nonempty compact set $\omega \subset S^{n-1}$ contained in some closed hemisphere.

PROOF. Assume that (5.18) does not hold. Then there exists $\varepsilon_0 > 0$, a sequence $\alpha_i \in (0, \frac{\pi}{2})$ converging to 0, and a sequence $\omega_i \subset S^{n-1}$ of nonempty compact sets contained in closed hemispheres so that

$$\lambda(\omega_{i,\frac{\pi}{2}}) - \lambda(\omega_{i,\frac{\pi}{2}-\alpha_i}) \geq \varepsilon_0 \tag{5.19}$$

for all i, where $\omega_{i,\alpha}$ is used to abbreviate $(\omega_i)_\alpha$. Since the set of compact subsets of S^{n-1} is compact when endowed with the topology of the Hausdorff metric, ω_i has a convergent subsequence, which we again denote by ω_i, that converges to a nonempty compact set ω in the Hausdorff metric. But $\omega_i \to \omega$ implies $\langle \omega_i \rangle \to \langle \omega \rangle$, which, together with (2.5) shows that $\omega_i^* \to \omega^*$.

From Lemma 5.8 and the fact that polar sets are spherically convex, it follows that

$$\lim_{i\to\infty} \lambda(\omega_i^*) = \lambda(\omega^*),$$

or via (5.2),

$$\lim_{i\to\infty} \lambda(S^{n-1} \setminus \omega_{i,\frac{\pi}{2}}) = \lim_{i\to\infty} \lambda(\omega_i^*) = \lambda(\omega^*). \tag{5.20}$$

Next we will show that

$$\lim_{i\to\infty} \lambda(S^{n-1} \setminus \omega_{i,\frac{\pi}{2}-\alpha_i}) = \lambda(\omega^*). \tag{5.21}$$

Since $\omega_i \to \omega$ and $\alpha_i \in (0, \frac{\pi}{2})$ is such that $\alpha_i \to 0$, Lemma 5.2 states precisely that

$$\omega^* = \bigcap_{j=1}^{\infty} (\omega^* \cup \eta_j) \quad \text{where } \eta_j = \bigcup_{i=j}^{\infty} \omega_{i,\alpha_i}^-, \tag{5.22}$$

where ω_{i,α_i}^- is used to abbreviate $(\omega_i)_{\alpha_i}^-$. Since η_j is a decreasing sequence, we have

$$\lambda\left(\bigcap_{j=1}^{\infty} (\omega^* \cup \eta_j) \right) = \lim_{j\to\infty} \lambda(\omega^* \cup \eta_j). \tag{5.23}$$

From (5.20), the fact that $\omega_j^* \subset \omega_{j,\alpha_j}^-$ from (5.5), the fact that $\omega_{j,\alpha_j}^- \subset \eta_j$ from the definition of η_j, (5.23), and (5.22), we have

$$\begin{aligned}\lambda(\omega^*) = \lim_{j\to\infty} \lambda(\omega_j^*) \le \lim_{j\to\infty} \lambda(\omega_{j,\alpha_j}^-) \le \lim_{j\to\infty} \lambda(\eta_j) \le \lim_{j\to\infty} \lambda(\omega^* \cup \eta_j) \\ = \lambda\Big(\bigcap_{j=1}^{\infty} (\omega^* \cup \eta_j)\Big) \\ = \lambda(\omega^*).\end{aligned}$$

This establishes

$$\lim_{i\to\infty} \lambda(\omega_{i,\alpha_i}^-) = \lambda(\omega^*).$$

But this establishes the promised (5.21), as can be seen after recalling that ω_{i,α_i}^- is defined in (5.4), to be $S^{n-1} \setminus \omega_{i,\frac{\pi}{2}-\alpha_i}$.

Together, (5.20) and (5.21), give

$$\lambda(\omega_{i,\frac{\pi}{2}}) - \lambda(\omega_{i,\frac{\pi}{2}-\alpha_i}) \;\to\; 0 \quad \text{as } i \to \infty.$$

This contradicts (5.19), and thus establishes (5.18). □

For $u \in S^{n-1}$ and $\varepsilon \in [0, \frac{\pi}{2}]$, let $\Omega_\varepsilon(u)$ denote the closed spherical cap of radius $\frac{\pi}{2} - \varepsilon$ centered at u. The open spherical cap of radius $\frac{\pi}{2} - \varepsilon$ centered at u will be denoted by $\Omega'_\varepsilon(u)$.

LEMMA 5.10. *Let λ be a Borel measure that is not concentrated in any closed hemisphere of S^{n-1}. Then there exist a real $c_0 > 0$ and a real $\varepsilon_0 > 0$ such that*

$$\lambda(\Omega_\varepsilon(u)) > c_0$$

for all $u \in S^{n-1}$ and for all $0 \le \varepsilon < \varepsilon_0$.

PROOF. It is sufficient to prove that there exist $c_0, \varepsilon_0 > 0$ such that

$$\lambda(\Omega_{\varepsilon_0}(u)) > c_0 \tag{5.24}$$

for all $u \in S^{n-1}$. To that end, suppose (5.24) does not hold. Then there exists a strictly decreasing, strictly positive sequence ε_i with limit 0, and a sequence of spherical caps $\Omega_{\varepsilon_i}(u_i)$, with $u_i \in S^{n-1}$, so that $\lambda(\Omega_{\varepsilon_i}(u_i)) \to 0$. A standard compactness argument allows us to conclude that $\Omega_{\varepsilon_i}(u_i)$ has a convergent subsequence, which we again denote by $\Omega_{\varepsilon_i}(u_i)$, that converges to a closed hemisphere $\Omega_0(u_0)$ in the Hausdorff metric. Let δ_j be a strictly decreasing, strictly positive sequence with limit 0. Since $\Omega_{\varepsilon_i}(u_i) \to \Omega_0(u_0)$, for every δ_j there is an $\varepsilon_{i_j} > 0$ so that $\Omega'_{\delta_j}(u_0) \subset \Omega_{\varepsilon_{i_j}}(u_{i_j})$. Then

$$0 \le \lambda(\Omega'_{\delta_j}(u_0)) \le \lambda(\Omega_{\varepsilon_{i_j}}(u_{i_j})) \to 0 \quad \text{as } j \to \infty.$$

Note that $\Omega'_0(u_0) = \bigcup_j \Omega'_{\delta_j}(u_0)$, and that the sequence $\Omega'_{\delta_k}(u_0)$ is increasing, with respect to set inclusion, as $\delta_k \downarrow 0$. Thus,

$$\lambda(\Omega'_0(u_0)) = \lim_{j\to\infty} \lambda(\Omega'_{\delta_j}(u_0)) = 0.$$

Thus, λ is concentrated in the closed hemisphere $S^{n-1} \setminus \Omega_0'(u_0)$, in contradiction to the hypothesis of the lemma. □

LEMMA 5.11. *Suppose μ and λ are Borel measures on S^{n-1} such that λ vanishes on the boundary of all compact, spherically convex sets. If μ and λ are Aleksandrov related, then there exist a $\delta = \delta(\mu, \lambda) \in (0, 1)$ and an $\alpha = \alpha(\mu, \lambda) \in (0, 1)$ such that*

$$\mu(\omega) < (1 - \delta)\lambda(\omega_{\frac{\pi}{2}-\alpha}) \tag{5.25}$$

for every nonempty compact set $\omega \subset S^{n-1}$ contained in some closed hemisphere.

PROOF. We first show that it is sufficient to demonstrate that there exists $\widetilde{\delta} \in (0, 1)$ so that

$$\mu(\omega) < (1 - \widetilde{\delta})\lambda(\omega_{\frac{\pi}{2}}) \tag{5.26}$$

for each compact, spherically convex set $\omega \subset S^{n-1}$.

From Lemma 5.10, we know that there exists a $c > 0$ so that

$$\lambda(\Omega) > c \tag{5.27}$$

for each closed hemisphere Ω. From definition (5.1), we see that $\omega_{\pi/2}$ always contains an open hemisphere. Since λ vanishes on all great hyperspheres, from (5.27), we see that

$$\lambda(\omega_{\frac{\pi}{2}}) > c. \tag{5.28}$$

Let $0 < \varepsilon < \frac{c}{2} \min\{1, \widetilde{\delta}/(1 - \widetilde{\delta})\}$ and $\delta = \widetilde{\delta} - \varepsilon(1 - \widetilde{\delta})\frac{2}{c} > 0$. Then, since obviously

$$(\widetilde{\delta} - \delta)\frac{c}{2} = (1 - \widetilde{\delta})\varepsilon, \tag{5.29}$$

Lemma 5.9 guarantees the existence of an $\alpha \in (0, \frac{\pi}{2})$ such that

$$\lambda(\omega_{\frac{\pi}{2}}) - \lambda(\omega_{\frac{\pi}{2}-\alpha}) < \varepsilon \tag{5.30}$$

for each nonempty compact set $\omega \subset S^{n-1}$ contained in some closed hemisphere. From (5.28), (5.30), and that $0 < \varepsilon < \frac{c}{2}$,

$$\lambda(\omega_{\frac{\pi}{2}-\alpha}) > \frac{c}{2}.$$

Rewriting this using (5.29) gives

$$(\widetilde{\delta} - \delta)\lambda\big(\omega_{\frac{\pi}{2}-\alpha}\big) > (1 - \widetilde{\delta})\varepsilon. \tag{5.31}$$

From (5.26), (5.3), (5.30), and (5.31), we have

$$\begin{aligned}\mu(\omega) \le \mu(\langle\omega\rangle) < (1 - \widetilde{\delta})\lambda(\langle\omega\rangle_{\frac{\pi}{2}}) &= (1 - \widetilde{\delta})\lambda(\omega_{\frac{\pi}{2}}) \\ &< (1 - \widetilde{\delta})(\varepsilon + \lambda(\omega_{\frac{\pi}{2}-\alpha})) <\end{aligned}$$

$$< (\widetilde{\delta} - \delta)\lambda(\omega_{\frac{\pi}{2}-\alpha}) + (1 - \widetilde{\delta})\lambda(\omega_{\frac{\pi}{2}-\alpha})$$
$$= (1 - \delta)\lambda(\omega_{\frac{\pi}{2}-\alpha}).$$

This shows (5.25) would follow from (5.26). It now remains to establish (5.26).

Suppose (5.26) does not hold. Since λ and μ are Aleksandrov related, there exists a sequence of compact, spherically convex $\omega_i \subset S^{n-1}$ such that

$$\lim_{i\to\infty} \frac{\lambda(\omega_{i,\frac{\pi}{2}})}{\mu(\omega_i)} = 1. \tag{5.32}$$

A standard compactness argument allows us to assume that

$$\omega_i \text{ converge to a compact, spherically convex set } \omega \subset S^{n-1}. \tag{5.33}$$

Then ω_i^* converges to ω^*. Using Lemma 5.8, we see that

$$\lim_{i\to\infty} \lambda(\omega_i) = \lambda(\omega) \quad \text{and} \quad \lim_{i\to\infty} \lambda(\omega_i^*) = \lambda(\omega^*).$$

The second of these together with (5.2) shows that

$$\lim_{i\to\infty} \lambda(\omega_{i,\frac{\pi}{2}}) = \lambda(\omega_{\frac{\pi}{2}}). \tag{5.34}$$

The spherical convex set ω can either satisfy $\omega \cap (-\omega) \neq \varnothing$ or be strongly spherically convex. We shall show that in both possible cases we are led to a contradiction.

First, suppose $\omega \cap (-\omega) \neq \varnothing$. Then (5.2) and Lemma 5.5 give

$$\lambda(S^{n-1} \setminus \omega^*) = \lambda(\omega_{\frac{\pi}{2}}) = \lambda(S^{n-1}). \tag{5.35}$$

This gives $\lambda(\omega^*) = 0$, and the hypothesis that μ and λ are Aleksandrov related gives $\mu(\omega) < \lambda(S^{n-1})$.

Since ω_α is a monotone nonincreasing sequence of Borel sets with a limit of ω as α decreases to 0 and μ is a finite Borel measure, we know that $\lim_{\alpha\to 0+} \mu(\omega_\alpha) = \mu(\omega)$. Now $\mu(\omega) < \lambda(S^{n-1})$ yields the existence of an $\alpha_0 > 0$ such that $\mu(\omega_{\alpha_0}) < \lambda(S^{n-1})$. Since $\omega_i \to \omega$ and ω_{α_0} is an open set containing ω, there exists an i_0 such that $\omega_i \subset \omega_{\alpha_0}$ for all $i \geq i_0$, and hence from (5.35), (5.34), and (5.32) coupled with (5.28), we have

$$\lambda(S^{n-1}) = \lambda(\omega_{\frac{\pi}{2}}) = \lim_{i\to\infty} \lambda(\omega_{i,\frac{\pi}{2}}) = \lim_{i\to\infty} \mu(\omega_i) \leq \mu(\omega_{\alpha_0}) < \lambda(S^{n-1}),$$

which provides the desired contradiction.

For the case that ω is strongly spherically convex when i is sufficiently large, the ω_i from (5.33) are also contained in the open hemisphere that contains ω. Let $\widetilde{\omega}_i = \langle \omega \cup \omega_i \rangle$, the spherical convex hull of $\omega \cup \omega_i$. Observe that $\widetilde{\omega}_i$ also converges to ω. There is a subsequence $\widetilde{\omega}_{i_k}$ so that

$$\omega \subset \widetilde{\omega}_{i_k} \subset \omega_{\frac{1}{k}}. \tag{5.36}$$

Since $\omega_{\frac{1}{k}}$ is a decreasing sequence that converges to ω, and μ is a finite measure, it follows that $\lim_{k\to\infty} \mu(\omega_{\frac{1}{k}}) = \mu(\omega)$, and thus, from (5.36), we see that $\mu(\omega) =$

$\lim_{k\to\infty} \mu(\widetilde{\omega}_{i_k})$. This, together with (5.32) coupled with (5.28), (5.34), and (5.2), gives

$$\mu(\omega) = \lim_{k\to\infty} \mu(\widetilde{\omega}_{i_k}) \geq \lim_{k\to\infty} \mu(\omega_{i_k}) = \lim_{k\to\infty} \lambda(\omega_{i_k,\frac{\pi}{2}}) = \lambda(\omega_{\frac{\pi}{2}}).$$

This shows that $\mu(\omega) = \lambda(S^{n-1} \setminus \omega^*)$, which contradicts the fact that λ and μ are Aleksandrov related.

We have shown that in both the cases where ω is strongly spherically convex and where it is not, we are led to a contradiction if (5.26) is presumed not to hold. □

6 Estimates for the Log-Volumes of Convex Bodies

This section presents estimates for the log-volumes of convex bodies with respect to a Borel measure. These estimates will be crucial to solving the problem of log-volume-product maximization.

LEMMA 6.1. *If μ is a Borel measure on S^{n-1}, then the set*

$$\Omega = \{u \in S^{n-1} : \mu(S^{n-1} \cap u^{\perp}) > 0\}$$

has spherical Lebesgue measure zero.

PROOF. Let $G_{n,k}$ be the Grassmann manifold of k-dimensional subspaces in $\mathbb{R}^n$, and for $k = 1, \ldots, n-1$, define Ω_k as

$$\{\xi \in G_{n,k} : \mu(\xi \cap S^{n-1}) > 0 \text{ but } \mu(\xi' \cap S^{n-1}) = 0 \text{ for each subspace } \xi' \subsetneq \xi\}.$$

For each $u \in S^{n-1}$ with $\mu(S^{n-1} \cap u^{\perp}) > 0$, there exists a subspace $\xi \subset u^{\perp}$ such that ξ belongs to some Ω_k. Using this and the observation that $\xi \subset u^{\perp}$ is equivalent to $u \in \xi^{\perp}$, we can write

$$\begin{aligned} \Omega &= \{u \in S^{n-1} : \mu(S^{n-1} \cap u^{\perp}) > 0\} \\ &= \bigcup_{k=1}^{n-1} \{u \in S^{n-1} : \xi \subset u^{\perp} \text{ for some } \xi \in \Omega_k\} \\ &= \bigcup_{k=1}^{n-1} \bigcup_{\xi\in\Omega_k} \{u \in S^{n-1} : u \in \xi^{\perp}\}. \end{aligned} \tag{6.1}$$

Obviously, for any $\xi \in G_{n,k}$, the set $\{u \in S^{n-1} : u \in \xi^{\perp}\}$ is of spherical Lebesgue measure zero. Thus, to show that the set Ω is of spherical Lebesgue measure zero, it suffices, by (6.1), to show that Ω_k is countable.

If $\xi_1, \ldots, \xi_m \in \Omega_k$ are distinct, then

$$|\mu| \geq \sum_{i=1}^{m} \mu(\xi_i \cap S^{n-1}). \tag{6.2}$$

To see this observe that

$$\begin{aligned}
|\mu| &\geq \mu\Big(\bigcup_{\xi\in\Omega_k} \xi \cap S^{n-1}\Big)\\
&\geq \mu\Big(\bigcup_{i=1}^{m}(\xi_i \cap S^{n-1})\Big)\\
&= \sum_{i=1}^{m}\mu(\xi_i \cap S^{n-1}) + \sum_{i=2}^{m}(-1)^{i-1}\sum_{1\leq j_1<\cdots<j_i\leq m}\mu(\xi_{j_1}\cap\cdots\cap\xi_{j_i}\cap S^{n-1})\\
&= \sum_{i=1}^{m}\mu(\xi_i \cap S^{n-1}),
\end{aligned}$$

where the last equality follows from the fact that $\xi_{j_1}\cap\cdots\cap\xi_{j_i}$ is a proper subspace of $\xi_{j_1}\in\Omega_k$. For any positive integer j, inequality (6.2) implies that the set

$$\{\xi\in\Omega_k : \mu(\xi\cap S^{n-1}) > |\mu|/j\}$$

cannot have more than j elements. Hence,

$$\Omega_k = \bigcup_{j=1}^{\infty}\{\xi\in\Omega_k : \mu(\xi\cap S^{n-1}) > |\mu|/j\}$$

is countable. □

Lemma 6.1 yields immediately the following lemma.

LEMMA 6.2. *Let μ be a Borel measure on S^{n-1}, and let ξ_0 be a codimension* 1 *subspace of $\mathbb{R}^n$. Then the set*

$$\mathcal{A} = \{A\in \mathrm{SO}(n) : \mu(A\xi_0\cap S^{n-1}) > 0\}$$

has Haar measure zero.

PROOF. As in the previous lemma, let

$$\Omega = \{u\in S^{n-1} : \mu(S^{n-1}\cap u^{\perp}) > 0\}.$$

Let $\xi_0 = u_0^{\perp}$ and for each $u\in S^{n-1}$, define

$$\mathcal{A}_u = \{A\in\mathrm{SO}(n) : A\xi_0 = u^{\perp}\} = \{A\in\mathrm{SO}(n) : Au_0^{\perp} = u^{\perp}\}.$$

Thus,

$$\mathcal{A} = \bigcup_{u\in\Omega}\mathcal{A}_u.$$

With the usual identifications, the space S^{n-1} is isometric to the quotient space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$. Let σ_n denote the Haar measure of $\mathrm{SO}(n)$, and let σ_{n-1} denote the Haar measure of $\mathrm{SO}(n-1)$ when transferred to $\mathcal{A}_u$. When suitably normalized, σ_n, σ_{n-1}, and spherical Lebesgue measure, denoted here by du, are related by

$$d\sigma_n = d\sigma_{n-1}du$$

(see, e.g., Santaló [48, (12.10)]). Each set $\mathcal{A}_u$ is a coset of $\mathrm{SO}(n-1)$, and thus $\sigma_{n-1}(\mathcal{A}_u) = \sigma_{n-1}(\mathrm{SO}(n-1))$. We have

$$\sigma_n(\mathcal{A}) = \int_{\mathcal{A}} d\sigma_n = \int_{\Omega} \left(\int_{\mathcal{A}_u} d\sigma_{n-1} \right) du = \sigma_{n-1}(\mathrm{SO}(n-1)) \int_{\Omega} du,$$

with Lemma 6.1 telling us that the last integral is 0. Hence, the Haar measure of $\mathcal{A}$ is 0, as claimed. □

To estimate the log-volumes of convex bodies with respect to a measure μ, we need to use a partition of closed hemispheres. The following lemma allows us to partition a closed hemisphere in a manner suitable for the measure μ.

LEMMA 6.3. *Let μ be a Borel measure on S^{n-1} and Ω be a closed hemisphere of S^{n-1}. Then for each $\varepsilon > 0$, there exist $m = m(\mu, \varepsilon, \Omega)$ compact, spherically convex subsets $\omega_1, \ldots, \omega_m$ such that*

$$\bigcup_{i=1}^{m} \omega_i = \Omega,$$

and, for each j,

$$|u - v| \leq \varepsilon \quad \text{for all } u, v \in \omega_j, \tag{6.3}$$

while

$$\mu\Big(\omega_j \cap \Big(\bigcup_{i \neq j} \omega_i \Big)\Big) = 0. \tag{6.4}$$

PROOF. Divide each of the $(n-1)$-dimensional faces of the n-dimensional cube $[-1,1]^n$ into $(2k)^{n-1}$ small $(n-1)$-dimensional cubes whose edge lengths are all $1/k$, where the integer k is chosen so that the diameter of each small cube is equal to $\sqrt{n-1}/k \leq \varepsilon$. Denote by $\mathcal{T}$ the collection of all these $(n-1)$-dimensional cubes on the boundary of the cube $[-1,1]^n$.

For each $(n-1)$-dimensional cube $C \in \mathcal{T}$, consider an $(n-2)$-dimensional face E of C. Since C is on one of the faces of the cube $[-1,1]^n$, we know that the subspace span E generated by E is of dimension $n-1$. Denote by $\mathcal{L}$ the set of all $(n-1)$-dimensional subspaces generated in this manner. Thus, an $(n-1)$-dimensional subspace $\xi \in \mathcal{L}$ if and only if there exists $C \in \mathcal{T}$ such that $\xi = \text{span } E$ for some $(n-2)$-dimensional face E of C. Obviously, $\mathcal{L}$ is a finite set.

For each $\xi \in \mathcal{L}$, let

$$\mathcal{A}_\xi = \{A \in \mathrm{SO}(n) : \mu(A\xi \cap S^{n-1}) > 0\},$$

which, by Lemma 6.2, has Haar measure 0. Since $\mathcal{L}$ is finite, the union $\bigcup_{\xi \in \mathcal{L}} \mathcal{A}_\xi$ has Haar measure 0 as well. Therefore there exists an $A_0 \in \mathrm{SO}(n)$ so that

$$\mu(A_0\xi \cap S^{n-1}) = 0 \quad \text{for all } \xi \in \mathcal{L}. \tag{6.5}$$

Define the partition

$$\mathcal{P} = \{\overline{C} \cap \Omega : C \in A_0\mathcal{T}\}.$$

where, as before, $\bar{\cdot} : [-1,1]^n \to S^{n-1}$ is the radial projection map. Note that $\mathcal{P}$ is a finite partition of Ω whose cardinality depends only on μ, ε, Ω.

The partition $\mathcal{P}$ satisfies (6.3) because the radial projection map is a contraction and the diameter of each $C \in A_0\mathcal{T}$ is at most ε.

In order to see that $\mathcal{P}$ satisfies (6.4), take any two elements $\omega = \overline{C} \cap \Omega$ and $\omega' = \overline{C'} \cap \Omega$ from $\mathcal{P}$, where $C, C' \in A_0\mathcal{T}$. Then

$$\omega \cap \omega' \subset \operatorname{span}(C \cap C') \cap S^{n-1}.$$

Note that $C \cap C'$ is contained in some $(n-2)$-dimensional face of C. Hence, (6.5) gives $\mu(\operatorname{span}(C \cap C') \cap S^{n-1}) = 0$, which in turn gives $\mu(\omega \cap \omega') = 0$. □

LEMMA 6.4. *Let λ be a Borel measure on S^{n-1} that is not concentrated in any closed hemisphere. Suppose $K_i \in \mathcal{K}_o^n$ is a sequence whose members are contained in the unit ball and such that $h_i = \min\{h_{K_i}(v) : v \in S^{n-1}\} \to 0$. Then there exists a $c > 0$ so that*

$$\liminf_{i\to\infty} \frac{\log \lambda_0(K_i)}{\log h_i} \geq c.$$

PROOF. Without loss of generality we may assume that none of the K_i is the unit ball B. For each K_i, choose a $v_i \in S^{n-1}$ such that $h_{K_i}(v_i) = h_i$, and let

$$\Omega_i = \{u \in S^{n-1} : u \cdot v_i > 1/\log \tfrac{1}{h_i}\}.$$

By using Lemma 5.10, we know that there exists a real $c_1 > 0$ so that

$$\lambda(\Omega_i) > c_1 \tag{6.6}$$

for sufficiently large i.

For $u \in \Omega_i$,

$$\rho_{K_i}(u)\frac{1}{\log \frac{1}{h_i}} < \rho_{K_i}(u) u \cdot v_i \leq h_{K_i}(v_i) = h_i, \tag{6.7}$$

where the first inequality comes from the definition of Ω_i, and the second from the definition of the support function and the fact that $\rho_{K_i}(u)u \in K_i$, from the definition of the radial function. From (6.7) we immediately obtain

$$\log \rho_{K_i}(u)^{-1} \geq \log \frac{1}{h_i \log \frac{1}{h_i}} = \log \frac{1}{h_i} - \log\log \frac{1}{h_i} \tag{6.8}$$

for all $u \in \Omega_i$.

From the fact that $K_i \subset B$, the definition of Ω_i, (6.8), $h_i \to 0$, and (6.6), we have

$$\begin{aligned}\liminf_{i\to\infty} \frac{\int_{S^{n-1}} \log \rho_{K_i}(u)\, d\lambda(u)}{\log h_i} &\geq \liminf_{i\to\infty} \frac{\int_{\Omega_i} \log \rho_{K_i}(u)^{-1}\, d\lambda(u)}{\log \frac{1}{h_i}} \\ &\geq \liminf_{i\to\infty} \frac{\int_{\Omega_i} \left(\log \frac{1}{h_i} - \log\log \frac{1}{h_i}\right) d\lambda(u)}{\log \frac{1}{h_i}} =\end{aligned}$$

$$= \liminf_{i\to\infty} \lambda(\Omega_i)\left(1 - \frac{\log\log\frac{1}{h_i}}{\log\frac{1}{h_i}}\right)$$

$$= \liminf_{i\to\infty} \lambda(\Omega_i)$$

$$\geq c_1. \qquad \square$$

LEMMA 6.5. *Let μ be a Borel measure on S^{n-1} and $K_i \in \mathcal{K}_o^n$ be a sequence such that $K_i \subset B$ and $h_i = \min\{h_{K_i}(v) : v \in S^{n-1}\} \to 0$. Assume that there exists a $c_0 > 0$, and there exist $x_i \in K_i$ so that $|x_i| \geq c_0 > 0$ for sufficiently large i and $x_i \to x$. Then*

$$\lim_{i\to\infty} \frac{\int_{\Omega_0'(x)} \log h_{K_i}(v) d\mu(v)}{\log h_i} = 0,$$

where $\Omega_0'(x) = \{v \in S^{n-1} : v \cdot x > 0\}$.

PROOF. Without loss of generality, we may assume that none of the K_i is B. For $i = 1, 2, \ldots$, define

$$\varepsilon_i = \max\left\{|\overline{x} - \overline{x_i}|, h_i^{\frac{1}{\log\log(1/h_i)}}\right\},$$

and let $\Omega_i = \{v \in S^{n-1} : v \cdot \overline{x_i} > \varepsilon_i\}$. Since $x_i \to x$ we know $\overline{x_i} \to \overline{x}$. Since $h_i \to 0$, we have $\lim_{i\to\infty} \varepsilon_i = 0$. Using the fact that $x_i \in K_i$ and the definition of ε_i, we see that for $v \in \Omega_i$,

$$v \cdot \overline{x} = v \cdot \overline{x_i} - v \cdot (\overline{x_i} - \overline{x}) > \varepsilon_i - |\overline{x} - \overline{x_i}| \geq 0,$$

which implies that $\Omega_i \subset \Omega_0'(x)$. Since we are given that $|x_i| \geq c_0 > 0$ for sufficiently large i, we see that for $v \in \Omega_i$,

$$h_{K_i}(v) \geq v \cdot x_i = |x_i|\, v \cdot \overline{x_i} > c_0\varepsilon_i \geq c_0\, h_i^{\frac{1}{\log\log(1/h_i)}}$$

for sufficiently large i. Thus,

$$\log\frac{1}{h_{K_i}(v)} \leq \log\frac{1}{c_0} + \frac{\log\frac{1}{h_i}}{\log\log\frac{1}{h_i}} \tag{6.9}$$

for sufficiently large i. The facts that $0 < h_i \leq h_{K_i}(v) \leq 1$ for all $v \in S^{n-1}$ and $h_i \to 0$, together with (6.9), implies that

$$\begin{aligned} 0 &\leq \lim_{i\to\infty} \frac{\int_{\Omega_i} \log\frac{1}{h_{K_i}(v)}\, d\mu(v)}{\log\frac{1}{h_i}} \\ &\leq \lim_{i\to\infty} \mu(\Omega_i)\left(\frac{\log\frac{1}{c_0}}{\log\frac{1}{h_i}} + \frac{1}{\log\log\frac{1}{h_i}}\right) = 0. \end{aligned} \tag{6.10}$$

Let δ_k be a strictly decreasing sequence of reals in the open interval $(0, 1)$ whose limit is 0, and let

$$\Omega_{\delta_k} = \{v \in S^{n-1} : v \cdot \overline{x} > \delta_k\}.$$

The Ω_{δ_k} are obviously monotone increasing, with respect to set inclusion, and their union is obviously $\Omega_0'(x)$, hence

$$\lim_{k\to\infty} \mu(\Omega_{\delta_k}) = \mu(\Omega_0'(x))$$

or

$$\lim_{k\to\infty} \mu(\Omega_0'(x) \setminus \Omega_{\delta_k}) = 0. \tag{6.11}$$

From the definition of Ω_i, it follows that for $v \in \Omega_0'(x) \setminus \Omega_i$,

$$0 < v \cdot \overline{x} = v \cdot \overline{x_i} + v \cdot (\overline{x} - \overline{x_i}) \leq \varepsilon_i + |\overline{x} - \overline{x_i}|.$$

Since $\lim_{i\to\infty}(\varepsilon_i + |\overline{x} - \overline{x_i}|) = 0$, it follows that for fixed k, when i is sufficiently large, $\varepsilon_i + |\overline{x} - \overline{x_i}| < \delta_k$. Thus, $v \cdot \overline{x} < \delta_k$, and $v \in \Omega_0'(x) \setminus \Omega_{\delta_k}$. Hence, for fixed k, when i is sufficiently large,

$$\Omega_0'(x) \setminus \Omega_i \subset \Omega_0'(x) \setminus \Omega_{\delta_k}.$$

In light of (6.11), this gives

$$\lim_{i\to\infty} \mu(\Omega_0'(x) \setminus \Omega_i) = 0. \tag{6.12}$$

Since $0 < h_i \leq h_{K_i}(v) \leq 1$ for all $v \in S^{n-1}$, we get

$$0 \leq \lim_{i\to\infty} \frac{\int_{\Omega_0'(x)\setminus\Omega_i} \log \frac{1}{h_{K_i}(v)}\, d\mu(v)}{\log \frac{1}{h_i}} \leq \lim_{i\to\infty} \mu(\Omega_0'(x) \setminus \Omega_i) = 0 \tag{6.13}$$

from (6.12).

To obtain our desired result, we now combine (6.10) and (6.13) to complete the proof. □

We shall require the fact that for $K \in \mathcal{K}_o^n$ such that $K \subset rB$, where $r > 0$, we have

$$|h_K(u) - h_K(v)| \leq r|u - v| \tag{6.14}$$

for all $u, v \in S^{n-1}$. That this is the case follows trivially from the fact that the support function $h_K : \mathbb{R}^n \to \mathbb{R}$ is always subadditive; specifically,

$$h_K(u) \leq h_K(u - v) + h_K(v) \leq h_{rB}(u - v) + h_K(v) = r|u - v| + h_K(v).$$

Note that the $\delta, \alpha \in (0, 1)$ in the hypothesis below are guaranteed to exist by appealing to Lemma 5.11.

LEMMA 6.6. *Suppose μ and λ are Borel measures on S^{n-1} such that λ vanishes on the boundary of all compact, spherically convex sets and suppose also that μ and λ are Aleksandrov related. Let $\delta, \alpha \in (0,1)$ be such that*

$$\mu(\omega) < (1-\delta)\lambda(\omega_{\frac{\pi}{2}-\alpha})$$

for every nonempty compact set $\omega \subset S^{n-1}$ contained in some closed hemisphere, and let

$$c_0 = \min\{e^{\frac{4}{\delta}\log\frac{\alpha}{8}}, e^{-1}\} \in (0,1). \tag{6.15}$$

Suppose also that $K_i \in \mathcal{K}^n_o$ is a sequence such that $K_i \subset c_0 B$ and

$$h_i = \min\{h_{K_i}(v) : v \in S^{n-1}\} \to 0.$$

Then for every closed hemisphere Ω, there exists an integer i_0 such that, for each $i > i_0$.

$$\int_\Omega \log\frac{1}{h_{K_i}(v)}\, d\mu(v) \le \left(1-\frac{\delta}{2}\right)\int_{S^{n-1}} \log\frac{1}{\rho_{K_i}(u)}\, d\lambda(u).$$

PROOF. Lemma 6.3 guarantees that for each positive integer i there exists a partition of Ω into m_i compact, spherically convex sets $\omega_{i,1}, \dots, \omega_{i,m_i}$ such that

$$|u-v| \le h_i^2 \quad \text{for all } u, v \in \omega_{i,j} \tag{6.16}$$

and

$$\mu\Big(\omega_{i,j} \cap \Big(\bigcup_{k\neq j} \omega_{i,k}\Big)\Big) = 0, \tag{6.17}$$

for $j = 1, \dots, m_i$.

Let $v_{i,j} \in \omega_{i,j}$, and abbreviate $h_{i,j} = h_{K_i}(v_{i,j})$. From the definition of h_i, the fact that $K_i \subset c_0 B$, and the definition of c_0, we have

$$0 < h_i \le h_{i,j} \le c_0 \le \frac{1}{e} < 1.$$

From this and the fact that $\lim_{i\to\infty} h_i = 0$, we get $h_{i,j} - c_0 h_i^2 > 0$ and

$$\lim_{i\to\infty}\left|\frac{\log\left(1-\frac{c_0h_i^2}{h_{i,j}}\right)}{\log h_{i,j}}\right| = \lim_{i\to\infty}\left|\frac{c_0h_i^2}{h_{i,j}\log h_{i,j}}\right| \le \lim_{i\to\infty}\left|\frac{c_0h_i^2}{h_i\log h_i}\right| = 0.$$

This and $h_i \to 0$ imply that there exists a positive integer i_0 such that when $i > i_0$,

$$0 < \frac{\log\left(1-\frac{c_0h_i^2}{h_{i,j}}\right)}{\log h_{i,j}} < \frac{\delta}{4} \tag{6.18}$$

and

$$h_i^2 < \frac{\alpha}{8}, \tag{6.19}$$

where α, δ are from the hypothesis.

Throughout the remainder of the proof, we will assume that $i > i_0$. From (6.16), we have

$$0 < h_{i,j} - c_0 h_i^2 \leq h_{i,j} - c_0|v_{i,j} - v| \quad \text{for } v \in \omega_{i,j}. \tag{6.20}$$

Now $v_{i,j} \in \omega_{i,j}$ and $h_{i,j} = h_{K_i}(v_{i,j})$, together with $K_i \subset c_0 B$ and (6.14), (6.20), and (6.18), show that for $v \in \omega_{i,j}$

$$\begin{aligned} \log \frac{1}{h_{K_i}(v)} = \log \frac{1}{h_{i,j} - (h_{i,j} - h_{K_i}(v))} &\leq \log \frac{1}{h_{i,j} - c_0|v_{i,j} - v|} \\ &\leq \log \frac{1}{h_{i,j} - c_0 h_i^2} \\ &= \log \frac{1}{h_{i,j}} - \log\left(1 - \frac{c_0 h_i^2}{h_{i,j}}\right) \\ &\leq \left(1 + \frac{\delta}{4}\right) \log \frac{1}{h_{i,j}}. \end{aligned} \tag{6.21}$$

Suppose $v \in \omega_{i,j,\frac{\pi}{2}-\alpha}$. Then by definition (5.1), there is some $u \in \omega_{i,j}$ such that $u \cdot v > \sin\alpha$. But from knowing $\alpha \in (0,1)$, an easy estimate shows that $\sin\alpha > \frac{\alpha}{4}$. Together with (6.16) and (6.19), we have

$$v \cdot v_{i,j} = v \cdot u + v \cdot (v_{i,j} - u) > \sin\alpha - |v_{i,j} - u| > \frac{\alpha}{4} - h_i^2 > \frac{\alpha}{8}. \tag{6.22}$$

Since $\rho_{K_i}(v)v \in K_i$, from the definition of support function, we have

$$(\rho_{K_i}(v)v) \cdot v_{i,j} \leq h_{K_i}(v_{i,j}) = h_{i,j}. \tag{6.23}$$

From the fact that $K_i \subset c_0 B$ and the definition of c_0, we have

$$h_{i,j} = h_{K_i}(v_{i,j}) \leq c_0 \leq e^{\frac{4}{\delta}\log\frac{\alpha}{8}}. \tag{6.24}$$

Now (6.23) together with (6.22) and (6.24) yield

$$\log \frac{1}{\rho_{K_i}(v)} \geq \log \frac{1}{h_{i,j}} + \log \frac{\alpha}{8} \geq \left(1 - \frac{\delta}{4}\right) \log \frac{1}{h_{i,j}} \tag{6.25}$$

for $v \in \omega_{i,j,\frac{\pi}{2}-\alpha}$.

For each i, we reindex in $\omega_{i,j}$ so that we have

$$\log \frac{1}{h_{i,1}} \geq \cdots \geq \log \frac{1}{h_{i,m_i}}. \tag{6.26}$$

For simplicity, abbreviate

$$\beta_{i,j} = \mu(\omega_{i,j}) \geq 0,$$

and hence (6.21) yields

$$\int_{\omega_{i,j}} \log \frac{1}{h_{K_i}(v)}\, d\mu(v) \leq \left(1 + \frac{\delta}{4}\right)\beta_{i,j} \log \frac{1}{h_{i,j}}. \tag{6.27}$$

Recalling that $\omega_{i,1},\dots,\omega_{i,m_i}$ is a partition of Ω into m_i compact, spherically convex sets and summing in (6.27) shows that for each i,

$$\int_\Omega \log \frac{1}{h_{K_i}(v)}\,d\mu(v) \le \left(1+\frac{\delta}{4}\right)\sum_{j=1}^{m_i}\beta_{i,j}\log\frac{1}{h_{i,j}}. \tag{6.28}$$

For each i, define

$$\begin{aligned}\Theta_{i,1} &= \omega_{i,1,\frac{\pi}{2}-\alpha},\\ \Theta_{i,j} &= \omega_{i,j,\frac{\pi}{2}-\alpha}\setminus\left(\bigcup_{l=1}^{j-1}\omega_{i,l,\frac{\pi}{2}-\alpha}\right),\quad j=2,\dots,m_i.\end{aligned}$$

Then for fixed i, the $\Theta_{i,j}$ are disjoint, and

$$\bigcup_{j=1}^{k}\Theta_{i,j} = \bigcup_{j=1}^{k}\omega_{i,j,\frac{\pi}{2}-\alpha} \tag{6.29}$$

for each $k=1,2,\dots,m_i$. Abbreviate

$$\gamma_{i,j}=\lambda(\Theta_{i,j})\ge 0.$$

From (6.25) we have

$$\int_{\Theta_{i,j}}\log\frac{1}{\rho_{K_i}(u)}\,d\lambda(u)\ge\left(1-\frac{\delta}{4}\right)\gamma_{i,j}\log\frac{1}{h_{i,j}}. \tag{6.30}$$

For fixed i, using (6.17), Lemma 5.11, Lemma 5.1, and (6.29), and the fact that the $\Theta_{i,j}$ are disjoint, we deduce that, for $k=1,\dots,m_i$,

$$\begin{aligned}\sum_{j=1}^{k}\beta_{i,j} &= \sum_{j=1}^{k}\mu(\omega_{ij}) = \mu\left(\bigcup_{j=1}^{k}\omega_{i,j}\right)\\ &< (1-\delta)\lambda\left(\left(\bigcup_{j=1}^{k}\omega_{i,j}\right)_{\frac{\pi}{2}-\alpha}\right)\\ &= (1-\delta)\lambda\left(\bigcup_{j=1}^{k}\omega_{i,j,\frac{\pi}{2}-\alpha}\right)\\ &= (1-\delta)\lambda\left(\bigcup_{j=1}^{k}\Theta_{i,j}\right) = (1-\delta)\sum_{j=1}^{k}\gamma_{i,j}.\end{aligned} \tag{6.31}$$

For fixed $i > i_0$, it follows from (6.28), (6.26), (6.31), (6.30), and the fact that the $\Theta_{i,1}, \dots, \Theta_{i,m_i} \subset S^{n-1}$ are disjoint that

$$\begin{aligned}
&\int_\Omega \log \frac{1}{h_{K_i}(v)}\, d\mu(v)\\
&\quad\le \left(1+\frac{\delta}{4}\right)\sum_{j=1}^{m_i}\beta_{i,j}\log\frac{1}{h_{i,j}}\\
&\quad= \left(1+\frac{\delta}{4}\right)\Bigg(\Bigg(\sum_{j=1}^{m_i}\beta_{i,j}\Bigg)\log\frac{1}{h_{i,m_i}}\\
&\qquad\qquad + \sum_{k=1}^{m_i-1}\sum_{j=1}^{k}\beta_{i,j}\left(\log\frac{1}{h_{i,k}}-\log\frac{1}{h_{i,k+1}}\right)\Bigg)\\
&\quad\le \left(1+\frac{\delta}{4}\right)(1-\delta)\Bigg(\Bigg(\sum_{j=1}^{m_i}\gamma_{i,j}\Bigg)\log\frac{1}{h_{i,m_i}}\\
&\qquad\qquad + \sum_{k=1}^{m_i-1}\sum_{j=1}^{k}\gamma_{i,j}\left(\log\frac{1}{h_{i,k}}-\log\frac{1}{h_{i,k+1}}\right)\Bigg)\\
&\quad= \left(1+\frac{\delta}{4}\right)(1-\delta)\sum_{j=1}^{m_i}\gamma_{i,j}\log\frac{1}{h_{i,j}}\\
&\quad\le \left(1+\frac{\delta}{4}\right)(1-\delta)\frac{1}{1-\frac{\delta}{4}}\sum_{j=1}^{m_i}\int_{\Theta_{i,j}}\log\frac{1}{\rho_{K_i}(u)}\, d\lambda(u)\\
&\quad\le \left(1-\frac{\delta}{2}\right)\int_{S^{n-1}}\log\frac{1}{\rho_{K_i}(u)}\, d\lambda(u),
\end{aligned}$$

which completes the proof. □

7 Maximizing the Log-Volume Product: Existence of Solutions

Let μ and λ be Borel measures on S^{n-1} with $|\mu| = |\lambda|$. For $K \in \mathcal{K}_o^n$, we define the functional $\Phi_{\mu,\lambda}\colon \mathcal{K}_o^n \to \mathbb{R}$ by $\Phi_{\mu,\lambda}(K) = \log \mu_0(K^*) + \log \lambda_0(K)$. However, since $|\mu| = |\lambda|$, we shall simply define it by

$$\Phi_{\mu,\lambda}(K) = -\int_{S^{n-1}} \log h_K\, d\mu + \int_{S^{n-1}} \log \rho_K\, d\lambda$$

and omit the $|\mu|$ and $|\lambda|$. Note that from the definition of $\Phi_{\mu,\lambda}(K)$ and (2.2), it follows immediately that

$$\Phi_{\mu,\lambda}(K) = \Phi_{\lambda,\mu}(K^*) \tag{7.1}$$

for all $K \in \mathcal{K}_o^n$.

Obviously, the functional $\Phi_{\mu,\lambda}$ is homogeneous of degree 0; i.e.,

$$\Phi_{\mu,\lambda}(aK) = \Phi_{\mu,\lambda}(K),$$

for every $a > 0$. Since the radial metric is equivalent to the Hausdorff metric on the space $\mathcal{K}_o^n$, the functional $\Phi_{\mu,\lambda}$ is continuous on $\mathcal{K}_o^n$.

Maximization of the log-volume-product. Let μ and λ be Borel measures on S^{n-1} with $|\mu| = |\lambda|$. Under what conditions does there exist a convex body $K_0 \in \mathcal{K}_o^n$ such that

$$\sup_{K\in\mathcal{K}_o^n} \Phi_{\mu,\lambda}(K) = \Phi_{\mu,\lambda}(K_0)?$$

Existence for this problem is provided by the following lemma.

LEMMA 7.1. *Let μ and λ be Borel measures on S^{n-1} that are Aleksandrov related. If either λ or μ vanishes on the boundary of all compact, spherically convex sets, then there exists $K_0 \in \mathcal{K}_o^n$ such that*

$$\Phi_{\mu,\lambda}(K_0) = \sup_{Q\in\mathcal{K}_o^n} \Phi_{\mu,\lambda}(Q).$$

PROOF. We begin with the trivial observation that for the unit ball $B \in \mathcal{K}_o^n$, we have $\Phi_{\mu,\lambda}(B) = 0$.

We first suppose that λ vanishes on the boundary of all compact, spherically convex sets. Let K_i be a maximizing sequence. Since $\Phi_{\mu,\lambda}$ is homogeneous of degree 0, we may dilate the K_i such that both $K_i \subset c_0 B^n$ and so that there exists an $x_i \in K_i \cap c_0 S^{n-1}$ where c_0 is defined by (6.15). By taking subsequences (twice), we may further assume that K_i converges to a nonempty compact convex set $K_0 \subset \mathbb{R}^n$ and that $x_i \to x \in c_0 S^{n-1}$.

If $o \in \operatorname{int} K_0$, then $K_0 \in \mathcal{K}_o^n$. The continuity of $\Phi_{\mu,\lambda}$ would assure us that $\Phi_{\mu,\lambda}(K_i) \to \Phi_{\mu,\lambda}(K_0)$, and we would be done. In order to show $o \in \operatorname{int} K_0$, we argue by contradiction. Assume that this is not the case; i.e., the origin $o \in \partial K_0$. Then, $h_i = \min\{h_{K_i}(v) : v \in S^{n-1}\}$ converges to 0.

Let

$$\Omega_- = \{v \in S^{n-1} : v \cdot x \le 0\}, \quad \Omega_+ = \{v \in S^{n-1} : v \cdot x > 0\}.$$

From Lemmas 6.4, 6.5, and 6.6, we easily deduce that there exist $c_1, \delta > 0$ such that when i is sufficiently large,

$$\begin{aligned}
\int_{S^{n-1}} \log \frac{1}{\rho_{K_i}(u)}\, d\lambda(u) &\ge \frac{c_1}{2} \log \frac{1}{h_i},\\
\int_{\Omega_+} \log \frac{1}{h_{K_i}(v)}\, d\mu(v) &\le \frac{c_1\delta}{8} \log \frac{1}{h_i},\\
\int_{\Omega_-} \log \frac{1}{h_{K_i}(v)}\, d\mu(v) &\le \left(1 - \frac{\delta}{2}\right) \int_{S^{n-1}} \log \frac{1}{\rho_{K_i}(u)}\, d\lambda(u),
\end{aligned}$$

where $c_1 > 0$ is a constant provided by Lemma 6.4 and δ is from Lemma 6.6. The above inequalities, together with the fact that $h_i \to 0$, imply that

$$\begin{aligned}
\Phi_{\mu,\lambda}(K_i) &= \int_{\Omega_-} \log \frac{1}{h_{K_i}(v)}\, d\mu(v) + \int_{\Omega_+} \log \frac{1}{h_{K_i}(v)}\, d\mu(v) \\
&\quad - \int_{S^{n-1}} \log \frac{1}{\rho_{K_i}(u)}\, d\lambda(u) \\
&\le \left(1 - \frac{\delta}{2}\right) \int_{S^{n-1}} \log \frac{1}{\rho_{K_i}(u)}\, d\lambda(u) + \frac{c_1\delta}{8} \log \frac{1}{h_i} \\
&\quad - \int_{S^{n-1}} \log \frac{1}{\rho_{K_i}(u)}\, d\lambda(u) \\
&= \frac{c_1\delta}{8} \log \frac{1}{h_i} - \frac{\delta}{2} \int_{S^{n-1}} \log \frac{1}{\rho_{K_i}(u)}\, d\lambda(u) \\
&\le \frac{c_1\delta}{8} \log \frac{1}{h_i} - \frac{c_1\delta}{4} \log \frac{1}{h_i} \\
&= -\frac{c_1\delta}{8} \log \frac{1}{h_i} \to -\infty.
\end{aligned}$$

This contradicts the fact that K_i is a maximizing sequence for $\Phi_{\mu,\lambda}$.

Having established Lemma 7.1 for the case where λ vanishes on the boundary of all spherical compact convex sets, we turn to the case where μ is the measure that vanishes on the boundary of all spherical compact convex sets. We now use the previously established case of Lemma 7.1 but with the maximum taken over all $Q^* \in \mathcal{K}_o^n$, together with (7.1), and the fact that a body in $\mathcal{K}_o^n$ is equal to the polar of its polar. □

An immediate consequence of Lemma 7.1 is the following:

THEOREM 7.2. *Suppose μ and λ are Borel measures on S^{n-1} that are Aleksandrov related. If either λ or μ is a measure that vanishes on the boundary of all spherical compact convex sets, then the maximum of the log-volume-product $\mu_0(Q)\lambda_0(Q^*)$, taken over all $Q \in \mathcal{K}_o^n$, is attained at a convex body in $\mathcal{K}_o^n$.*

In the symmetric case, in view of Lemma 5.6, arguments similar to those in the proof of Theorem 7.2 give the following result:

THEOREM 7.3. *Let μ and λ be even Borel measures on S^{n-1} satisfying $|\mu| = |\lambda|$. Suppose that μ is not concentrated on any great hypersphere and λ vanishes on the boundary of all compact, spherically convex sets and that it is strictly positive on all nonempty open sets. Then the log-volume-product $\mu_0(Q^*)\lambda_0(Q)$, taken over all $Q \in \mathcal{K}_e^n$, attains its maximum at a body in $\mathcal{K}_e^n$.*

This theorem is easily proved in a manner almost identical to that of Theorem 7.2, but Lemma 5.6 is required to conclude that μ and λ are Aleksandrov related and thus justify our ability to invoke Lemma 6.6.

8 The Gauss Image Problem: Existence of Solutions

Let λ be a finite measure defined on the σ-algebra of Lebesgue measurable sets on S^{n-1}, and μ be a Borel measure on S^{n-1} such that $|\mu| = |\lambda|$. Recall that $C^+(S^{n-1})$ is the class of strictly positive continuous functions on S^{n-1}. Define the functional

$$\Phi_{\mu,\lambda}: C^+(S^{n-1}) \to \mathbb{R}$$

for $f \in C^+(S^{n-1})$ by letting

$$\Phi_{\mu,\lambda}(f) = \int_{S^{n-1}} \log f(u) d\mu(u) - \int_{S^{n-1}} \log h_{\langle f\rangle}(v) d\lambda(v), \tag{8.1}$$

where $\langle f\rangle = \text{conv}\{f(u)u : u \in S^{n-1}\} \in \mathcal{K}^n_o$ since f is strictly positive. Observe that, from (2.2) we have

$$\Phi_{\mu,\lambda}(f) = |\mu| \log \mu_0(f) + |\lambda| \log \lambda_0(\rho_{\langle f\rangle^*}).$$

Since $\langle af\rangle = a\langle f\rangle$ for $a > 0$, and thus $h_{\langle af\rangle} = ah_{\langle f\rangle}$, it follows from the definition of $\Phi_{\mu,\lambda}$ that $\Phi_{\mu,\lambda}(af) = \Phi_{\mu,\lambda}(f)$; i.e., $\Phi_{\mu,\lambda}$ is homogeneous of degree 0. The continuity of $\Phi_{\mu,\lambda}$ follows immediately from (2.4).

LEMMA 8.1. *Suppose λ and μ are Borel measures defined on S^{n-1}. The supremum, taken over all $Q \in \mathcal{K}^n_o$, of*

$$\int_{S^{n-1}} \log \rho_Q(u) d\mu(u) - \int_{S^{n-1}} \log h_Q(u) d\lambda(u)$$

is attained at $K \in \mathcal{K}^n_o$ if and only if

$$\sup\{\Phi_{\mu,\lambda}(f) : f \in C^+(S^{n-1})\} = \Phi_{\mu,\lambda}(\rho_K).$$

PROOF. Note that in the maximization problem

$$\sup\{\Phi_{\mu,\lambda}(f) : f \in C^+(S^{n-1})\}, \tag{8.2}$$

we have for the convex hull $\langle f\rangle = \text{conv}\{f(u)u : u \in S^{n-1}\}$ that $\rho_{\langle f\rangle} \geq f$ and that $\langle \rho_{\langle f\rangle}\rangle = \langle f\rangle$ so $h_{\langle \rho_{\langle f\rangle}\rangle} = h_{\langle f\rangle}$ for each $f \in C^+(S^{n-1})$. Thus, directly from definition (8.1), it follows that

$$\Phi_{\mu,\lambda}(f) \leq \Phi_{\mu,\lambda}(\rho_{\langle f\rangle}).$$

This tells us that in searching for the supremum in (8.2) we can restrict our attention to the radial functions of bodies in $\mathcal{K}^n_o$; i.e.,

$$\sup\{\Phi_{\mu,\lambda}(f) : f \in C^+(S^{n-1})\} = \sup\{\Phi_{\mu,\lambda}(\rho_Q) : Q \in \mathcal{K}^n_o\}.$$

This yields the desired result. □

THEOREM 8.2. *Let μ be a Borel measure on S^{n-1}, and let λ be a Borel measure on S^{n-1} that is absolutely continuous. If the supremum, taken over all $Q \in \mathcal{K}^n_o$ of*

$$\int_{S^{n-1}} \log \rho_Q(u) d\mu(u) - \int_{S^{n-1}} \log h_Q(u) d\lambda(u)$$

is attained at $K_0 \in \mathcal{K}_o^n$, *then*

$$\mu = \lambda(K_0, \cdot).$$

PROOF. Since $\langle \rho_Q \rangle = Q$, for each $Q \in \mathcal{K}_o^n$ the fact that K_0 is a solution of the maximization problem can be rewritten, in light of (8.1), as

$$\Phi_{\mu,\lambda}(\rho_{K_0}) = \sup\{\Phi_{\mu,\lambda}(\rho_Q) : Q \in \mathcal{K}_o^n\}. \tag{8.3}$$

Lemma 8.1 and the fact that K_0 is a solution of the maximization problem (8.3) tells us that

$$\Phi_{\mu,\lambda}(\rho_{K_0}) = \sup\{\Phi_{\mu,\lambda}(f) : f \in C^+(S^{n-1})\}.$$

Suppose $g \in C^+(S^{n-1})$ is fixed. For real t, define $\rho_t : S^{n-1} \to (0, \infty)$ by

$$\rho_t = \rho(t, \cdot) = \rho_{K_0} e^{tg},$$

that is,

$$\log \rho_t = \log \rho_{K_0} + tg. \tag{8.4}$$

From Lemma 4.2 we know

$$|\lambda| \frac{d}{dt} \log \lambda_0((\langle \rho_t \rangle^*)\Big|_{t=0} = -\int_{S^{n-1}} g(u) d\lambda(K_0, u). \tag{8.5}$$

From (8.4) we see that

$$|\mu| \log \mu_0(\rho_t) = \int_{S^{n-1}} \log \rho_t \, d\mu = \int_{S^{n-1}} (tg + \log \rho_{K_0}) d\mu.$$

Therefore,

$$|\mu| \frac{d}{dt} \log \mu_0(\rho_t)\Big|_{t=0} = \int_{S^{n-1}} g(u) d\mu(u). \tag{8.6}$$

The Euler-Lagrange equation,

$$\frac{d}{dt} \Phi_{\mu,\lambda}(\rho_t)\Big|_{t=0} = \frac{d}{dt}\big(|\lambda| \log \lambda_0((\langle \rho_t \rangle^*) + |\mu| \log \mu_0(\rho_t))\big)\Big|_{t=0} = 0,$$

together with (8.5) and (8.6), gives

$$\int_{S^{n-1}} g(u) d\lambda(K_0, u) = \int_{S^{n-1}} g(u) d\mu(u). \tag{8.7}$$

Since (8.7) holds for all positive g, it holds for differences of these functions and thus for all continuous functions. The conclusion is that $\mu = \lambda(K_0, \cdot)$. □

THEOREM 8.3. *Suppose* μ *is a Borel measure on* S^{n-1}, *while* λ *is a Borel measure on* S^{n-1} *that is absolutely continuous. If* μ *is Aleksandrov related to* λ, *then there exists a convex body* $K_0 \in \mathcal{K}_o^n$ *so that* $\mu = \lambda(K_0, \cdot)$. *Moreover, if* λ *is strictly positive on nonempty open sets, then the convex body* K_0 *is unique up to dilation.*

PROOF. Theorem 7.2 and the fact that μ is Aleksandrov related to λ tells us that the log-volume-product $\mu_0(Q)\lambda_0(Q^*)$, taken over all $Q \in \mathcal{K}_o^n$, attains a maximum at a convex body $K \in \mathcal{K}_o^n$. From Theorem 8.2, together with $|\mu| = |\lambda|$ (since μ and λ are Aleksandrov related), we know that $\mu = \lambda(K, \cdot)$. Uniqueness follows from Lemma 3.8. □

If the measure λ assumes positive values on all nonempty open sets, then the following statements are equivalent:

THEOREM 8.4. *Suppose μ is a Borel measure on S^{n-1}, while λ is a Borel measure on S^{n-1} that is absolutely continuous and strictly positive on nonempty open sets. If $|\mu| = |\lambda|$, then the following statements are equivalent*:

(1) *There exists a body $K_0 \in \mathcal{K}_o^n$ such that $\lambda(K_0, \cdot) = \mu$.*
(2) *There exists a body $K_0 \in \mathcal{K}_o^n$ such that*

$$\sup_{Q \in \mathcal{K}_o^n} \mu_0(Q)\lambda_0(Q^*) = \mu_0(K_0)\lambda_0(K_0^*).$$

(3) *The measures μ and λ are Aleksandrov related.*

Moreover, the convex body K_0 is unique up to dilation.

PROOF. Theorem 7.2 gives (3) ⇒ (2). Theorem 8.2 gives (2) ⇒ (1). Lemma 3.7 gives (1) ⇒ (3). Uniqueness follows from Lemma 3.8. □

For the origin-symmetric case, in view of Lemma 5.6 and Theorem 7.3, we have the following:

THEOREM 8.5. *Suppose μ is an even Borel measure on S^{n-1} that is not concentrated on any great hypersphere, and λ is an even Borel measure on S^{n-1} that is absolutely continuous and strictly positive on nonempty open sets. If $|\mu| = |\lambda|$, then there exists an origin-symmetric convex body $K_0 \in \mathcal{K}_e^n$, unique up to dilation, so that both*

(1) *the Gauss image measure $\lambda(K_0, \cdot) = \mu$ and*
(2) *the log-volume-product $\mu_0(Q)\lambda_0(Q^*)$, taken over $Q \in \mathcal{K}_e^n$, attains its maximum at $K_0 \in \mathcal{K}_o^n$.*

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