Thermal equilibration on the edges of topological liquids

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Thermal conductance has emerged as a powerful probe of topological order in the quantum Hall effect and beyond. The interpretation of experiments crucially depends on the ratio of the sample size and the equilibration length, on which energy exchange among contra-propagating chiral modes becomes significant. We show that at low temperatures the equilibration length diverges as $1/T^2$ for almost all Abelian and non-Abelian topological orders. A faster $1/T^4$ divergence is present on the edges of the non-Abelian PH-Pfaffian and negative-flux Read-Rezayi liquids. We address experimental consequences of the $1/T^2$ and $1/T^4$ laws in a sample, shorter than the equilibration length.

Introduction - The universal properties of a gapped state of matter are known as its topological order [1]. In the language of bulk physics, the order reduces to the list of possible anyons. In terms of the edge, the order tells about gapless chiral edge modes. Topologically protected edge modes give rise to quantized transport. In particular, a quantized electrical conductance is a defining feature of the fractional quantum Hall (FQH) effect. Yet, the electrical conductance alone is insufficient to determine the order. For example, all orders of Kitaev's sixteen-fold way [2, 3] exhibit the same conductance. Which of them is present at the filling factor 5/2 in GaAs is hotly debated [3, 4]. Moreover, electric transport is not a useful probe of magnetic materials, such as candidate Kitaev spin liquids [2].

Another quantized transport coefficient is the thermal conductance [5–7]. It gives way more information than the electrical conductance but is also harder to measure [4]. Only very recently have thermal-conductance data become available in FQHE [8, 9] and $RuCl_3$ [10]. The interpretation of such data is straightforward on chiral edges, where all modes propagate in the same direction and have the same temperature. If both up- and down-stream contra-propagating modes are present, the quantization depends on equilibration among the modes [8, 11, 12]. This does not pose a challenge for the electric transport since the observed voltage equilibration length is believed to be on the order of a micron or shorter [13]. The temperature equilibration length $\ell_{\rm eq}$ extends to tens of microns and can be comparable to or greater than the size of a mesoscopic device [8]. As a result, large finite-size effects have been observed in low-temperature thermal transport. Hence, the interpretation of the data requires the knowledge of the temperature dependence of the equilibration length. We find it in this paper in the limit of low temperatures.

The results are remarkably universal. We predict the same $1/T^2$ dependence for almost all Abelian and non-Abelian orders. Exceptions from the $1/T^2$-law are the PH-Pfaffian order [14–18] and a family of orders [19], related to the Read-Rezayi (RR) states [20], where a different universal $1/T^4$ dependence is found. The universality comes from strong restrictions on possible scaling dimensions of Bose operators responsible for energy exchange between the up- and down-stream modes in the edge conformal field theory (CFT). We address an experimental test of the predicted universality in a geometry with an edge, shorter than $\ell_{\rm eq}$. At a small number of filling factors such as $\nu = \frac{3n-1}{4n-1}$, $n > 1$, the predicted low-temperature behavior of the equilibration length is non-universal.

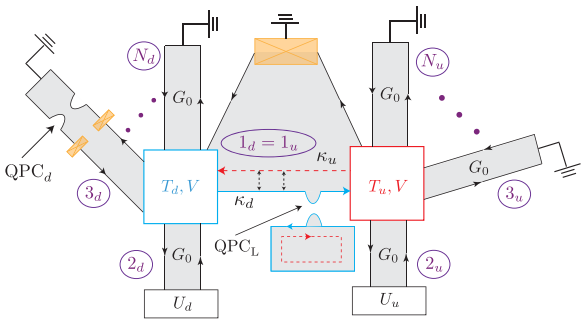

FIG. 1: Thermal equilibration of downstream (solid blue line) and upstream (dashed red) channels is investigated. Gray areas contain FQH liquid. White and yellow boxes are floating contacts.

Thermal equilibration length - Three length scales are important: the equilibration length $\ell_{\rm eq}$, the edge length L, and the thermal length $\ell_{\rm th} \sim \hbar v/T$, where v is on the order of the velocities of the edge modes. The physics at the scales $< \ell_{\rm th}$ is dominated by quantum interference and does not depend on the temperature. The upstream and downstream modes emerge from two reservoirs (Fig. 1) at the temperatures $T_{\rm u}$ and $T_{\rm d}$ and exchange heat. We assume that $\ell_{\rm th} \ll L \ll \ell_{\rm eq}$. This assumption means that $T_{\rm u,d}$ remain approximately constant along the edge and allows treating thermal exchange perturbatively. For the convenience of the scaling analysis we will assume that the ratio $T_{\rm u}/T_{\rm d}$ remains fixed in the limit $T \to 0$.

Down- and up-stream modes are coupled by random

and non-random interactions. When contra-propagating edge modes have a temperature difference ΔT, a heat flux of $\kappa_0 T\Delta T$ flows between them on a length segment $\ell_{\rm eq}$, where $\kappa_0 = \pi^2 k_{\rm B}^2/3h$ defines a thermal conductance quantum $\kappa_0 T$. We employ the renormalization group (RG) technique to analyze the temperature dependence of $\ell_{\rm eq}$.

Random interaction is included in the Hamiltonian as

$$H_{\rm r} = \int_0^L \xi(x)\hat{O}(x)\, dx. \tag{1}$$

Here, $\xi(x)$ is random, and $\hat{O}(x)$ is a product of two operators acting on upstream and downstream modes. The explicit form of $\hat{O}(x)$ depends on the edge structure of the system, but the operator must always be bosonic. We assume that the correlation length of the random potential is shorter than $\ell_{\rm th}$ and set $\langle \xi(x)\xi(x')\rangle = W\delta(x-x')$.

We will see that $H_{\rm r}$ is typically irrelevant in the RG sense. Hence, it is sufficient to use the leading order RG equation for W [21]

$$\frac{dW}{d\ell} = (3-2\Delta)W, \tag{2}$$

where Δ is the scaling dimension of the operator $\hat{O}$. The RG flow stops at $\ell_{\rm th}$. From Eq. (2), the effective coupling (after renormalization) takes the form $W_{\rm eff} \sim W\ell_{\rm th}^{3-2\Delta}$. The energy flux per unit length between the up- and down-stream modes can be found from Fermi's golden rule [22]:

$$\mathcal{J} \sim \frac{W_{\rm eff}}{\ell_{\rm th}}\,(\kappa_0 T\Delta T) \sim W\ell_{\rm th}^{2-2\Delta}\,(\kappa_0 T\Delta T)\,. \tag{3}$$

The thermal equilibration length $\ell_{\rm eq}$ is defined by $\kappa_0 T\Delta T \sim \mathcal{J}\ell_{\rm eq}$ so that

$$\ell_{\rm eq} \sim T^{2-2\Delta}. \tag{4}$$

Thus, our problem reduces to computing Δ, that is, finding the most relevant possible operator $\hat{O}(x)$, Eq. (1).

Non-random interactions $H_{\rm N} = \xi\int_0^L \hat{O}(x)\, dx$ can be analyzed in a similar way [23]. However, only operators $\hat{O}(x)$ that couple at least three one-dimensional edge modes with three different velocities should be considered. Otherwise, it is impossible to conserve both momentum and energy in a scattering event. Besides, non-random operators that transfer a finite charge between modes are ineffective in equilibration due to the momentum mismatch between the modes [24]. Indeed, charge tunneling in a magnetic field involves a finite momentum boost, which is incompatible with the energy conservation for low-energy excitations at $T \to 0$. Thus, the list of non-random operators that can transfer energy between the modes is very limited, and in all cases, the equilibration length turns out to be determined by random operators (1).

Jain states - A generic Lagrangian density of an edge of an Abelian liquid at the filling factor ν without disorder [1] is

$$L = \frac{1}{4\pi\nu}[\partial_t\phi_c\partial_x\phi_c - v_c(\partial_x\phi_c)^2] - \sum_i w_i\partial_x\phi_c\partial_x\phi_i + \sum_{ij}\frac{1}{4\pi}[\tilde{K}_{ij}\partial_t\phi_i\partial_x\phi_j - V_{ij}\partial_x\phi_i\partial_x\phi_j], \tag{5}$$

where the charge mode ϕ_c describes the total charge density $e\partial_x\phi_c/2\pi$ on the edge and ϕ_i are the neutral modes. v_c is the speed of the downstream charge mode. $\tilde{K}$ determines the commutators of the neutral mode operators and V determines the velocities of the neutral modes. Their propagation directions depend on the signs of the eigenvalues of $\tilde{K}$ and are upstream for negative eigenvalues. We focus on edges defined by chemical etching and assume that any screening gates are far away. Since the charge mode participates in the long-range Coulomb interaction and the neutral modes do not, we expect v_c to be much greater than the velocities of the neutral modes and the intermode interactions V_{ij} and w_i.

The Jain states [25] possess upstream edge modes at the filling factors $\nu = \frac{n}{2np-1}$ with $n > 1$. As was discussed by Kane and Fisher [21], disorder gives rise to charge tunneling operators in the action. In our limit of large v_c, those operators are always relevant in the RG sense. They are responsible for voltage equilibration. Following Kane and Fisher, we change variables in the action so that those operators disappear. In the new variables, the $(n-1)$ upstream neutral modes propagate at the same speed and exhibit the SU(n) symmetry. Their interaction with the charge mode involves only irrelevant random couplings (1) with $\hat{O} = \hat{O}_c\hat{O}_n$ [26], where $\hat{O}_{c,n}$ act on the charge and neutral modes. The conservation of the electric charge demands that $\hat{O}_c$ be a combination of derivatives of ϕ_c. Any such operator is bosonic. Since $\hat{O}$ must satisfy the Bose statistics, $\hat{O}_n$ is also bosonic. We next observe that upstream and downstream modes are described by two chiral CFTs. The scaling dimensions of bosonic operators in chiral CFTs are integers, which must be greater than zero unless the operators are trivial constants. It follows that the scaling dimensions of $\hat{O}_{n,c}$ cannot be less than 1 and hence $\Delta \geq 2$. Operators with $\Delta = 2$ do exist, e.g., $\hat{O} = \partial_x\phi_c\partial_x\phi_i$. Thus, $\ell_{\rm eq} \sim 1/T^2$ from Eq. (4).

Non-Abelian orders - We first consider topological orders of the sixteen-fold way at half-integer filling factors [3]. These orders include the non-Abelian Moore-Read Pfaffian state and its two non-Abelian relatives, anti-Pfaffian and PH-Pfaffian, which are seen as the leading candidates at $\nu = 5/2$. Five more orders of the sixteen-fold way are non-Abelian, and eight orders are Abelian. In all cases, the edge theory has the same general structure [3] with one downstream charged boson and several neutral Majorana modes. It is conventional in the litera-

ture to combine pairs of Majoranas into neutral bosons, but we will not do so. The number of the Majorana modes is even for Abelian orders and odd for non-Abelian orders. In the absence of disorder, the edge Lagrangian density contains a bosonic charge field ϕ_c and Majorana fermions ψ_k:

$$L = \frac{2}{4\pi}\partial_x\phi_c(\partial_t - v_c\partial_x)\phi_c + \sum_{k=1}^{|C|} i\psi_k(\partial_t - u\ \mathrm{sign}[C]\partial_x)\psi_k, \qquad (6)$$

where the Chern number C sets the number $|C|$ of the neutral Majoranas, $k = 1, \ldots, |C|$.

The Pfaffian order corresponds to $C = 1$. This reflects a single Majorana co-propagating downstream with the charged mode. We are not interested in the Pfaffian state and any other order with $C \geq 0$ since all modes have the same temperature automatically at $C \geq 0$. $C = -3$ in the anti-Pfaffian state, which is of interest for this paper. *A priori*, the velocities of the $|C|$ Majoranas do not have to be the same. As discussed in Refs. [16, 27], random charge tunneling, responsible for charge equilibration, makes the velocities of the Majorana modes identical. This comes from exactly the same mechanism as in the Jain states [21]. A similar physical picture [28] applies for other $C < -1$: all upstream Majoranas propagate at the same speed. Some subtleties [28, 29] are present at $C = -4$ (the anti-331 state) and $C = -2$ (the 113 state). In particular, the anti-331 edge action contains a four-fermion contribution $\sim \psi_1\psi_2\psi_3\psi_4$. This, however, does not affect the discussion below. On the other hand, the physics of the PH-Pfaffian order with $C = -1$ ends up entirely different since only one neutral Majorana is present on the edge. We will start with the case of $C < -1$.

The argument from the discussion of the Jain states applies almost verbatim at $C < -1$. The only change is a different choice of an operator $\hat{O}$ with $\Delta = 2$: $\hat{O} = i\partial_x\phi_c\psi_n\psi_m$, $n \neq m$, where the imaginary-unit factor ensures the hermiticity of $\hat{O}$. Thus, $\ell_{\rm eq} \sim 1/T^2$ again.

We now turn to the anti-RR states at $\nu = 2/(k+2)$, $k > 2$. Their edge structures are addressed in detail in Ref. [30]. The bosonic charge mode ϕ_c runs downstream; the upstream neutral modes have identical velocities and are described by the $\mathrm{SU}(2)_k$ chiral Wess-Zumino-Witten model [31]. All states in that model can be organized into representations of the affine SU(2) algebra of the currents J^x, J^y, J^z at level k [31]. The expressions for the currents can be found in Ref. [30]. They are Bose operators of scaling dimension 1. To compute $\ell_{\rm eq}$ we repeat again the same argument as for the Jain states and just change the expression for $\hat{O} = \partial_x\phi_c J^\alpha$. Again, $\ell_{\rm eq} \sim 1/T^2$.

At this point, one might think that the $1/T^2$ scaling is a general rule. Yet, exceptions exist. The simplest exception is the PH-Pfaffian order [14–18] with the edge structure (6) with $C = -1$. There is only one upstream neutral Majorana mode ψ. One easily sees that no local Bose operator $\hat{O} = \partial_x\phi_c\hat{O}_n$ with scaling dimension 2 can be found. Indeed, we do not have enough fields to build a product of two Majoranas, as in the other states of the sixteen-fold way. The square of a single Majorana is a trivial constant. This leaves us with $\hat{O}_n = i\psi\partial_x\psi$ as the option, most relevant [32] in the RG sense. It follows that $\Delta = 3$ and $\ell_{\rm eq} \sim 1/T^4$. Unusually rapid growth of the equilibration length at low temperatures in the PH-Pfaffian state is of interest [9] for the interpretation of recent thermal conductance data.

The PH-Pfaffian order is a relative of the negative-flux RR states [19] at the filling factors $\nu = \frac{k}{3k-2}$, $k > 2$. The edge contains a downstream Bose charge mode and an upstream sector described by the $\mathrm{SU}(2)_k/\mathrm{U}(1)$ coset chiral CFT [33] also known as the parafermion theory. As shown in Supplemental Material [22], the equilibration length follows the $\ell_{\rm eq} \sim 1/T^4$ law for those states. To demonstrate that, we prove that the upstream sector has no spin-1 fields mutually local with all primaries, just like no such field exists in the PH-Pfaffian state.

Anti-Jain states - The particle-hole conjugates of the Jain states occur at $\nu = 1 - \frac{n}{2np\pm1}$. As we will see, in most anti-Jain states, energy equilibration is dominated by the same operators as in the Jain states, and the scaling remains $\ell_{\rm eq} \sim 1/T^2$ at most filling factors. We start with $\nu = 1 - \frac{n}{2np+1}$ and only consider $p > 1$ since $\nu = 1 - \frac{n}{2n+1} = \frac{n+1}{2(n+1)-1}$ were addressed above. The Lagrangian density of the edge without disorder is

$$L = \frac{\partial_x\phi_1(\partial_t - v_1\partial_x)\phi_1}{4\pi} - \frac{\partial_x\phi_\rho(\partial_t + v_\rho\partial_x)\phi_\rho}{4\pi(1-\nu)} - w\partial_x\phi_1\partial_x\phi_\rho + L_n, \qquad (7)$$

where ϕ_1 is a downstream charge field with the charge density $e\partial_x\phi_1/2\pi$, ϕ_ρ is an upstream charge field with the charge density $e\partial_x\phi_\rho/2\pi$, and L_n describes $(n-1)$ upstream neutral modes, moving at the same velocity by the Kane-Fisher mechanism [21]. The interaction w is strong, and we switch to the overall downstream charge mode $\phi_c = \phi_\rho + \phi_1$ and an additional upstream neutral mode $\phi_n = \phi_1 + \frac{\phi_\rho}{1-\nu}$. The only downstream mode ϕ_c is much faster than all other modes. The Lagrangian density becomes

$$L = \frac{\partial_x\phi_c(\partial_t - v_c\partial_x)\phi_c}{4\pi\nu} - \frac{1-\nu}{4\pi\nu}\partial_x\phi_n(\partial_t + v_n\partial_x)\phi_n - \tilde{w}\partial_x\phi_n\partial_x\phi_c + L_n, \qquad (8)$$

where $\tilde{w}, v_n \ll v_c$. The coupling $\tilde{w}$ disappears in the language of the normal modes $\tilde{\phi}_{n,c}$:

$$\phi_c = \tilde{\phi}_c \cosh\theta + \tilde{\phi}_n \sqrt{1-\nu} \sinh\theta; \quad (9)$$

$$\phi_n = \tilde{\phi}_n \cosh\theta + \tilde{\phi}_c \frac{\sinh\theta}{\sqrt{1-\nu}}; \quad (10)$$

$$\tanh 2\theta = -\frac{4\pi\nu}{\sqrt{1-\nu}} \frac{\tilde{w}}{v_c + v_n}. \quad (11)$$

In contrast to all states considered above, random charge tunneling between the downstream mode ϕ_1 and the upstream modes is irrelevant in the RG sense. This can be seen from computing the scaling dimension of the charge tunneling operators $T_e = \exp(i\phi_1)\Psi_e$, where Ψ_e is an electron operator that places an electron charge into the mode ϕ_ρ and excites neutral modes in the L_n sector. Such random tunneling is inevitably present on the edge and establishes charge equilibration of ϕ_1 and ϕ_ρ. We will assume that the temperature is low and that the renormalized amplitude $A(\ell_{\rm th})$ of such random tunneling operators is small at the thermal length $\ell_{\rm th}$. The voltage equilibration length is the scale on which the charge exchange is significant between different channels. It is thus much longer than $\ell_{\rm th}$. We will assume below that both terminals have the same chemical potential so that the voltage is equilibrium despite a long voltage equilibration scale.

Operators that transfer charge between modes also transfer energy. We thus have to consider the effect of irrelevant operators that transfer charge me between upstream and downstream modes on energy equilibration. For example, one electron charge is transferred by $\hat{O} = \exp(i\phi_n)\psi_n$, where ψ_n acts in the L_n sector and has the scaling dimension [21] $\Delta_\psi = p - \nu/[2(1-\nu)]$. The scaling dimension of $\hat{O}$ depends on θ, Eq. (11), and is bounded from below by $p \geq 2$. The most relevant equilibration operator $\partial_x\phi_c\partial_x\phi_n$ transfers zero charge and has the scaling dimension $\Delta = 2$ so that $\ell_{\rm eq} \sim 1/T^2$.

The filling factors $\nu = 1 - \frac{n}{2np-1}$, $p > 1$ pose a greater challenge and do not always exhibit universal behavior. We set $n > 1$ since $\nu = 1 - \frac{1}{2p-1} = 1 - \frac{1}{2(p-1)+1}$ were addressed above. The action is similar to (7-11) but now L_n describes $(n-1)$ downstream modes and only ϕ_n (10) runs upstream. The only upstream mode is not much faster than the rest of the modes and hence the charge transfer operators, responsible for the equilibration of the L_n sector, may or may not be relevant. We will assume that they are relevant so that the Kane-Fisher mechanism [21] makes equal the mode velocities in the L_n sector. The same analysis as at $\nu = 1 - \frac{n}{2np+1}$ predicts then $\ell_{\rm eq} \sim 1/T^2$ at $p > 2$.

The answer changes at $p = 2$, $n > 1$. The most important operator $\hat{O} = \exp(i\phi_n)\psi_n$, where ψ_n acts in the L_n sector, transfers a single electron from the ϕ_1 channel. The scaling dimension of ψ_n is known [21] and equals $1/2 - 1/2n$. Hence, the scaling dimension of $\hat{O}$, $\Delta = 2 - 1/n + O(\tilde{w}^2)$, where $O(\tilde{w}^2)$ is a small positive correction. We discover $\ell_{\rm eq} \sim 1/T^\alpha$ with a non-universal $\alpha \approx 2 - 2/n$.

Experimental setup is sketched in Fig. 1. Thermal transport along the lower edge of the arm labelled $1_d = 1_u$ is probed. The edge connects two floating contacts at the temperatures T_d and T_u and the same voltage V. To heat the contacts we use Joule heat [34]. Each floating contact is connected to several FQH arms. Arms 2_d and 2_u are connected to sources at the voltages $U_{d,u}$. All other arms $3_d, \ldots, N_d$ and $3_u, \ldots, N_u$ are grounded. The balance of the currents, entering and leaving the floating contacts, yields $G_0U_d = N_dG_0V$ and $G_0U_u + G_0V = N_uG_0V$, where G_0 is the conductance of an arm. This defines the voltages $U_{d,u}$. The Joule heat dissipated in each contact is $Q_d = G_0(U_d^2 - N_dV^2)/2$, $Q_u = G_0(U_u^2 + V^2 - N_uV^2)/2$. If needed, the dissipated heat can be changed continuously by partially closing point contact QPC$_d$. We assume that all edges, except the lower edge of arm $1_d = 1_u$, are so long that they reach thermal equilibrium with the thermal conductance $KT = |K_u - K_d|T$, where $K_{d,u}T$ are the combined thermal conductances of the downstream and upstream modes respectively. We chose $K_u > K_d$. Changes are minor in the opposite case. The temperatures T_d and T_u can be found from Nyquist noise measurements and satisfy the energy balance equations

$$Q_d = (N_d - 1)\frac{K(T_d^2 - T_0^2)}{2} + \frac{(K + K_d)T_d^2 - K_uT_u^2}{2} + \mathcal{J}L, \quad (12)$$

$$Q_u = (N_u - 1)\frac{K(T_u^2 - T_0^2)}{2} - \frac{KT_0^2 + K_dT_d^2 - K_uT_u^2}{2} - \mathcal{J}L, \quad (13)$$

where T_0 is the ambient temperature and $\mathcal{J}$ the thermal flux (3) between the upstream and downstream modes on the lower edge of arm $1_d = 1_u$. Extracting $\mathcal{J}$ from the above equations, one finds the equilibration length.

The above discussion assumed ideal contacts: normal modes leave contacts at their temperatures. To handle possible non-ideality, QPC$_L$ allows changing the length L of edge $1_d = 1_u$. The equilibration length can be extracted from the L-dependence of the data.

Conclusions - A promising place to look for non-Abelian orders is $\nu > 2$. In particular, the states of the sixteen-fold way were proposed at $\nu = 5/2$. An anti-RR order is a candidate at $\nu = 12/5$. Negative-flux RR orders might explain the observed $\nu = 12/5$ and $19/8$ [35, 36]. Our results at $\nu < 1$ can easily be extended to higher ν. The reason consists in partial decoupling of the integer and fractional channels [12] (see Ref. 37 for a related discussion in the integer QHE). At sufficiently short scales only the latter participate in equilibration and their equilibration length exhibits the same temperature dependence as at the filling factor $\nu - 2$. A much longer equilibration length of the fractional and integer channels will be addressed elsewhere.

In summary, we propose a way to probe the thermal

equilibration length on the edges of FQH liquids. The same universal $1/T^2$ behavior is present in most topological orders. Some orders exhibit the $1/T^4$ law. All those orders are non-Abelian and thus, the unusually fast $\sim 1/T^4$ growth of the equilibration length at low temperatures would constitute evidence of non-Abelian statistics.

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