

Spectral properties of reducible conical metrics

Bin Xu and Xuwen Zhu

.....

Abstract We show that the monodromy of a spherical conical metric g is reducible if and only if the metric g has a real-valued eigenfunction with eigenvalue 2 for the holomorphic extension Δ_g^{Hol} of the associated Laplace–Beltrami operator. Such an eigenfunction produces a meromorphic vector field, which is then related to the developing maps of the conical metric. We also give a lower bound of the first nonzero eigenvalue of Δ_g^{Hol} , together with a complete classification of the dimension of the space of real-valued 2-eigenfunctions for Δ_g^{Hol} depending on the monodromy of the metric g . This paper can be seen as a new connection between the complex analysis method and the PDE approach in the study of spherical conical metrics.

1. Introduction

The study of the interplay between the geometry and the spectrum of geometrically related operators has a long history and has produced a lot of interesting results. In this paper, we study how the monodromy of a spherical conical metric influences the spectrum of the associated Laplace–Beltrami operator. Our results connect two areas of research which attracted considerable attention in recent years. One is the study of metrics of constant positive curvature with conical singularities on compact Riemann surfaces (which we call “spherical conical metrics”), and another is the spectral theory of the Laplace–Beltrami operators on singular surfaces. For the theory of spherical conical metrics, reducible monodromy is expected to occur at the singular points of the moduli space of such metrics. For the theory of Laplace operators, metrics with conical singularities give an interesting class of examples and exhibit many surprising phenomena. This paper appears to be the first to establish a connection between those two areas.

We start by introducing the basic setup. Let Σ be a compact Riemann surface, $\mathbf{p} = (P_1, \dots, P_n)$ be an n -tuple of distinct points on Σ , and $\vec{\beta} = (\beta_1, \dots, \beta_n) \in (\mathbb{R}_+ \setminus \{1\})^n$ be an n -dimensional vector. We say g is a *conical metric* representing the divisor $D = \sum_{j=1}^n (\beta_j - 1)[P_j]$ on Σ if g is a smooth conformal metric on the punctured surface $\Sigma \setminus \text{supp } D = \Sigma \setminus \{P_1, \dots, P_n\}$ and has conical singularities of angle $2\pi\beta_j$ at P_j for $j = 1, \dots, n$. The latter condition means that near P_j there is a complex coordinate z such that $z(P_j) = 0$, and g can be written as $g = e^{2u}|dz|^2$, where $u - (\beta_j - 1)\log|z|$

Illinois Journal of Mathematics

Advance publication—final volume, issue, and page numbers to be assigned.

DOI [10.1215/00192082-9043431](https://doi.org/10.1215/00192082-9043431), © 2021 by the University of Illinois at Urbana–Champaign

Received March 13, 2020. Received in final form January 13, 2021.

2020 *Mathematics Subject Classification*: Primary 34M35; Secondary 53C21.

extends to $z = 0$ continuously. We say a conical metric g representing D is *spherical* if g has constant curvature one on the punctured surface $\Sigma \setminus \text{supp } D$.

A classical way to view spherical conical metrics is through their developing maps. For any such metric g , there exists a multi-valued locally univalent meromorphic function $f : \Sigma \setminus \text{supp } D \rightarrow \overline{\mathbb{C}}$, called a *developing map* of g , such that g is given by the pullback by f of the standard spherical metric. Such a developing map has the following three properties (cf. [14, Lemmas 2.1 and 3.1]):

1. (Pull-back) Denote the standard metric on the sphere by $g_{\text{st}} = \frac{4|dw|^2}{(1+|w|^2)^2}$ for $w \in \overline{\mathbb{C}}$, then $g = f^*g_{\text{st}}$ on $\Sigma \setminus \text{supp } D$;
2. (Monodromy) The monodromy of f is contained in $\text{PSU}(2) = \left\{ w \mapsto \frac{aw + b}{-bw + \bar{a}} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$;
3. (Cone angle) Near each P_j , the principal singular term of the Schwarzian derivative of f is given by $\frac{1-\beta_j^2}{2z^2}$.

We note here that for a given spherical conical metric, its developing map is not unique, and all such maps are related by Möbius transforms in $\text{PSU}(2)$. So for a given metric, the monodromy of all its developing maps are contained in the same conjugacy class of $\text{PSU}(2)$. In this paper, we are in particular interested in the following class of metrics.

DEFINITION 1

A spherical conical metric g is called *reducible* if there exists for the metric g a developing map with monodromy in $\text{U}(1) = \{w \mapsto e^{i\theta}w : \theta \in [0, 2\pi)\}$. Such a developing map of g is called *multiplicative*. The metric g is called *trivially reducible* if the monodromy of its developing map is trivial.

One can also view a spherical conical metric as a solution to the following singular Liouville equation:

$$(1) \quad \Delta_{g_0} u - e^{2u} + K_{g_0} = 0,$$

where g_0 is a conical metric with the prescribed conical singularities but not necessarily with constant curvature one, and $g = e^{2u}g_0$ gives the sought-after spherical conical metric in the same conformal class. Here, Δ_{g_0} is the associated Laplace–Beltrami operator of g_0 . When some of the cone angles are bigger than 2π , the existence and uniqueness of solutions to (1) is still not completely understood. One approach is via perturbation near a given spherical conical metric g , which leads one to study the linearized operator of the above equation, given by

$$\Delta_g - 2.$$

Unlike for complete metrics, in order for Δ_g of a conical metric to be self-adjoint, boundary conditions are needed. One common choice is the Friedrichs extension Δ_g^{Fr} , which is the extension such that the domain consists only of bounded functions on Σ . This is also the extension one uses to solve the perturbation problem of (1). It is known

that when all cone angles are less than 2π , the first nonzero eigenvalue λ_1 of Δ_g^{Fr} is bounded below by 2, and $\lambda_1 = 2$ if and only if g is a spherical football [33, 42]. However, when some of the cone angles are bigger than 2π , 2 is no longer the lower bound, and the deformation is obstructed exactly when 2 is in the spectrum of Δ_g^{Fr} . In [44], it is shown that the deformation can be unobstructed by “splitting” cone points, and there is a trichotomy of deformation rigidity depending on the dimension of eigenspace with eigenvalue 2. In addition to the Friedrichs extension, in this paper we also consider another extension, called the holomorphic extension Δ_g^{Hol} , which was introduced in [27] in the case of flat conical metrics. We show that the functions in the domain of this extension are also closely related to the spectral geometry of reducible conical metrics.

Now we state the main result of this paper, which is a spectral characterization of spherical conical metrics with reducible monodromy. Denote by Δ_g the Laplace–Beltrami operator of a spherical conical metric g , and $\mathcal{D}^{\text{Hol}}$ (resp. $\mathcal{D}^{\text{Fr}}$) the domain of the holomorphic (resp. Friedrichs) extension of Δ_g . The main theorem is stated below (for a more precise statement, see Theorems 1 and 2).

THEOREM A

A spherical conical metric g has reducible monodromy if and only if there is a real-valued eigenfunction $\phi \in \mathcal{D}^{\text{Hol}}$ satisfying

$$\Delta_g \phi = 2\phi.$$

There has been a lot of recent development in understanding spherical conical metrics. One of the features of this problem is that it can be approached from many aspects of mathematics, including complex analysis, min-max theory, integrable systems, synthetic geometry, etc. (see [2, 5, 10, 12, 14, 16, 17, 19–21, 29, 33, 34, 37, 40–44, 48, 50, 51, 53] and the references therein). This paper can be seen as a new connection between the complex analysis method and the PDE approach.

The study of reducible conical metrics was initiated in [51], and has seen a lot of development recently [14, 20, 49]. One feature of reducible metrics is that there exist multiplicative developing maps of such a metric which give meromorphic differentials (sometimes called “character one-forms”) that are dual to meromorphic vector fields ([14], also see Section 4.2 herein). One expects that there will be constraints on the divisor $D = \sum_{j=1}^n (\beta_j - 1)[P_j]$ for reducible metrics. When Σ is the Riemann sphere, Song and the first author [49] determined the angle constraints when all the angles are in $2\pi\mathbb{Q}$. Recently, Eremenko [20] gave a complete answer on the angle constraint problem on the Riemann sphere. There is ongoing work of the first author and his collaborators [11] on the case when the genus of Σ is positive. In [52], the local rigidity of one family of such metrics was shown by using synthetic geometry which exemplifies the constraints on $\text{supp } D$.

The number 2 also appears as the upper bound for the first nonzero eigenvalue in the eigenvalue isoperimetric problem among all smooth metrics on $\mathbb{S}^2$, where the standard spherical metric is the only extremal metric for λ_1 [26, 30]. There are also two

analogues in Kähler geometry. One is that a positive lower bound on Ricci curvature gives a lower bound on the first nonzero eigenvalue [32, 45]; the other is that the complex gradient vector fields of 2-eigenfunctions of a Kähler–Einstein metric on a Fano manifold are holomorphic and form a reductive Lie algebra [36]. Our proof can be seen as an analogue of [32, 36, 45], where a similar Bochner technique is used to obtain a lower bound for the first nonzero eigenvalue and, in the case of equality, produce meromorphic vector fields from 2-eigenfunctions (see Section 4.1). Here we prove the following result (see Theorem 3 and Proposition 2).

THEOREM B

For any spherical conical metric g , the first nonzero eigenvalue of Δ_g^{Hol} satisfies $\lambda_1 \geq 2$. If there is a 2-eigenfunction $\phi \in \mathcal{D}^{\text{Hol}}$, then the complex gradient vector field $X := \phi^z \partial_z$ is meromorphic on Σ .

The existence of such meromorphic vector fields indicate some symmetry of the metric itself. These vector fields can also be viewed as generators of gauge transformations, which are obstructions in solving the nonlinear uniformization problem (see examples and discussion in Section 4).

If, in addition, the 2-eigenfunction is real-valued, we then show that the metric is reducible by relating the meromorphic vector field to a developing map. As an application, we are also able to show that the dimension of the space of real-valued 2-eigenfunctions is completely determined by the monodromy (see Theorem 4).

THEOREM C

For a reducible spherical metric g , the dimension of real-valued 2-eigenfunctions of Δ_g^{Hol} is either 1 or 3. The dimension equals 3 if and only if g is trivially reducible.

We point out here that the real-valued 2-eigenfunctions we find are actually in the domains of both the Friedrichs extension and the holomorphic extension. Therefore, all reducible metrics are included in the obstructed case as discussed in [44]. However, having 2 in the spectrum of the Friedrichs Laplacian does *not* imply that the monodromy is reducible, and in fact there is evidence that there exist irreducible metrics with 2 in their Friedrichs spectrum. In addition, the assumption in Theorem A that the eigenfunction is real-valued is also essential (see more discussion in Section 5).

We also mention another type of metrics called HCMU metrics. The Gaussian curvature functions of these metrics behave similarly to the real-valued eigenfunctions discussed above. There is also a corresponding existence of meromorphic vector fields and character 1-forms (see [6, 8, 13, 15, 35] and the references therein for details).

This paper is organized as follows. In Section 2, we describe various self-adjoint extensions of the Laplace–Beltrami operator of a spherical conical metric. In Section 3, we construct appropriate eigenfunctions assuming the metric is reducible. In Section 4, we prove a lower bound for eigenvalues, and in the case of equality we prove the reducible monodromy property by producing a meromorphic vector field from a real-valued 2-eigenfunction which is then related to a developing map. In Section 5, we discuss the relation of our work to existing works and open problems.

2. Self-adjoint extensions of the operator Δ_g

Consider the Laplace–Beltrami operator of a spherical conical metric g , denoted by Δ_g , acting on $\mathcal{C}_c^\infty(\Sigma \setminus \text{supp } D)$. Locally near a cone point of angle $2\pi\beta$, the metric is given by the geodesic coordinates as

$$g = d\tau^2 + \beta^2 \sin^2 \tau d\theta^2, \quad (\tau, \theta) \in (0, \epsilon) \times \mathbb{R}/2\pi\mathbb{Z}.$$

Under these coordinates the Laplace–Beltrami operator is locally given by

$$\Delta_g = -\partial_\tau^2 - \frac{\cos \tau}{\sin \tau} \partial_\tau - \frac{1}{\beta^2} \frac{1}{\sin^2 \tau} \partial_\theta^2.$$

This operator is of conical type, which has been extensively studied [3, 4, 7, 39]. Conical operators can be viewed as rescaled versions of b-operators [38]. We now briefly recall some notation here. Let $\Sigma_D := [\Sigma; \text{supp } D]$ be the surface Σ with cone points blown up—that is, each puncture is replaced by a circle, and polar coordinates are introduced near the cone points. Denote by $\mathcal{V}_b$ the b-vector fields on Σ_D , which are smooth in the interior and locally given by a basis of $\{\tau \partial_\tau, \partial_\theta\}$ near the punctures. Let $\text{Diff}_b^m(\Sigma_D)$ be the space of b-differential operators of order not exceeding m , locally of the form

$$A = \sum_{j+\ell \leq m} a_{j\ell}(\tau, \theta) (\tau \partial_\tau)^j \partial_\theta^\ell, \quad a_{j\ell} \in \mathcal{C}^\infty(\Sigma_D).$$

A conical operator is a rescaled version of a b-operator, given by elements in $\tau^{-m} \text{Diff}_b^m(\Sigma_D)$. In particular, the Laplace operator Δ_g can be written as $-\tau^{-2}[(\tau \partial_\tau)^2 + \beta^{-2} \partial_\theta^2] + \dots$, where the remaining terms are smooth multiples of $\tau^2 \partial_\tau$ and $\tau \partial_\theta$ —hence, lower order. Therefore, $\Delta_g \in \tau^{-2} \text{Diff}_b^2(\Sigma_D)$. Let $L_b^2(\Sigma_D)$ be the L^2 space with respect to the b-measure which is locally given by $\frac{d\tau}{\tau} \otimes d\theta$. Note that it is related to the L^2 space associated with $d\text{Vol}_g$ by the following relation: $\tau^{-1} L_b^2(\Sigma_D) = L^2(\Sigma_D, d\text{Vol}_g)$. We also denote by $H_b^\ell(\Sigma_D)$ the b-Sobolev space with respect to the b-operators; that is,

$$H_b^\ell(\Sigma_D) = \{u \in L_b^2(\Sigma_D) \mid Vu \in L_b^2(\Sigma_D), \forall V \in \text{Diff}_b^\ell(\Sigma_D)\}.$$

Using such b-based spaces for conic operators has certain advantages, as these functions satisfy dilation invariance properties.

There is a well-developed theory of self-adjoint extensions of symmetric operators in the setting of manifolds with conical singularities (cf. [23–25, 31]). For the case of Laplace–Beltrami operators in this paper, we also refer to [27, 28] for the theory on flat conical surfaces. Since such extensions only concern the local behavior near each cone point and the leading part of Δ_g is the same as in the flat case, the expansions later in this section follow from exactly the same computation.

The closure of Δ_g in $\tau^{-1} L_b^2(\Sigma_D)$ with respect to the graph norm is a symmetric operator,

$$\Delta_g^{\min} : \mathcal{D}^{\min} \rightarrow \tau^{-1} L_b^2,$$

$$\mathcal{D}^{\min} = \overline{\mathcal{C}_c^\infty(\Sigma \setminus \text{supp } D)} \quad \text{with respect to } \|u\|_{\tau^{-1} L_b^2} + \|\Delta_g u\|_{\tau^{-1} L_b^2},$$

while there is another domain,

$$\begin{aligned}\Delta_g^{\max} : \mathcal{D}^{\max} &\rightarrow \mathfrak{r}^{-1} L_b^2, \\ \mathcal{D}^{\max} &= \{u \in \mathfrak{r}^{-1} L_b^2 : \Delta_g u \in \mathfrak{r}^{-1} L_b^2 \text{ in the distributional sense; i.e.,} \\ &\quad \exists \phi \in \mathfrak{r}^{-1} L_b^2, \text{ s.t. } \forall v \in \mathcal{D}^{\min}, (\Delta_g v, u) = (v, \phi)\}.\end{aligned}$$

In other words, $\mathcal{D}^{\max}$ is the dual space of $\mathcal{D}^{\min}$ with respect to the L^2 product.

There is a complete description of $\mathcal{D}^{\min}$ and $\mathcal{D}^{\max}$ in [25]. In particular, we have the following proposition.

PROPOSITION 1 ([25, Lemma 3.5, Proposition 3.6, Lemma 3.11])

The minimal and maximal domains of Δ_g satisfy the following conditions:

1. $\mathcal{D}^{\min} = \mathcal{D}^{\max} \cap (\bigcap_{\epsilon > 0} \mathfrak{r}^{1-\epsilon} H_b^2(\Sigma_D))$;
2. $\mathfrak{r} H_b^2(\Sigma_D) \subset \mathcal{D}^{\min}$;
3. $\exists \epsilon > 0$ such that $\mathcal{D}^{\max} \hookrightarrow \mathfrak{r}^{-1+\epsilon} H_b^2(\Sigma_D)$.

Near a cone point of angle $2\pi\beta$, we can write out the expansion of an element in $\mathcal{D}^{\max}$ as follows:

$$\begin{aligned}(2) \quad u_{\max} &= a_0 + b_0 \log \mathfrak{r} \\ &+ \sum_{1 \leq |k| \leq J} |k|^{-1/2} a_k \mathfrak{r}^{|k|/\beta} e^{ik\theta} + \sum_{1 \leq |k| \leq J} |k|^{-1/2} b_k \mathfrak{r}^{-|k|/\beta} e^{ik\theta} + \tilde{u}, \\ &\tilde{u} \in \mathcal{D}^{\min}, a_k, b_k \in \mathbb{C},\end{aligned}$$

where $J = [\beta]$ if $\beta \notin \mathbb{N}$ and $J = \beta - 1$ if $\beta \in \mathbb{N}$. When there are multiple cone points, we use the notation $(a_k^i, b_k^i)_{-J_i \leq k \leq J_i}$ for the expansion near P_i . We refer to [27, Proposition 3.3] for the explicit computation that justifies the above expansion. Note that if $\beta < 1$, the only coefficients remaining are (a_0, b_0) .

The classical von Neumann theory [46, 47] shows that any self-adjoint extension of Δ_g is a space between $\mathcal{D}^{\min}$ and $\mathcal{D}^{\max}$ and has a one-to-one correspondence with the Lagrangian in the space of coefficients:

$$\bigcup_{\substack{0 \leq |k| \leq J_i \\ 1 \leq i \leq n}} \{a_k^i, b_k^i\} \in \mathbb{C}^{2J}, \quad J = \sum_{i=1}^n (2J_i + 1).$$

Here the symplectic pairing is given by

$$\begin{aligned}(3) \quad \Omega(\vec{A}, \vec{A}') &:= \langle A_+, A'_- \rangle - \langle A_-, A'_+ \rangle = \sum (a_k^i \overline{b_k^{i'}} - b_k^i \overline{a_k^{i'}}), \\ \vec{A} &= (a_{-J_1}^1, \dots, a_k^i, \dots, a_{J_n}^n, b_{-J_1}^1, \dots, b_k^i, \dots, b_{J_n}^n) =: (A_+, A_-),\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{C}^J$.

In particular, there are two self-adjoint extensions we are going to use.

DEFINITION 2

The domain of the Friedrichs extension $\mathcal{D}^{\text{Fr}}$ consists of all bounded elements $u \in \mathcal{D}^{\text{max}}$, i.e., any function with an expansion,

$$u = a_0 + \sum_{1 \leq |k| \leq J} |k|^{-1/2} a_k \mathfrak{r}^{|k|/\beta} e^{ik\theta} + \tilde{u}, \quad \tilde{u} \in \mathcal{D}^{\text{min}}, a_k \in \mathbb{C}.$$

DEFINITION 3

The domain of holomorphic extension $\mathcal{D}^{\text{Hol}}$ consists of functions with expansion,

$$(4) \quad u = a_0 + \sum_{1 \leq k \leq J} |k|^{-1/2} a_k \mathfrak{r}^{|k|/\beta} e^{ik\theta} + \sum_{1 \leq -k \leq J} |k|^{-1/2} b_k \mathfrak{r}^{-|k|/\beta} e^{ik\theta} + \tilde{u},$$

$$\tilde{u} \in \mathcal{D}^{\text{min}}, a_k, b_k \in \mathbb{C}.$$

We denote by Δ_g^{Fr} and Δ_g^{Hol} the two self-adjoint operators associated to the domain $\mathcal{D}^{\text{Fr}}$ and $\mathcal{D}^{\text{Hol}}$.

In terms of complex coordinate $z = |z|e^{i\theta}$, where $|z| \sim \mathfrak{r}^{1/\beta}$, the two expansions above can be rewritten as

$$(5) \quad u \in \mathcal{D}^{\text{Fr}} \quad \Leftrightarrow \quad u = a_0 + \sum_{1 \leq k \leq J} (a_k z^k + b_k \bar{z}^k) + \mathcal{O}(|z|^{J+1}),$$

$$(6) \quad u \in \mathcal{D}^{\text{Hol}} \quad \Leftrightarrow \quad u = a_0 + \sum_{1 \leq |k| \leq J} a_k z^k + \mathcal{O}(|z|^{J+1}).$$

Notice that if all $\beta_i < 1$, then the only nontrivial coefficient is a_0 , in which case $\mathcal{D}^{\text{Fr}} = \mathcal{D}^{\text{Hol}}$.

3. A 2-eigenfunction in the domain

In this section, we construct a 2-eigenfunction for reducible metrics.

THEOREM 1

If g is a spherical conical metric on Σ with reducible monodromy, then $2 \in \text{Spec}(\Delta_g^{\text{Fr}}) \cap \text{Spec}(\Delta_g^{\text{Hol}})$.

Let g be a reducible metric representing D on Σ . Then there exists a developing map of g , written as $f : \Sigma \setminus \text{supp } D \rightarrow \overline{\mathbb{C}}$, such that the monodromy of f is contained in $U(1)$. Hence, the metric g can be written as the pullback by f of the standard spherical metric $g_{\text{st}} = \frac{4|dw|^2}{(1+|w|^2)^2}$; i.e.,

$$(7) \quad g = \frac{4|f'|^2}{(1+|f|^2)^2} |dz|^2.$$

Its Laplace–Beltrami operator is then given by

$$(8) \quad \Delta_g = -\frac{(1+|f|^2)^2}{|f'|^2} \partial_z \partial_{\bar{z}}.$$

Consider the following function on $\Sigma \setminus \text{supp } D$:

$$(9) \quad \phi = \frac{1 - |f|^2}{1 + |f|^2}.$$

Since f is a multi-valued developing map, ϕ is also potentially multi-valued on $\Sigma \setminus \text{supp } D$. Moreover, a priori ϕ is only defined on $\Sigma \setminus \text{supp } D$. However, we have the following lemma for reducible metrics.

LEMMA 1

If g has reducible monodromy, then ϕ is a well-defined single-valued function on Σ .

Proof

Take a representative $\mathfrak{f}$ of f outside the branching points of f . Since the monodromy of f is contained in $U(1)$, it is straightforward to check that $|\mathfrak{f}|$, and hence ϕ , is single-valued on Σ outside the branching points of f . In addition, the definition of ϕ is independent of choice of $\mathfrak{f}$. By [14, Theorem 1.4], ϕ extends continuously to Σ . $\square$

REMARK

When g is irreducible, the function ϕ defined above is *not* single-valued.

LEMMA 2

Near a cone point of angle $2\pi\beta$, if $\beta \notin \mathbb{N}$, then there exists one complex coordinate z and one representative $\mathfrak{f}$ of f such that $\mathfrak{f} = z^\beta$; if $\beta \in \mathbb{N}$, then there exists one complex coordinate z , a $\text{PSL}(2, \mathbb{C})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and one representative $\mathfrak{f}$ of f such that $\mathfrak{f} = \frac{az^\beta + b}{cz^\beta + d}$.

Proof

It follows from the proof of Theorem 1.4 in [14]. $\square$

LEMMA 3

The function ϕ defined in (9) is contained in $\mathcal{D}^{\text{Fr}} \cap \mathcal{D}^{\text{Hol}}$.

Proof

Near a cone point with noninteger cone angle, using the expression $f = z^\beta$, we get the local expression of ϕ as

$$\phi = \frac{1 - |z|^{2\beta}}{1 + |z|^{2\beta}},$$

and hence ϕ has the following expansion:

$$\phi \sim 1 - 2|z|^{2\beta} + \mathcal{O}(|z|^{2\beta+\epsilon}).$$

Near a cone point with integer cone angle $2\pi n$, the developing map f has a different expression $f = \frac{az^n+b}{cz^n+d}$, where $ad-bc=1$, and hence ϕ is given by

$$(10) \quad \begin{aligned} \phi \sim & \frac{|b|^2 - |d|^2}{|b|^2 + |d|^2} - \frac{2\bar{b}d}{(|b|^2 + |d|^2)^2} z^n - \frac{2bd}{(|b|^2 + |d|^2)^2} \bar{z}^n \\ & + \frac{2\bar{b}d(\bar{a}b + c\bar{d})}{(|b|^2 + |d|^2)^3} z^{2n} + \frac{2bd(\bar{a}b + \bar{c}d)}{(|b|^2 + |d|^2)^3} \bar{z}^{2n} \\ & + \frac{2(|b|^2 - |d|^2)(|a|^2 + |c|^2)}{(|b|^2 + |d|^2)^2} |z|^{2n} + \mathcal{O}(|z|^{2n+\epsilon}). \end{aligned}$$

In either case, by comparing with (5) and (6), we see that ϕ is in $\mathcal{D}^{\text{Fr}} \cap \mathcal{D}^{\text{Hol}}$. $\square$

Using the explicit expression of Δ_g in (8), it is easy to check that ϕ satisfies

$$\Delta_g \phi = 2\phi.$$

This completes the proof of Theorem 1.

EXAMPLE 1

Consider the standard sphere $\mathbb{S}^2$ with the spherical metric $g_{\text{st}} = d\mathfrak{r}^2 + \sin^2 \mathfrak{r} d\theta^2$, which is related to the metric (7) in conformal coordinates (where the developing map is $f(z) = z, z = re^{iw}$) by the following relation

$$\mathfrak{r} = 2 \arctan r, \quad \theta = w.$$

Then one of the 2-eigenfunctions of g_{st} is given by $\cos r$, which is exactly the same as ϕ in (9). The other two eigenfunctions $\sin r \cos \theta$ and $\sin r \sin \theta$ are identified with the real and imaginary parts of

$$\frac{2f}{1 + |f|^2}.$$

4. A spectral condition for reducible metrics

In this section we prove the following theorem.

THEOREM 2

If there is a real-valued function $\phi \in \mathcal{D}^{\text{Hol}}$ satisfying $\Delta_g \phi = 2\phi$, then g is a reducible metric.

Let $\phi \in \mathcal{D}^{\text{Hol}}$ be an eigenfunction with $\Delta_g \phi = \lambda \phi$. Define the complex gradient of ϕ as

$$X = \phi^{\cdot z} \partial_z.$$

We will use this vector field to show $\lambda \geq 2$, and if $\lambda = 2$, then X is a meromorphic vector field, which will then be related to a developing map of g provided that the 2-eigenfunction ϕ is real-valued. Before the proof, we first give two examples.

EXAMPLE 2

Take the spherical football with angles $(2\pi\beta, 2\pi\beta)$, where $\beta \notin \mathbb{N}$, and let z be the (global) conformal coordinate centered at one of the cone points. The eigenfunction defined in (9) is given by $\frac{1-|z|^{2\beta}}{1+|z|^{2\beta}}$, and its corresponding gradient vector field is given by

$$X = -z\partial_z.$$

EXAMPLE 3

Take the three 2-eigenfunctions on $\mathbb{S}^2$ described in Example 1, then the three corresponding vector fields are holomorphic on the sphere:

$$(11) \quad X_1 = -\frac{1}{2}z\partial_z, \quad X_2 = \frac{1}{4}(1-z^2)\partial_z, \quad X_3 = \frac{1}{4}i(1+z^2)\partial_z.$$

Consider the double cover of a sphere using the pullback by $f : z \rightarrow z^2$, which gives a spherical conical metric with two antipodal cone points each with angle 4π . The three eigenfunctions from the sphere lift to the double cover, and the three meromorphic vector fields are given by

$$(12) \quad X_1 = -\frac{1}{4}z\partial_z, \quad X_2 = \left(\frac{1}{8}z^{-1} - \frac{1}{8}z^3\right)\partial_z, \quad X_3 = i\left(\frac{1}{8}z^{-1} + \frac{1}{8}z^3\right)\partial_z.$$

REMARK

Each of the meromorphic vector fields above can be viewed as a generator of an infinitesimal diffeomorphism on the twice-punctured sphere. In particular, X_1 is associated to the conformal dilations on the sphere and generates a diffeomorphic family of spherical conical metrics with the same conical data. Such conformal dilations can be seen as gauge actions, which shows that the presence of 2-eigenfunctions creates obstructions in solving the nonlinear Liouville equation (1) that was studied in detail in [44].

4.1. A meromorphic vector field

We will prove the following theorem.

THEOREM 3

For any spherical conical metric g , the first nonzero eigenvalue of Δ_g^{Hol} satisfies $\lambda_1 \geq 2$.

If the equality holds, then we have the following proposition for any 2-eigenfunction in the holomorphic extension.

PROPOSITION 2

Let $\phi \in \mathcal{D}^{\text{Hol}}$ be an eigenfunction (not necessarily real-valued) satisfying $\Delta_g\phi = 2\phi$. Then its complex gradient vector field defined as $X = \phi^z\partial_z$ is meromorphic on Σ . Moreover, X has the following properties:

1. At any point $p \notin \text{supp } D$, X is holomorphic;
2. At a point $p \in \text{supp } D$ with $\beta \notin \mathbb{N}$ and $\beta < 1$, X has a zero;
3. At a point $p \in \text{supp } D$ with $\beta > 1$, X can have a pole of order at most $[\beta] - 1$.

We start with an arbitrary eigenfunction $\phi \in \mathcal{D}^{\text{Hol}}$ satisfying

$$\Delta_g \phi = \lambda \phi, \quad \lambda > 0.$$

Locally the metric is given by $g = e^{2u}|dz|^2$ as in (7), then we can write the gradient vector field X of ϕ as

$$X = e^{-2u} \frac{\partial \phi}{\partial \bar{z}} \partial_z = \frac{4|f'|^2}{(1 + |f|^2)^2} \frac{\partial \phi}{\partial \bar{z}} \partial_z.$$

Away from cone points, by elliptic regularity it is easy to see that ϕ is smooth. Hence, X is a smooth vector field on $\Sigma \setminus \text{supp } D$.

Using Bochner's identity (see, for example, [1, Exercise 1.38]), we have the following pointwise identity for X :

$$(13) \quad \nabla^* \nabla X = \lambda X - \text{Ric } X,$$

where ∇^* is the formal adjoint operator of ∇ . Recall for the spherical metric g considered here, $\text{Ric } X = X$. Hence, by [1, Equation (4.80)],

$$(14) \quad \nabla^* \nabla^{(0,1)} X = \frac{1}{2} (\nabla^* \nabla X - \text{Ric } X) = \frac{1}{2} (\lambda - 2) X.$$

We now show that the following integration by parts is valid:

$$(15) \quad \int_{\Sigma} |\nabla^{(0,1)} X|^2 = \int_{\Sigma} (\nabla^* \nabla^{(0,1)} X, X).$$

Note that all we need to show is that the integral of the left-hand side converges; in other words, $\nabla^{(0,1)} X$ is in L^2 . Then using $\nabla X = \nabla^{(1,0)} X + \nabla^{(0,1)} X$ and pointwise $(\nabla^{(1,0)} X, \nabla^{(0,1)} X) = 0$, one can apply integration by parts to get the expression on the right-hand side. If (15) holds, then from (14) we immediately get $\lambda \geq 2$, and $\lambda = 2$ if and only if $\nabla^{(0,1)} X = 0$.

To check that $\nabla^{(0,1)} X$ is indeed square integrable, we compute the decay rate of $\nabla^{(0,1)} X$ near each cone point. To do this, we take $z = r e^{i\theta}$ to be the complex coordinate near a cone point of angle $2\pi\beta$ and decompose the eigenfunction ϕ locally into a Fourier series $\phi = \sum_{k \in \mathbb{Z}} \phi_k(r) e^{ik\theta}$. We compute the functions $\phi_k(r)$ and then use them to express $\nabla^{(0,1)} X$. If all ϕ_k decay fast enough near the cone points, then $\nabla^{(0,1)} X$ will be square integrable. There is a slight difference between the two cases $\beta \notin \mathbb{N}$ and $\beta \in \mathbb{N}$, so we carry out the computation in the first case in detail and then point out the difference in the second case.

4.1.1. Developing maps near a cone point of $\beta \notin \mathbb{N}$. Near such a point, the developing map f can be written as

$$f = z^\beta,$$

so the metric is given by

$$g = \frac{4\beta^2 |z|^{2(\beta-1)}}{(1 + |z|^{2\beta})^2} |dz|^2.$$

Recall $z = r e^{i\theta}$. Then the equation $\Delta_g \phi = \lambda \phi$ is given by

$$-\frac{(1+r^{2\beta})^2}{4\beta^2 r^{2(\beta-1)}} \frac{1}{r^2} ((r\partial_r)^2 + \partial_\theta^2) \phi = \lambda \phi.$$

Write $\phi = \sum_{k=-\infty}^{\infty} \phi_k(r) e^{ik\theta}$. The Fourier decomposition of the equation gives a sequence of regular-singular ODEs:

$$-\frac{(1+r^{2\beta})^2}{4\beta^2 r^{2(\beta-1)}} \frac{1}{r^2} ((r\partial_r)^2 - k^2) \phi_k = \lambda \phi_k, \quad k \in \mathbb{Z}.$$

Each ODE has two linearly independent solutions, with leading term r^k and r^{-k} , respectively. By choosing $\phi \in \mathcal{D}^{\text{Hol}}$, we require that for $-J \leq k \leq J$, each ϕ_k has an expansion with leading term r^k . Putting the leading term r^k into the equation, we see that in order to match the right-hand side, the next term should be given by $r^{k+2\beta}$. This gives an $r^{k+2\beta}$ term to match on the right-hand side; hence, the next term in the expansion is given by $r^{k+4\beta}$. Iteratively, we get

$$(16) \quad \phi_k(r) = C_{k,0} r^k + C_{k,1} r^{k+2\beta} + C_{k,2} r^{k+4\beta} + \dots + C_{k,j} r^{k+2j\beta} + \dots,$$

where $C_{k,i}$ are determined iteratively. In particular, if $\lambda = 2$, then there is the following iteration:

$$(17) \quad \begin{aligned} -8\beta^2 C_{k,j-1} &= ((k+2j\beta)^2 - k^2) C_{k,j} \\ &+ 2((k+2(j-1)\beta)^2 - k^2) C_{k,j-1} \\ &+ ((k+2(j-2)\beta)^2 - k^2) C_{k,j-2}. \end{aligned}$$

(When $j = 1$, the $C_{k,j-2}$ term is removed.) It is straightforward to check that if $\lambda = 2$, then $C_{k,j}$ are actually given by

$$(18) \quad C_{k,j} = (-1)^j \frac{2\beta}{k+\beta} C_{k,0}, \quad j = 1, 2, \dots$$

On the other hand, for $|k| > J$, the assumption that ϕ is in $L^2(\Sigma, d\text{Vol}_g)$ requires the solution ϕ_k to start with $r^{|k|}$:

$$(19) \quad \phi_k(r) = C_{k,0} r^{|k|} + C_{k,1} r^{|k|+2\beta} + C_{k,2} r^{|k|+4\beta} + \dots + C_{k,j} r^{|k|+2j\beta} + \dots,$$

where the coefficients $C_{k,j}$, $j > 0$ can be determined again by $C_{k,0}$ in the same fashion. In particular, when $\lambda = 2$, we have a similar expression as (18), except k is replaced by $|k|$.

Now we compute X and $\nabla^{(0,1)} X$ using this local expansion.

The vector field X

Compute the complex gradient of ϕ as

$$X = \left(e^{-2u} \frac{\partial \phi}{\partial \bar{z}} \right) \partial_z = \left(\sum_{k \in \mathbb{Z}} X_k(r) e^{i(k+1)\theta} \right) \partial_z.$$

Here we use the local expression of g and $\partial_{\bar{z}} = \frac{e^{i\theta}}{2} (\partial_r - \frac{1}{ir} \partial_\theta)$ to get

$$(20) \quad X_k = e^{-i(k+1)\theta} \frac{(1+r^{2\beta})^2}{4\beta^2 r^{2(\beta-1)}} \frac{e^{i\theta}}{2} \left(\partial_r - \frac{1}{ir} \partial_\theta \right) (\phi_k(r) e^{ik\theta}).$$

It simplifies to the following:

$$X_k = \frac{(1 + r^{2\beta})^2}{8\beta^2 r^{2(\beta-1)}} \left(\phi'_k - \frac{k}{r} \phi_k \right).$$

Recall when $-J \leq k \leq J$, ϕ_k is given by (16). The first term in ϕ_k is $C_{k,0}r^k$, which is eliminated by the operator $\partial_r - k/r$. Therefore,

$$\phi'_k - \frac{k}{r} \phi_k = \sum_{j \geq 1} 2j\beta C_{k,j} r^{k+2j\beta-1}.$$

This gives

$$(21) \quad X_k = \frac{(1 + r^{2\beta})^2}{8\beta^2} \sum_{j \geq 1} 2j\beta C_{k,j} r^{k+(2j-2)\beta+1} = \frac{1}{8\beta^2} \sum_{j \geq 1} \tilde{C}_{k,j} r^{k+(2j-2)\beta+1},$$

where $\tilde{C}_{k,j} = 2j\beta C_{k,j} + 4(j-1)\beta C_{k,j-1} + 2(j-2)\beta C_{k,j-2}$ (for $j = 1$, the last term is removed). The same computation above applies to $k > J$ as well.

On the other hand, when $k < -J$, the first term in ϕ_k is not eliminated, and we get

$$\phi'_k - \frac{k}{r} \phi_k = \sum_{j \geq 0} (-2k + 2j\beta) C_{k,j} r^{|k|+2j\beta-1},$$

which gives

$$(22) \quad \begin{aligned} X_k &= \frac{(1 + r^{2\beta})^2}{8\beta^2} \sum_{j \geq 0} (-2k + 2j\beta) C_{k,j} r^{|k|+(2j-2)\beta+1} \\ &= \frac{1}{8\beta^2} \sum_{j \geq 0} \tilde{C}_{k,j} r^{|k|+(2j-2)\beta+1}, \end{aligned}$$

where $\tilde{C}_{k,j}$ satisfies

$$\tilde{C}_{k,j} = (-2k + 2j\beta) C_{k,j} + 2(-2k + 2(j-1)\beta) C_{k,j-1} + (-2k + 2(j-2)\beta) C_{k,j-2}.$$

Again for $j = 1$, the last term is removed.

Here we give the following observation of $\tilde{C}_{k,j}$, which will be used later.

LEMMA 4

If $\lambda = 2$, then the coefficients $\tilde{C}_{k,j}$ in (21) and (22) satisfy

$$(23) \quad \tilde{C}_{k,j} = 0, \quad \forall j \geq 2.$$

Proof

It follows directly by substituting (18) into the expression of $\tilde{C}_{k,j}$. □

Expression of $\nabla^{(0,1)} X$

Now we compute $\nabla^{(0,1)} X$ which is a (1,1)-tensor, locally given by

$$\sum_{k \in \mathbb{Z}} \partial_{\bar{z}} (X_k e^{i(k+1)\theta}) \partial_z \otimes d\bar{z}.$$

Using the expression of X_k obtained above, we compute $\partial_{\bar{z}}(X_k e^{i(k+1)\theta}) = \frac{e^{i\theta}}{2}(\partial_r - \frac{1}{ir}\partial_\theta)(X_k(r)e^{i(k+1)\theta})$ for each k .

When $-J \leq k \leq J$, the first term in (21) is given by $\frac{1}{8\beta^2}\tilde{C}_{k,1}r^{k+1}$ and it is again eliminated by the operator $\partial_r - (k+1)/r$. So we have

$$(24) \quad \partial_{\bar{z}}(X_k e^{i(k+1)\theta}) = \frac{e^{i(k+2)\theta}}{16\beta^2} \sum_{j \geq 2} (2j-2)\beta \tilde{C}_{k,j} r^{k+(2j-2)\beta}.$$

Since $J = [\beta]$ and β is not integer, we have $-\beta < J$. Therefore, we have a leading term $r^{2\beta+k}$, where $2\beta+k \geq \beta > 0$. Moreover, if $\lambda = 2$, then by Lemma 4, we simply have

$$(25) \quad \partial_{\bar{z}}(X_k e^{i(k+1)\theta}) = \frac{e^{i(k+2)\theta}}{16\beta^2} \sum_{j \geq 2} \tilde{C}_{k,j} (2j-2)\beta r^{k+(2j-2)\beta} = 0.$$

Note that when $k > J$, the same computation as in (24) and (25) applies.

On the other hand, when $k < -J$, the leading term in (22) is not eliminated, and we get

$$(26) \quad \partial_{\bar{z}}(X_k e^{i(k+1)\theta}) = \frac{e^{i(k+2)\theta}}{16\beta^2} \sum_{j \geq 0} \tilde{C}_{k,j} (2|k| + (2j-2)\beta) r^{|k|+(2j-2)\beta}.$$

Since $\beta \notin \mathbb{N}$, we have $|k| > J > \beta$, so the leading term in (26) is given by $r^{|k|-2\beta}$ which satisfies $|k| - 2\beta > -\beta$. Recall that $r = |z|$, so in terms of the geodesic coordinates (τ, θ) for which the associated volume form is $\tau d\tau d\theta$, the computation above implies that each term decays slower than τ^{-1} .

Combining the analysis above, we have justified the integration by parts in (15) near a cone point with $\beta \notin \mathbb{N}$.

4.1.2. Developing maps near a cone point with $\beta = n \in \mathbb{N}$. Near such a cone point the developing map f is given by

$$f = \frac{az^n + b}{cz^n + d}$$

for some $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$. By choosing a suitable coordinate and a representative of the developing map the metric is given by the same form as the noninteger case [22]:

$$g = \frac{4n^2 |z|^{2(n-1)}}{(1 + |z|^{2n})^2} |dz|^2.$$

Therefore, we can again obtain the expansion of ϕ_k as follows:

$$(27) \quad \begin{aligned} \phi_k &= \sum_{j \geq 0} C_{k,j} r^{k+2jn}, & -J \leq k; \\ \phi_k &= \sum_{j \geq 0} C_{k,j} r^{|k|+2jn}, & -J > k. \end{aligned}$$

The computation of X and $\nabla^{(0,1)}X$ can now be applied verbatim, and we list the results here. First of all,

$$X = \left(\sum_{k \in \mathbb{Z}} X_k(r) e^{i(k+1)\theta} \right) \partial_z.$$

In terms of local coordinates, it is given by

$$(28) \quad X_k = e^{-i(k+1)\theta} \frac{(1+r^{2n})^2}{4n^2 r^{2(n-1)}} \frac{e^{i\theta}}{2} \left(\partial_r - \frac{1}{ir} \partial_\theta \right) (\phi_k(r) e^{ik\theta}).$$

It immediately follows that

$$(29) \quad \begin{aligned} X_k(r) &= \sum_{j \geq 1} \tilde{C}_{k,j} r^{k+(2j-2)n+1}, \quad -J \leq k, \\ X_k(r) &= \sum_{j \geq 0} \tilde{C}_{k,j} r^{|k|+(2j-2)n+1}, \quad -J > k. \end{aligned}$$

We then compute $\nabla^{(0,1)}X$ which is locally given by

$$\sum_{k \in \mathbb{Z}} \partial_{\bar{z}} (X_k e^{i(k+1)\theta}) \partial_z \otimes d\bar{z}.$$

As before, we compute $\partial_{\bar{z}} (X_k e^{i(k+1)\theta}) = \frac{e^{i\theta}}{2} (\partial_r - \frac{1}{ir} \partial_\theta) (X_k(r) e^{i(k+1)\theta})$ to get

$$(30) \quad \begin{aligned} \partial_{\bar{z}} (X_k e^{i(k+1)\theta}) &= e^{i(k+2)\theta} \sum_{j \geq 2} (2j-2)n \tilde{C}_{k,j} r^{k+(2j-2)n}, \quad -J \leq k, \\ \partial_{\bar{z}} (X_k e^{i(k+1)\theta}) &= e^{i(k+2)\theta} \sum_{j \geq 0} (-2k + (2j-2)n) \tilde{C}_{k,j} r^{|k|+(2j-2)n}, \quad -J > k. \end{aligned}$$

Notice that $J = n - 1$, so we can check again that in both cases above the term has enough decay to be integrable. In the first case, the smallest exponent $k + 2n > 0$. In the second case, we have $|k| - 2n > -n$ except when $k = -(J+1) = -n$. If $|k| - 2n > -n$, then again notice that $r^{-n} \sim r^{-1}$, so the term is integrable. If $k = -n$, then we have $-2k - 2n = 0$. In this case, the leading term vanishes in $\nabla^{(0,1)}X$ and the next term in the expansion is integrable again.

4.1.3. Justification of the integration by parts. Now we are ready to prove Theorem 3 and Proposition 2.

Proof of Theorem 3

From the previous computation of $\nabla^{(0,1)}X$ near each conical point, we know that $|\nabla^{(0,1)}X|^2$ is integrable. Therefore, using (14) and (15) we have

$$(31) \quad 0 \leq \int |\nabla^{(0,1)}X|^2 = \frac{1}{2}(\lambda - 2) \int |X|^2,$$

which immediately shows that $\lambda \geq 2$. □

Now we look at the case when $\lambda = 2$.

Proof of Proposition 2

We combine (15) and (14) to get

$$(32) \quad \nabla^{(0,1)} X = 0,$$

which shows that $\nabla^{(0,1)} X$ is holomorphic on $\Sigma \setminus \text{supp } D$.

We next show that X can be extended to a meromorphic vector field on Σ . From the expansion of X near each cone point (see Expressions (21), (22), and (29)), we see that X is bounded by $|z|^{-J}$; therefore, there cannot be any essential singularities. Hence, X must be meromorphic on Σ .

Moreover, we can see the worst order of pole of X from the expansion. For the behavior near a smooth point, X is smooth since ϕ is smooth. Near a cone point, we look at the expansion of X and $\nabla^{(0,1)} X$. If $\beta \notin \mathbb{N}$, then all coefficients $\tilde{C}_{k,j}$ for $k < -J$ in (26) have to vanish because $\nabla^{(0,1)} X = 0$, and this shows that the worst decay of X is given by the $k = -J$ mode, which is r^{-J+1} by (21). In particular, if $\beta < 1$, then the worst decay is r^1 , so it is actually a zero for X ; if $\beta > 1$, then the order of the pole is given by $r^{-[\beta]+1}$. On the other hand, if $\beta = n \in \mathbb{N}$, then the coefficient $C_{-J-1,0}$ in (30) might not vanish (since this term corresponds to $k = -n$ and does not appear in $\nabla^{(0,1)} X$); therefore, the worst possible decay in X is of the order r^{-n+1} . $\square$

We remark here that the worst order of decay of X described in the proposition above applies to any 2-eigenfunction. In the next step, we assume in addition that ϕ is real-valued, and the decay estimate will be improved by relating to developing maps.

4.2. From vector fields to reducible metrics

From now on, we assume that the eigenfunction ϕ is real-valued.

LEMMA 5

The algebraic dual one-form of X , denoted by Ω , is a meromorphic one-form on Σ .

Proof

This follows from Proposition 2 that X is meromorphic. $\square$

We denote by $D_X := (\Omega)$ the divisor associated to the meromorphic one-form Ω . Let (U, z) be a coordinate chart that does not intersect $\text{supp } D_X \cup \text{supp } D$. Denote $g = e^{2u}|dz|^2$ on U , and let $X = \frac{1}{4}F(z)\partial_z$; that is, $F(z) = 4e^{-2u}\phi_{\bar{z}}$.

LEMMA 6

There exists a global positive constant $C = C_\phi$ which is independent of U , such that

$$(33) \quad \phi_z = -\frac{\phi^2 - C^2}{F(z)}, \quad e^{2u} = -4\frac{\phi^2 - C^2}{|F(z)|^2}.$$

Moreover, the real part of the one-form Ω is exact on $\Sigma \setminus (\text{supp } D_X \cup \text{supp } D)$.

Proof

Since ϕ is a 2-eigenfunction of Δ_g , in local coordinates we have

$$-4e^{-2u}\phi_{z\bar{z}} = 2\phi \quad \text{and} \quad \phi_{z\bar{z}} = -\frac{\phi e^{2u}}{2} = -\frac{2\phi\phi_{\bar{z}}}{F} = -\left(\frac{\phi^2}{F}\right)_{\bar{z}}.$$

Since F does not vanish anywhere in U , there exists a holomorphic function $g(z)$ on U such that

$$\phi_z = -\frac{\phi^2}{F(z)} + g(z).$$

Since ϕ is real-valued, we also have

$$\overline{\phi_{\bar{z}}} = -\frac{\phi^2}{F(z)} + g(z).$$

Combining with $F(z) = 4e^{-2u}\phi_{\bar{z}}$, we find that

$$e^{2u} = \frac{4\overline{\phi_{\bar{z}}}}{F(z)} = -\frac{4\phi^2}{|F(z)|^2} + \frac{4g(z)}{F(z)} = -\frac{4\phi^2}{|F(z)|^2} + \frac{4g(z)F(z)}{|F(z)|^2}.$$

Therefore, the holomorphic function $4g(z)F(z)$ satisfies

$$4g(z)F(z) = (|F(z)|^2 e^{2u} + 4\phi^2),$$

where the right-hand side is a positive real function. It follows that $g(z)F(z) = C^2$ for some positive constant $C = C_U$ on U . Since $\Sigma \setminus (\text{supp } D_X \cup \text{supp } D)$ is connected, the constant $C = C_U$ does not depend on U . This proves (33).

By the first equality in (33), we have

$$(34) \quad 2\Re \frac{dz}{F} = \frac{dz}{F} + \frac{d\bar{z}}{\bar{F}} = \frac{d\phi}{C^2 - \phi^2} = d\left(\frac{1}{2C} \ln \frac{C + \phi}{C - \phi}\right).$$

Since $\Omega = 4\frac{dz}{F}$ and C does not depend on U , we have that $\Re\Omega$ is exact on $\Sigma \setminus (\text{supp } D_X \cup \text{supp } D)$. $\square$

LEMMA 7

Define the following one-form on Σ :

$$(35) \quad \omega := -2C \frac{dz}{F}.$$

Consider the multi-valued holomorphic function on $\Sigma \setminus \{\text{poles of } \omega\}$ defined by

$$f(z) := \exp\left(\int^z \omega\right),$$

which satisfies that:

$$f : \Sigma \setminus (\text{supp}(\omega) \cup \text{supp } D) \rightarrow \mathbb{C}^*$$

is a multi-valued holomorphic function with monodromy in $U(1)$. Moreover, on $\Sigma \setminus (\text{supp}(\omega) \cup \text{supp } D)$, we have

$$f^* g_{\text{st}} = g$$

and

$$(36) \quad \phi = C \cdot \frac{1 - |f|^2}{1 + |f|^2}.$$

Proof

We first observe that the divisor $D_X := (\Omega)$ is equal to (ω) since Ω is a multiple of ω . The monodromy property of f follows from the previous lemma that $\Re \omega$ is exact on $\Sigma \setminus (\text{supp}(\omega) \cup \text{supp} D)$. Using (34), we have

$$|f|^2 = \exp\left(\int^z 2\Re \omega\right) = \frac{C - \phi}{C + \phi},$$

which proves (36).

In order to show $f^*g_{\text{st}} = g$, it suffices to show that in any complex coordinate chart (U, z) not intersecting $\text{supp} D_X \cup \text{supp} D$, we have

$$\frac{4|f'(z)|^2}{(1 + |f|^2)^2} = 4 \frac{C^2 - \phi^2}{|F|^2}.$$

Since $|f'(z)|^2 = \frac{4C^2|f|^2}{|F|^2}$ and $|f|^2 = \frac{C - \phi}{C + \phi}$, the above equation follows from a direct computation. $\square$

LEMMA 8

The one-form ω defined in (35) has only simple poles with real residues, and its real part is exact outside its poles. Moreover, the $\mathbb{R}$ -divisor D represented by the metric g and the $\mathbb{Z}$ -divisor (ω) associated to ω are related by

$$D = (\omega)_0 + \sum_{P: \text{pole of } \omega} (|\text{Res}_P(\omega)| - 1)[P],$$

where $(\omega)_0$ is the zero divisor of the meromorphic one-form ω . In particular, a cone point of noninteger angle $2\pi\beta$ must be a simple pole with residue $\pm\beta$ of ω , and a cone point of integer angle $2\pi n$ may be either a simple pole with residue $\pm n$ or a zero of ω with multiplicity $n - 1$.

Proof

We divide the proof into the following three cases.

Case 1: A smooth point

Let P be a smooth point of the metric g where X vanishes. We show that P is a simple pole of ω such that $\text{Res}_P(\omega)$ equals either -1 or 1 . Take an open disc U centered at P such that $U^* := U \setminus \{P\}$ does not intersect $\text{supp}(\omega) \cup \text{supp} D$. Since g is smooth on U , by [14, Lemma 3.2], we can choose a developing map of $g|_{U^*}$, denoted by $h : U^* \rightarrow \overline{\mathbb{C}}$, and a suitable coordinate z with $z(P) = 0$, such that $h(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$. On the other hand, by Lemma 7, the restriction $f|_{U^*}$ is a developing map of g . Since f has trivial local monodromy around P , f extends to U and coincides with h up to a $\text{PSU}(2)$ transformation. Therefore, any representative $\mathfrak{f}$ of the developing map f is also given by the form $\mathfrak{f}(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$ on U . Hence, on U

we have

$$\omega = \frac{df}{f} = \frac{d\mathfrak{f}}{\mathfrak{f}} = \frac{dz}{(az+b)(cz+d)}.$$

Since ω has a pole at $z = 0$, we have that $bd = 0$, but b and d cannot both be 0 because $ad - bc = 1$, which implies that P is a simple pole of ω and has residue ± 1 .

Case 2: A cone point of noninteger angle

Let P be a cone point of angle $2\pi\beta$, where $\beta > 0$ is *not* an integer. We show that P is a simple pole of ω such that $\text{Res}_P(\omega)$ equals either $-\beta$ or β . Similar to the previous case, we take an open disc U centered at P such that $U^* = U \setminus \{P\}$ does not intersect $\text{supp}(\omega) \cup \text{supp} D$. By Lemma 7, the restriction $f|_{U^*}$ is a developing map of $g|_{U^*}$. Since $f|_{U^*}$ has monodromy in $U(1)$ and $g|_{U^*}$ has a cone point of angle $2\pi\beta$ at P , by the proof of Theorem 1.4 in [14], there exists in U a complex coordinate z which is centered at P , such that in any disc contained in U^* , each representative $\mathfrak{f}$ of f , has the form of either $\mathfrak{f}(z) = \mu z^\beta$ or $\mathfrak{f}(z) = \tilde{\mu} z^{-\beta}$, where μ and $\tilde{\mu}$ are nonzero constants. Therefore, in a neighborhood of P , we have $\omega = \frac{d\mathfrak{f}}{\mathfrak{f}} = \pm \beta \frac{dz}{z}$.

Case 3: A cone point of integer angle

Let P be a cone point of angle $2\pi n$, where $n > 1$ is an integer. We show that P is either a simple pole with residue $\pm n$ or a zero of ω with multiplicity $(n - 1)$. Take an open disc U centered at P such that $U^* = U \setminus \{P\}$ does not intersect $\text{supp}(\omega) \cup \text{supp} D$. Since g is smooth on U^* and has a cone point with angle $2\pi n \in 2\pi\mathbb{Z}$ at P , by [14, Lemma 3.2], we can choose a developing map $h : U \rightarrow \overline{\mathbb{C}}$ for $g|_U$ and a suitable coordinate z with $z(P) = 0$ such that h is given by $h(z) = \frac{az^n + b}{cz^n + d}$ with $ad - bc = 1$. On the other hand, by Lemma 7, the restriction $f|_{U^*}$ is a developing map of $g|_{U^*}$. Since f has trivial monodromy around P , f extends to U and coincides with h up to a $\text{PSU}(2)$ transformation. Therefore, each representative $\mathfrak{f}$ of f also has the form of $\mathfrak{f}(z) = \frac{az^n + b}{cz^n + d}$ with $ad - bc = 1$ on U . Hence, on U we have

$$\omega = \frac{df}{f} = \frac{d\mathfrak{f}}{\mathfrak{f}} = \frac{nz^{n-1}}{(az^n + b)(cz^n + d)} dz.$$

If $bd \neq 0$, then P is a zero of ω with multiplicity $(n - 1)$. Otherwise, P is a simple pole of ω with residue $\pm n$. $\square$

Proof of Theorem 2

It follows immediately from Lemma 8. Moreover, ω is a character one-form ([14, Definition 1.3]) of the reducible metric g . $\square$

COROLLARY 1

The function ϕ extends continuously to Σ and is smooth outside those poles with non-integer residues of ω . The positive constant C in Lemma 6 equals $\max_\Sigma |\phi|$. Moreover, ϕ achieves the maximum C (resp. the minimum $-C$) at each pole of ω with positive (resp. negative) residue. Each zero of ω is a saddle point of ϕ .

Proof

It follows from (36), Lemma 8, and the local behavior of f near cone points described in the proof of Lemma 8. $\square$

REMARK

This corollary shows that ϕ is also in the Friedrichs extension.

*4.3. The dimension of the 2-eigenspace***THEOREM 4**

Let $\mathbb{E}_{\mathbb{R}}^{\text{Hol}}$ be the real vector space of real 2-eigenfunctions of Δ_g^{Hol} for a reducible metric g . Then $\dim \mathbb{E}_{\mathbb{R}}^{\text{Hol}}$ equals either 1 or 3. Moreover, $\dim \mathbb{E}_{\mathbb{R}}^{\text{Hol}} = 3$ if and only if g is the pullback metric f^*g_{st} by a branched cover $f : \Sigma \rightarrow \overline{\mathbb{C}}$.

Proof

Choose a multiplicative developing map f of the reducible metric g .

Case 1: (Nontrivial) reducible monodromy

Suppose the monodromy of f is nontrivial. Then there are exactly two multiplicative developing maps, f and $1/f$, for the metric g . Since each real-valued 2-eigenfunction ϕ of Δ_g^{Hol} can be expressed in terms of such a developing map as (36), we can see that the dimension of $\mathbb{E}_{\mathbb{R}}^{\text{Hol}}$ equals 1.

Case 2: Trivial monodromy

Suppose that g is the pullback metric f^*g_{st} by a branched cover $f : \Sigma \rightarrow \overline{\mathbb{C}}$. First we know that $\dim \mathbb{E}_{\mathbb{R}}^{\text{Hol}} \geq 3$, as the three eigenfunctions on $\mathbb{S}^2$ lift to Σ via the pullback, and they give three independent eigenfunctions. On the other hand, for a real-valued 2-eigenfunction ϕ with maximum 1, by (36), there exists constants a, b such that

$$|a|^2 + |b|^2 = 1 \quad \text{and} \quad \phi = \frac{1 - |g|^2}{1 + |g|^2} \quad \text{for } g = \frac{af + b}{-\bar{b}f + \bar{a}}.$$

By a simple computation, we find that ϕ is a linear combination of the following three 2-eigenfunctions:

$$\frac{1 - |f|^2}{1 + |f|^2}, \quad \Re \frac{2f}{1 + |f|^2} \quad \text{and} \quad \Im \frac{2f}{1 + |f|^2}.$$

This proves the dimension of $\mathbb{E}_{\mathbb{R}}^{\text{Hol}}$ in this case is 3. □

REMARK

In some cases, one can obtain more information including all complex-valued 2-eigenfunctions. When Σ is a branched cover of the sphere obtained by covering map $f(z) = z^n$, $n \in \mathbb{N}$, explicit computation shows that the complex dimension of complex-valued 2-eigenfunctions in $\mathcal{D}^{\text{Hol}}$ and $\mathcal{D}^{\text{Fr}}$ are both equal to 3 (see [44, Lemma 2] for the computation). In this case, the eigenfunctions in the domain of the two extensions coincide and are given by the pullback by f of the three eigenfunctions on the sphere.

5. Further discussion

The spectral condition in Theorem A uses a real eigenfunction in the holomorphic extension, which then is automatically a function in the Friedrichs extension—i.e., the coefficients a_k in (6) all vanish. However, one cannot replace the statement of

the theorem by using a real eigenfunction in the Friedrichs extension. In fact, works by Mondello–Panov [41], Eremenko–Gabrielov–Tarasov [18], and Chen [9] suggest that there exist irreducible metrics with eigenvalue 2 in the spectrum of the Friedrichs Laplacian. There is ongoing work to find an explicit example of such an eigenfunction and understand the behavior of its associated complex gradient vector field, which is no longer guaranteed to be meromorphic.

One thing to notice here is that, even though the Laplace–Beltrami operator is real, Friedrichs extension is the only extension that respects the real splitting—that is, we have the following relation for a function $\phi = u + iv$:

$$\Delta_g \phi = 2\phi \quad \Longleftrightarrow \quad \Delta_g u = 2u, \quad \Delta_g v = 2v,$$

and

$$\phi \in \mathcal{D}^{\text{Fr}} \quad \Longleftrightarrow \quad u \in \mathcal{D}^{\text{Fr}}, \quad v \in \mathcal{D}^{\text{Fr}}.$$

However, for the holomorphic extension, we have only

$$\phi \in \mathcal{D}^{\text{Hol}} \quad \Longleftarrow \quad u \in \mathcal{D}^{\text{Hol}}, \quad v \in \mathcal{D}^{\text{Hol}}.$$

The observation above justifies the choice of a real eigenfunction in the statement of the theorem. It is unknown whether for a reducible metric there exists any nontrivial complex-valued eigenfunction in the holomorphic extension, such that its real or imaginary part is not in the same extension. Corresponding to Theorem 4, one may ask the question about the complex dimension of all such functions.

Similarly, there is a question whether an irreducible metric can have a nontrivial complex-valued eigenfunction in the holomorphic extension, which is not excluded by our theorem. By Proposition 2, any such eigenfunction would produce a meromorphic vector field, and it is an interesting question whether there is any geometric implication if such a metric exists.

Acknowledgments. The authors would like to thank Rafe Mazzeo and Song Sun for many insightful discussions and for their hospitality during B. X.’s visit to Stanford University and UC Berkeley in spring 2019. Part of the work was completed when both authors visited SCMS, Fudan University in summer 2019. The authors would also like to thank Pierre Albin, Zhijie Chen, Alexandre Eremenko, Luc Hillairet, Misha Karpukhin, Chris Kottke, Chang-Shou Lin, Gerardo Mendoza, Gabriele Mondello, Dmitri Panov, Chin-Lung Wang, Yingyi Wu, Hao Yin, and the anonymous referees for helpful suggestions and comments. X. Z. is partially supported by the NSF grant DMS-2041823. Part of the work was supported by the NSF grant DMS-1440140 while X. Z. was in residence at MSRI in Berkeley, California, during the fall 2019 semester. B. X. is partially supported by the National Natural Science Foundation of China (Grant Nos. 11571330, 11971450, and 12071449) and the Fundamental Research Funds for the Central Universities. Part of the work was done while B. X. was visiting the Institute of Mathematical Sciences at Shanghai Tech University in spring 2019.

References

- [1] W. Ballmann, *Lectures on Kähler Manifolds*, ESI Lect. Math. Phys. European Mathematical Society (EMS), Zürich, 2006. [MR 2243012](#). [DOI 10.4171/025](#).
- [2] D. Bartolucci, F. De Marchis, and A. Malchiodi, *Supercritical conformal metrics on surfaces with conical singularities*, Int. Math. Res. Not. IMRN **24** (2011), 5625–5643. [MR 2863376](#). [DOI 10.1093/imrn/rnq285](#).
- [3] J. Brüning and R. Seeley, *Regular singular asymptotics*, Adv. in Math. **58** (1985), no. 2, 133–148. [MR 0814748](#). [DOI 10.1016/0001-8708\(85\)90114-8](#).
- [4] J. Brüning and R. Seeley, *An index theorem for first order regular singular operators*, Amer. J. Math. **110** (1988), no. 4, 659–714. [MR 0955293](#). [DOI 10.2307/2374646](#).
- [5] A. Carlotto, *On the solvability of singular Liouville equations on compact surfaces of arbitrary genus*, Trans. Amer. Math. Soc. **366** (2014), no. 3, 1237–1256. [MR 3145730](#). [DOI 10.1090/S0002-9947-2013-05847-3](#).
- [6] Q. Chen, X. Chen, and Y. Wu, *The structure of HCMU metric in a K -surface*, Int. Math. Res. Not. **16** (2005), 941–958. [MR 2146186](#). [DOI 10.1155/IMRN.2005.941](#).
- [7] J. Cheeger, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. USA **76** (1979), no. 5, 2103–2106. [MR 0530173](#). [DOI 10.1073/pnas.76.5.2103](#).
- [8] X. Chen, *Obstruction to the existence of metric whose curvature has umbilical Hessian in a K -surface*, Comm. Anal. Geom. **8** (2000), no. 2, 267–299. [MR 1753319](#). [DOI 10.4310/CAG.2000.v8.n2.a2](#).
- [9] Z. Chen, personal communication, 2019.
- [10] Z. Chen, T.-J. Kuo, and C.-S. Lin, *Existence and non-existence of solutions of the mean field equations on flat tori*, Proc. Amer. Math. Soc. **145** (2017), no. 9, 3989–3996. [MR 3665050](#). [DOI 10.1090/proc/13543](#).
- [11] Q. Chen, B. Li, J. Song, and B. Xu, *Jenkins–Strebel differentials and reducible cone spherical metrics on compact Riemann surfaces*, in preparation.
- [12] C.-L. Chai, C.-S. Lin, and C.-L. Wang, *Mean field equations, hyperelliptic curves and modular forms: I*, Camb. J. Math. **3** (2015), nos. 1–2, 127–274. [MR 3356357](#). [DOI 10.4310/CJM.2015.v3.n1.a3](#).
- [13] Q. Chen and Y. Wu, *Character 1-form and the existence of an HCMU metric*, Math. Ann. **351** (2011), no. 2, 327–345. [MR 2836661](#). [DOI 10.1007/s00208-010-0598-z](#).
- [14] Q. Chen, W. Wang, Y. Wu, and B. Xu, *Conformal metrics with constant curvature one and finitely many conical singularities on compact Riemann surfaces*, Pacific J. Math. **273** (2015), no. 1, 75–100. [MR 3290445](#). [DOI 10.2140/pjm.2015.273.75](#).
- [15] Q. Chen, Y. Wu, and B. Xu, *On one-dimensional and singular Calabi’s extremal metrics whose Gauss curvatures have nonzero umbilical Hessians*, Israel J. Math. **208** (2015), no. 1, 385–412. [MR 3416925](#). [DOI 10.1007/s11856-015-1204-6](#).
- [16] S. Dey, *Spherical metrics with conical singularities on 2-spheres*, Geom. Dedicata **196** (2018), 53–61. [MR 3853628](#). [DOI 10.1007/s10711-017-0306-1](#).
- [17] A. Eremenko and A. Gabrielov, *On metrics of curvature 1 with four conic singularities on tori and on the sphere*, Illinois J. Math. **59** (2015), no. 4, 925–947. [MR 3628295](#). [DOI 10.1215/ijm/1488186015](#).

- [18] A. Eremenko, A. Gabrielov, and V. Tarasov, *Spherical quadrilaterals with three non-integer angles*, Zh. Mat. Fiz. Anal. Geom. **12** (2016), no. 2, 134–167. MR 3498735. DOI 10.15407/mag12.02.134.
- [19] A. Eremenko, *Metrics of positive curvature with conic singularities on the sphere*, Proc. Amer. Math. Soc. **132** (2004), no. 11, 3349–3355. MR 2073312. DOI 10.1090/S0002-9939-04-07439-8.
- [20] A. Eremenko, *Co-axial monodromy*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **XX** (2020), no. 2, 619–634. MR 4105912. DOI 10.2422/2036-2145.201706_022.
- [21] A. Eremenko, *Metrics of constant positive curvature with four conic singularities on the sphere*, Proc. Amer. Math. Soc. **148** (2020), no. 9, 3957–3965. MR 4127839. DOI 10.1090/proc/15012.
- [22] Y. Feng, Y. Q. Shi, and B. Xu, *On the explicit expression of a conformal metric of constant curvature one near a conical singularity*, J. Univ. Sci. Technol. China **47** (2017), no. 6, 455–458. MR 3903099. DOI 10.3969/j.issn.0253-2778.2017.06.001.
- [23] J. B. Gil, T. Krainer, and G. A. Mendoza, *Resolvents of elliptic cone operators*, J. Funct. Anal. **241** (2006), no. 1, 1–55. MR 2264246. DOI 10.1016/j.jfa.2006.07.010.
- [24] J. B. Gil, T. Krainer, and G. A. Mendoza, *Geometry and spectra of closed extensions of elliptic cone operators*, Canad. J. Math. **59** (2007), no. 4, 742–794. MR 2338233. DOI 10.4153/CJM-2007-033-7.
- [25] J. B. Gil and G. A. Mendoza, *Adjoint of elliptic cone operators*, Amer. J. Math. **125** (2003), no. 2, 357–408. MR 1963689. DOI 10.1353/ajm.2003.0012.
- [26] J. Hersch, *Quatre propriétés isopérimétriques de membranes sphériques homogènes*, C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A1645–A1648. MR 0292357.
- [27] L. Hillairet, “Spectral theory of translation surfaces: A short introduction” in *Actes du Séminaire de Théorie Spectrale et Géométrie, Année 2009–2010*, Sémin. Théor. Spectr. Géom. **28**, Univ. Grenoble I, Saint-Martin-d’Hères, 2010, 51–62. MR 2848211. DOI 10.5802/tsg.278.
- [28] L. Hillairet and A. Kokotov, *Isospectrality, comparison formulas for determinants of Laplacian and flat metrics with non-trivial holonomy*, Proc. Amer. Math. Soc. **145** (2017), no. 9, 3915–3928. MR 3665043. DOI 10.1090/proc/13494.
- [29] M. Kapovich, *Branched covers between spheres and polygonal inequalities in simplicial trees*, <https://math.ucdavis.edu/~kapovich/EPR/covers.pdf>, 2017.
- [30] M. Karpukhin, N. Nadirashvili, A. V. Penskoi, and I. Polterovich, *An isoperimetric inequality for Laplace eigenvalues on the sphere*, accepted J. Differential Geom.
- [31] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics] **136**, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1997. MR 1449639.
- [32] A. Lichnerowicz, *Géométrie des groupes de transformations*, Travaux et Recherches Mathématiques III. Dunod, Paris, 1958. MR 0124009.
- [33] F. Luo and G. Tian, *Liouville equation and spherical convex polytopes*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 1119–1129. MR 1137227. DOI 10.2307/2159498.

- [34] C.-S. Lin and C.-L. Wang, *Elliptic functions, Green functions and the mean field equations on tori*, Ann. of Math. (2), **172** (2010), no. 2, 911–954. MR 2680484. DOI 10.4007/annals.2010.172.911.
- [35] C.-S. Lin and X. Zhu, *Explicit construction of extremal Hermitian metrics with finite conical singularities on S^2* , Comm. Anal. Geom. **10** (2002), no. 1, 177–216. MR 1894145. DOI 10.4310/CAG.2002.v10.n1.a8.
- [36] Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kaehlerienne*, Nagoya Math. J. **11** (1957), 145–150. MR 0094478. DOI 10.1017/S0027763000002026.
- [37] R. C. McOwen, *Point singularities and conformal metrics on Riemann surfaces*, Proc. Amer. Math. Soc. **103** (1988), no. 1, 222–224. MR 0938672. DOI 10.2307/2047555.
- [38] R. B. Melrose, *The Atiyah–Patodi–Singer index theorem*, Res. Notes Math. **4**, A. K. Peters, Wellesley, MA, 1993. MR 1348401. DOI 10.1016/0377-0257(93)80040-i.
- [39] E. A. Mooers, *Heat kernel asymptotics on manifolds with conic singularities*, J. Anal. Math. **78** (1999), 1–36. MR 1714065. DOI 10.1007/BF02791127.
- [40] G. Mondello and D. Panov, *Spherical metrics with conical singularities on a 2-sphere: Angle constraints*, Int. Math. Res. Not. IMRN **16** (2016), 4937–4995. MR 3556430. DOI 10.1093/imrn/rnv300.
- [41] G. Mondello and D. Panov, *Spherical surfaces with conical points: Systole inequality and moduli spaces with many connected components*, Geom. Funct. Anal. **29** (2019), no. 4, 1110–1193. MR 3990195. DOI 10.1007/s00039-019-00506-3.
- [42] R. Mazzeo and H. Weiss, “Teichmüller theory for conic surfaces” in *Geometry, Analysis and Probability*, Progr. Math. **310**, Birkhäuser/Springer, Cham, 2017. MR 3821926.
- [43] R. Mazzeo and X. Zhu, *Conical metrics on Riemann surfaces, I: The compactified configuration space and regularity*, Geom. Topol. **24** (2020), no. 1, 309–372. MR 4080484. DOI 10.2140/gt.2020.24.309.
- [44] R. Mazzeo and X. Zhu, *Conical metrics on Riemann surfaces, II: Spherical metrics*, accepted Int. Math. Res. Not.
- [45] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14**, 333–340. MR 0142086. DOI 10.2969/jmsj/01430333.
- [46] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975. MR 0493420.
- [47] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I*, 2nd ed. Academic Press, New York, 1980. Functional analysis. MR 0751959.
- [48] J. Song, Y. Cheng, B. Li, and B. Xu, *Drawing cone spherical metrics via Strebel differentials*, Int. Math. Res. Not. IMRN **11** (2020), 3341–3363. MR 4123106. DOI 10.1093/imrn/rny103.
- [49] J. Song and B. Xu, *On rational functions with more than three branch points*, Algebra Colloq. **27** (2020), no. 2, 231–246. MR 4095937. DOI 10.1142/S100538672000019X.
- [50] M. Troyanov, *Prescribing curvature on compact surfaces with conical singularities*, Trans. Amer. Math. Soc. **324** (1991), no. 2, 793–821. MR 1005085. DOI 10.2307/2001742.

- [51] M. Umehara and K. Yamada, *Metrics of constant curvature 1 with three conical singularities on the 2-sphere*, Illinois J. Math. **44** (2000), no. 1, 72–94. [MR 1731382](#). [DOI 10.1215/ijm/1255984594](#).
- [52] X. Zhu, *Rigidity of a family of spherical conical metrics*, New York J. Math. **26** (2020), 272–284. [MR 4075251](#).
- [53] X. Zhu, *Spherical conic metrics and realizability of branched covers*, Proc. Amer. Math. Soc. **147** (2019), no. 4, 1805–1815. [MR 3910445](#). [DOI 10.1090/proc/14318](#).

Xu: CAS Wu Wen-Tsun Key Laboratory of Mathematics and School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, China; bxu@ustc.edu.cn

Zhu: Department of Mathematics, Northeastern University, Boston, Massachusetts, USA; x.zhu@northeastern.edu