

Stability of Gradient Learning Dynamics in Continuous Games: Scalar Action Spaces

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Abstract—Learning processes in games explain how players grapple with one another in seeking an equilibrium. We study a natural model of learning based on individual gradients in two-player continuous games. In such games, the arguably natural notion of a local equilibrium is a differential Nash equilibrium. However, the set of locally exponentially stable equilibria of the learning dynamics do not necessarily coincide with the set of differential Nash equilibria of the corresponding game. To characterize this gap, we provide formal guarantees for the stability or instability of such fixed points by leveraging the spectrum of the linearized game dynamics. We provide a comprehensive understanding of scalar games and find that equilibria that are both stable and Nash are robust to variations in learning rates.

I. INTRODUCTION

The study of learning in games is experiencing a resurgence in the control theory [19], [21], [22], optimization [11], [13], and machine learning [4]–[6], [8], [14] communities. Partly driving this resurgence is the prospect for game-theoretic analysis to yield machine learning algorithms that generalize better or are more robust. Towards understanding the optimization landscape in such formulations, dynamical systems theory is emerging as a principal tool for analysis and ultimately synthesis [1]–[3], [11], [12]. A predominant learning paradigm used across these different domains is gradient-based learning. Updates in large decision spaces can be performed locally with minimal information, while still guaranteeing local convergence in many problems [5], [13].

One of the primary means to understand the optimization landscape of games is the eigenstructure and spectrum of the Jacobian of the learning dynamics in a neighborhood of a stationary point. In particular, for a zero-sum continuous game $(f, -f)$ with some continuously-differentiable f , the Nash equilibria are saddle points of the function f . As the example in Fig. 1 demonstrates, not all saddle points are relevant. Loosely speaking, the equilibrium conditions for the game correspond to constraints on the curvature directions of the cost function and hence, on the eigenstructure of the Jacobian nearby equilibria.

The local stability of a hyperbolic fixed point in a nonlinear system can be assessed by examining the eigenstructure of the linearized dynamics [9], [20]. However, in a

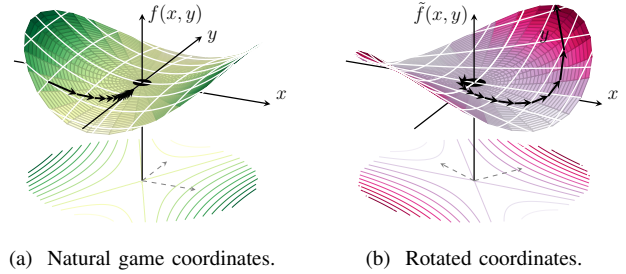


Fig. 1: *Cost landscape is crucial to understanding dynamics.* The zero-sum game defined by $f(x, y) = \frac{1}{2}x^2 - \frac{1}{8}y^2$ has a Nash equilibrium at the origin, which is a stable saddle point of gradient play (1). If the cost function is rotated to $\tilde{f}(x, y) = \frac{1}{32}x^2 + \frac{11}{32}y^2 - \frac{5\sqrt{3}}{16}xy$ —a rotation by $\frac{\pi}{3}$ —then the origin is no longer a Nash equilibrium, and is *unstable* under gradient play.

game context there are extra constraints coming from the underlying game—that is, players are constrained to move only along directions over which they have control. They can only control their individual actions, as opposed to the entire state of the dynamical system corresponding to the learning rules being applied by the agents. It has been observed in earlier work that not all stable attractors of gradient play are local Nash equilibria and not all local Nash equilibria are stable attractors of gradient play [11]. Furthermore, changes in players’ learning rates—which corresponds to scaling rows of the Jacobian—can change an equilibrium from being stable to unstable and vice versa [5].

To summarize, there is a subtle but extremely important difference between game dynamics and traditional nonlinear dynamical systems: alignment conditions are important for distinguishing between equilibria that have game-theoretic meaning versus those which are simply stable attractors of learning rules, and features of learning dynamics such as learning rates can play an important role in shaping not only equilibria but also alignment properties. Motivated by this observation along with the recent resurgence of applications of learning in games in control, optimization, and machine learning, in this paper we provide an in-depth analysis of the spectral properties of gradient-based learning in two-player continuous games.

Contributions. This paper characterizes the spectral properties of structured 2×2 matrices and analyzes the stability of equilibria in continuous games. Having a complete algebraic

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understanding of the spectrum of the game Jacobian is fundamental to understanding when Nash equilibria coincide with stable equilibria. Many of our results are geometric in nature and are accompanied by diagrams.

It is known that the quadratic numerical range of a block operator matrix contains the operator's (point) spectrum [23]. Thus, it serves as an important tool for quantifying the spectrum of two-player game dynamics. The method for obtaining the quadratic numerical range is by reducing a block matrix to 2×2 matrices.

Towards this end, we decompose the 2×2 game Jacobian into coordinates that reflect the interaction between the players. The decomposition provides insights on games and vector fields in general, which permits us to provide a complete characterization of the stability of equilibria in two-player gradient learning dynamics.

Organization. In Section II, we describe the gradient-based learning paradigm and analyze the spectral properties of block operator matrices using the quadratic numerical range [23]. In Section III, we analyze the spectral properties of two-player continuous games on scalar action spaces. Our main results are on general-sum games, with insights drawn from specific classes of games. In Section IV, we certify the stability of Nash and non-Nash equilibria in two-player scalar games. A key finding is that in the scalar case, equilibria that are both stable and Nash are robust to variations in learning rates; in the vector case, they are not. We provide an example in Section V and conclude in Section VI.

II. PRELIMINARIES

This section contains game-theoretic preliminaries, mathematical formalism, and a description of the gradient-based learning paradigm studied in this paper.

A. Game-Theoretic Preliminaries

A 2-player *continuous game* $\mathcal{G} = (f_1, f_2)$ is a collection of costs defined on $X = X_1 \times X_2$ where player (agent) $i \in \mathcal{I} = \{1, 2\}$ has cost $f_i : X \rightarrow \mathbb{R}$. In this paper, the results apply to games with sufficiently smooth costs $f_i \in C^r(X, \mathbb{R})$ for some $r \geq 0$. Agent i 's set of feasible actions is the d_i -dimensional precompact set $X_i \subseteq \mathbb{R}^{d_i}$. The notation x_{-i} denotes the action of player i 's competitor; that is, $x_{-i} = x_j$ where $j \in \mathcal{I} \setminus \{i\}$.¹

The most common and arguably natural notion of an equilibrium in continuous games is due to Nash [16].

Definition 1 (Local Nash equilibrium): A joint action profile $x = (x_1, x_2) \in W_1 \times W_2 \subset X_1 \times X_2$ is a local Nash equilibrium on $W_1 \times W_2$ if, for each player $i \in \mathcal{I}$, $f_i(x_i, x_{-i}) \leq f_i(x'_i, x_{-i})$, $\forall x'_i \in W_i$.

A local Nash equilibrium can equivalently be defined as in terms of best response maps: $x_i \in \arg \min_y f_i(y, x_{-i})$. From this perspective, local optimality conditions for players' optimization problems give rise to the notion of a differential Nash equilibrium [18], [19]; non-degenerate differential Nash are known to be generic and structurally stable amongst

local Nash equilibria in sufficiently smooth games [17]. Let $D_i f_i$ denote the derivative of f_i with respect to x_i and, analogously, let $D_i(D_i f_i) \equiv D_i^2 f_i$ be player i 's individual Hessian.

Definition 2: For continuous game $\mathcal{G} = (f_1, f_2)$ where $f_i \in C^2(X_1 \times X_2, \mathbb{R})$, a joint action profile $(x_1, x_2) \in X_1 \times X_2$ is a *differential Nash equilibrium* if $D_i f_i(x_1, x_2) = 0$ and $D_i^2 f_i(x_1, x_2) > 0$ for each $i \in \mathcal{I}$.

A differential Nash equilibrium is a strict local Nash equilibrium [18, Thm. 1]. Furthermore, the conditions $D_i f_i(x) = 0$ and $D_i^2 f_i(x) \geq 0$ are necessary for a local Nash equilibrium [18, Prop. 2].

Learning processes in games, and their study, arose as one of the explanations for how players grapple with one another in seeking an equilibrium [7]. In the case of sufficiently smooth games, gradient-based learning is a natural learning rule for myopic players².

B. Gradient-based Learning as a Dynamical System

At time t , a myopic agent i updates its current action $x_i(t)$ by following the gradient of its individual cost f_i given the decisions of its competitors x_{-i} . The synchronous adaptive process that arises is the discrete-time dynamical system

$$x_i(t+1) = x_i(t) - \gamma_i D_i f_i(x_i(t), x_{-i}(t)) \quad (1)$$

for each $i \in \mathcal{I}$ where $D_i f_i$ is the gradient of player i 's cost with respect to x_i and γ_i is player i 's learning rate.

a) *Stability:* Recall that a matrix A is called Hurwitz if its spectrum lies in the open left-half complex plane $\mathbb{C}_-^o$. Furthermore, we often say such a matrix is *stable* in particular when A corresponds to the dynamics of a linear system $\dot{x} = Ax$ or the linearization of a nonlinear system around a fixed point of the dynamics.³

It is known that (1) will converge locally asymptotically to a differential Nash equilibrium if the local linearization is a contraction [5]. Let

$$g(x) = (D_1 f_1(x), D_2 f_2(x)) \quad (2)$$

be the vector of individual gradients and let $Dg(x)$ be its Jacobian—i.e., the *game Jacobian*. Further, let $\sigma_p(A) \subset \mathbb{C}$ denote the *point spectrum* (or *spectrum*) of the matrix A , and $\rho(A)$ its *spectral radius*. Then, x is *locally exponentially stable* if and only if $\rho(I - \Gamma Dg(x)) < 1$, where $\Gamma = \text{blockdiag}(\gamma_1 I_{d_1}, \gamma_2 I_{d_2})$ is a diagonal matrix and I_{d_i} is the identity matrix of dimension d_i . The map $I - \Gamma Dg(x)$ is the local linearization of (1). Hence, to study stability (and, in turn, convergence) properties it is useful to analyze the spectrum of not only the map $I - \Gamma Dg(x)$ but also $Dg(x)$ itself.

For instance, when $\gamma = \gamma_1 = \gamma_2$, the spectral mapping theorem tells us that $\rho(I - \gamma Dg(x)) = \max_{\lambda \in \sigma_p(Dg(x))} |1 - \gamma \lambda|$ so that understanding the spectrum of $Dg(x)$ is imperative

²A myopic player effectively believes it cannot influence its opponent's future behavior, and reacts only to local information about its cost.

³The Hartman-Grobman theorem [20] states that around any hyperbolic fixed point of a nonlinear system, there is a neighborhood on which the nonlinear system is stable if the spectrum of Jacobian lies in $\mathbb{C}_-^o$.

¹For 2-player games, $x_{-1} = x_2$ and $x_{-2} = x_1$.

for understanding convergence of the discrete time update. On the other hand, when $\gamma_1 \neq \gamma_2$, we write the local linearization as $I - \gamma_1 \Lambda Dg(x)$ where $\Lambda = \text{blockdiag}(I_{d_1}, \tau I_{d_2})$ and $\tau = \gamma_2/\gamma_1$ is the learning rate ratio. Again, via the spectral mapping theorem, when $I - \gamma_1 \Lambda Dg(x)$ is a contraction for different choices of learning rate γ_1 is determined by the spectrum of $\Lambda Dg(x)$. Hence, given a *fixed point* x (i.e., $g(x) = 0$), we study the stability properties of the limiting continuous time dynamical system—i.e., $\dot{x} = -g(x)$ when $\gamma_1 = \gamma_2$ and $\dot{x} = -\Lambda g(x)$ otherwise. From here forward, we will simply refer to the system $\dot{x} = -\Lambda g(x)$ and point out when $\Lambda = I_{d_1+d_2}$ if not clear from context.

b) *Partitioning the Game Jacobian*: Let $x = (x_1, x_2)$ be a joint action profile such that $g(x) = 0$. Towards better understanding the spectral properties of $Dg(x)$ (respectively, $\Lambda Dg(x)$), we partition $Dg(x)$ into blocks:

$$J(x) = \begin{bmatrix} -D_1^2 f_1(x) & -D_{12} f_1(x) \\ -D_{21} f_2(x) & -D_2^2 f_2(x) \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}. \quad (3)$$

A differential Nash equilibrium (the second order conditions of which are sufficient for a local Nash equilibrium) is such that $J_{11} < 0$ and $J_{22} < 0$. On the other hand, as noted above, J is Hurwitz or stable if its point spectrum $\sigma_p(J) \subset \mathbb{C}_-^o$. Moreover, since the diagonal blocks are symmetric, J is similar to the matrix in Fig 2. For the remainder of the paper, we will study the Dg at a given fixed point x as defined in (3).

$$J(x, y) \sim \begin{bmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{bmatrix}$$

Fig. 2: *Similarity*: the game Jacobian in (3) is similar to a matrix with diagonal block-diagonals.

c) *Classes of Games*: Different classes of games can be characterized via J . For instance, a *zero-sum game*, where $f_1 \equiv -f_2$, is such that $J_{12} = -J_{21}^\top$. On the other hand, a game $\mathcal{G} = (f_1, f_2)$ is a *potential game* if and only if $D_{12} f_1 \equiv D_{21} f_2^\top$ [15, Thm. 4.5], which implies that $J_{12} = J_{21}^\top$.

C. Spectrum of Block Matrices

One useful tool for characterizing the spectrum of a block operator matrix is the numerical range and quadratic numerical range, both of which contain the operator's spectrum [23] and therefore all of its eigenvalues. The *numerical range* of J is defined by

$$W(J) = \{\langle Jz, z \rangle : z \in \mathbb{C}^{d_1+d_2}, \|z\|_2 = 1\},$$

and is convex. Given a block operator J , let

$$J_{v,w} = \begin{bmatrix} \langle J_{11}v, v \rangle & \langle J_{12}w, v \rangle \\ \langle J_{21}v, w \rangle & \langle J_{22}w, w \rangle \end{bmatrix} \quad (4)$$

where $v \in \mathbb{C}^{d_1}$ and $w \in \mathbb{C}^{d_2}$. The *quadratic numerical range* of J , defined by

$$W^2(J) = \bigcup_{v \in \mathcal{S}_1, w \in \mathcal{S}_2} \sigma_p(J_{v,w}), \quad (5)$$

is the union of the spectra of (4) where $\sigma_p(\cdot)$ denotes the (point) spectrum of its argument and $\mathcal{S}_i = \{z \in \mathbb{C}^{d_i} : \|z\|_2 = 1\}$. It is, in general, a non-convex subset of $\mathbb{C}$.

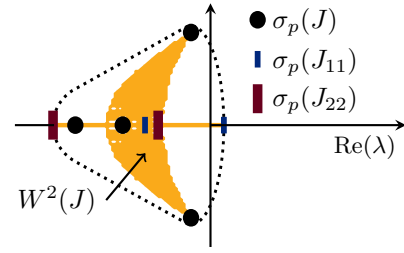


Fig. 3: *Spectrum of a stable equilibrium that is not Nash*. The spectrum of J , J_{11} , and J_{22} in Example 1 are contained in the numerical range (convex dashed region) and quadratic numerical range (non-convex region) of J . The eigenvalues of J are in the left plane, hence the fixed point is stable under gradient play (1). However, the first player's J_{11} is indefinite, hence the fixed point is not a Nash equilibrium.

The quadratic numerical range (5) is equivalent to the set of solutions of the characteristic polynomial

$$\lambda^2 - \lambda(\langle J_{11}v, v \rangle + \langle J_{22}w, w \rangle) + \langle J_{11}v, v \rangle \langle J_{22}w, w \rangle - \langle J_{12}v, w \rangle \langle J_{21}w, v \rangle = 0 \quad (6)$$

for $v \in \mathcal{S}_1$ and $w \in \mathcal{S}_2$. We use the notation $\langle Jx, y \rangle = x^* Jy$ to denote the inner product. Note that $W^2(J)$ is a subset of $W(J)$ and, as previously noted, contains $\sigma_p(J)$. Albeit non-convex, $W^2(J)$ provides a tighter characterization of the spectrum⁴.

Example 1: Consider the game Jacobian of the zero-sum game $(f, -f)$ defined by cost $f : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$f(x, y) = -\frac{1}{2}x_1^2 + \frac{5}{2}x_2^2 + 7y_1x_1 - 3y_2x_2 - 2y_1^2 - 6y_2^2.$$

The numerical range, quadratic numerical range, spectrum and diagonal entries of J , defined using the origin as the fixed point, are plotted in Fig. 3. In this example, the origin is not a differential Nash equilibrium since $D_1^2 f_1(0, 0)$ is indefinite, yet it is an exponentially stable equilibrium of $\dot{x} = -g(x)$ since all the eigenvalues of J are all negative.

Observing that the quadratic numerical range for a block 2×2 matrix J derived from a game on a finite dimensional Euclidean space reduces to characterizing the spectrum of 2×2 matrices, we first characterize stability properties of scalar 2-player continuous games.

III. STABILITY OF 2-PLAYER SCALAR GAMES

We characterize the stability of differential Nash equilibria in 2-player scalar continuous games. Consider a game (f_1, f_2) with action spaces $X_1, X_2 \subseteq \mathbb{R}$. Let x be a fixed point of (2) such that $g(x) = 0$. We decompose its game Jacobian (3) into components that reflect the dynamic interaction between the players.

⁴There are numerous computational approaches for estimating the numerical ranges $W(\cdot)$ and $W^2(\cdot)$ (see, e.g., [10, Sec. 6]).

A. Jacobian Decomposition: 2×2 case

Consider the decomposition of a $\mathbb{R}^{2 \times 2}$ game Jacobian

$$J(x) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} m & -z \\ z & m \end{bmatrix} + \begin{bmatrix} h & p \\ p & -h \end{bmatrix} \quad (7)$$

where $m = \frac{1}{2}(a + d)$, $h = \frac{1}{2}(a - d)$, $p = \frac{1}{2}(b + c)$, $z = \frac{1}{2}(c - b)$. Let $\text{tr}(J)$ be its trace, $\det(J)$ be its determinant, and $\text{disc}(J)$ be the discriminant of its characteristic polynomial.⁵ Several directly verifiable quantities are stated.

Statement 1: Given a matrix $J \in \mathbb{R}^{2 \times 2}$ and its spectrum $\sigma_p(J) = \{\lambda_1, \lambda_2\}$, the above decomposition gives rise to the following conditions:

$$\text{tr}(J) = \lambda_1 + \lambda_2 = a + d = 2m,$$

$$\det(J) = \lambda_1 \lambda_2 = ad - bc = (m^2 + z^2) - (h^2 + p^2),$$

$$\text{disc}(J) = (\lambda_1 + \lambda_2)^2 - 4\lambda_1 \lambda_2 = 4(h^2 + p^2 - z^2),$$

$$\lambda_{1,2} = \frac{1}{2}(\text{tr}(J) \mp \sqrt{\text{disc}(J)}) = m \mp \sqrt{h^2 + p^2 - z^2}.$$

The change of coordinates from (a, b, c, d) to (m, h, p, z) in Statement 1 provides important insights into linear vector fields and, in particular, to games. The stability of vector field $\dot{x} = Jx$ is given by the trace and determinant conditions.

Proposition 1: The matrix $J \in \mathbb{R}^{2 \times 2}$ is stable if and only if $m^2 + z^2 > h^2 + p^2$ and $m < 0$.

Proof: Statement 1 and direct computation show that these conditions are equivalent to $\lambda_1 + \lambda_2 < 0$ and $\lambda_1 \lambda_2 > 0$, well-known conditions for stability of 2×2 systems (illustrated in Fig. 5b). ■

B. Discussion of Decomposition

The purpose of the decomposition into the alternative coordinates is to geometrically—and thus more directly—assess the conditions for stability of a differential Nash equilibrium.

a) *Relationship to complex plane:* Fig. 4 plots the coordinates of m, z, h, p relative to each other to illustrate the decomposition in Statement 1. If $h = 0, p = 0$, then the eigenvalues of J are $\lambda_{1,2} = m \mp zi$. Fig. 4a corresponds to a plot of eigenvalues in the complex plane. Stability is given by the familiar open-left half plane condition: $\sigma_p(J) \subset \mathbb{C}_-^o$. If $h \neq 0$ or $p \neq 0$ a circular region in the center of the plane expands the values of m, z for which the eigenvalues of the matrix are purely real. Fig. 4b shows that the eigenvalues are purely real if and only if $z^2 \leq h^2 + p^2$.

b) *Effect of rotation in game vector fields:* Note the similarity between (7) and the well-known symmetric/skew-symmetric (Helmholtz) decomposition

$$J(x) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} m+h & p \\ p & m-h \end{bmatrix} + \begin{bmatrix} 0 & -z \\ z & 0 \end{bmatrix}. \quad (8)$$

Assuming that $m < 0$, from Proposition 1 we can see that increasing the rotational component of the Jacobian helps stability. Increasing the relative magnitude of p , the non-rotational interaction term hurts stability. If there is no

⁵The characteristic polynomial of J is $\lambda \mapsto \det(J - \lambda I)$ and its discriminant is $\text{tr}(J)^2 - 4\det(J)$ for $J \in \mathbb{R}^{2 \times 2}$.

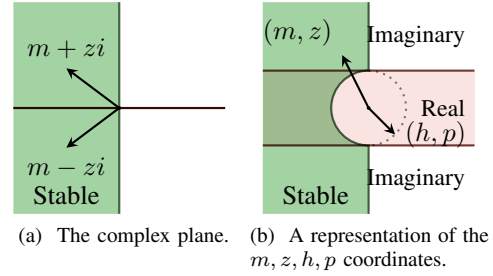


Fig. 4: Visualization of Statement 1: If h and p are zero, then the eigenvalues of J are $\lambda_{1,2} = m \mp zi$. If h and/or p are non-zero, then a circle centered around the origin with radius $\sqrt{h^2 + p^2}$ is excluded from left-half stability region.

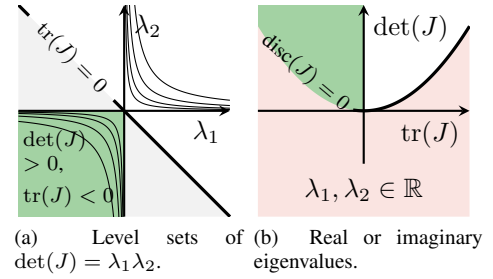


Fig. 5: Visualization of Proposition 1: $\dot{y} = J(x)y$ is **stable** $\iff \det(J) > 0$ and $\text{tr}(J) < 0$.

rotational component, ie. J is symmetric, p 's negative impact on stability can be seen directly from the Schur complement⁶. In this case J is stable iff $J < 0$ and thus stability requires that both the diagonals and the Schur complement are negative: $a < 0$, $d < 0$, and $a - p^2 d^{-1} < 0$. If $d < 0$, increasing p can only increase the Schur complement.

C. Types of Games

The decomposition also provides a natural classification of 2-player scalar games into four types based on specific coordinates being zero, as illustrated in Fig. 7.

a) *Potential games* ($z = 0$): The point (m, z) lives on the horizontal axis in Fig. 7a, thus stable fixed points are a subset of Nash equilibria. Since $z = 0$, Proposition 1 indicates that increasing p , the interaction term between the players, and increasing h , the difference in curvature between the two players both only hurt stability.

b) *Zero-sum games* ($p = 0$): The point (h, p) lives on the horizontal axis in Fig. 7b, thus all Nash equilibria are stable, but not all stable fixed points are Nash. The magnitude of the interaction term z helps stability and may make a fixed point stable even if it is not Nash.

c) *Hamiltonian games* ($m = 0$): The point (m, z) lives on the vertical axis in Fig. 7c, thus no strict Nash equilibria can exist. At best these games are marginally stable if $|z|$ is large enough relative to the magnitude of (h, p) .

⁶The Schur complements of the matrix in (3) are $J_{11} - J_{12} J_{22}^{-1} J_{21}$ (where J_{22} is invertible) and $J_{22} - J_{21} J_{11}^{-1} J_{12}$ (where J_{11} is invertible).

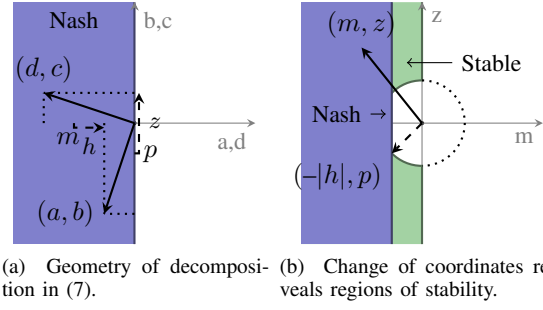


Fig. 6: *Decomposition of a general scalar game.* The rows vectors of J are plotted in (a) and the same matrix with a change of coordinates is plotted in (b). **Nash** regions ($m < -|h|$) and **stability** regions ($m < 0, m^2 + z^2 > h^2 + p^2$) are visible. Their set differences characterize the conditions for a stable non-Nash and unstable Nash equilibria.

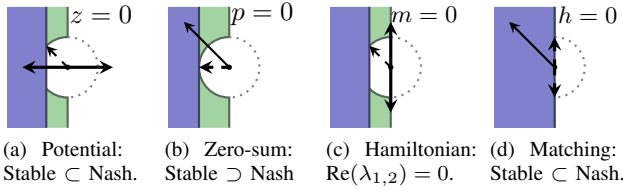


Fig. 7: *Stability and Nash for different classes of games.* (a) Potential games: symmetric interaction term only hurts stability. (b) Zero-sum games: rotation can compensate for unhappy player. (c) Hamiltonian games: players have zero total curvature, $a + d = 0$. (d) Matching, $a = d$: there are no stable non-Nash equilibria.

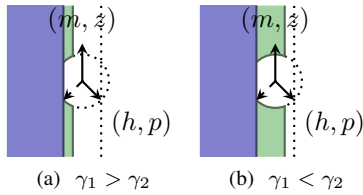


Fig. 8: *Time-scale separation affects stability.* The learning rate ratio $\tau = \gamma_2/\gamma_1 > 0$ affects the stability of the game dynamics. The factor $\beta = \frac{\tau-1}{\tau+1}$ expands or shrinks the region for stability.

d) *Constant-curvature games*, ($h = 0$): The point (h, p) lives on the vertical axis in Fig. 7d, so any stable point is also a Nash. Any fixed point with a, d having the same sign can be rescaled to have constant curvature $\gamma_1 a = \gamma_2 d$ by a choice of non-uniform learning rates $\gamma_1, \gamma_2 > 0$.

IV. CERTIFICATES FOR STABILITY OF GAME DYNAMICS

A. Stability: Uniform Learning Rates

For a game $\mathcal{G} = (f_1, f_2)$, let the set of differential Nash equilibria be denoted $\text{DNE}(\mathcal{G})$ and let the stable points of $\dot{x} = -g(x)$ be $\text{S}(\mathcal{G})$. Let $\overline{\text{DNE}}(\mathcal{G})$ and $\overline{\text{S}}(\mathcal{G})$ be their respective complements. The intersections of these sets characterize the stability/instability of Nash/non-Nash equilibria.

Theorem 1 (Certificates for 2-Player Scalar Games): Consider a game $\mathcal{G} = (f_1, f_2)$ on $X_1 \times X_2 \subseteq \mathbb{R}^2$. Let x be a fixed point of (2) and let m, h, p, z be defined by (7). The following equivalences hold:

- (i) $x \in \text{DNE}(\mathcal{G}) \cap \text{S}(\mathcal{G}) \iff \{m < -|h|\} \wedge \{m^2 + z^2 > h^2 + p^2\}.$
- (ii) $x \in \text{DNE}(\mathcal{G}) \cap \overline{\text{S}}(\mathcal{G}) \iff \{m < -|h|\} \wedge \{m^2 + z^2 \leq h^2 + p^2\}.$
- (iii) $x \in \overline{\text{DNE}}(\mathcal{G}) \cap \text{S}(\mathcal{G}) \iff \{0 > m \geq -|h|\} \wedge \{m^2 + z^2 > h^2 + p^2\}.$
- (iv) $x \in \overline{\text{DNE}}(\mathcal{G}) \cap \overline{\text{S}}(\mathcal{G}) \iff \{\{0 > m \geq -|h|\} \wedge \{m^2 + z^2 \leq h^2 + p^2\}\} \vee \{m \geq 0\}.$

The contributions to the stability of a non-Nash equilibrium or the instability of a Nash equilibrium are stated in (ii) and (iii). We illustrate the geometry of these two cases with the shaded regions in Fig. 6b.

B. Stability: Non-Uniform Learning Rates

Consider players updating their actions according to gradient play as defined in (1) with individual learning rates $\gamma_1, \gamma_2 > 0$, not necessarily equal. We study how the players' ratio $\tau = \gamma_2/\gamma_1$ affects the stability of fixed point x under the learning dynamics by analyzing the game Jacobian

$$J(x) = \begin{bmatrix} a & b \\ \tau c & \tau d \end{bmatrix}. \quad (9)$$

Learning rates do not affect whether a fixed point is a Nash equilibrium. They do, however, affect whether it is stable.

Corollary 1 (Stability in General-Sum Scalar Games): Consider a game $\mathcal{G} = (f_1, f_2)$ on $X_1 \times X_2 \subseteq \mathbb{R}^2$ and a fixed point x . Suppose players perform gradient play (1) with learning rate ratio $\tau = \gamma_2/\gamma_1$. Then, the following are true.

- (i) If a Nash equilibrium is stable for some τ , then it is stable for all τ .
- (ii) If a non-Nash equilibrium is stable, then there exists some τ that makes it unstable.
- (iii) If a fixed point is non-Nash, the determinant of its game Jacobian is positive and $m < |h|$, then there exists some τ that makes it stable.

Proof: To prove (i), we observe that if $m < -|h|$, then $m \leq \beta h$ for all β such that $|\beta| < 1$. To prove (ii), choose $\tau < \left|\frac{a}{d}\right|$. Without loss of generality, assume $a < 0$ and $d > 0$. Then, it directly follows that $a + \tau d < 0$. To prove (iii), note that a matrix J is stable if and only if the determinant of J is positive and $m < 0$. Hence, without loss of generality, let $d < 0$. Then there is a learning rate τ such that $\tau|d| > |a|$ so that $m < 0$. ■

Stable Nash equilibria in scalar games are robust to variations in learning rates and non-Nash equilibria are not. For continuous games with vector action spaces, Corollary 1(i) no longer holds, demonstrating that Nash equilibria are not robust, in general, to variations in learning rates.

V. AN ILLUSTRATIVE EXAMPLE

We demonstrate our main results below and in Fig. 9.

Example 2 (Nonlinear torus game): Consider a game $\mathcal{G} = (f_1, f_2)$ defined on $\mathbb{S}^1 \times \mathbb{S}^1$ with costs

$$f_1(x, y) = \frac{2}{a} \cos\left(\frac{a}{2}x\right) + \frac{2}{a} \cos\left(\frac{a}{2}x + by\right),$$

$$f_2(x, y) = \frac{2}{d} \cos\left(\frac{d}{2}y\right) + \frac{2}{d} \cos\left(\frac{d}{2}y + cx\right).$$

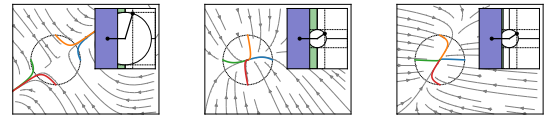
There is a fixed point of the learning dynamics at the origin. Its linearized game Jacobian is $J(0) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. First, to show Corollary 1(i), we start with an unstable, Nash fixed point of a potential game ($a = -0.4, b = 1, c = 1, d = -1$). We decrease $p = \frac{1}{2}(b + c)$ until it becomes stable ($b = 0.2, c = 0.2$). Then, we decrease τ from 1 to 0.1 while maintaining stability. Second, to show Corollary 1(ii), we start with an unstable, non-Nash fixed point of a zero-sum game ($a = 0.4, b = -0.2, c = 0.2, d = -1$). We increase $z = \frac{1}{2}(c - b)$ until it becomes stable ($b = -1, c = 1$). Then, we decrease τ from 1 to 0.01 making it unstable again. Third, to show Corollary 1(iii), we start with an unstable, non-Nash fixed point of a Hamiltonian game ($a = 0.5, b = 0.1, c = 0.5, d = -0.5$). We increase the interaction term $z = \frac{1}{2}(c - b)$ until it becomes marginally stable ($b = -0.5, c = 1.1$). Then, we increase τ slightly from 1 to 2, making the fixed point stable.

VI. CONCLUSION

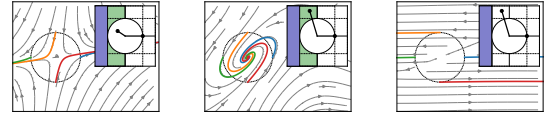
We characterize the local stability and Nash optimality of fixed points of 2-player general-sum gradient learning dynamics. Our results give valuable insights into the interaction of algorithms in settings most accurately modeled as games, for example, when agents lack trust or reliable communication. In the sequel, we characterize continuous games defined on vector action spaces.

REFERENCES

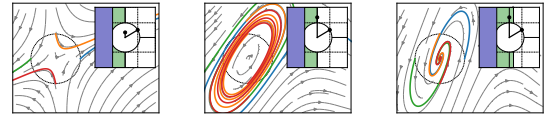
- [1] David Balduzzi, Wojciech M Czarnecki, Thomas W Anthony, Ian M Gemp, Edward Hughes, Joel Z Leibo, Georgios Piliouras, and Thore Graepel. Smooth markets: A basic mechanism for organizing gradient-based learners. *Proc. Inter. Conf. Learning Representations*, 2020.
- [2] Hugo Berard, Gauthier Gidel, Amjad Almahairi, Pascal Vincent, and Simon Lacoste-Julien. A closer look at the optimization landscapes of generative adversarial networks. *Proc. Inter. Conf. Learning Representations*, 2020.
- [3] Victor Boone and Georgios Piliouras. From Darwin to Poincaré and von Neumann: Recurrence and Cycles in Evolutionary and Algorithmic Game Theory. In *Inter. Conf. Web and Internet Economics*, pages 85–99, 2019.
- [4] Jingjing Bu, Lillian J Ratliff, and Mehran Mesbahi. Global convergence of policy gradient for sequential zero-sum linear quadratic dynamic games. *arXiv preprint arXiv:1911.04672*, 2019.
- [5] Benjamin Chasnov, Lillian Ratliff, Eric Mazumdar, and Samuel Burden. Convergence Analysis of Gradient-Based Learning in Continuous Games. In *Proc. Uncertainty in Artificial Intelligence*, 2019.
- [6] Tanner Fiez, Benjamin Chasnov, and Lillian J Ratliff. Implicit Learning Dynamics in Stackelberg Games: Equilibria Characterization, Convergence Analysis, and Empirical Study. *Proc. Inter. Conf. Machine Learning*, 2020.
- [7] Drew Fudenberg, Fudenberg Drew, David K Levine, and David K Levine. *The theory of learning in games*. MIT press, 1998.
- [8] Ian J. Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative Adversarial Nets. In *Advances in Neural Information Processing Systems*, 2014.
- [9] Hassan K Khalil. *Nonlinear systems theory*. Prentice Hall, 2002.
- [10] Heinz Langer, A Markus, V Matsaev, and C Tretter. A new concept for block operator matrices: the quadratic numerical range. *Linear algebra and its applications*, 330(1-3):89–112, 2001.
- [11] Eric Mazumdar, Lillian J Ratliff, and Shankar Sastry. On gradient-based learning in continuous games. *SIAM Journal on Mathematics of Data Science*, 2(1):103–131, 2020.
- [12] Panayotis Mertikopoulos, Christos Papadimitriou, and Georgios Piliouras. Cycles in adversarial regularized learning. In *Proc. 29th Ann. ACM-SIAM Symp. Discrete Algorithms*, pages 2703–2717. SIAM, 2018.
- [13] Panayotis Mertikopoulos and Zhengyuan Zhou. Learning in games with continuous action sets and unknown payoff functions. *Mathematical Programming*, 173(1-2):465–507, 2019.
- [14] Luke Metz, Ben Poole, David Pfau, and Jascha Sohl-Dickstein. Unrolled generative adversarial networks. *Proc. Inter. Conf. Learning Representations*, 2017.
- [15] Dov Monderer and Lloyd S Shapley. Potential games. *Games and economic behavior*, 14(1):124–143, 1996.
- [16] John Nash. Non-cooperative games. *Ann. Math.*, pages 286–295, 1951.
- [17] L. J. Ratliff, S. A. Burden, and S. S. Sastry. Genericity and structural stability of non-degenerate differential Nash equilibria. In *Proc. Amer. Control Conf.*, pages 3990–3995, 2014.
- [18] Lillian J Ratliff, Samuel A Burden, and S Shankar Sastry. Characterization and computation of local Nash equilibria in continuous games. In *Proc. 51st Ann. Allerton Conf. Communication, Control, and Computing*, pages 917–924. IEEE, 2013.
- [19] Lillian J Ratliff, Samuel A Burden, and S Shankar Sastry. On the Characterization of Local Nash Equilibria in Continuous Games. *IEEE Trans Automa. Control*, 61(8):2301–2307, 2016.
- [20] S. Shankar Sastry. *Nonlinear systems: analysis, stability, and control*. Springer-Verlag New York, 1999.
- [21] Yujie Tang and Na Li. Distributed zero-order algorithms for nonconvex multi-agent optimization. In *Proc. 57th Allerton Conf. Communication, Control, and Computing*, pages 781–786, 2019.
- [22] T. Tatarenko and M. Kamgarpour. Learning Nash Equilibria in Monotone Games. In *Proc. IEEE Conf. Decision and Control*, pages 3104–3109, 2019.
- [23] Christiane Tretter. *Spectral theory of block operator matrices and applications*. World Scientific, 2008.



(a) Potential game: a Nash goes from unstable to stable, and remains stable with time-scale separation.



(b) Zero-sum game: a non-Nash goes from unstable to stable, and destabilizes with decreasing τ .



(c) Hamiltonian game: a non-Nash goes from unstable to marginally stable, and stabilizes with increasing τ .

Fig. 9: *Demonstration of Corollary 1: vector field plots of the three scenarios from Example 2.*