Analog Subspace Coding: A New Approach to Coding for Non-Coherent Wireless Networks

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***Abstract*— We provide a precise framework to study subspace codes for non-coherent communications in wireless networks. To this end, an *analog operator channel* is defined with inputs and outputs being subspaces of ^{n}. Then a certain distance is defined to capture the performance of subspace codes in terms of their capability to recover from interference and rank-deficiency of the network. We also study the robustness of the proposed model with respect to additive noise. Furthermore, we propose a new approach to construct subspace codes in the analog domain, also regarded as Grassmann codes, by leveraging polynomial evaluations over finite fields together with characters associated to finite fields that map their elements to the unit circle in the complex plane. The constructed codes, referred to as character-polynomial (CP) codes, are shown to perform better compared to other existing constructions of Grassmann codes in terms of the trade-off between the rate and the normalized minimum distance, for a wide range of values for n.**

I. Introduction

Wireless networks are rapidly growing in size, are becoming more hierarchical, and are becoming increasingly distributed. While the efforts for 5G standardization are still ongoing, several new features have been introduced in the recent releases of the Long-Term Evolution (LTE) standard to start supporting the diverse requirements of the wide range of use cases in 5G. Started with Release 10 the deployment of small cells in LTE is becoming increasingly popular to deliver enhanced spectral capacity and extended network coverage [1], which is also fundamental to enhanced mobile broadband (eMBB) and massive machine type communications (mMTC) scenarios in 5G. Moreover, features such as coordinated multipoint (CoMP) transmission and reception [2] together with enhanced intercell interference coordination (eICIC) [3] have been introduced and used since Release 10 and evolved since then.

The aforementioned techniques are, however, difficult to scale as the number of small cells, that can be also regarded as relays, keeps increasing and as more layers are added in the hierarchical network. Motivated by the emergence of such massive networks we study coding for wireless networks consisting of many relays operating in a non-coherent fashion, where the network nodes are oblivious to the channel gains of the point-to-point wireless links as well as the structure of the network. In a sense, this resembles a random linear network coding scenario, though completely in the physical layer, where physical-layer transport blocks are linearly combined in the relay nodes as they receive the spatial sum of blocks sent by the neighboring nodes. This holds assuming omni-directional radio frequency (RF) transmitter and receiver antennas are deployed at the network nodes. Also, in the considered setup, the relay nodes, such as small cells, do not attempt to decode messages and only amplify and forward the received physical-layer blocks.

In this paper, we define a new framework for reliable communications over wireless networks in a non-coherent fashion, as discussed above, using *analog subspace codes*. Let $\mathcal{W}$ denote an ambient vector space of dimension n over a field $\mathbb{L}$, i.e., $\mathcal{W} = \mathbb{L}^n$. A subspace code in $\mathcal{W}$ is a non-empty subset of the set of all the subspaces of $\mathcal{W}$. We observe that subspace codes in the analog domain, where the underlying field $\mathbb{L}$ is $\mathbb{R}$ or , become relevant for conveying information across networks in such a scenario. This work is mainly inspired by the seminal work by Koetter and Kschischang [4]. In a sense, we develop a counterpart for Koetter-Kschischang's operator channel, introduced in [4], in the analog domain, referred to as *analog operator channel*. More specifically, the analog operator channel models the *rank deficiency* of the network, caused by relay failures or lacking a sufficient number of active relays, as subspace erasures. Also, it models the interference from neighboring cells/small cells as subspace errors. Furthermore, we propose a novel construction of analog subspace codes by leveraging characters associated to Abelian groups and mapping them to the complex plane.

Analog subspace codes can be also viewed as codes in Grassmann space, also referred to as Grassmann codes, provided that the dimensions of all the subspace codewords are equal. There is a long history on studying bounds [5]–[9], using packing and covering arguments, and capacity analysis in Grassmann space, mostly motivated by space-time coding for multiple-input multiple-output (MIMO) wireless systems [10]–[12]. In such systems, a separate block code is needed to guarantee the reliability regardless, and the space-time code can be interpreted as the means of improving the reliability by exploiting the diversity the MIMO channel offers. However, we arrive at the problem of constructing subspace codes from the analog operator channel. In other words, subspace codes are used for reliable communications over analog operator channels the same way block codes are conventionally used for reliable communications over point-to-point links. A detailed overview of prior works on Grassmann codes and their relations to our approach can be found in the extended version of this paper [13].

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II. Preliminaries

A. Notation Convention

Let $[n]$ denote the set of positive integers less than or equal to n, i.e., $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. Also, for $x \in \mathbb{R}$, $(x)_+ \stackrel{\text{def}}{=} \max(0, x)$.

In this paper, matrices are represented by bold capital letters. The row space of a matrix $\boldsymbol{X}$ is denoted by $\langle \boldsymbol{X} \rangle$. Also, for a square matrix $\boldsymbol{X}$, the trace of $\boldsymbol{X}$, denoted by $\text{tr}(\boldsymbol{X})$, is defined to be the sum of elements of $\boldsymbol{X}$ on the main diagonal. For a matrix $\boldsymbol{A}$, $\|\boldsymbol{A}\|$ and $\|\boldsymbol{A}\|_2$ denote the Frobenius and spectral norm of $\boldsymbol{A}$, respectively. Also, $\boldsymbol{A}^+$ and $\kappa_{\boldsymbol{A}}$ denote the pseudoinverse and condition number of $\boldsymbol{A}$, respectively.

The ambient vector space is denoted by W. The parameter n is reserved for the dimension of W throughout the paper. Also, we have $W = \mathbb{L}^n$, where $\mathbb{L}$ can be either $\mathbb{R}$ or . Let $\mathcal{P}(W)$ denote the set of all subspaces of W. For a subspace $V \in \mathcal{P}(W)$, the dimension of V is denoted by $\dim(V)$. The sum of two subspaces $U, V \in \mathcal{P}(W)$ is defined as

$$U + V \stackrel{\text{def}}{=} \{u + v : u \in U, v \in V\}, \tag{1}$$

and the direct sum is defined as

$$U \oplus V \stackrel{\text{def}}{=} \{(u, v) : u \in U, v \in V\}. \tag{2}$$

Note that if U and V intersect trivially, i.e., $U \cap V = \{\mathbf{0}\}$, where $\mathbf{0}$ is the all-zero vector, then $U + V = U \oplus V$.

The set of all m-dimensional subspaces of $\mathbb{L}^n$ is denoted by $G_{m,n}(\mathbb{L})$, which is referred to as Grassmann space or Grassmannian in the literature.

For a set M, a σ-quasimetric on M is a function $d : M \times M \to \mathbb{R}$ that satisfies all the conditions of a metric except the triangle inequality being relaxed to

$$\forall x, y, z \in M, \quad d(x, z) < \sigma\big(d(x, y) + d(y, z)\big), \tag{3}$$

for a constant $\sigma > 1$. This inequality is referred to as *σ-relaxed triangle inequality*.

B. Analog operator channel

This model is motivated by non-coherent communications over wireless networks, as discussed in Section I. Hence, each piece of the model is followed by a brief explanation from this perspective. Let $\boldsymbol{x}_i \in^n$, for $i \in [m]$, denote the input vectors. The input vectors, as physical layer transport blocks, can be sent by several antennas of a transmitter, e.g., a cellular base station, at different time frames. By discarding the interference and the additive noise, the output of the channel is a set of vectors $\boldsymbol{y}_j = \sum_{i=1}^{m} h_{j,i} \boldsymbol{x}_i$, where $j \in [l]$. Each vector $\boldsymbol{y}_j$ is the received transport block by an antenna of the receiver at a certain time frame. Note that a time-frame-level synchronization is assumed across the wireless links, e.g., by employing specific patterns in a designated subset of orthogonal frequency-division multiplexing (OFDM) symbols in each time frame as in LTE networks [14]. Also, the relays in the network, e.g., small cells, are assumed to be amplify-and-forward relays. They can forward a transport block, received during a certain time frame, in a subsequent time frame. This is because the communication is assumed to be done in the unit of time frame, i.e., the relay has to wait for the current time frame to end before it can begin forwarding what it received. Then, due to the different delays, in the unit of time frames, that transport blocks may encounter as they are propagated through the network, the received $\boldsymbol{y}_j$'s can be the combination of transmitted $\boldsymbol{x}_i$'s across different antennas and time frames. Under a non-coherent scenario, both the transmitter and the receiver are oblivious to $h_{j,i}$'s, the topology of the network, and the link-level channel gains. It is possible that several interference blocks, e.g., up to t of them, from neighboring cells/small cells are also received by the receiver. Hence, we have

$$\boldsymbol{Y}_{l \times n} = \boldsymbol{H}_{l \times m} \boldsymbol{X}_{m \times n} + \boldsymbol{G}_{l \times t} \boldsymbol{E}_{t \times n}, \tag{4}$$

where $\boldsymbol{X}$'s rows are the transmitted blocks $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$, $\boldsymbol{E}$'s rows are the interference blocks $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_t$, $\boldsymbol{Y}$'s rows are the received blocks $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_l$, and $\boldsymbol{H} = [h_{j,i}]_{l \times m}$ and $\boldsymbol{G} = [g_{j,i}]_{l \times t}$ are assumed to be unknown to the transmitter and the receiver. Note that both $\boldsymbol{H}$ and $\boldsymbol{G}$ depend on the network topology as well as the link-level channel gains, however, $\boldsymbol{G}$ also depends on the specific nodes where the interference blocks have entered the network. An example of the communication scenario, described by (4), is illustrated in [13, Figure 1].

In the scenario described by (4), even in the absence of the interference blocks $\boldsymbol{E}$, the only way to convey information to the receiver is through the subspace spanned by the rows of $\boldsymbol{X}$. This is mainly due to the underlying assumption on non-coherent communications, where $\boldsymbol{H}$ is assumed to be completely unknown to both the transmitter and the receiver. Furthermore, $\boldsymbol{H}$ may not be full column rank, e.g., when $l < n$, which implies that $\langle \boldsymbol{X} \rangle$ can not be fully recovered. In order to capture the rank deficiency of $\boldsymbol{H}$, a stochastic erasure operator is defined as follows. For some $k \geqslant 0$, $\mathcal{H}_k(U)$ returns a random k-dimensional subspace of U, if $\dim(U) > k$, and returns U otherwise. Then the analog operator channel is defined as follows:

Definition 1: An analog operator channel associated with W is a channel with input $U \in \mathcal{P}(W)$ and output $V \in \mathcal{P}(W)$ together with the following input-output relation:

$$V = \mathcal{H}_k(U) \oplus E, \tag{5}$$

where E is the interference subspace, also referred to as the error subspace, with $E \cap U = \{\mathbf{0}\}$. Then $\rho = \dim(U) - k$ is referred to as the dimension of erasures and $t = \dim(E)$ is referred to as the dimension of errors.

In the communication scenario described by (4), the additive noise of the physical layer, often modeled as additive white Gaussian noise (AWGN), is discarded. Note that the intermediate relay nodes in the wireless network, such as small cells, are not often limited by power constraints as the end mobile users are. Hence, it is natural to assume that the relay nodes operate at high signal-to-noise ratio (SNR). Nevertheless, it is essential to investigate the effect of additive noise as a perturbation of the transformation described by (4). This will be discussed in Section V.

III. Analog Metric Space, Subspace Codes, and Error Correction

In this section we provide a precise description of *chordal* distance defined for Grassmann space. Then we extend and modify the *chordal* distance to arrive at a new notion of distance, defined for the set of all subspaces of the ambient space, i.e., $\mathcal{P}(W)$, and show that it conveniently captures the error-correction capability of subspace codes when used over analog operator channels.

The chordal distance $d_c : G_{m,n}(\mathbb{L}) \times G_{m,n}(\mathbb{L}) \to \mathbb{R}$ was first introduced for $\mathbb{L} = \mathbb{R}$ in [15] and was extended to $\mathbb{L} =$ in [6]. Consider two m-planes U and V. Let $\boldsymbol{Z}$ denote an orthonormal matrix spanning $V \in G_{m,n}(\mathbb{L})$, i.e. ,

$$V = \langle \boldsymbol{Z} \rangle, \quad \boldsymbol{Z}\boldsymbol{Z}^{\mathrm{H}} = \boldsymbol{I}_m.$$

Then, the matrix $\boldsymbol{P}_V = \boldsymbol{Z}^{\mathrm{H}}\boldsymbol{Z}$ is an orthogonal projection operator from $\mathbb{L}^n$ on V. Similarly, let $\boldsymbol{P}_U$ denote the orthogonal projection operator on U. Then the chordal distance between U and V is defined as follows:

$$d_c(U,V) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \left\| \boldsymbol{P}_U - \boldsymbol{P}_V \right\|. \tag{6}$$

Since the Frobenius norm induces a metric on the set of all $n \times n$ matrices, regardless of whether they are projection matrices or not, one can use (6) to generalize the notion of chordal distance to subspaces of different dimensions. This generalized chordal distance, is further modified to arrive at a new notion of *distance* over $\mathcal{P}(W)$, defined as follows.

Definition 2: The *distance* $d : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathbb{R}$ is defined as

$$d(U,V) \stackrel{\text{def}}{=} \|\boldsymbol{P}_U - \boldsymbol{P}_V\|^2 = \mathrm{tr}\big((\boldsymbol{P}_U - \boldsymbol{P}_V)^2\big), \tag{7}$$

where $U, V \in \mathcal{P}(W)$ and $\boldsymbol{P}_U, \boldsymbol{P}_V$ are the projection matrices associated to U, V, respectively.

Note that $d(.,.) = 2d_c(.,.)^2$ by (6) and (7). It is shown in [13, Lemma 13] that the square of a metric is a 2-quasimetric, where a quasimetric is defined in Section II-A. Hence, $d(.,.)$ is a 2-quasimetric. It is further shown in [13, Lemma 14] that for $U, V, T \in \mathcal{P}(W)$, $d(.,.)$ satisfies the triangle inequality, i.e., $\sigma = 1$ in (3), as long as P_U and P_V are simultaneously diagonalizable, i.e., one can find a basis in which both P_U and P_V are diagonal matrices. This property is later utilized to characterize the error-and-erasure correction capability of codes used over analog operator channels in terms of their *minimum distance* the same way it is done given an underlying metric. Hence, we refer to $d(.,.)$ as a distance through the rest of this paper.

Definition 3: An analog subspace code $\mathcal{C}$ is a subset of $\mathcal{P}(W)$. The size of $\mathcal{C}$ is denoted by $|\mathcal{C}|$. The minimum distance of $\mathcal{C}$ is defined as

$$d_{\min}(\mathcal{C}) \stackrel{\text{def}}{=} \min_{U,V \in \mathcal{C}, U \neq V} d(U,V),$$

where $d(.,.)$ is defined in Definition 2. The maximum dimension of the codewords of $\mathcal{C}$ is denoted by

$$l(\mathcal{C}) \stackrel{\text{def}}{=} \max_{U \in \mathcal{C}} \dim(U).$$

The code $\mathcal{C}$ is then referred to as an $[n, l(\mathcal{C}), |\mathcal{C}|, d_{\min}(\mathcal{C})]$ subspace code, where n is the dimension of the ambient space W.

If the dimension of all codewords in $\mathcal{C}$ are equal, then the code is referred to as a *constant-dimension* code, which is also called a code on Grassmannian or a Grassmann code in the literature.

Definition 4: Let $\mathcal{C}$ be an $[n, l, M, d_{\min}(\mathcal{C})]$ subspace code. The normalized weight λ, the rate R, and the normalized minimum distance δ of $\mathcal{C}$ are defined as follows:

$$\lambda \stackrel{\text{def}}{=} \frac{l}{n}, \quad R \stackrel{\text{def}}{=} \frac{\ln M}{n}, \quad \delta \stackrel{\text{def}}{=} \frac{d_{\min}(\mathcal{C})}{2l}.$$

As in conventional block codes, one can associate a minimum distance decoder to a subspace code $\mathcal{C}$, e.g., when used for communication over an analog operator channel, in order to recover from subspace errors and erasures. Such a decoder returns the nearest codeword $V \in \mathcal{C}$ given $U \in \mathcal{P}(W)$ as its input, i.e., for any $V' \in \mathcal{C}$, $d(U,V) \leqslant d(U,V')$. The following theorem relates the minimum distance of $\mathcal{C}$ to its error-and-erasure correction capability under minimum distance decoding.

Theorem 1: Consider a subspace code $\mathcal{C}$ used for communication over an analog operator channel, as defined in Definition 1, i.e., the input to the channel is $U \in \mathcal{C}$. Let t and ρ denote the dimension of errors and erasures, respectively. Then the minimum distance decoder successfully recovers the transmitted codeword $U \in \mathcal{C}$ from the received subspace V if

$$2(\rho + t) < d_{\min}(\mathcal{C}), \tag{8}$$

where $d_{\min}(\mathcal{C})$ is the minimum distance of $\mathcal{C}$ defined in Definition 3.

The proof is omitted due to space constraints, please see [13] for the proof.

Theorem 1 implies that erasures and errors have equal costs in the subspace domain as far as the minimum-distance decoder is concerned. In other words, the minimum-distance decoder for a code $\mathcal{C}$ can correct up to $\left\lfloor \frac{d_{\min}(\mathcal{C})-1}{2} \right\rfloor$ errors and erasures.

Remark 1. If one uses the chordal distance, instead of the distance $d(.,.)$ defined in Definition 2, a result similar to Theorem 1 can be obtained while the condition in (8) is replaced by $\sqrt{2}(\sqrt{\rho} + \sqrt{t})$ being strictly less than the minimum chordal distance of the code. Since $d(.,.) = 2d_c(.,.)^2$, where $d_c(.,.)$ is the generalized chordal distance, this condition can be expressed in terms of $d_{\min}(\mathcal{C})$ as follows:

$$4(\sqrt{\rho} + \sqrt{t})^2 < d_{\min}(\mathcal{C}). \tag{9}$$

Note that the left hand side of (9) is greater than or equal to that of (8) by a multiplicative factor that is between 2 and 4. This shows the clear advantage in using the new distance $d(.,.)$ instead of the chordal distance in characterizing the error-and-erasure correction capability of analog subspace codes. The advantage is due to the fact that although $d(.,.)$ does not always satisfy the triangle inequality, it exhibits properties of a metric when dealing with inputs and outputs of analog operator channels.

IV. Character-Polynomial Subspace Codes

Consider a cyclic group G of order $|G|$. A character χ associated to G is a homomorphism from G to the unit circle in the complex plane with the regular multiplication of complex numbers, i.e.,

$$\chi(g_1 g_2) = \chi(g_1)\chi(g_2) \tag{10}$$

for all $g_1, g_2 \in G$. The set of all characters associated to G are described as follows [16]:

$$\chi_j(g'^k) = \mathrm{e}(\frac{jk}{|G|}), \tag{11}$$

for $k = 0, 1, \ldots, |G| - 1$, where g' is a generator of G and $\mathrm{e}(x) \stackrel{\text{def}}{=} \exp(2\pi i x)$. For a finite field $\mathbb{F}_q$ the *additive* characters, using (11), are denoted by χ_j, for $j = 0, 1, \ldots, q-1$, and are described as follows [16]:

$$\chi_j(\alpha) = \mathrm{e}(\frac{\mathrm{tr_a}(j\alpha)}{p}) \tag{12}$$

for $j \in \mathbb{F}_q$, where p is the characteristic of $\mathbb{F}_q$, and

$$\mathrm{tr_a}(\gamma) \stackrel{\text{def}}{=} \gamma + \gamma^p + \cdots + \gamma^{p^{m-1}}$$

is the *absolute* trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$, where $q = p^m$. Note that (12) implies that $\chi_j(\alpha) = \chi_1(j\alpha)$ and the trivial additive character is $\chi_0(\alpha) = 1$ for $\alpha \in \mathbb{F}_q$.

Next, for some $k < q$, let

$$\mathcal{F} \stackrel{\text{def}}{=} \{f \in \mathbb{F}_q[x] : f(x) = \sum_{i \in [k], i \bmod p \neq 0} f_i x^i\}. \tag{13}$$

Note that $|\mathcal{F}| = q^{\lceil k(p-1)/p \rceil}$. We fix $n = q - 1$ in our construction.

Definition 5: The code $\mathcal{C}(\mathcal{F}) \subseteq G_{1,n}()$, referred to as a character-polynomial (CP) code, is defined as follows:

$$\mathcal{C}(\mathcal{F}) \stackrel{\text{def}}{=} \{\langle (c_1, c_2, \ldots, c_n) \rangle : c_i = \chi\big(f(\alpha_i)\big), \forall f \in \mathcal{F}\}, \tag{14}$$

where χ is a fixed nontrivial additive character of $\mathbb{F}_q$, and α_i's are distinct non-zero elements of $\mathbb{F}_q$.

The following theorem provides a lower bound on the normalized minimum distance of $\mathcal{C}(\mathcal{F})$ in terms of q and d.

Theorem 2: The code $\mathcal{C}(\mathcal{F})$ has size $q^{\lceil k(p-1)/p \rceil}$ and

$$\delta \geqslant 1 - \frac{\big((k-1)\sqrt{q} + 1\big)^2}{n^2}, \tag{15}$$

where $\delta = d_{\min}/2m$ (here $m = 1$) is the normalized minimum distance of the code.

Proof: The main ingredient in the proof is the result, due to Weil [17], on the summations over characters. Please see [13] for more details. ■

In Figure 1, we compare the trade-off between the rate R and the normalized minimum distance δ that the CP codes offer at different values of n with Shannon's lower bound [5], for $n \to \infty$, and lower bounds derived by Henkel [18, Theorem 4.2] for finite values of n. Note that these lower bounds are of the same type as Gilbert-Varshamov bound and do not yield explicit constructions. Nevertheless, it can be observed that CP codes can outperform these lower bounds at low rates thereby improving these bounds while providing explicit constructions.

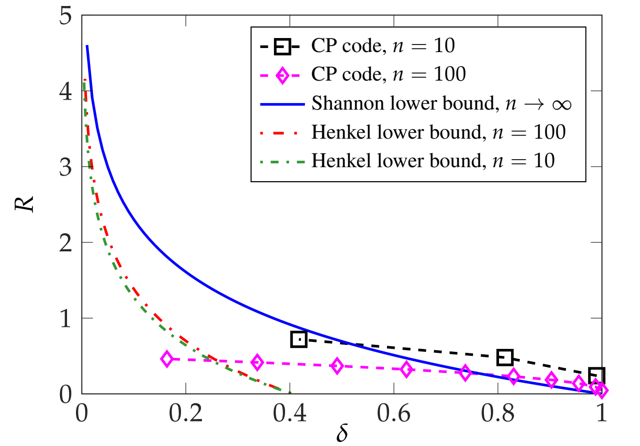

Fig. 1: Comparison of CP codes with lower-bounds in terms of the trade-off between R and δ.

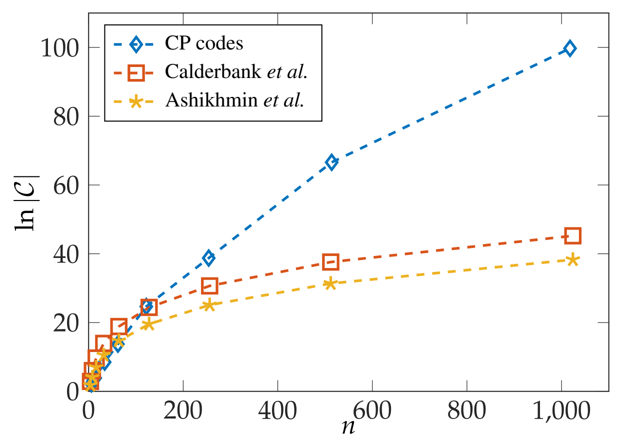

Fig. 2: Comparison of the codes in $G_{2,n}(\mathbb{R})$ obtained from our proposed CP codes in $G_{1,\frac{n}{2}}()$ with the codes constructed by Calderbank *et al.* [19] and Ashikhmin *et al.* [20].

Remark 2. Given a subspace code in $\mathcal{C} \subseteq G_{m,n}()$ one can construct a code in $G_{2m,2n}(\mathbb{R})$ by mapping $\boldsymbol{C}_i \in \mathcal{C}$ to

$$\begin{bmatrix} \Re(\boldsymbol{C}_i) & \Im(\boldsymbol{C}_i) \\ -\Im(\boldsymbol{C}_i) & \Re(\boldsymbol{C}_i) \end{bmatrix}, \tag{16}$$

where $\Re(.)$ and $\Im(.)$ represent the real part and the imaginary part of their input, respectively. It can be observed that this mapping preserves the normalized distance between the codewords. This mapping enables us to construct codes in $G_{2,n}(\mathbb{R})$ using the proposed CP codes, while keeping the normalized minimum distance and the size of the code the same, in order to have fair comparisons with existing code constructions in the real Grassmann space.

In Figure 2, we compare CP codes with two existing constructions of Grassmann codes, that are constructed explicitly for a wide range of n, in the literature. In [19], Calderbank *et al.* introduce a group-theoretic framework for packing in $G_{2^i,2^k}(\mathbb{R})$ for any pair of integers (i,k) with $i \leqslant k$. In another prior work, Ashikhmin *et al.* [20] provide a code construction method in $G_{2^i,2^k}()$ based on binary Reed-Muller (RM) codes. By utilizing the mapping specified in (16) for both CP codes and the codes constructed in [20] with $i = 0$, we compare the log-size of the codes obtained in $G_{2,n}(\mathbb{R})$ with that of codes in $G_{2,n}(\mathbb{R})$ from [19], while fixing $\delta = \frac{1}{2}$ for codes from [19] [20] and having $\delta \geqslant \frac{1}{2}$ for CP codes, for all considered values of n. The results are shown in Figure 2. Note that n is equal to 2^k for the constructions in [19] and [20], while for CP codes we pick $n = 2p$, where p is the largest p with $p < 2^{k-1}$, for $k \in \{3, \dots, 10\}$. Also, since the rates, for all considered codes, is decreasing with n, we plot the log-sizes of these codes instead of their rates. It can be observed that CP codes offer significantly larger code size and, consequently, rate comparing to the other explicit constructions, as n grows large.

V. Robustness Against Additive Noise

In this section, we analyze the robustness of the analog operator channel in the presence of an additive noise. The additive noise is denoted by $\boldsymbol{N} \in^{l\times n}$, referred to as the *noise matrix*. In the presence of the additive noise, the transform equation described in (4) is extended as follows:

$$\boldsymbol{Y}_{l\times n} = \boldsymbol{H}_{l\times m}\boldsymbol{X}_{m\times n} + \boldsymbol{G}_{l\times t}\boldsymbol{E}_{t\times n} + \boldsymbol{N}_{l\times n}. \tag{17}$$

More specifically, the effect of all the noise terms added to the blocks across the wireless network is included in the noise matrix $\boldsymbol{N}$. For ease of notation, let $\boldsymbol{A}$ denote the term $\boldsymbol{HX} + \boldsymbol{GE}$, consisting of terms associated to the transmitted blocks and the interference blocks, referred to as the *signal matrix*.

In the rest of this section, we aim at characterizing the perturbation imposed by the additive noise in terms of the subspace distance. Let $r = \text{rank}(\boldsymbol{A})$ and r_d denote the rank deficiency of $\boldsymbol{A}$, i.e.,

$$r_d \overset{\text{def}}{=} l - r. \tag{18}$$

Also, let $\boldsymbol{A}_1$ be an $r \times n$ full row rank sub-matrix of $\boldsymbol{A}$ and $\boldsymbol{A}_2$ denote the sub-matrix of $\boldsymbol{A}$ consisting of its remaining rows. Let $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ be sub-matrices of $\boldsymbol{N}$ with row indices associated with row indices of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, respectively. Without loss of generality one can write

$$A = \left[\frac{\boldsymbol{A}_1}{\boldsymbol{A}_2}\right], \quad N = \left[\frac{\boldsymbol{N}_1}{\boldsymbol{N}_2}\right], \tag{19}$$

where both $\boldsymbol{A}_1$ and $\boldsymbol{N}_1$ are $r \times n$ matrices and both $\boldsymbol{A}_2$ and $\boldsymbol{N}_2$ are $r_d \times n$ matrices. Then we have the following theorem.

Theorem 3: Let $\boldsymbol{A} \in^{l\times n}$ with $\text{rank}(\boldsymbol{A}) = r$. Let $\boldsymbol{A}_1 \in^{r\times n}$ denote a foll row rank sub-matrix of $\boldsymbol{A}$. Then for any $\boldsymbol{N} \in^{l\times n}$ that satisfies $\|\boldsymbol{A}_1^+\|_2 \|\boldsymbol{N}\|_2 < 1$ we have

$$d(\langle \boldsymbol{A}\rangle, \langle \boldsymbol{A}+\boldsymbol{N}\rangle) \leqslant (\sqrt{r_d} + \sqrt{\Delta})^2, \tag{20}$$

where r_d is the rank deficiency of $\boldsymbol{A}$, as defined in (18). Also,

$$\Delta \overset{\text{def}}{=} 2\epsilon + \epsilon^2, \tag{21}$$

where

$$\epsilon \overset{\text{def}}{=} \left(\frac{(1+\sqrt{2})\kappa(\boldsymbol{A}_1)}{1 - \|\boldsymbol{A}_1^+\|_2 \|\boldsymbol{N}\|_2} \cdot \frac{\|\boldsymbol{N}\|}{\|\boldsymbol{A}_1\|_2}\right)^2. \tag{22}$$

The proof is omitted due to space constraints, please see [13] for the proof.

Theorem 3 shows that the additive noise affects the output of the analog operator channel in two ways. It, in a sense, *rotates* the output subspace by a value upper bounded by Δ and also implicitly induces an extra interference term of dimension upper bounded by r_d (For simplicity, we consider the worst case scenario where this dimension is r_d). This motivates us to define the *noisy* analog operator channel as follows. First, we define a stochastic operator $\mathcal{R}_\Delta$, called *rotation operator*, which takes a subspace $U \in \mathcal{P}(W)$ as the input and returns a random subspace $V \in \mathcal{P}(W)$ with $\dim(V) = \dim(U)$ as the output such that $d(U, V) \leqslant \Delta$.

Definition 6: A noisy analog operator channel associated with the analog ambient space W is a channel with input U and output V, where $U, V \in \mathcal{P}(W)$, with the following input-output relation:

$$V = \mathcal{R}_\Delta\big(\mathcal{H}_k(U) \oplus E\big) \oplus F, \tag{23}$$

where $\mathcal{H}_k(U)$ and E induce subspace erasures and errors, respectively, as in the analog operator channel model, defined in Definition 1, $\mathcal{R}_\Delta$ is the rotation operation defined above, and F is the implicit interference caused by the additive noise.

The following theorem extends the result of Theorem 1 to take into account the effect of the additive noise as well as subspace errors and erasures.

Theorem 4: Consider a subspace code $\mathcal{C}$ used for communication over a noisy analog operator channel, as defined in Definition 6, i.e., the input to the channel is $U \in \mathcal{C}$. Let t, ρ, and r_d denote the dimension of errors, erasures, and the implicit noise interference F, respectively. Then the minimum distance decoder successfully recovers the transmitted codeword $U \in \mathcal{C}$ from the received subspace V if

$$\rho + t + (\sqrt{\rho + t + \Delta} + \sqrt{\Delta} + 2\sqrt{r_d})^2 < d_{\min}(\mathcal{C}). \tag{24}$$

The proof is omitted due to space constraints, please see [13] for the proof.

Remark 3. Note that Theorem 4 reduces to Theorem 1 by setting $r_d = \Delta = 0$. In other words, Theorem 4 *properly* extends the result of Theorem 1, on relating the minimum distance of analog subspace codes to their error-and-erasure correction capability, to the noisy analog operator channel scenario. In practice, the implicit noise interference term F and, consequently, the term r_d in (24) can be potentially removed by simply discarding a certain number of received blocks at the receiver. However, this requires knowing the rank of the received signal by the receiver which may not be readily available due to the assumptions on non-coherent communications. This can be further explored when considering a practical wireless networking scenario to see whether such information, i.e., the rank of the received signal, can be obtained or well-approximated, e.g., using principal component analysis (PCA) methods, by the receiver. Also, as shown in Theorem 3, the other term, besides r_d, resulting from the additive noise that affects the output subspace is Δ. Note that for a fixed signal matrix $\boldsymbol{A}$, as $\|\boldsymbol{N}\| \to 0$, we have $\epsilon \to 0$ as well as $\Delta \to 0$, where ϵ and Δ are specified in (22) and (21), respectively. This together with a procedure to remove the r_d term, as discussed above, show that the analog operator channel can be made *robust* with respect to the additive noise, i.e., the subspace distance perturbation in the received signal matrix, caused by the additive noise, goes to zero as $\|\boldsymbol{N}\| \to 0$.

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