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Correction to: The Landau Equation with the Specular Reflection Boundary Condition

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In the paper [5, Section 6], we quoted a crucial lemma in [7, Proposition 4.1] for the proof of the main L^2 decay estimates [5, Theorem 13]. Unfortunately, an omission was recently discovered in its proof. We now present an alternative proof for [5, Theorem 13] based on the methodology in [3,4] without using the method of [7, Proposition 4.1].

We consider the following linearized *Landau equation*

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(g, f). \tag{1}$$

The initial-boundary condition of f is given by

$$\begin{cases} f(0, x, v) = f_0(x, v), \text{ if } x \in \Omega \text{ and } v \in \mathbb{R}^3, \\ f(t, x, v) = f(t, x, v - 2(v \cdot n_x)n_x), \text{ if } x \in \partial\Omega \text{ and } v \cdot n_x < 0 \\ \|f_0\|_{\infty,\vartheta+m} < \epsilon \end{cases} \tag{2}$$

for some small $\epsilon > 0$, $\vartheta \geq 0$ and $m > \frac{3}{2}$. We say that the domain Ω is rotationally symmetric if there exist vectors x_0 and w such that

$$((x - x_0) \times w) \cdot n_x = 0$$

for all $x \in \partial\Omega$. Without loss of generality, we assume that the conservation laws of total mass and energy for $t \geq 0$ terms of the perturbation f apply:

$$\int_{\Omega\times\mathbb{R}^3} f(t, x, v)\sqrt{\mu}\, \mathrm{d}x\, \mathrm{d}v = 0, \quad \int_{\Omega\times\mathbb{R}^3} |v|^2 f(t, x, v)\sqrt{\mu}\, \mathrm{d}x\, \mathrm{d}v = 0. \tag{3}$$

In addition, we assume that the conservation of total angular momentum if Ω is rotationally symmetric:

$$\int_{\Omega\times\mathbb{R}^3} ((x - x_0) \times w) \cdot f(t, x, v)v\sqrt{\mu}\, \mathrm{d}x\, \mathrm{d}v = 0. \tag{4}$$

Define the energy

$$\mathcal{E}_\vartheta(f(t)) \stackrel{\text{def}}{=} \left|f(t)\right|_{2,\vartheta}^2 + \int_0^t \left|f(s)\right|_{\sigma,\vartheta}^2 \, \mathrm{d}s. \tag{5}$$

In order to prove the main L^2 decay theorem (Theorem 3), we first intend to prove the following positivity of L:

Proposition 1. *Let f be a weak solution of (1)–(4) with $\mathcal{E}_\vartheta(f(0))$ bounded for some $\vartheta \geq 0$. Then there exists a sufficiently small positive constant $\epsilon > 0$ such that if*

$$\|g\|_{\infty,m} \leq \epsilon \tag{6}$$

for some $m > \frac{3}{2}$, then we have $\delta_\epsilon > 0$ such that

$$\int_0^1 (Lf, f)\, \mathrm{d}s \geq \delta_\epsilon \int_0^1 \|f\|_\sigma^2 \, \mathrm{d}s.$$

In order to prove the positivity of L, it suffices to prove the following proposition as we have Lemma 5 of [2]:

Proposition 2. *Let f be a weak solution of (1)–(4) with $\mathcal{E}_\vartheta(f(0))$ bounded for some $\vartheta \geq 0$. Then there exists a sufficiently small $\epsilon > 0$ such that if $\|g\|_{\infty,m} \leq \epsilon$ for some $m > \frac{3}{2}$, we have $C_\epsilon > 0$ such that*

$$\int_0^1 \|Pf(\tau)\|_\sigma^2 \, \mathrm{d}s \leq C_\epsilon \int_0^1 \|(I-P)f(\tau)\|_\sigma^2 \, \mathrm{d}s.$$

Proof. If the proposition is not true, then there exist a sequence of family g_n and a sequence of solutions f_n to (1)–(4) with $g = g_n$ and $f = f_n$ such that

$$\|g_n\|_{\infty,m} \leq \frac{1}{n} \tag{7}$$

for some $m > \frac{3}{2}$, but

$$\int_0^1 \|(I-P)f_n(\tau)\|_\sigma^2 \, \mathrm{d}s \leq \frac{1}{n} \int_0^1 \|Pf_n(\tau)\|_\sigma^2 \, \mathrm{d}s \tag{8}$$

for any n.

We first prove the weak compactness of f_n. We first reformulate the Equation (1) as

$$f_t + v \cdot \nabla_x f = \bar{A}_g f + \bar{K}_g f, \tag{9}$$

$$\begin{aligned} \bar{A}_g f &:= \partial_i \left[\left\{\phi^{ij} * [\mu + \mu^{1/2} g]\right\} \partial_j f\right] \\ &\quad - \left\{\phi^{ij} * [v_i \mu^{1/2} g]\right\} \partial_j f - \left\{\phi^{ij} * [\mu^{1/2} \partial_j g]\right\} \partial_i f \\ &=: \nabla_v \cdot (\sigma_G \nabla_v f) + a_g \cdot \nabla_v f, \end{aligned} \tag{10}$$

$$
\begin{aligned}
\bar{K}_g f &:= Kf + \partial_i \sigma^i f - \sigma^{ij} v_i v_j f \\
&\qquad - \partial_i \left\{ \phi^{ij} * [\mu^{1/2} \partial_j g] \right\} f + \left\{ \phi^{ij} * [v_i \mu^{1/2} \partial_j g] \right\} f,
\end{aligned} \tag{11}
$$

where $G = \mu + \sqrt{\mu} g$,

$$
Kf := -\mu^{-1/2} \partial_i \left\{ \mu \left[\phi^{ij} * \left\{ \mu^{1/2} [\partial_j f + v_j f] \right\} \right] \right\}, \tag{12}
$$

and $\sigma^{ij} = \sigma^{ij}_{\mu}, \quad \sigma^i = \sigma^{ij} v_j$, with

$$
\sigma^{ij}_u(v) \stackrel{\text{def}}{=} \phi^{ij} * u = \int_{\mathbb{R}^3} \phi^{ij}(v - v') u(v') \, \mathrm{d}v'.
$$

Note that the eigenvalues $\lambda(v)$ of $\sigma(v)$ satisfy [2, Lemma 3]

$$
(1 + |v|)^{-3} \lesssim \lambda(v) \lesssim (1 + |v|)^{-1}. \tag{13}
$$

For any fixed $l < 0$, we multiply (1) for $f = f_n$ and $g = g_n$ by $(1 + |v|)^{2l} f_n$ and integrate both sides of the resulting equation and obtain

$$
\begin{aligned}
&\iint_{\Omega \times \mathbb{R}^3} \frac{1}{2} \left((1 + |v|)^{2l} f_n^2(t, x, v) - (1 + |v|)^{2l} f_n^2(t_0, x, v) \right) \mathrm{d}x \, \mathrm{d}v \\
&+ \int_{t_0}^{t} \iint_{\Omega \times \mathbb{R}^3} (1 + |v|)^{2l} (Lf_n) f_n \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s \\
&= \int_{t_0}^{t} \iint_{\Omega \times \mathbb{R}^3} (1 + |v|)^{2l} \Gamma(g, f_n) f_n \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s \\
&\leq \int_{t_0}^{t} \|g_n\|_\infty \|f_n\|^2_{\sigma, l} \, \mathrm{d}s,
\end{aligned}
$$

by Theorem 2.8 of [6]. Also, since $l < 0$, we deduce by Lemma 6 of [2] that

$$
\begin{aligned}
&\int_{t_0}^{t} \iint_{\Omega \times \mathbb{R}^3} (1 + |v|)^{2l} (Lf_n) f_n \, \mathrm{d}x \, \mathrm{d}v \, \mathrm{d}s \\
&\quad \geq \int_{t_0}^{t} \mathrm{d}s \left(\frac{1}{2} \|f_n(s)\|^2_{\sigma, l} - C_l \|(1 + |v|)^l f_n(s)\|^2_{L^2} \right).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\frac{1}{2} \|(1 + |v|)^l f_n(t)\|^2_{L^2} + \int_{t_0}^{t} \mathrm{d}s \, \frac{1}{4} \|f_n(s)\|^2_{\sigma, l} \leq \|(1 + |v|)^l f_n(t_0)\|^2_{L^2} \\
&+ C \int_{t_0}^{t} \mathrm{d}s \|(1 + |v|)^l f_n(s)\|^2_{L^2}.
\end{aligned}
$$

Thus, by (7) and Grönwall's inequality, we obtain that

$$
\|(1 + |v|)^l f_n(t)\|^2_{L^2} + \int_{t_0}^{t} \|f_n(s)\|^2_{\sigma, l} \, \mathrm{d}s \leq C e^{t - t_0} \|(1 + |v|)^l f_n(t_0)\|^2_{L^2}. \tag{14}
$$

On the other hand, we note that

$$\|f\|_\sigma \geq C\|(1+|v|)^{-1/2} f\|_{L^2},$$

by (13). Thus we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} \|f_n(s)\|_\sigma^2\,\mathrm{d}s &= \|f_n(t)\|_\sigma^2 \geq C\|(1+|v|)^{-1/2} f_n(t)\|_{L^2}^2\\
&\geq C\|(1+|v|)^{-1/2} f_n(t_0)\|_{L^2}^2 - 2C\int_{t_0}^{t}\iint_{\Omega\times\mathbb{R}^3}(1+|v|)^{-1}(Lf_n)f_n\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s\\
&\quad -2C\int_{t_0}^{t}\|g_n\|_\infty\|f_n\|_{\sigma,-1/2}^2\,\mathrm{d}s\\
&\geq C\|(1+|v|)^{-1/2} f_n(t_0)\|_{L^2}^2 - 2C\int_{t_0}^{t}\left(\frac{3}{2}\|f(s)\|_{\sigma,-1/2}^2 - C\|f(s)\|_\sigma^2\right)\mathrm{d}s\\
&\quad -2C\int_{t_0}^{t}\|g_n\|_\infty\|f_n\|_{\sigma,-1/2}^2\,\mathrm{d}s\\
&\geq C\|(1+|v|)^{-1/2} f_n(t_0)\|_{L^2}^2 - C'\int_{t_0}^{t}\|f(s)\|_\sigma^2\,\mathrm{d}s,
\end{aligned}$$

for some $C' > 0$ by Lemma 2.7 of [6]. By (7) and Grönwall's inequality, we obtain that

$$\int_{t_0}^{t}\|f_n(s)\|_\sigma^2\,\mathrm{d}s \geq C(1-e^{-C'(t-t_0)})\|(1+|v|)^{-1/2} f_n(t_0)\|_{L^2}^2. \tag{15}$$

Now we define the normalized term Z_n of f_n as

$$Z_n \stackrel{\text{def}}{\equiv} \frac{f_n}{\sqrt{\int_0^1\|Pf_n\|_\sigma^2\,\mathrm{d}s}}.$$

For $s \in [0, 1]$, we have

$$\begin{aligned}
\|(1+|v|)^{-1/2} Z_n(s)\|_{L^2}^2 &= \frac{\|(1+|v|)^{-1/2} f_n(s)\|_{L^2}^2}{\sqrt{\int_0^1\|Pf_n\|_\sigma^2\,\mathrm{d}\tau}}\\
&\leq \frac{Ce^s\|(1+|v|)^{-1/2} f_n(0)\|_{L^2}^2}{\int_0^1\|Pf_n\|_\sigma^2\,\mathrm{d}\tau},
\end{aligned}$$

by (14) for $l = -1/2$. On the other hand, by the assumption (8) we have

$$\begin{aligned}
(n+1)\int_0^1\|Pf_n\|_\sigma^2\,\mathrm{d}\tau &\geq n\int_0^1\|Pf_n\|_\sigma^2\,\mathrm{d}\tau + n\int_0^1\|(1-P)f_n\|_\sigma^2\,\mathrm{d}\tau\\
&\geq n\int_0^1\|f_n\|_\sigma^2\,\mathrm{d}\tau.
\end{aligned}$$

Thus,

$$\|(1+|v|)^{-1/2}Z_n(s)\|_{L^2}^2 = \frac{\|(1+|v|)^{-1/2}f_n(s)\|_{L^2}^2}{\sqrt{\int_0^1 \|Pf_n\|_\sigma^2\,\mathrm{d}\tau}}$$

$$\leq \frac{n+1}{n}\frac{Ce^s\|(1+|v|)^{-1/2}f_n(0)\|_{L^2}^2}{\int_0^1 \|f_n\|_\sigma^2\,\mathrm{d}\tau} \leq 2\frac{C\|(1+|v|)^{-1/2}f_n(0)\|_{L^2}^2}{\int_0^1 \|f_n\|_\sigma^2\,\mathrm{d}\tau},$$

for any $n \geq 1$. Now, by (15), we have

$$\int_0^s \|f_n(\tau)\|_\sigma^2\,\mathrm{d}\tau \geq C(1-e^{-Cs})\|(1+|v|)^{-1/2}f_n(0)\|_{L^2}^2.$$

Thus, we obtain the uniform bound

$$\sup_{0\leq s\leq 1}\|(1+|v|)^{-1/2}Z_n(s)\|_{L^2}^2 \leq C$$

for some $C > 0$. Also, by the normalization we already had $\int_0^1 \|Z_n(s)\|_\sigma^2\,\mathrm{d}s = 1$. Note that this will also imply that there is no concentration in time. Therefore, there exists the weak limit Z of Z_n in $\int_0^1 \|\cdot\|_\sigma^2\,\mathrm{d}s$.
Also, by (8), we have

$$\int_0^1 \|(I-P)Z_n\|_\sigma^2\,\mathrm{d}s \leq \frac{1}{n} \to 0. \tag{16}$$

By the triangle inequality, we also have that $\int_0^1 \|PZ_n(s)\|_\sigma^2\,\mathrm{d}s$ is uniformly bounded from above. In addition, the norm $\|\cdot\|_\sigma$ is an anisotropic Sobolev norm with respect to direction of the velocity v by definition. Since the eigenvalues $\lambda(v)$ of the matrix $\sigma(v)$ satisfies the bound (13), the normed vector space with the norm $\|\cdot\|_\sigma$ can be understood as a weighted L^2 Sobolev space and is reflexive. Then by Alaoglu's theorem and Eberlein–šmulian's theorem, PZ_n converges weakly to PZ in $\int_0^1 \|\cdot\|_\sigma^2\,\mathrm{d}s$ up to a subsequence. Thus, we conclude that $(I-P)Z = 0$ and $Z = PZ$. Thus, we can write $Z(t,x,v)$ as

$$Z(t,x,v) = (a(t,x) + b(t,x)\cdot v + c(t,x)|v|^2)\sqrt{\mu}.$$

Also, by taking the limit $n \to \infty$, we note that the limit Z satisfies

$$\partial_t Z + v\cdot\nabla_x Z = \Gamma(g_\infty, Z) = 0 \tag{17}$$

in the sense of distribution, as the condition (7) makes $g_\infty = 0$ a.e. outside a null set that results in the vanishing integral $\int \Gamma(g_\infty, Z)\phi$ via an integration by parts and we also have $\int LZ\phi$ vanishes as $Z = PZ$, for a test function $\phi \in C_c^1$.

Now our main strategy is to show that Z has to be zero by (16), the specular reflection boundary conditions, (17), and the conservation laws (3) and (4). On the other hand, we will show the strong convergence of Z_n to Z in $\int_0^1 \|\cdot\|_\sigma^2\,\mathrm{d}s$ by proving the compactness. This will lead us to a contradiction.

We first introduce the following lemma, which provides more information on the form of Z:

Lemma 1. (Lemma 6 of [4]) *There exist constants a_0, c_1, c_2, and constant vectors b_0, b_1 and $\bar{w}$such that $Z(t, x, v)$ takes the form*

$$\left(\left(\frac{c_0}{2}|x|^2 - b_0 \cdot x + a_0\right) + (-c_0 t x - c_1 x + \bar{w} \times x + b_0 t + b_1)\right.$$
$$\left. \times v + \left(\frac{c_0 t^2}{2} + c_1 t + c_2\right)|v|^2\right)\sqrt{\mu}.$$

Moreover, these constants are finite.

Our case also shares the same transport equation (17) for Z that deduces the same macroscopic equations as (72)–(76) of [4] with $Z = PZ$ and the lemma holds. Moreover, a better bound (14) provides that the coefficients are finite.

0.1. Plan for the proof of the strong convergence

We first show the strong convergence of Z_n to Z in $\int_0^1 \|\cdot\|_\sigma^2 \, \mathrm{d}s$. First of all, we note that we have seen already that there is no concentration in time-boundary at $s = 0$ or $s = 1$ by (14). Then regarding the remainder of the domain $(\varepsilon, 1-\varepsilon) \times \Omega \times \mathbb{R}^3$ for some $\varepsilon > 0$, we split it into three parts; we define the interior D^ε_{int}, the non-grazing set D^ε_{ng}, and the singular grazing set D^ε_{sg} so that

$$(\varepsilon, 1-\varepsilon) \times \Omega \times \mathbb{R}^3 = D^\varepsilon_{int} \cup D^\varepsilon_{lv} \cup D^\varepsilon_{ng} \cup D^\varepsilon_{sg}.$$

More precisely, we define the interior D^ε_{int} as

$$D^\varepsilon_{int} \stackrel{\text{def}}{=} (\varepsilon, 1-\varepsilon) \times S_\varepsilon,$$

where

$$S_\varepsilon = \left\{(x, v) \in \Omega \times \mathbb{R}^3 : \zeta(x) < -\varepsilon^4 \text{ and } |v| \leq \frac{4}{\varepsilon}\right\}.$$

Then we define sets of the compliment. Firstly, define the set of large velocity D^ε_{lv} as

$$D^\varepsilon_{lv} \stackrel{\text{def}}{=} (\varepsilon, 1-\varepsilon) \times \Omega \times \left\{|v| > \frac{4}{\varepsilon}\right\} \stackrel{\text{def}}{=} (\varepsilon, 1-\varepsilon) \times S^c_{\varepsilon,0}.$$

We define the singular grazing set D^ε_{sg} as

$$D^\varepsilon_{sg} \stackrel{\text{def}}{=} (\varepsilon, 1-\varepsilon) \times S^c_{\varepsilon,1},$$

where

$$S^c_{\varepsilon,1} = \left\{(x, v) \in \Omega \times \mathbb{R}^3 : \zeta(x) \geq -\varepsilon^4 \text{ and } \left[|n_x \cdot v| < \frac{\varepsilon}{2} \text{ or } |v| > \frac{1}{\varepsilon}\right]\right\}.$$

Lastly, we define the non-grazing set D^ε_{ng} as

$$D^\varepsilon_{ng} \stackrel{\text{def}}{=} (\varepsilon, 1-\varepsilon) \times S^c_{\varepsilon,2},$$

where

$$S^c_{\varepsilon,2} = \left\{(x, v) \in \Omega \times \mathbb{R}^3 : \zeta(x) \geq -\varepsilon^4 \text{ and } \left[|n_x \cdot v| \geq \frac{\varepsilon}{2} \text{ and } |v| \leq \frac{1}{\varepsilon}\right]\right\}.$$

Here recall that $\zeta(x)$ is the smooth function such that $\Omega = \{x : \zeta(x) < 0\}$.

To prove the strong convergence in $\int_0^1 \| \cdot \|^2_\sigma \, \mathrm{d}s$, it suffices to show

$$\sum_{1\leq j\leq 5} \int_0^1 \mathrm{d}s \|\langle Z_n, e_j\rangle e_j - \langle Z, e_j\rangle e_j\|^2_\sigma \to 0,$$

where e_j are an orthonormal basis for

$$\text{span}\{\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}\},$$

as we have (16). Since $e_j(v)$ is smooth and the 0^{th} and the 1^{st} derivatives are exponentially decaying for large $|v|$, it suffices to prove

$$\int_0^1 \mathrm{d}s \int_\Omega \mathrm{d}x \; |\langle Z_n, e_j\rangle - \langle Z, e_j\rangle|^2 \to 0.$$

We establish this by considering the decomposition of the domain as above.

0.2. Interior compactness on D^ε_{int}

Suppose χ_1 is a smooth cutoff function that is supported on D^ε_{int} and consider

$$Z_n = (1 - \chi_1)Z_n + \chi_1 Z_n.$$

In this subsection, we will consider the contribution $\chi_1 Z_n$ via the averaging lemma. We define another smooth cutoff function $\tilde{\chi}_1$ such that $\tilde{\chi}_1 = 1$ on D^ε_{int} and $\tilde{\chi}_1 = 0$ outside $D^{\varepsilon/2}_{int}$. Then $\tilde{\chi}_1$ has a larger support than χ_1 and $\tilde{\chi}_1 = 1$ on D^ε_{int}. The reason that we additionally define $\tilde{\chi}_1$ with a larger support than χ_1 is in order to make $(1 - \chi_1)Z_n = Z_n$ outside D^ε_{int} and to make $\tilde{\chi}_1 Z_n = Z_n$ on D^ε_{int}.

We first observe that $\tilde{\chi}_1 Z_n$ satisfies the following equation:

$$(\partial_t + v \cdot \nabla_x)(\tilde{\chi}_1(1 + |v|)^{-1/2} Z_n) = (1 + |v|)^{-1/2} \\ (-\tilde{\chi}_1 L[Z_n] + Z_n[\partial_t + v \cdot \nabla_x]\tilde{\chi}_1 + \tilde{\chi}_1 \Gamma(g_n, Z_n)) .$$

We claim that the right-hand side is uniformly bounded in $L^2([0, 1] \times \Omega \times \mathbb{R}^3)$. We observe that the second term is easily uniformly bounded by the L^2 norm of $(1 + |v|)^{-1/2} Z_n$, which is uniformly bounded by (14). We also observe that the L^2 norms of the first and the third terms are bounded as follows. By Lemma 1 of [2], $\tilde{\chi}_1 L Z_n$ can be written as

$$(1 + |v|)^{-1/2}\tilde{\chi}_1 L Z_n = \Big(- \partial_i(\sigma^{ij}\partial_j Z_n \tilde{\chi}_1) + \sigma^{ij}\partial_j Z_n \partial_i \tilde{\chi}_1 - \partial_i \sigma^i Z_n \tilde{\chi}_1 \\ +\sigma^{ij} v_i v_j Z_n \tilde{\chi}_1 + \partial_i(\mu^{1/2}(\phi^{ij} * (\mu^{1/2}(\partial_j Z_n + v_j Z_n)))\tilde{\chi}_1)$$

$$-\mu^{1/2}(\phi^{ij} * (\mu^{1/2}(\partial_j Z_n + v_j Z_n)))\partial_i \tilde{\chi}_1 - v_i \mu^{1/2}(\phi^{ij}$$
$$*(\mu^{1/2}(\partial_j Z_n + v_j Z_n)))\tilde{\chi}_1\Big)(1+|v|)^{-1/2} \equiv \partial_i g_1 + g_2, \tag{18}$$

where $\tilde{\chi}_1$ has a compact support and $g_1, g_2 \in L^2([0,1]\times\Omega\times\mathbb{R}^3)$ as

$$\|g_1\|_{L^2} + \|g_2\|_{L^2} \lesssim \|(I-P)Z_n\|_\sigma.$$

Also, we apply Lemma 7 at (56) of [2] to estimate $\tilde{\chi}_1\Gamma(g_n, Z_n)$ with g_1 there is our g_n and $g_2 = Z_n$ to see that

$$(1+|v|)^{-1/2}\tilde{\chi}_1\Gamma(g_n, Z_n) = \partial_{ij}g^{ij} + \partial_i g^i + g,$$

where

$$\|g^{ij}\|_{L^2} + \|g^i\|_{L^2} + \|g\|_{L^2} \lesssim \|g_n\|_{L^2}\|Z_n\|_\sigma \lesssim \|g_n\|_{\infty,m}\|Z_n\|_\sigma,$$

as $m > \frac{3}{2}$ by the assumption (7). Therefore, we have

$$(\partial_t + v\cdot\nabla_x)(\tilde{\chi}_1(1+|v|)^{-1/2}Z_n) = h,$$

where $h \in L^2([0,1]\times\Omega; H^{-2}(\mathbb{R}^3))$. Then by the averaging lemma [1, Theorem 5], we have

$$\langle \tilde{\chi}_1(1+|v|)^{-1/2}Z_n, e_j\rangle \in H^{1/6}([0,1]\times\Omega),$$

which holds uniformly in n. Thus, up to a subsequence, we have the convergence

$$\langle \tilde{\chi}_1(1+|v|)^{-1/2}Z_n, e_j\rangle \to \langle \tilde{\chi}_1(1+|v|)^{-1/2}Z, e_j\rangle \text{ in } L^2([0,1]\times\Omega). \tag{19}$$

0.3. Near the time-boundary and the grazing set D^ε_{sg}

Now, note that the leftover from the previous section is now

$$\int_0^1 \mathrm{d}s \int_\Omega \mathrm{d}x |\langle (1-\chi_1)(Z_n - Z), e_j\rangle|^2.$$

Regarding the contribution, we note that

$$\begin{aligned}
&\int_0^1 \mathrm{d}s \int_\Omega \mathrm{d}x |\langle (1-\chi_1)|Z_n - Z|, e_j\rangle|^2 \\
&\quad\le \int_0^1 \mathrm{d}s \int_\Omega \mathrm{d}x \int_{\mathbb{R}^3} \mathrm{d}v (1-\chi_1)^2 |Z_n - Z|^2 e_j \\
&\quad= \left(\int_0^\varepsilon + \int_{1-\varepsilon}^1\right)\mathrm{d}s \int_{\Omega\times\mathbb{R}^3} \mathrm{d}x\,\mathrm{d}v + \sum_{j=0}^2 \int_0^1 \mathrm{d}s \int_{S^c_{\varepsilon,j}} \mathrm{d}x\,\mathrm{d}v
\end{aligned} \tag{20}$$

In this subsection, we only consider the contribution

$$\left(\int_0^\varepsilon + \int_{1-\varepsilon}^1\right)\mathrm{d}s \int_{\Omega\times\mathbb{R}^3} \mathrm{d}x\,\mathrm{d}v + \sum_{j=0}^1 \int_0^1 \mathrm{d}s \int_{S^c_{\varepsilon,j}} \mathrm{d}x\,\mathrm{d}v \tag{21}$$

near the time-boundary and the grazing set D^{ε}_{sg}.

The first integral of (21) is bounded as

$$\left(\int_0^{\varepsilon}+\int_{1-\varepsilon}^1\right) \mathrm{d}s \int_{\Omega\times\mathbb{R}^3} \mathrm{d}x\, \mathrm{d}v\, (1-\chi_1)^2|Z_n-Z|^2 e_j$$
$$\leq 2\varepsilon \sup_{0\leq s\leq 1} (\|(1+|v|)^{-1/2}Z_n(s)\|_2^2+\|(1+|v|)^{-1/2}Z(s)\|_2^2).$$

Note that we have the uniform boundedness

$$\sup_{0\leq s\leq 1,\ n\geq 1} \|(1+|v|)^{-1/2}Z_n(s)\|_{L^2}^2 < C,$$

by (14) and that $\|(1+|v|)^{-1/2}Z(s)\|_2^2 = \|(1+|v|)^{-1/2}Z(0)\|_2^2$, by the transport equation (17). Then this rules out the possible concentration at $t=0$ or $t=1$.

Regarding another term in (21), we observe that

$$\int_0^1 \mathrm{d}s \int_{S^c_{\varepsilon,0}} \mathrm{d}x\, \mathrm{d}v\, (1-\chi_1)^2|Z_n-Z|^2 e_j$$
$$\leq \int_0^1 \mathrm{d}s \int_{\Omega} \mathrm{d}x \int_{|v|\geq\frac{4}{\varepsilon}} \mathrm{d}v\, (1+|v|)^{-1/2}(|Z_n|^2+|Z|^2)(1+|v|)^{2+1/2}\sqrt{\mu}.$$

Then for a sufficiently small $\varepsilon \ll 1$, we have

$$(1+|v|)^{2+1/2}\sqrt{\mu} \approx (1+|v|)^{2+1/2}\exp(-|v|^2/2) \lesssim \exp(-c|v|^2)$$
$$\lesssim \exp\left(-\frac{16c}{\varepsilon^2}\right) \lesssim \varepsilon,$$

for some uniform constant $0<c<\frac{1}{2}$. Therefore, we have

$$\int_0^1 \mathrm{d}s \int_{S^c_{\varepsilon,0}} \mathrm{d}x\, \mathrm{d}v\, (1-\chi_1)^2|Z_n-Z|^2 e_j$$
$$\leq \int_0^1 \mathrm{d}s \int_{\Omega} \mathrm{d}x \int_{|v|\geq\frac{4}{\varepsilon}} \mathrm{d}v\, (1+|v|)^{-1/2}(|Z_n|^2+|Z|^2)(1+|v|)^{2+1/2}\sqrt{\mu}$$
$$\lesssim \varepsilon \sup_{0\leq s\leq 1} (\|(1+|v|)^{-1/2}Z_n(s)\|_2^2+\|(1+|v|)^{-1/2}Z(s)\|_2^2).$$

Note that we have the uniform boundedness

$$\sup_{0\leq s\leq 1,\ n\geq 1} \|(1+|v|)^{-1/2}Z_n(s)\|_{L^2}^2 < C,$$

by (14) and that $\|(1+|v|)^{-1/2}Z(s)\|_2^2 = \|(1+|v|)^{-1/2}Z(0)\|_2^2$, by the transport equation (17).

On the other hand, for the other remainder term in (21), we observe that

$$\int_0^1 \mathrm{d}s \int_{S^c_{\varepsilon,1}} \mathrm{d}x\, \mathrm{d}v (1-\chi_1)^2|Z_n-Z|^2 e_j$$

$$\leq \int_0^1 \mathrm{d}s \int_{S^c_{\varepsilon,1}} \mathrm{d}x\,\mathrm{d}v(1-\chi_1)^2 \left(|(I-P)(Z_n - Z)|^2 + |PZ_n - PZ|^2 \right) e_j$$
$$= \int_{S^c_{\varepsilon,1}} \mathrm{d}x\,\mathrm{d}v(1-\chi_1)^2 \left(|(I-P)Z_n|^2 + |PZ_n - PZ|^2 \right) e_j, \tag{22}$$

as $(I-P)Z = 0$. Note that by the additional exponential decay e_j with respect to $|v|$, we have

$$\int_{S^c_{\varepsilon,1}} \mathrm{d}x\,\mathrm{d}v(1-\chi_1)^2 |(I-P)Z_n|^2 e_j \lesssim \|(I-P)Z_n\|_\sigma \lesssim \frac{1}{n}.$$

In addition, we define

$$PZ_n = a_n(t,x) + \boldsymbol{b}_n(t,x)\cdot(e_2,e_3,e_4) + c_n(t,x)e_5,$$

and

$$PZ = a(t,x) + \boldsymbol{b}(t,x)\cdot(e_2,e_3,e_4) + c(t,x)e_5,$$

for $\{e_j\}_{j=1,\ldots,5}$ is the orthonormal basis of span$\{\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}\}$. Note that a_n, b_n, c_n, a, b, and c are functions of t and x. Then, we observe that the remainder term satisfies

$$\begin{aligned}
&\int_{S^c_{\varepsilon,1}} \mathrm{d}x\,\mathrm{d}v(1-\chi_1)^2 |PZ_n - PZ|^2 e_j \\
&\lesssim \int_0^1 \mathrm{d}s \int_{\Omega\setminus\Omega_\varepsilon} \mathrm{d}x \left(|a_n - a|^2 + |\boldsymbol{b}_n - \boldsymbol{b}|^2 + |c_n - c|^2 \right) \\
&\int_{|v\cdot n_x| < \frac{\varepsilon}{2} \text{ or } |v| > \frac{1}{\varepsilon}} \mathrm{d}v\ (1+|v|)^l \sqrt{\mu} \\
&\lesssim \int_{|v\cdot n_x| < \frac{\varepsilon}{2} \text{ or } |v| > \frac{1}{\varepsilon}} \mathrm{d}v\ (1+|v|)^l \sqrt{\mu}
\end{aligned} \tag{23}$$

for some $l \geq 2$ by

$$\int_0^1 \|PZ_n\|_\sigma^2\,\mathrm{d}s \approx \int_0^1 \left(\|a_n(s,\cdot)\|_2^2 + \|\boldsymbol{b}_n(s,\cdot)\|_2^2 + \|c_n(s,\cdot)\|_2^2 \right) \mathrm{d}s \lesssim 1,$$

and

$$\int_0^1 \|PZ\|_\sigma^2\,\mathrm{d}s \approx \int_0^1 \left(\|a(s,\cdot)\|_2^2 + \|\boldsymbol{b}(s,\cdot)\|_2^2 + \|c(s,\cdot)\|_2^2 \right) \mathrm{d}s \lesssim 1,$$

from the linear independency of e_j. Then, if $|v| > \frac{1}{\varepsilon}$, then $(1+|v|)^l\sqrt{\mu} \leq C\varepsilon$, for $|v| > \frac{1}{\varepsilon}$, if ε is sufficiently smnall. On the other hand, if $|v\cdot n_x| < \frac{\varepsilon}{2}$, we have

$$\int_{|v\cdot n_x| < \frac{\varepsilon}{2}} \mathrm{d}v\ (1+|v|)^l\sqrt{\mu} \lesssim \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \mathrm{d}v_{||} \int_{\mathbb{R}^2} \mathrm{d}v_\perp e^{-|v_\perp|^2/8} \lesssim \varepsilon,$$

where $v_{||} \overset{\text{def}}{=} (n_x \cdot v)n_x$, and $v_\perp = v - v_{||}$ for $|n_x \cdot v| \le \frac{\varepsilon}{2}$. Then the (LHS) of (22) is bounded from above by

$$\int_0^1 \mathrm{d}s \int_{S^c_{\varepsilon,1}} \mathrm{d}x\, \mathrm{d}v (1-\chi_1)^2 |Z_n - Z|^2 e_j \lesssim \varepsilon.$$

0.4. On the non-grazing set D^ε_{ng}

Finally, we are now left with the L^2 norm for the non-grazing set D^ε_{ng} from (20)

$$\int_0^1 \mathrm{d}s \int_{S^c_{\varepsilon,2}} \mathrm{d}x\, \mathrm{d}v\ (1-\chi_1)^2 |Z_n - Z|^2 e_j.$$

In this subsection, we will prove that there is no concentration at the boundary, so that we can conclude that Z_n converges strongly to Z in $[0,1] \times \bar{\Omega} \times \mathbb{R}^3$. The main strategy in this section is to show that the non-grazing set part $\chi_\pm Z_n$ can be controlled by the inner boundary part $Z_n|_{\gamma_\varepsilon}$, which will be further controlled by the interior compactness. Here the inner boundary is defined as $\gamma^\varepsilon \overset{\text{def}}{=} \{x : \zeta(x) = -\varepsilon^4\} \times \mathbb{R}^3$. Now we fix $(s, x, v) \in D^\varepsilon_{ng}$. Then we define backward/forward in time characteristic trajectories $\chi_\pm$ as

$$\begin{aligned} \chi_+(t,x,v) &= 1_{\Omega\setminus\Omega_\varepsilon}(x - v(t-s)) 1_{\{|v|\le 1/\varepsilon,\ n_{x-v(t-s)}\cdot v > \varepsilon\}}(v),\ \text{for } 0 \le t \le s,\\ \chi_-(t,x,v) &= 1_{\Omega\setminus\Omega_\varepsilon}(x - v(t-s)) 1_{\{|v|\le 1/\varepsilon,\ n_{x-v(t-s)}\cdot v < -\varepsilon\}}(v),\ \text{for } 0 \le s \le t, \end{aligned} \tag{24}$$

where $\Omega_\varepsilon \overset{\text{def}}{=} \{x \in \Omega : \zeta(x) \le -\varepsilon^4\}$. Note that $\chi_\pm$ solves the transport equation $(\partial_t + v \cdot \nabla_x)\chi_\pm = 0$ with

$$\chi_\pm(s,x,v) = 1_{\Omega\setminus\Omega_\varepsilon}(x) 1_{\{|v|\le 1/\varepsilon,\ n_x\cdot v \lesseqgtr \pm\varepsilon\}}(v),$$

and that it satisfies the following lemma:

Lemma 2. (Lemma 10 of [4]) $\chi_\pm$ *satisfies the followings:*

(1) *For* $0 \le s - \varepsilon^2 \le t \le s$, *if* $\chi_+(t,x,v) \ne 0$ *then* $n_x \cdot v > \frac{\varepsilon}{2} > 0$. *Moreover,* $\chi_+(s-\varepsilon^2, x, v) = 0$, *for* $\zeta(x) \ge -\varepsilon^4$.
(2) *For* $s \le t \le s + \varepsilon^2 \le 1$, *if* $\chi_-(t,x,v) \ne 0$, *then* $n_x \cdot v < -\frac{\varepsilon}{2} < 0$. *Moreover,* $\chi_-(s+\varepsilon^2, x, v) = 0$, *for* $\zeta(x) \ge -\varepsilon^4$.

We now observe that $\chi_\pm Z_n$ satisfies the following equation

$$\begin{aligned} &(\partial_t + v \cdot \nabla_x)(\chi_\pm Z_n)\\ &\quad = -\chi_\pm L[Z_n] + \chi_\pm \Gamma(g_n, Z_n). \end{aligned}$$

We claim that

$$\int_{S^c_{\varepsilon,2}} |Z_n|^2 \,\mathrm{d}x\, \mathrm{d}v \lesssim \varepsilon,$$

if n is sufficiently large. To see this, we first observe the L^2 estimate for χ_+ part over $[s-\varepsilon^2, s] \times S^c_{\varepsilon,2}$ that for the inner boundary $\gamma^\varepsilon \overset{\text{def}}{=} \{x : \zeta(x) = -\varepsilon^4\} \times \mathbb{R}^3$,

$$\|\chi_+ Z_n(s)\|^2_{L^2(S^c_{\varepsilon,2})} + \int_{s-\varepsilon^2}^{s} \|\chi_+ Z_n(t)\|^2_{\gamma_+}\,\mathrm{d}t - \int_{s-\varepsilon^2}^{s} \|\chi_+ Z_n(t)\|^2_{\gamma_+^\varepsilon}\,\mathrm{d}t$$
$$= \|\chi_+ Z_n(s-\varepsilon^2)\|^2_{L^2(S^c_{\varepsilon,2})} + \int_{s-\varepsilon^2}^{s} \|\chi_+ Z_n(t)\|^2_{\gamma_-}\,\mathrm{d}t - \int_{s-\varepsilon^2}^{s} \|\chi_+ Z_n(t)\|^2_{\gamma_-^\varepsilon}\,\mathrm{d}t$$
$$-2\int_{s-\varepsilon^2}^{s} (\chi_+ L[Z_n], \chi_+ Z_n)\,\mathrm{d}t + 2\int_{s-\varepsilon^2}^{s} (\chi_+ \Gamma(g_n, Z_n), \chi_+ Z_n)\,\mathrm{d}t,$$

where $(\cdot,\cdot)$ is the L^2 inner product on $S^c_{\varepsilon,2}$. By Lemma 2, $\chi_+ Z_n(s-\varepsilon^2) = 0$. Also, $\chi_+ Z_n = 0$ on γ_- and γ_-^ε by the support condition of χ_+. On the other hand, by (16) and Lemma 6 of [2], we have

$$\int_{s-\varepsilon^2}^{s} (\chi_+ L[Z_n], \chi_+ Z_n)\,\mathrm{d}t = \int_{s-\varepsilon^2}^{s} (L[Z_n], \chi_+^2 Z_n)\,\mathrm{d}t$$
$$= \int_{s-\varepsilon^2}^{s} (L[(1-\chi_+^2+\chi_+^2)Z_n], \chi_+^2 Z_n)\,\mathrm{d}t = \int_{s-\varepsilon^2}^{s} (L[\chi_+^2 Z_n], \chi_+^2 Z_n)\,\mathrm{d}t,$$

by the support condition of χ_+. Thus,

$$\int_{s-\varepsilon^2}^{s} (\chi_+ L[Z_n], \chi_+ Z_n)\,\mathrm{d}t = \int_{s-\varepsilon^2}^{s} (L[\chi_+^2 Z_n], \chi_+^2 Z_n)\,\mathrm{d}t$$
$$\le C\int_0^1 \|(I-P)\chi_+^2 Z_n\|^2_\sigma\,\mathrm{d}t \le C\int_0^1 \|(I-P)Z_n\|^2_\sigma\,\mathrm{d}t = \frac{C}{n}.$$

Finally, we observe that, by Theorem 2.8 at (2.16) of [6], (7), and (14), we have

$$\int_{s-\varepsilon^2}^{s} (\chi_+ \Gamma(g_n, Z_n), \chi_+ Z_n)\,\mathrm{d}t = \int_{s-\varepsilon^2}^{s} (\Gamma(g_n, Z_n), \chi_+^2 Z_n)\,\mathrm{d}t$$
$$\le C\|g_n\|_\infty \int_{s-\varepsilon^2}^{s} \|Z_n\|_\sigma \|\chi_+^2 Z_n\|_\sigma\,\mathrm{d}t \le C\|g_n\|_\infty \int_{s-\varepsilon^2}^{s} \|Z_n\|^2_\sigma\,\mathrm{d}t \le \frac{C}{n}.$$

Altogether, we have

$$\|\chi_+ Z_n(s)\|^2_{L^2} + \int_{s-\varepsilon^2}^{s} \|\chi_+ Z_n(t)\|^2_{\gamma_+}\,\mathrm{d}t - \int_{s-\varepsilon^2}^{s} \|\chi_+ Z_n(t)\|^2_{\gamma_+^\varepsilon}\,\mathrm{d}t \le \frac{C}{n}. \quad (25)$$

Here, we note that by definition

$$\chi_+ Z_n(s,x,v) = 1_{\Omega\setminus\Omega_\varepsilon}(x) 1_{\{|v|\le 1/\varepsilon,\ n_x\cdot v>\varepsilon\}}(v) Z_n(s,x,v).$$

Similarly, we obtain for the part $\chi_- Z_n$

$$\|\chi_- Z_n(s)\|^2_{L^2} + \int_{s}^{s+\varepsilon^2} \|\chi_- Z_n(t)\|^2_{\gamma_-}\,\mathrm{d}t - \int_{s}^{s+\varepsilon^2} \|\chi_- Z_n(t)\|^2_{\gamma_-^\varepsilon}\,\mathrm{d}t \le \frac{C}{n}. \quad (26)$$

Altogether, we have

$$\|Z_n(s)\|^2_{L^2(S^c_{\varepsilon,2})} \le \int_{s-\varepsilon^2}^{s} \|\chi_+ Z_n(t)\|^2_{\gamma^\varepsilon_+}\,\mathrm{d}t + \int_{s}^{s+\varepsilon^2} \|\chi_- Z_n(t)\|^2_{\gamma^\varepsilon_-}\,\mathrm{d}t + \frac{C}{n}. \tag{27}$$

Now we will prove that the right-hand side of (27) can be arbitrarily small by showing that the right-hand side can further be bounded via the interior compactness inside S_ε. In order to control the trace norm on the non-grazing set, we are going to derive a trace theorem for the Landau equation to $1_{\{|v|\le\frac{1}{\varepsilon}\}}(Z_n - Z)$ over the domain $\bar{S}_\varepsilon$. We first consider the estimate for $t \in (s-\varepsilon^2, s)$. Recall that χ_+ from (24) indeed satisfies

$$\begin{aligned} &\partial_t \chi_+ + v\cdot\nabla_x \chi_+ = 0,\\ &\chi_+(s-\varepsilon^2, x, v) = 0 \text{ for } \mathrm{dist}(x, \partial\Omega_\varepsilon) \le \varepsilon, \end{aligned} \tag{28}$$

where $\Omega_\varepsilon \stackrel{\text{def}}{=} \{x \in \Omega : \zeta(x) = -\varepsilon^4\}$. We choose a smooth cutoff function $\chi_b = \chi_b^\varepsilon(x)$ near $\partial\Omega_\varepsilon$ such that $\chi_b \equiv 1$ if $\mathrm{dist}(x,\partial\Omega_\varepsilon) \le \frac{\varepsilon^4}{4}$, $\chi_b \equiv 0$ if $\mathrm{dist}(x,\partial\Omega_\varepsilon) \ge \varepsilon^4$, and the growth is up to $|\nabla_x \chi_b| \lesssim \varepsilon^{-3/2}$. We also choose a smooth cutoff function $\chi_2 = \chi_2(v)$ such that $\chi_2 = 1$ for $|v| \le \frac{1}{\varepsilon}$ and $= 0$ for $|v| \ge \frac{4}{\varepsilon}$ and

$$|\chi_+\chi_2| + |\nabla_v(\chi_+\chi_2)| + |\nabla_v^2(\chi_+\chi_2)| \lesssim \mu\left(\frac{|v|}{4}\right). \tag{29}$$

Note that $\chi_2(v)$ has a larger support than $1_{|v|\le\frac{1}{\varepsilon}}$. We then take $\bar\chi = \chi_2\chi_b\chi_+$, such that $\bar\chi(s-\varepsilon^2, x, v) = 0$ for $\mathrm{dist}(x,\partial\Omega_\varepsilon) \le \varepsilon$ and

$$(\partial_t + v\cdot\nabla_x)\bar\chi = \chi_+\chi_2 v\cdot\nabla_x\chi_b.$$

Now consider the following rearranged Equation (9) for this argument:

$$\partial_t Z_n + v\cdot\nabla_x Z_n = \nabla_v\cdot(\sigma_{G_n}\nabla_v Z_n) + a_{g_n}\cdot\nabla_v Z_n + \bar K_{g_n} Z_n,$$

where $G_n = \mu + \sqrt{\mu} g_n$. Then, note that $\bar\chi Z_n$ satisfies the equation

$$\begin{aligned} (\partial_t + v\cdot\nabla_x)(\bar\chi Z_n) &= \chi_2\chi_+ Z_n v\cdot\nabla_x\chi_b + \nabla_v\cdot(\sigma_{G_n}\nabla_v(\bar\chi Z_n))\\ &-\sigma_{G_n} Z_n\Delta_v\bar\chi - 2\sigma_{G_n}\nabla_v Z_n\cdot\nabla_v\bar\chi - Z_n\nabla_v(\sigma_{G_n})\cdot\nabla_v\bar\chi\\ &+\bar\chi a_{g_n}\cdot\nabla_v Z_n + \bar K_{g_n}(\bar\chi Z_n). \end{aligned} \tag{30}$$

We multiply $\bar\chi Z_n$ and integrate on $(s-\varepsilon^2, s)\times S_\varepsilon$ to obtain that

$$\begin{aligned} &\frac{1}{2}\left(\|\bar\chi Z_n(s)\|^2_{L^2(S_\varepsilon)} - \|\bar\chi Z_n(s-\varepsilon^2)\|^2_{L^2(S_\varepsilon)}\right) + \int_{s-\varepsilon^2}^{s}\mathrm{d}t\,\|\bar\chi Z_n\|^2_{\gamma^\varepsilon}\\ &= -\int_{s-\varepsilon^2}^{s}\mathrm{d}t\iint_{S_\varepsilon}\mathrm{d}x\,\mathrm{d}v\,\sigma_{G_n}|\nabla_v(\bar\chi Z_n)|^2\\ &+\int_{s-\varepsilon^2}^{s}\mathrm{d}t\iint_{S_\varepsilon}\mathrm{d}x\,\mathrm{d}v\,\Big[\bar\chi Z_n\Big(\chi_2\chi_+ Z_n v\cdot\nabla_x\chi_b\\ &-\sigma_{G_n} Z_n\Delta_v\bar\chi - 2\sigma_{G_n}\nabla_v Z_n\cdot\nabla_v\bar\chi - Z_n\nabla_v(\sigma_{G_n})\cdot\nabla_v\bar\chi \end{aligned}$$

$$+\bar{\chi} a_{g_n} \cdot \nabla_v Z_n + \bar{K}_{g_n}(\bar{\chi} Z_n)\Big)\Big], \tag{31}$$

by the integration by parts. Note that $\bar{\chi} Z_n = 0$ on γ_-^ε by the support condition of χ_+. By (28) and the support condition of χ_b, we also have $\bar{\chi} Z_n(s-\varepsilon^2) = 0$. Thus, we have

$$\begin{aligned}
&\frac{1}{2}\|\bar{\chi} Z_n(s)\|^2_{L^2(S_\varepsilon)} + \int_{s-\varepsilon^2}^{s} \mathrm{d}t \|\bar{\chi} Z_n\|^2_{\gamma_+^\varepsilon} + \int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\, \sigma_{G_n} |\nabla_v(\bar{\chi} Z_n)|^2 \\
&= \int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v \Big[\bar{\chi} Z_n\Big(\chi_2\chi_+ Z_n v \cdot \nabla_x \chi_b - \sigma_{G_n} Z_n \Delta_v \bar{\chi} \\
&\quad -2\sigma_{G_n}\nabla_v Z_n \cdot \nabla_v \bar{\chi} - Z_n \nabla_v(\sigma_{G_n}) \cdot \nabla_v \bar{\chi} + \bar{\chi} a_{g_n} \cdot \nabla_v Z_n + \bar{K}_{g_n}(\bar{\chi} Z_n)\Big)\Big].
\end{aligned} \tag{32}$$

We estimate the upper bound of each term of the right-hand side. We first observe that

$$\begin{aligned}
&\left|\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\, \bar{\chi} Z_n \chi_2 \chi_+ Z_n v \cdot \nabla_x \chi_b\right| \\
&\lesssim \int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\, \mu\left(\frac{|v|}{4}\right) |Z_n|^2 |\nabla_x \chi_b| \\
&\lesssim \varepsilon^{-3/2} \int_{s-\varepsilon^2}^{s} \mathrm{d}t\, \|(1+|v|)^{-1/2} Z_n\|^2_{L^2(S_\varepsilon)},
\end{aligned}$$

by the assumption of χ_b. Also, by (11), Lemma 3 and Lemma 6 of [2], we have

$$\begin{aligned}
&\left|\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\, \bar{\chi} Z_n \bar{K}_{g_n}(\bar{\chi} Z_n)\right| \\
&\le \int_{s-\varepsilon^2}^{s} \mathrm{d}t\, (\eta\|\bar{\chi} Z_n\|_\sigma + C_\eta \|\bar{\chi} Z_n\|_{L^2(S_\varepsilon)}) \|(\bar{\chi} Z_n)\|_\sigma \\
&\le \int_{s-\varepsilon^2}^{s} \mathrm{d}t\, (\eta' \|Z_n\|^2_\sigma + C_{\eta'} \|(1+|v|)^{-1/2} Z_n\|^2_{L^2(S_\varepsilon)})
\end{aligned} \tag{33}$$

for a sufficiently small η' by Young's inequality. We also note that by Lemma 3 of [2], we have $\sigma^{ij} v_i v_j = \lambda_1 |v|^2$, where $\lambda_1 \approx (1+|v|)^{-3}$. Therefore,

$$\iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\, |\sigma^{ij} v_i v_j (\bar{\chi} Z_n)^2| \le \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\, \frac{(\bar{\chi} Z_n)^2}{1+|v|}.$$

Here, note that by Lemma 3 of [2] and Lemma 2.4 of [6], if n is sufficiently large so that $\|g_n\|_{L^\infty} \ll 1$, then

$$\sigma^{ij} \partial_i (\bar{\chi} Z_n) \partial_j (\bar{\chi} Z_n) \approx \sigma^{ij}_{G_n} \partial_i (\bar{\chi} Z_n) \partial_j (\bar{\chi} Z_n),$$

where $G_n = \mu + \sqrt{\mu} g_n$. Then, by (29), Lemma 2.4 of [6] and Lemma 3 of [2], we have

$$\left|\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\, (\bar{\chi} Z_n) \sigma_{G_n} Z_n \Delta_v \bar{\chi}\right| \lesssim \int_{s-\varepsilon^2}^{s} \mathrm{d}t$$

$$\iint_{\zeta<-\varepsilon^4 \text{ and } |v|\le\frac{4}{\varepsilon}} \mathrm{d}x\,\mathrm{d}v\;\mu\left(\frac{|v|}{4}\right)\frac{|Z_n|^2}{1+|v|}$$
$$\lesssim \int_{s-\varepsilon^2}^{s} \mathrm{d}t\;\|(1+|v|)^{-1/2}Z_n\|^2_{L^2(S_\varepsilon)}.$$

Similarly, we have

$$\left|\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\; 2(\bar{\chi}Z_n)\sigma_{G_n}\nabla_v Z_n\cdot\nabla_v\bar{\chi}\right|$$
$$\lesssim \int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{\zeta<-\varepsilon^4 \text{ and } |v|\le\frac{4}{\varepsilon}} \mathrm{d}x\,\mathrm{d}v\;\mu\left(\frac{|v|}{4}\right)\sigma_{G_n}|Z_n||\nabla_v Z_n|$$
$$\lesssim \eta\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\;\sigma_{G_n}|\nabla_v Z_n|^2$$
$$+C_\eta\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\;\mu\left(\frac{|v|}{2}\right)\sigma_{G_n}|Z_n|^2$$
$$\lesssim \eta\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\;\sigma_{G_n}|\nabla_v Z_n|^2$$
$$+C_\eta\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\;\mu\left(\frac{|v|}{2}\right)\frac{|Z_n|^2}{1+|v|}$$
$$\lesssim \eta\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\;\sigma_{G_n}|\nabla_v Z_n|^2$$
$$+C_\eta\int_{s-\varepsilon^2}^{s} \mathrm{d}t\;\|(1+|v|)^{-1/2}Z_n\|^2_{L^2(S_\varepsilon)}$$

for any small $\eta>0$, by Young's inequality. In addition, we have

$$\left|\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\;(\bar{\chi}Z_n)Z_n\nabla_v\sigma_{G_n}\cdot\nabla_v\bar{\chi}\right|$$
$$\lesssim \int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{\zeta<-\varepsilon^4 \text{ and } |v|\le\frac{4}{\varepsilon}} \mathrm{d}x\,\mathrm{d}v\;\mu\left(\frac{|v|}{4}\right)\frac{|(\bar{\chi}Z_n)||Z_n|}{(1+|v|)^2}$$
$$\lesssim \int_{s-\varepsilon^2}^{s} \mathrm{d}t\;\|(1+|v|)^{-1/2}Z_n\|^2_{L^2(S_\varepsilon)}.$$

Also, by (29) and the definition of a_{g_n} from (10), we observe that

$$\left|\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\;(\bar{\chi}Z_n)\bar{\chi}a_{g_n}\cdot\nabla_v Z_n\right|$$
$$\lesssim \int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\;|Z_n||\nabla_v Z_n|\left(|\phi^{ij}*(v_i\mu^{1/2}g_n)|+|\phi^{ij}*(\mu^{1/2}\partial_j g_n)|\right)$$
$$\lesssim \int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{S_\varepsilon} \mathrm{d}x\,\mathrm{d}v\;|Z_n||\nabla_v Z_n|\left(2|\phi^{ij}*(\mu^{1/4}g_n)|+|\partial_j\phi^{ij}*(\mu^{1/2}g_n)|\right)$$
$$\lesssim \|g_n\|_{L^\infty}\int_{s-\varepsilon^2}^{s} \mathrm{d}t \iint_{\zeta<-\varepsilon^4 \text{ and } |v|\le\frac{4}{\varepsilon}} \mathrm{d}x\,\mathrm{d}v\;\mu\left(\frac{|v|}{4}\right)\frac{|Z_n||\nabla_v Z_n|}{(1+|v|)}$$

$$\lesssim \eta \int_{s-\varepsilon^2}^{s} \mathrm{d}t\ \|Z_n\|_\sigma^2 + \frac{C_\eta}{n^2} \int_{s-\varepsilon^2}^{s} \mathrm{d}t\ \|(1+|v|)^{-1/2} Z_n\|_{L^2(S_\varepsilon)}^2$$

for a sufficiently small $\eta > 0$, by Young's inequality.

Altogether, we have

$$\begin{aligned}
\int_{s-\varepsilon^2}^{s} \|\chi_+ Z_n(t)\|_{\gamma_+^\varepsilon}^2 \,\mathrm{d}t &\lesssim (C_\eta + \varepsilon^{-3/2}) \\
&\int_{s-\varepsilon^2}^{s} \left\|(1+|v|)^{-1/2} Z_n\right\|_{L^2(S_\varepsilon)}^2 \mathrm{d}t + \eta \int_{s-\varepsilon^2}^{s} \mathrm{d}t\ \|Z_n\|_\sigma^2 \\
&\lesssim (C_\eta + \varepsilon^{-3/2}) \int_{s-\varepsilon^2}^{s} \left\|(1+|v|)^{-1/2} Z\right\|_{L^2(S_\varepsilon)}^2 \mathrm{d}t \\
&+ (C_\eta + \varepsilon^{-3/2}) \int_{s-\varepsilon^2}^{s} \left\|(1+|v|)^{-1/2} (Z_n - Z)\right\|_{L^2(S_\varepsilon)}^2 \mathrm{d}t \\
&+ \eta \int_{s-\varepsilon^2}^{s} \mathrm{d}t\ \|Z_n\|_\sigma^2
\end{aligned}$$

for any small $\eta > 0$. We repeat the same argument for the part $\int_s^{s+\varepsilon^2} \|\chi_- Z_n(t)\|_{\gamma_-^\varepsilon}^2 \,\mathrm{d}t$ of (27) using χ_-, instead of χ_+. Note that, by the interior compactness, we have for a fixed $\varepsilon > 0$

$$\lim_{n\to\infty} \int_{s-\varepsilon^2}^{s} \left\|(1+|v|)^{-1/2} (Z_n - Z)\right\|_{L^2(S_\varepsilon)}^2 \mathrm{d}t = 0.$$

Then, by (27), we have for a small $\eta \sim \sqrt{\varepsilon}$ such that $C_\eta \lesssim \varepsilon^{-3/2}$ and for a sufficiently large $n > 0$,

$$\begin{aligned}
\|Z_n(s)\|_{L^2(S_{\varepsilon,2}^c)}^2 &\lesssim (C_\eta + \varepsilon^{-3/2}) \int_{s-\varepsilon^2}^{s+\varepsilon^2} \left\|(1+|v|)^{-1/2} Z\right\|_{L^2(S_\varepsilon)}^2 \mathrm{d}t \\
&+ (C_\eta + \varepsilon^{-3/2}) \int_{s-\varepsilon^2}^{s+\varepsilon^2} \left\|(1+|v|)^{-1/2} (Z_n - Z)\right\|_{L^2(S_\varepsilon)}^2 \mathrm{d}t \\
&+\eta \int_{s-\varepsilon^2}^{s+\varepsilon^2} \mathrm{d}t\ \|Z_n\|_\sigma^2 + \frac{C}{n} \\
&\lesssim 2(C_\eta + \varepsilon^{-3/2})\varepsilon^2 \sup_{t\in[0,1]} \left\|(1+|v|)^{-1/2} Z(t)\right\|_{L^2(S_\varepsilon)}^2 \\
&+ (C_\eta + \varepsilon^{-3/2})\varepsilon^2 + \eta \int_{s-\varepsilon^2}^{s+\varepsilon^2} \mathrm{d}t\ \|Z_n\|_\sigma^2 + \frac{C}{n} \lesssim C'\sqrt{\varepsilon},
\end{aligned}$$

by (14) where $C' > 0$ depends on $a_0, c_0, c_1, c_2, b_0, b_1$, and $\bar{w}$ of Lemma 1. Therefore, for any small $\varepsilon > 0$, we have

$$\|Z_n(s)\|_{L^2(S_{\varepsilon,2}^c)}^2 \lesssim C'\sqrt{\varepsilon}, \tag{34}$$

for large n.

0.5. Strong convergence and the non-zero PZ

By (19)–(21), (14), (22), and (34), we obtain

$$\int_0^1 \mathrm{d}s \int_\Omega \mathrm{d}x \, |\langle Z_n, e_j\rangle - \langle Z, e_j\rangle|^2 \to 0,$$

where e_j are an orthonormal basis for span$\{\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}\}$. Since $e_j(v)$ is smooth and the 0^{th} and the 1^{st} derivatives are exponentially decaying for large $|v|$, we obtain that

$$\sum_{1\le j\le 5} \int_0^1 \mathrm{d}s \|\langle Z_n, e_j\rangle e_j - \langle Z, e_j\rangle e_j\|_\sigma^2 \to 0.$$

Finally, note that

$$Z_n = \sum_{1\le j\le 5} \langle Z_n, e_j\rangle e_j + (I-P)Z_n,$$

and we have (16). Therefore, we obtain the strong convergence of Z_n to Z in $\int_0^1 \mathrm{d}s \|\cdot\|_\sigma^2$, and we have

$$\int_0^1 \mathrm{d}s \, \|PZ\|_\sigma^2 = 1.$$

Also, recall that the specular reflection condition for Z_n is $Z_n(t,x,v) = Z_n(t,x,R_x(v))$. By taking $n\to\infty$, we can observe that Z satisfies the same condition for $|v\cdot n_x| \ge \varepsilon/2$. By continuity of Z, we obtain $Z(t,x,v) = Z(t,x,R_x(v))$.

0.6. Z is indeed zero

On the other hand, we show below that PZ is indeed zero, which will lead us to a contradiction. The proof will be done via the use of the specular boundary conditions, (17), and the conservation laws (3) and (4). Recall that, by the conservation laws (3), we first obtain

$$\int Z\sqrt{\mu} = \int Z|v|^2\sqrt{\mu} = 0.$$

On the other hand, Lemma 1 implies that, for any $s\in[0,1]$, we obtain the conservation laws in the form of

$$\int \left(\left(\frac{c_0}{2}|x|^2 - b_0\cdot x + a_0\right) + \left(\frac{c_0 t^2}{2} + c_1 s + c_2\right)|v|^2 \right)\sqrt{\mu} = 0, \tag{35}$$

and

$$\int \left(\left(\frac{c_0}{2}|x|^2 - b_0\cdot x + a_0\right)|v|^2 + \left(\frac{c_0 t^2}{2} + c_1 s + c_2\right)|v|^4 \right)\sqrt{\mu} = 0. \tag{36}$$

This implies $c_0 = c_1 = 0$. Also, by the specular reflection condition that $Z(s, x, v) = Z(s, x, R_x(v))$, we have for any $x \in \partial\Omega$ that

$$b \cdot n_x = 0 \text{ or } (\bar{w} \times x + b_0 s + b_1) \cdot n_x = 0.$$

First of all, the coefficient b_0 of the time-variable s is zero, which gives

$$b \cdot n_x = 0 \text{ or } (\bar{w} \times x + b_1) \cdot n_x = 0. \tag{37}$$

If $\bar{w} = 0$, then $b_1 \cdot n_x = 0$ on $\partial\Omega$. Then we can choose a point $x' \in \partial\Omega$ such that $b_1 \parallel n_{x'}$ via taking the minimizer of $\min_{\zeta(x)} b_1 \cdot x$. Then this gives $b_1 \cdot n_{x'} = 0$ and $b_1 = 0$. If $\bar{w} \neq 0$, then we decompose b_1 as

$$b_1 = \beta_1 \frac{\bar{w}}{|\bar{w}|} + \beta_2 \eta,$$

where $|\eta| = 1$ and $\eta \perp \bar{w}$. Then

$$\eta = \left(\frac{\bar{w}}{|\bar{w}|} \times \eta \right) \times \frac{\bar{w}}{|\bar{w}|}.$$

Therefore, we get

$$b_1 = \beta_1 \frac{\bar{w}}{|\bar{w}|} + \beta_2 \left(\frac{\bar{w}}{|\bar{w}|} \times \eta \right) \times \frac{\bar{w}}{|\bar{w}|} = \beta_1 \frac{\bar{w}}{|\bar{w}|} - x_0 \times \bar{w},$$

where $x_0 = -\beta_2 \left(\frac{\bar{w}}{|\bar{w}|} \times \eta \right) \frac{1}{|\bar{w}|}$. Therefore, by (37) we have

$$\beta_1 \frac{\bar{w}}{|\bar{w}|} n_x + ((x - x_0) \times \bar{w}) \cdot n_x = 0.$$

Now note that we can choose a point $x' \in \partial\Omega$ such that $\bar{w} \parallel n_{x'}$. Then we deduce $\bar{w} \times (n_{x'} \times (x' - x_0)) = 0$ and obtain $\beta_1 = 0$. Therefore, we obtain

$$Z = \bar{w} \times (x - x_0) \cdot v\sqrt{\mu}$$

and $\bar{w} \times (x - x_0) \cdot n_x = 0$. If Ω is not rotationally symmetric, then no nonzero $\bar{w}$ and x_0 exist, which provides $Z = 0$ from the former case that $\bar{w} = 0$. If Ω is indeed rotationally symmetric and there are nonzero $\bar{w}$ and x_0 such that

$$Z = \bar{w} \times (x - x_0) \cdot v\sqrt{\mu} \text{ and } \bar{w} \times (x - x_0) \cdot n_x = 0.$$

Now we use the conservation of total angular momentum (4) that

$$\int_{\Omega \times \mathbb{R}^3} ((x - x_0) \times \bar{w}) \cdot Z v \sqrt{\mu} \, dx \, dv = 0,$$

which is equivalent to saying that

$$\int_{\Omega \times \mathbb{R}^3} (\bar{w} \times (x - x_0) \cdot v)^2 \mu \, dx \, dv = 0.$$

Therefore, $\bar{w} \times (x - x_0) \cdot v = 0$. Thus we conclude that $Z = 0$ and this leads to a contradiction. □

This finishes the proof for the positivity on a fixed time interval [0, 1]. In the next section, we prove the main L^2 decay theorem in the interval $[0, t]$.

We are now ready to prove our main theorem on the L^2 decay estimates for the solutions f to (1).

Theorem 3. (Theorem 13 of [5]) *Let f be the weak solution of (1) with initial-boundary value conditions (2), which satisfies the conservation laws (3), and (4) if Ω has a rotational symmetry. Suppose that $\|f_0\|_{\infty,\vartheta+m} < \epsilon$ and $\|g\|_{\infty,m} < \epsilon$ for some small $\epsilon > 0$ and $m > \frac{3}{2}$. For any $\vartheta \in 2^{-1}\mathbb{N} \cup \{0\}$, there exist C and $\epsilon = \epsilon(\vartheta) > 0$ such that*

$$\sup_{0 \le s < \infty} \mathcal{E}_\vartheta(f(s)) \le C 2^{2\vartheta} \mathcal{E}_\vartheta(f_0), \tag{38}$$

and

$$\|f(t)\|_{2,\vartheta} \le C_{\vartheta,k} \left(\mathcal{E}_{\vartheta+\frac{k}{2}}(0)\right)^{1/2} \left(1 + \frac{t}{k}\right)^{-k/2} \tag{39}$$

for any $t > 0$ and $k \in \mathbb{N}$, where $\mathcal{E}_\vartheta(f(t))$ is defined as (5).

Proof. Define

$$T = \sup_t \left(t : \sup_{0 \le s \le t} \mathcal{E}_\vartheta(f(s)) \le 1 \right) > 0 \tag{40}$$

for some $\vartheta \ge 0$. For $0 \le t \le T$, let $0 \le N \le t \le N+1$, for some non-negative integer N. We split $[0, t] = \left(\cup_{j=0}^{N-1} [j, j+1]\right) \cup [N, t]$. On each interval $[j, j+1]$ for $j = 0, 1, \ldots, N-1$, we define $f^j(s, x, v) \overset{\text{def}}{=} f(s + j, x, v)$. Then clearly $f^j(s, x, v)$ is a weak solution of (1)–(4) on the time interval $s \in [0, 1]$ with the new initial condition $f^j(0, x, v) = f(j, x, v)$. Note that since we only consider $t \in [0, T]$ for T from (40), $\mathcal{E}_\vartheta(f^j(0))$ is uniformly bounded from above. We take the L^2 energy estimate over $0 \le s \le N$ to obtain

$$\|f(N)\|_2^2 + \int_0^N \mathrm{d}s \; (Lf, f) = \|f(0)\|_2^2 + \int_0^N \mathrm{d}s \; (\Gamma(g, f), f),$$

by the specular reflection boundary condition. Equivalently, we have

$$\|f(N)\|_2^2 + \sum_{j=0}^{N-1} \int_0^1 \mathrm{d}s \; \left(Lf^j, f^j\right) = \|f(0)\|_2^2 + \int_0^N \mathrm{d}s \; (\Gamma(g, f), f).$$

Then we use Proposition 1 and obtain

$$\|f(N)\|_2^2 + \sum_{j=0}^{N-1} \delta_{\epsilon,j} \int_0^1 \mathrm{d}s \; \|f^j\|_\sigma^2 \le \|f(0)\|_2^2 + \int_0^N \mathrm{d}s \; (\Gamma(g, f), f).$$

Thus,

$$\|f(N)\|_2^2 + \min_{\{j=0,\dots,N-1\}} \delta_{\epsilon,j} \int_0^N \mathrm{d}s\ \|f\|_\sigma^2 \le \|f(0)\|_2^2 + \int_0^N \mathrm{d}s\ (\Gamma(g,f), f(41)$$

By Theorem 2.8 of [6], we obtain the energy inequality over $[0, N]$

$$\|f(N)\|_2^2 + \min_{\{j=0,\dots,N-1\}} \delta_{\epsilon,j} \int_0^N \mathrm{d}s\ \|f(s)\|_\sigma^2 \le \|f(0)\|_2^2 + C_0 \int_0^N \mathrm{d}s\ \|g(s)\|_\infty \|f(s)\|_\sigma^2. \tag{42}$$

This completes the derivation of the energy inequality for the base case $\vartheta = 0$ in the interval $[0, N]$. For $\vartheta \ge 0$, we multiply $(1+|v|)^{2\vartheta}(v) f(t,x,v)$ and take the L^2 energy estimate over $0 \le s \le N$ to obtain

$$\|f(N)\|_{2,\vartheta}^2 + \int_0^N \mathrm{d}s\ \left((1+|v|)^{2\vartheta} Lf, f\right) = \|f(0)\|_{2,\vartheta}^2 + \int_0^N \mathrm{d}s\ ((1+|v|)^{2\vartheta}\Gamma(g,f), f),$$

by the specular reflection boundary condition. By Lemma 2.7 and Theorem 2.8 of [6], we have for some $C_\vartheta > 0$

$$\|f(N)\|_{2,\vartheta}^2 + \int_0^N \mathrm{d}s\ \left(\frac{1}{2}\|f(s)\|_{\sigma,\vartheta}^2 - C_\vartheta \|f(s)\|_\sigma^2\right) \le \|f(0)\|_{2,\vartheta}^2 + C_\vartheta \int_0^N \mathrm{d}s\ \|g(s)\|_\infty \|f(s)\|_{\sigma,\vartheta}^2. \tag{43}$$

This completes the derivation of the energy inequality for $\vartheta \ge 0$ in the interval $[0, N]$. Therefore, by the ingredients (42) for the base case $\vartheta = 0$ and (43) for a general $\vartheta \ge 0$, we obtain (4.36) of [6] by the same proof via the induction on ϑ for $\eta \equiv 0$, $s = 0$ and $t = N$. Then by the same proof of Theorem 1.2 of [6], we obtain (38) and (39) in the time interval $s \in [0, N]$; for any $\vartheta \in 2^{-1}\mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$, there exist C and $\epsilon = \epsilon(\vartheta) > 0$ such that

$$\sup_{0 \le s \le N} \mathcal{E}_\vartheta(f(s)) \le C 2^{2\vartheta} \mathcal{E}_\vartheta(f_0),$$

and

$$\|f(N)\|_{2,\vartheta} \le C_{\vartheta,k} \left(\mathcal{E}_{\vartheta+\frac{k}{2}}(0)\right)^{1/2} \left(1 + \frac{N}{k}\right)^{-k/2}.$$

Now we consider the local interval $[N, t]$ where we have $0 \le t - N \le 1$ and $t \le T$. We recall that if $\|g\|_{\infty,m} \le \epsilon$ for a sufficiently small ϵ, we have

$$\|(1+|v|)^\vartheta f(t)\|_{L^2}^2 + \int_N^t \|f(s)\|_{\sigma,\vartheta}^2\ \mathrm{d}s \le C e^{t-N} \|(1+|v|)^\vartheta f(N)\|_{L^2}^2, \tag{44}$$

by (14) for $l = \vartheta$ on $[N, t]$. Note that (44) holds for a solution to (1) under (6) and (2)–(4) by the local L^2 energy inequality and the Grönwall inequality as in (14) and we do not need the additional assumption (8) for (14). Then we observe that

$$\mathcal{E}_\vartheta(f(t)) \le Ce^{t-N}\mathcal{E}_\vartheta(f(N)) \le C'e^{t-N}2^{2\vartheta}\mathcal{E}_\vartheta(f_0) \le C'e2^{2\vartheta}\mathcal{E}_\vartheta(f_0)$$

for some $C' > 0$ and

$$\begin{aligned}\|f(t)\|_{2,\vartheta} &\le Ce^{t-N}\|f(N)\|_{2,\vartheta} \le Ce^{t-N}C_{\vartheta,k}\left(\mathcal{E}_{\vartheta+\frac{k}{2}}(0)\right)^{1/2}\left(1+\frac{N}{k}\right)^{-k/2}\\ &\le CeC_{\vartheta,k}\left(\mathcal{E}_{\vartheta+\frac{k}{2}}(0)\right)^{1/2}2^{k/2}\left(1+\frac{t}{k}\right)^{-k/2},\end{aligned}$$

since

$$\left(1+\frac{N}{k}\right)^{-k/2} \le 2^{k/2}\left(1+\frac{t}{k}\right)^{-k/2}$$

for $N \le t \le N+1$ and $k \ge 1$. Therefore, we obtain (38) and (39) for the time interval $[0, t]$ for any $0 \le t \le T$ where T is defined as (40).

We finally choose initially

$$\mathcal{E}_\vartheta(f_0) \le \epsilon_0 \le \frac{1}{2C2^{2\vartheta}},$$

and we define

$$T_2 = \sup_t\left(t : \sup_{0\le s\le t}\mathcal{E}_\vartheta(f(s)) \le \frac{1}{2}\right) > 0.$$

Since $0 \le t \le T_2 \le T$, we have, from (38), that

$$\sup_{0\le s\le T}\mathcal{E}_\vartheta(f(s)) \le C2^{2\vartheta}\mathcal{E}_\vartheta(f_0) \le \frac{1}{2}.$$

Thus, we deduce that $T_2 = \infty$ from the continuity of $\mathcal{E}_\vartheta$, and the theorem follows. □

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