VANDERMONDES IN SUPERSPACE

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Abstract. Superspace of rank n is a $\mathbb{Q}$-algebra with n commuting generators $x_1, \ldots, x_n$ and n anticommuting generators $\theta_1, \ldots, \theta_n$. We present an extension of the Vandermonde determinant to superspace which depends on a sequence $\mathbf{a} = (a_1, \ldots, a_r)$ of nonnegative integers of length $r \leq n$. We use superspace Vandermondes to construct graded representations of the symmetric group. This construction recovers hook-shaped Tanisaki quotients, the coinvariant ring for the Delta Conjecture constructed by Haglund, Rhoades, and Shimozono, and a superspace quotient related to positroids and Chern plethysm constructed by Billey, Rhoades, and Tewari. We define a notion of partial differentiation with respect to anticommuting variables to construct doubly graded modules from superspace Vandermondes. These doubly graded modules carry a natural ring structure which satisfies a 2-dimensional version of Poincaré duality. The application of polarization operators gives rise to other bigraded modules which give a conjectural module for the symmetric function $\Delta'_{e_{k-1}} e_n$ appearing in the Delta Conjecture of Haglund, Remmel, and Wilson.

1. Introduction

In this paper we extend the Vandermonde from the classical polynomial ring in n variables to a noncommutative deformation of this ring called *superspace*. We use superspace Vandermondes to generate interesting graded symmetric group modules including

- a family $R_{n,k}$ of quotient rings introduced by Haglund, Rhoades, and Shimozono [11] with connections to the cohomology of Pawlowski-Rhoades moduli spaces of spanning line configurations [14, 18] (Theorems 3.6 and 4.2),
- a class of quotient rings studied by Billey, Rhoades, and Tewari [5] related to positroids and Chern plethysm (Proposition 4.5),
- the Tanisaki quotients R_λ corresponding to hook-shaped partitions $\lambda \vdash n$ which present the cohomology of the corresponding Springer fiber $\mathcal{B}_\lambda$ [20] (Proposition 4.3) and conjecturally other Tanisaki quotients R_λ (Conjecture 6.1),
- a class of *doubly* graded modules with a bigraded multiplication which exhibit a kind of rotational symmetry (Corollary 5.6) and a 2-dimensional version of Poincaré duality (Corollary 5.9), and
- a class of doubly graded modules whose bigraded Frobenius image is conjecturally given by the expression $\Delta'_{e_{k-1}} e_n$ appearing in the Delta Conjecture of Haglund, Remmel, and Wilson [10] (Conjecture 6.3).

Let $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \ldots, x_n]$ be the polynomial ring in n variables equipped with the action of the symmetric group $\mathfrak{S}_n$ by subscript permutation. The *Vandermonde* Δ_n is an important element of $\mathbb{Q}[\mathbf{x}_n]$ with several equivalent definitions. If we let $\varepsilon_n \in \mathbb{Q}[\mathfrak{S}_n]$ be the group algebra element

$$\varepsilon_n := \sum_{w \in \mathfrak{S}_n} \operatorname{sign}(w) \cdot w, \tag{1.1}$$

Key words and phrases. Vandermonde, harmonic space, delta operator, superspace.

we have

$$\Delta_n := \prod_{1 \le i < j \le n} (x_i - x_j) = \det \begin{pmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ & & \vdots & \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{pmatrix} = \varepsilon_n \cdot (x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 x_n^0). \tag{1.2}$$

For a positive integer n, *superspace* of rank n is the unital associative $\mathbb{Q}$-algebra with generators $x_1, x_2, \dots, x_n$ and $\theta_1, \theta_2, \dots, \theta_n$ subject to the relations

$$x_i x_j = x_j x_i, \quad x_i \theta_j = \theta_j x_i, \quad \theta_i \theta_j = -\theta_j \theta_i \tag{1.3}$$

for all $1 \le i, j \le n$. Abusing notation, we use $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ to denote superspace of rank n, with the understanding that the θ-variables anticommute. The 'super' refers to supersymmetry in physics; the x-variables correspond to bosons whereas the θ-variables correspond to fermions (see e.g. [15]). Extending coefficients to $\mathbb{R}$, superspace is the ring of polynomial-valued differential forms on Euclidean n-space; in this setting the variable θ_i would be more commonly written dx_i. We also have the tensor product model $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n] = \mathrm{Sym}(V^*) \otimes \bigwedge(V^*)$, where V is an n-dimensional vector space and V^* is its dual space. Superspace carries a natural bigrading by considering x-degree and θ-degree separately.

We endow $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ with the diagonal action of $\mathfrak{S}_n$:

$$w \cdot x_i := x_{w(i)}, \quad w \cdot \theta_i := \theta_{w(i)} \quad \text{for } w \in \mathfrak{S}_n \text{ and } 1 \le i \le n. \tag{1.4}$$

This action of $\mathfrak{S}_n$ on $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ has been used in [5, 22] to build interesting graded $\mathfrak{S}_n$-modules connected to Chern plethysm and delta operators. We use the last formulation in (1.2) of the Vandermonde determinant to extend Vandermondes to superspace and construct graded $\mathfrak{S}_n$-modules of our own. The following superspace elements will be our object of study.

Definition 1.1. *Let $k, r \ge 0$ with $n = k + r$ and let $\mathbf{a} = (a_1, \dots, a_r)$ be a list of r nonnegative integers. The* $\mathbf{a}$-superspace Vandermonde *is the following element of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$:*

$$\Delta_n(\mathbf{a}) := \varepsilon_n \cdot (x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r} x_{r+1}^{k-1} x_{r+2}^{k-2} \cdots x_{n-1}^1 x_n^0 \cdot \theta_1 \theta_2 \cdots \theta_r).$$

Example 1.2. *Let $n = 3$. Using the anticommutativity of the θ-variables,*

$$\begin{aligned} \Delta_3(1,1) &= 2x_1x_2\theta_1\theta_2 - 2x_1x_3\theta_1\theta_3 + 2x_2x_3\theta_2\theta_3 \\ \Delta_3(2,0) &= x_1^2\theta_1\theta_2 + x_2^2\theta_1\theta_2 - x_1^2\theta_1\theta_3 - x_3^2\theta_1\theta_3 + x_2^2\theta_2\theta_3 + x_3^2\theta_2\theta_3 \\ \Delta_3(1) &= x_1x_2\theta_1 - x_1x_2\theta_2 - x_1x_3\theta_1 - x_2x_3\theta_3 + x_2x_3\theta_2 + x_1x_3\theta_3. \end{aligned}$$

Example 1.2 illustrates that $\Delta_n(\mathbf{a}) \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ is nonzero even when the sequence $\mathbf{a}$ has repeated entries. Indeed, the case where $\mathbf{a}$ is a constant sequence will be the primary focus of this paper.

Definition 1.1 specializes to the classical Vandermonde Δ_n when $\mathbf{a} = \varnothing$ is the empty sequence of length zero. If $\mathbf{a} = (a_1, \dots, a_r)$ is a rearrangement of $\mathbf{b} = (b_1, \dots, b_r)$, the anticommutativity of the θ-variables implies $\Delta_n(\mathbf{a}) = \Delta_n(\mathbf{b})$.

The superpolynomial $\Delta_n(\mathbf{a})$ can be viewed as a (noncommutative) determinant. If $A = (A_{i,j})_{1 \le i,j \le n}$ is a matrix whose elements lie in $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$, we define $\det(A) \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ by

$$\det(A) := \sum_{w \in \mathfrak{S}_n} \mathrm{sign}(w) A_{1,w(1)} A_{2,w(2)} \cdots A_{n,w(n)}, \tag{1.5}$$

where the terms are multiplied in the specified order. For $\mathbf{a} = (a_1, \dots, a_r)$ with $n = k + r$ we have

$$\Delta_n(\mathbf{a}) = \det \begin{pmatrix} x_1^{a_1}\theta_1 & x_2^{a_1}\theta_2 & \cdots & x_n^{a_1}\theta_n \\ x_1^{a_2}\theta_1 & x_2^{a_2}\theta_2 & \cdots & x_n^{a_2}\theta_n \\ & & \ddots & \\ x_1^{a_r}\theta_1 & x_2^{a_r}\theta_2 & \cdots & x_n^{a_r}\theta_n \\ \hline x_1^{k-1} & x_2^{k-1} & \cdots & x_n^{k-1} \\ x_1^{k-2} & x_2^{k-2} & \cdots & x_n^{k-2} \\ & & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{1.6}$$

The authors are unaware of a superspace extension of the factorization $\Delta_n = \prod_{1 \leq i < j \leq n}(x_i - x_j)$.

We use an action of partial derivatives on superspace to build $\mathfrak{S}_n$-modules. For $1 \leq i \leq n$, let $\partial_i : \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n] \to \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ be the unique linear operator which satisfies

$$\partial_i(\theta_j) = 0 \quad \text{and} \quad \partial_i(x_j) = \delta_{i,j} \tag{1.7}$$

for all $1 \leq j \leq n$ (where $\delta_{i,j}$ is the Kronecker delta) together with the Leibniz Rule

$$\partial_i(fg) = f\partial_i(g) + \partial_i(f)g \quad \text{for all } f, g \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]. \tag{1.8}$$

The operator ∂_i is partial differentiation with respect to x_i where the θ-variables are regarded as constants.

Definition 1.3. *Suppose $n = k + r$ for $k, r \geq 0$ and let $\mathbf{a} = (a_1, \dots, a_r) \in (\mathbb{Z}_{\geq 0})^r$. Let $V_n(\mathbf{a})$ be the smallest $\mathbb{Q}$-linear subspace of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ containing $\Delta_n(\mathbf{a})$ which is closed under the n partial derivative operators $\partial_1, \dots, \partial_n$.*

Since the superpolynomial $\Delta_n(\mathbf{a})$ is *alternating*:

$$w \cdot \Delta_n(\mathbf{a}) = \mathrm{sign}(w)\Delta_n(\mathbf{a}) \quad \text{for all } w \in \mathfrak{S}_n \tag{1.9}$$

it will follow that $V_n(\mathbf{a})$ is closed under the action of $\mathfrak{S}_n$. The space $V_n(\mathbf{a})$ is concentrated in θ-degree r and is a graded vector space with respect to x-degree. Ignoring the (constant) θ-degree, we regard $V_n(\mathbf{a})$ as a singly graded $\mathfrak{S}_n$-module by considering x-degree.

The classical Vandermonde Δ_n gives a model for the coinvariant ring of $\mathfrak{S}_n$. For $1 \leq i \leq n$, let $e_d = e_d(\mathbf{x}_n)$ be the degree d *elementary symmetric polynomial*:

$$e_d := \sum_{1 \leq i_1 < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}. \tag{1.10}$$

The *invariant ideal* $I_n \subseteq \mathbb{Q}[\mathbf{x}_n]$ is the ideal $I_n := \langle e_1, e_2, \dots, e_n \rangle$ generated by these polynomials. Equivalently, the ideal I_n is generated by the vector space $\mathbb{Q}[\mathbf{x}_n]_+^{\mathfrak{S}_n}$ of $\mathfrak{S}_n$-invariant polynomials with vanishing constant term. The *coinvariant ring* is the quotient $R_n := \mathbb{Q}[\mathbf{x}_n]/I_n$. The ring R_n has the structure of a graded $\mathfrak{S}_n$-module.

The coinvariant ring is one of the most important representations in algebraic combinatorics. Chevalley proved [6] that $R_n \cong \mathbb{Q}[\mathfrak{S}_n]$ as ungraded $\mathfrak{S}_n$-modules, so that R_n gives a graded refinement of the regular representation of $\mathfrak{S}_n$. Borel showed [4] that R_n presents the cohomology of the variety $\mathcal{F}\ell_n$ of complete flags in $\mathbb{C}^n$. If $\mathbf{a} = \varnothing$ is the empty sequence so that $\Delta_n(\mathbf{a}) = \Delta_n$ is the classical Vandermonde, we have an isomorphism (see [1]) of graded $\mathfrak{S}_n$-modules

$$R_n \cong V_n(\varnothing) = \mathrm{span}_{\mathbb{Q}}\{\partial_1^{b_1} \cdots \partial_n^{b_n} \Delta_n \,:\, b_1, \dots, b_n \geq 0\}. \tag{1.11}$$

The isomorphism (1.11) gives two ways of viewing the coinvariant algebra, each with virtues and defects. The space $R_n = \mathbb{Q}[\mathbf{x}_n]/I_n$ has a natural multiplication structure, but as a quotient space, deciding whether two polynomials $f, g \in \mathbb{Q}[\mathbf{x}_n]$ are equal in R_n can be difficult. The graded vector space $V_n(\varnothing)$ is not closed under multiplication, but its elements are honest polynomials (not cosets), so calculating invariants like dimension is more conceptually straightforward. In this paper we use the spaces $V_n(\mathbf{a})$ to extend (1.11) to a wider class of graded $\mathfrak{S}_n$-modules.

- If $n = k + r$ and $\mathbf{a} = (k-1, \dots, k-1)$ is a length r sequence of $(k-1)$'s, then $V_n(\mathbf{a})$ is isomorphic as a graded $\mathfrak{S}_n$-module (up to sign twist and grading reversal) to the quotient $R_{n,k} := \mathbb{Q}[\mathbf{x}_n]/I_{n,k}$ where $I_{n,k} \subseteq \mathbb{Q}[x_1, \dots, x_n]$ is the ideal generated by $x_1^k, x_2^k, \dots, x_n^k$ together with the top k elementary symmetric polynomials $e_n, e_{n-1}, \dots, e_{n-k+1}$ (Theorem 3.6). The ring $R_{n,k}$ was defined by Haglund, Rhoades, and Shimozono [11] in connection with the *Delta Conjecture* [10] of Macdonald theory.
- If $n = k + r$, $k \leq s$, and $\mathbf{a} = (s-1, \dots, s-1)$ is a length r sequence of $(s-1)$'s, then $V_n(\mathbf{a})$ is isomorphic (up to sign twist and grading reversal) to a two-parameter family $R_{n,k,s}$ of quotient rings defined in [11] and further studied from a geometric perspective in [14] (Theorem 4.2).
- If $r \leq n$ and $\mathbf{a} = (0, \dots, 0)$ is a length r sequence of zeros, then $V_n(\mathbf{a})$ is isomorphic (up to sign twist and grading reversal) to the Tanisaki quotient R_λ corresponding to the hook-shaped partition $\lambda = (r+1, 1, \dots, 1) \vdash n$ (Proposition 4.3).

The modules $V_n(\mathbf{a})$ of Definition 1.3 are defined using classical partial derivative operators acting on the commuting x-variables. We introduce the following partial differentiation operators which act on the anticommuting θ-variables. Given $1 \leq i \leq n$, we define a $\mathbb{Q}[\mathbf{x}_n]$-endomorphism of superspace by

$$\partial_i^\theta(\theta_{j_1} \cdots \theta_{j_r}) := \begin{cases} (-1)^{s-1} \theta_{j_1} \cdots \widehat{\theta_{j_s}} \cdots \theta_{j_r} & \text{if } j_s = i, \\ 0 & \text{if } i \notin \{j_1, \dots, j_r\}, \end{cases} \tag{1.12}$$

for any $1 \leq j_1 < \cdots < j_r \leq n$, where $\widehat{\cdot}$ denotes omission. The operator ∂_i^θ lowers θ-degree by 1 while leaving x-degree unchanged. We use these operators to build the following class of doubly-graded vector spaces.

Definition 1.4. *Suppose $n = k + r$ for $k, r \geq 0$ and let $\mathbf{a} = (a_1, \dots, a_r) \in (\mathbb{Z}_{\geq 0})^r$. Let $W_n(\mathbf{a})$ be the smallest $\mathbb{Q}$-linear subspace of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ containing $\Delta_n(\mathbf{a})$ which is closed under the n partial derivative operators $\partial_1, \dots, \partial_n$ as well as the n operators $\partial_1^\theta, \dots, \partial_n^\theta$.*

The space $W_n(\mathbf{a})$ has the structure of a doubly graded $\mathfrak{S}_n$-module. By restricting $W_n(\mathbf{a})$ to the top θ-degree component, we recover the singly graded module $V_n(\mathbf{a})$. In contrast to the spaces $V_n(\mathbf{a})$, there is a natural way to put a ring structure on $W_n(\mathbf{a})$.

It will be shown in Lemma 5.1 that the operators ∂_i and ∂_i^θ satisfy the same relations as the generators x_i and θ_i of superspace:

$$\partial_i \partial_j = \partial_j \partial_i, \quad \partial_i \partial_j^\theta = \partial_j^\theta \partial_i, \quad \partial_i^\theta \partial_j^\theta = -\partial_j^\theta \partial_i^\theta \tag{1.13}$$

for all $1 \leq i, j \leq n$. Given $f \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$, we therefore have a well-defined operator $\partial(f)$ obtained by replacing each x_i in f with a ∂_i and each θ_i in f with a ∂_i^θ. For example, we have

$$\partial(x_1^2 \theta_1 \theta_2 - x_1 x_3 \theta_1) = \partial_1^2 \partial_1^\theta \partial_2^\theta - \partial_1 \partial_3 \partial_1^\theta.$$

We have an action of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ on itself given by $f \cdot g := \partial(f)(g)$. We use this action to define the following family of bigraded quotient rings.

Definition 1.5. *Suppose $n = k + r$ for $k, r \geq 0$ and let $\mathbf{a} = (a_1, \dots, a_r) \in (\mathbb{Z}_{\geq 0})^r$. Let $I_n(\mathbf{a}) \subseteq \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ be the ideal*

$$I_n(\mathbf{a}) := \{ f \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n] \,:\, f \cdot \Delta_n(\mathbf{a}) = 0 \} \tag{1.14}$$

and let

$$R_n(\mathbf{a}) := \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]/I_n(\mathbf{a}) \tag{1.15}$$

be the corresponding quotient ring.

We will show (Corollary 5.7) that $R_n(\mathbf{a})$ is isomorphic to $W_n(\mathbf{a})$ as bigraded $\mathfrak{S}_n$-modules. The ring $R_n(\mathbf{a})$ enjoys a 2-dimensional kind of duality (Theorem 5.5, Corollary 5.6) which states that twisting $R_n(\mathbf{a})$ by the sign representation is equivalent to 'rotating' its bigrading. We prove that $R_n(\mathbf{a})$ satisfies a 2-dimensional version of Poincaré duality (Corollary 5.9) which is implied in the case $\mathbf{a} = \varnothing$ by the fact that $\mathcal{F}\ell_n$ is a compact smooth projective complex variety. We propose the problem of finding a geometric interpretation of the 2-dimensional duality satisfied by the rings $R_n(\mathbf{a})$. We further conjecture a 2-dimensional unimodality property of the bigraded Hilbert series of $R_n(\mathbf{a})$ (Conjecture 6.5) which is implied by the Hard Lefschetz Theorem when $\mathbf{a} = \varnothing$.

For $k \leq n$, Pawlowski and Rhoades [14] defined the moduli space $X_{n,k}$ of n-tuples $(\ell_1, \dots, \ell_n)$ of lines in $\mathbb{C}^k$ such that $\ell_1 + \cdots + \ell_n = \mathbb{C}^k$. This space is homotopy equivalent to $\mathcal{F}\ell_n$ when $k = n$ and is a Zariski open subset of the n-fold product $(\mathbb{P}^{k-1})^n$ of $(k-1)$-dimensional complex projective space with itself. Although $X_{n,k}$ is a smooth complex manifold, it is almost never compact and so does not satisfy the hypotheses of Poincaré duality; correspondingly, the Hilbert series of the cohomology ring $H^\bullet(X_{n,k})$ is not palindromic. When $\mathbf{a} = (k-1, \dots, k-1)$ is a length $n-k$ sequence of $(k-1)$'s, the θ-degree zero piece of $R_n(\mathbf{a})$ presents the cohomology $H^\bullet(X_{n,k})$. The results and conjectures of the previous paragraph suggest that although $H^\bullet(X_{n,k})$ does not satisfy desired properties such as Poincaré duality and Hard Lefschetz, it is a 1-dimensional slice of a 2-dimensional object that does.

The paper is structured as follows. In **Section 2** we give background material related to combinatorics and the representation theory of $\mathfrak{S}_n$. In **Section 3** we calculate the graded isomorphism type of $V_n(\mathbf{a})$ for certain constant sequences $\mathbf{a}$ to give a new model for the coinvariant algebra for the Delta Conjecture introduced in [11]. In **Section 4** we generalize the results in Section 3 to other constant sequences $\mathbf{a}$, giving a Vandermonde model for the hook-shaped Tanisaki quotients in the process. We also describe a subspace model for a quotient of superspace introduced in [5] which gives a bigraded refinement of a symmetric group action on positroids. In **Section 5** we define partial differentiation operators on superspace with respect to antisymmetric variables, prove the relevant duality result, and discuss a possible connection to the superspace coinvariant ring. We close in **Section 6** with some open problems, including an extension of the module $V_n(\mathbf{a})$ to two sets of commuting variables (with one set of skew-commuting variables) with conjectural doubly graded Frobenius image equal the symmetric function $\Delta'_{e_{k-1}} e_n$ appearing in the Delta Conjecture.

2. Background

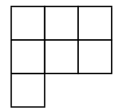
2.1. **Combinatorics.** It will often be convenient for us to assert identities up to a nonzero scalar. To this end, suppose f and g are elements of the polynomial ring $\mathbb{Q}[\mathbf{x}_n]$ or of superspace $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$. We use the notation $f \doteq g$ to indicate that there is a nonzero rational number $a \in \mathbb{Q} - \{0\}$ such that $f = ag$.

Let R be a ring and let $R[q]$ be the ring of polynomials in q with coefficients in R. Given a polynomial $f = r_d q^d + r_{d-1} q^{d-1} + \cdots + r_1 q + r_0 \in R[q]$ with the $r_i \in R$ and $r_d \neq 0$, the *q-reversal* of f is given by

$$\mathrm{rev}_q f := r_0 q^d + r_1 q^{d-1} \cdots + r_{d-1} q^1 + r_d \in R[q]. \tag{2.1}$$

Let $n \geq 0$. A *partition* λ of n is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ of positive integers with $\lambda_1 + \cdots + \lambda_k = n$. We write $\ell(\lambda) = k$ to indicate the number of parts of λ and $\lambda \vdash n$ to indicate that λ is a partition of n. For $1 \leq i \leq n$, we write $m_i(\lambda)$ for the multiplicity of i as a part of λ.

We identify a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ with its *Ferrers diagram* consisting of λ_i left justified boxes in row i. The Ferrers diagram of $(3,3,1) \vdash 7$ is shown below.



Given $\lambda \vdash n$, the *conjugate* λ' is the partition whose Ferrers diagram is obtained from that of λ by reflection across the main diagonal $y = x$. For example, we have $(3,3,1)' = (3,2,2)$.

We will make use of the following standard q-analog notation. For $n \geq k \geq 0$ we have the *q-number, q-factorial,* and *q-binomial coefficient*:

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]!_q := [n]_q [n-1]_q \cdots [1]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q \cdot [n-k]!_q}. \tag{2.2}$$

If $k_1 + \cdots + k_r = n$, we also have the *q-multinomial coefficient*

$$\begin{bmatrix} n \\ k_1, \dots, k_r \end{bmatrix}_q := \frac{[n]!_q}{[k_1]!_q \cdots [k_r]!_q}. \tag{2.3}$$

For $n \geq k \geq 0$, let $\mathrm{Stir}(n,k)$ be the *(signless) Stirling number of the second kind* counting k-block set partitions of $[n] := \{1, 2, \dots, n\}$. The *$q$-Stirling number* $\mathrm{Stir}_q(n,k)$ is defined by the recursion

$$\mathrm{Stir}_q(n,k) = \mathrm{Stir}_q(n-1,k-1) + [k]_q \cdot \mathrm{Stir}_q(n-1,k) \tag{2.4}$$

together with the initial condition $\mathrm{Stir}_q(0,k) = \delta_{k,0}$ (Kronecker delta).

An *ordered set partition* of $[n]$ is a sequence $(B_1, \dots, B_k)$ of nonempty subsets of $[n]$ such that we have the disjoint union $[n] := B_1 \sqcup \cdots \sqcup B_k$. The number of k-block ordered set partitions of $[n]$ is $k! \cdot \mathrm{Stir}(n,k)$.

2.2. **Representation theory.** A *supermonomial* in $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ is a product $x_1^{a_1} \cdots x_n^{a_n} \theta_{i_1} \cdots \theta_{i_r}$ for some $a_1, \dots, a_n \geq 0$ and $1 \leq i_1 < \cdots < i_r \leq n$. The *$x$-degree* of this supermonomial is $a_1 + \cdots + a_n$ and the *θ-degree* is r.

The family of supermonomials forms a basis for $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$; we call elements of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ *superpolynomials.* If f is a superpolynomial, the *x-degree* of f is the largest x-degree of the terms appearing in f. We say f is *x-homogeneous* if all of its terms have the same x-degree. The terms *θ-degree* and *θ-homogeneous* have analogous meanings. If f is simultaneously x-homogeneous and θ-homogeneous, we call f *homogeneous.*

Let M be a $\mathbb{Q}[\mathbf{x}_n]$-module. For $m \in M$, the *annihilator* of m is the subset

$$\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]}(m) := \{r \in \mathbb{Q}[\mathbf{x}_n] \,:\, r \cdot m = 0\} \subseteq \mathbb{Q}[\mathbf{x}_n]. \tag{2.5}$$

The subset $\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]}(m) \subseteq \mathbb{Q}[\mathbf{x}_n]$ is an ideal.

Let $V = \bigoplus_{d \geq 0} V_d$ be a graded vector space with each graded piece V_d finite-dimensional. The *Hilbert series* of V is the power series

$$\mathrm{Hilb}(V;q) := \sum_{d \geq 0} \dim(V_d) q^d. \tag{2.6}$$

Let Λ be the algebra of symmetric functions over the ground field $\mathbb{Q}(q,t)$ in an infinite variable set $\mathbf{x} = (x_1, x_2, \dots)$. The ring Λ is graded by degree; we let Λ_n be the homogeneous piece of degree n.

For any $\lambda \vdash n$, we have the *Schur function* $s_\lambda = s_\lambda(\mathbf{x}) \in \Lambda_n$. The family $\{s_\lambda \,:\, \lambda \vdash n\}$ of all such symmetric functions forms a basis for Λ_n. The *omega involution* is the linear map $\omega : \Lambda \to \Lambda$ defined on the Schur basis by $\omega(s_\lambda) := s_{\lambda'}$. It can be shown that ω is a ring homomorphism.

The irreducible representations of $\mathfrak{S}_n$ over the field $\mathbb{Q}$ are indexed by partitions of n. If $\lambda \vdash n$, let S^λ be the corresponding irreducible representation of $\mathfrak{S}_n$. For example, the trivial representation of $\mathfrak{S}_n$ is $S^{(n)}$ and the sign representation of $\mathfrak{S}_n$ is $S^{(1^n)}$.

The *Frobenius map* gives a relationship between the Schur basis and the representation theory of $\mathfrak{S}_n$. Given any finite-dimensional $\mathfrak{S}_n$-module V, there are unique multiplicities $m_\lambda \geq 0$ such that

$$V \cong_{\mathfrak{S}_n} \bigoplus_{\lambda \vdash n} m_\lambda S^\lambda. \tag{2.7}$$

The *Frobenius image* $\mathrm{Frob}(V) \in \Lambda_n$ is the symmetric function

$$\mathrm{Frob}(V) := \sum_{\lambda \vdash n} m_\lambda s_\lambda. \tag{2.8}$$

If V is a finite-dimensional $\mathfrak{S}_n$-module and sign denotes the 1-dimensional sign representation of $\mathfrak{S}_n$, the tensor product $\mathrm{sign} \otimes V$ is another $\mathfrak{S}_n$-module. The effect of tensoring with the sign representation on Frobenius image is the application of the omega involution, that is

$$\mathrm{Frob}(\mathrm{sign} \otimes V) = \omega(\mathrm{Frob}(V)). \tag{2.9}$$

Most of the modules we consider in this paper will be graded. If $V = \bigoplus_{i \geq 0} V_i$ is a graded $\mathfrak{S}_n$-module with each piece V_i finite-dimensional, the *graded Frobenius image* of V is the series

$$\mathrm{grFrob}(V; q) := \sum_{i \geq 0} \mathrm{Frob}(V_i) \cdot q^i. \tag{2.10}$$

Similarly, if $V = \bigoplus_{i,j \geq 0} V_{i,j}$ is a bigraded $\mathfrak{S}_n$-module with each bigraded piece $V_{i,j}$ finite-dimensional, we set

$$\mathrm{grFrob}(V; q, t) := \sum_{i,j \geq 0} \mathrm{Frob}(V_{i,j}) \cdot q^i t^j. \tag{2.11}$$

The bigraded Frobenius image (2.11) can be extended to define multigraded Frobenius images $\mathrm{grFrob}(V; q_1, q_2, q_3, \dots)$ in the obvious way.

We will need the induction product of symmetric group modules. Let G be a group and let H be a subgroup of G. If V is a representation of H, let $V \uparrow_H^G$ be the induction of V from H to G. If V is a $\mathfrak{S}_n$-module and W is a $\mathfrak{S}_m$-module, the tensor product $V \otimes W$ is naturally a $\mathfrak{S}_n \times \mathfrak{S}_m$-module. Viewing $\mathfrak{S}_n \times \mathfrak{S}_m$ as a subgroup of $\mathfrak{S}_{n+m}$ where $\mathfrak{S}_n$ acts on the first n letters and $\mathfrak{S}_m$ acts on the last m letters, the *induction product* of V and W is the $\mathfrak{S}_{n+m}$-module

$$V \circ W := (V \otimes W) \uparrow_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}}. \tag{2.12}$$

The corresponding effect on Frobenius images is

$$\mathrm{Frob}(V \circ W) = \mathrm{Frob}(V) \cdot \mathrm{Frob}(W). \tag{2.13}$$

For $\lambda \vdash n$, let $\widetilde{H}_\lambda = \widetilde{H}_\lambda(\mathbf{x}; q, t) \in \Lambda_n$ be the associated *modified Macdonald symmetric function.* As with Schur functions, the set $\{\widetilde{H}_\lambda : \lambda \vdash n\}$ forms a basis for Λ_n.

Given any symmetric function F, the (primed and unprimed) delta operators $\Delta_F, \Delta'_F : \Lambda \to \Lambda$ are the Macdonald eigenoperators defined by

$$\Delta_F : \widetilde{H}_\lambda \mapsto F[B_\lambda(q,t)] \cdot \widetilde{H}_\lambda \tag{2.14}$$

$$\Delta'_F : \widetilde{H}_\lambda \mapsto F[B_\lambda(q,t) - 1] \cdot \widetilde{H}_\lambda \tag{2.15}$$

The eigenvalue $F[B_\lambda(q,t)] \in \mathbb{Q}(q,t)$ involved in Δ_F is the plethystic shorthand

$$F[B_\lambda(q,t)] := F(\dots, q^i t^j, \dots), \tag{2.16}$$

where (i,j) range over all pairs of nonnegative integers such that $i < \lambda_{j+1}$; all remaining arguments of F are set to 0. The eigenvalue $F[B_\lambda(q,t) - 1] \in \mathbb{Q}(q,t)$ involved in Δ'_F has the same definiton as $F[B_\lambda(q,t)]$ except that $1 = q^0t^0$ does not appear as an argument.[1] By linearity, the operators Δ_F and Δ'_F extend to operators on the full vector space Λ of symmetric functions.

Let e_n be the degree n elementary symmetric function. For $k \leq n$, the *Delta Conjecture* of Haglund, Remmel, and Wilson [10] predicts the monomial expansion of the symmetric function $\Delta'_{e_{k-1}} e_n$. It reads

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(\mathbf{x}; q, t) = \mathrm{Val}_{n,k}(\mathbf{x}; q, t), \tag{2.17}$$

where Rise and Val are certain formal power series involving the combinatorics of lattice paths. For more details, see [10].

The Delta Conjecture asserts the equality of three formal power series involving the infinite set of variables $\mathbf{x}$ together with the two additional parameters q and t. This conjecture remains open, but is known to be true when one of these parameters is set to zero. Combining results of [8, 10, 11, 16, 21] we have

$$\Delta'_{e_{k-1}} e_n \mid_{t=0} = \mathrm{Rise}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Rise}_{n,k}(\mathbf{x}; 0, q) = \mathrm{Val}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Val}_{n,k}(\mathbf{x}; 0, q). \tag{2.18}$$

Let $C_{n,k}(\mathbf{x}; q)$ be the common symmetric function of Equation (2.18).

For $\lambda \vdash n$, we will need the Hall-Littlewood Q'-function $Q'_\lambda(\mathbf{x}; q)$. This may be defined in terms of the modified Macdonald polynomials by

$$Q'_\lambda(\mathbf{x}; q) := \mathrm{rev}_q \widetilde{H}_\lambda(\mathbf{x}; q, 0). \tag{2.19}$$

In the special case $\lambda = (1^n)$, the Hall-Littlewood function gives the graded isomorphism type of the coinvariant ring R_n attached to $\mathfrak{S}_n$, up to grading reversal:

$$\mathrm{grFrob}(R_n; q) = \mathrm{rev}_q Q'_{(1^n)}(\mathbf{x}; q). \tag{2.20}$$

3. Vandermondes and the Delta Conjecture

3.1. **Vandermondes and annihilators.** In this paper we study the graded Frobenius images $\mathrm{grFrob}(V_n(\mathbf{a}); q)$ for various sequences $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$. In order to do this, we use the following action of the polynomial ring $\mathbb{Q}[\mathbf{x}_n]$ on superspace $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$.

Recall from Section 1 that we have an action of the partial derivative operator ∂_i on $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ for each $1 \leq i \leq n$. Since $\partial_i \partial_j = \partial_j \partial_i$ for all $1 \leq i, j \leq n$, given any polynomial $f \in \mathbb{Q}[\mathbf{x}_n]$ we may define $\partial(f)$ to be the differential operator on $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ obtained from f by replacing each x_i by ∂_i. The action of $\mathbb{Q}[\mathbf{x}_n]$ on $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ is given by

$$f \cdot g := \partial(f)(g) \quad \text{for all } f \in \mathbb{Q}[\mathbf{x}_n] \text{ and } g \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]. \tag{3.1}$$

This is related to the action of $\mathfrak{S}_n$ in that

$$w \cdot (f \cdot g) = (w \cdot f) \cdot (w \cdot g) \tag{3.2}$$

for all $w \in \mathfrak{S}_n$, $f \in \mathbb{Q}[\mathbf{x}_n]$, and $g \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$.

For any $r \leq n$ and any sequence $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$, the annihilator

$$\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a}) = \{f \in \mathbb{Q}[\mathbf{x}_n] \,:\, f \cdot \Delta_n(\mathbf{a}) = 0\} \tag{3.3}$$

in $\mathbb{Q}[\mathbf{x}_n]$ of the $\mathbf{a}$-superspace Vandermonde is an ideal in $\mathbb{Q}[\mathbf{x}_n]$. Since $\Delta_n(\mathbf{a})$ is homogeneous in the x-variables, the annihilator $\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})$ is homogeneous. Equation (3.2) and the fact that $\Delta_n(\mathbf{a})$ is alternating imply that $\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})$ is closed under the action of $\mathfrak{S}_n$. The quotient $\mathbb{Q}[\mathbf{x}_n]/\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})$ therefore has the structure of a graded $\mathfrak{S}_n$-module. The graded $\mathfrak{S}_n$-modules $V_n(\mathbf{a})$ and $\mathbb{Q}[\mathbf{x}_n]/\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})$ are related as follows.

[1]The Macdonald eigenoperators Δ_F and Δ'_F are not to be confused with the Vandermonde Δ_n and its superspace extension $\Delta_n(\mathbf{a})$.

Proposition 3.1. *Let $r, k \geq 0$ with $n = r + k$ and let $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$. We have*

$$\mathrm{grFrob}(\mathbb{Q}[\mathbf{x}_n]/\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]}\Delta_n(\mathbf{a}); q) = (\mathrm{rev}_q \circ \omega)\mathrm{grFrob}(V_n(\mathbf{a}); q). \tag{3.4}$$

Proposition 3.1 is standard, but we include a proof for completeness.

Proof. The action of $\mathbb{Q}[\mathbf{x}_n]$ on $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ gives a canonical map $\varphi : \mathbb{Q}[\mathbf{x}_n]/\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]}\Delta_n(\mathbf{a}) \to V_n(\mathbf{a})$:

$$\varphi : f \mapsto f \cdot \Delta_n(\mathbf{a}) = \partial(f)(\Delta_n(\mathbf{a})). \tag{3.5}$$

The definitions of $V_n(\mathbf{a})$ and $\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]}\Delta_n(\mathbf{a})$ guarantee that φ is well-defined and bijective. Since $\Delta_n(\mathbf{a})$ is an alternant, for $w \in \mathfrak{S}_n$ we have

$$\varphi(w \cdot f) = (w \cdot f) \cdot \Delta_n(\mathbf{a}) = \mathrm{sign}(w)(w \cdot f) \cdot (w \cdot \Delta_n(\mathbf{a})) = \mathrm{sign}(w) w \cdot \varphi(f), \tag{3.6}$$

so that φ twists by the sign representation. The degree reversal comes from the fact that φ is defined using an action of partial derivatives. □

3.2. **A vanishing lemma.** Proposition 3.1 is our basic tool for identifying the graded modules $V_n(\mathbf{a})$. Our first example is inspired by the Delta Conjecture.

For positive integers $k \leq n$, following [11] we define an ideal $I_{n,k} \subseteq \mathbb{Q}[\mathbf{x}_n]$ by

$$I_{n,k} := \langle x_1^k, x_2^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle \tag{3.7}$$

and let

$$R_{n,k} := \mathbb{Q}[\mathbf{x}_n]/I_{n,k} \tag{3.8}$$

be the corresponding quotient. The ring $R_{n,k}$ specializes to the classical coinvariant ring $R_n = \mathbb{Q}[\mathbf{x}_n]/I_n$ when $k = n$ and plays the role of the coinvariant ring for the Delta Conjecture. Haglund, Rhoades, and Shimozono proved [11, 12] that

$$\mathrm{grFrob}(R_{n,k}; q) = (\mathrm{rev}_q \circ \omega) C_{n,k}(\mathbf{x}; q) = (\mathrm{rev}_q \circ \omega) \Delta'_{e_{k-1}} e_n \mid_{t=0} . \tag{3.9}$$

On the geometric side, Pawlowski and Rhoades [14] showed that $R_{n,k}$ presents the cohomology of the space

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) \,:\, \ell_i \text{ is a 1-dimensional subspace of } \mathbb{C}^k \text{ and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}. \tag{3.10}$$

of spanning configurations of n lines in $\mathbb{C}^k$.

Equation (3.9) says that $R_{n,k}$ has graded Frobenius characteristic equal to the $t = 0$ Delta Conjecture **upon applying the twist** $\mathrm{rev}_q \circ \omega$. Our first main result (Theorem 3.6 below) uses superspace Vandermondes to remove this twist. If $\mathbf{a} = (k-1, \ldots, k-1)$ is a length r sequence of $(k-1)$'s and $k + r = n$ we will show that

$$\mathrm{grFrob}(V_n(\mathbf{a}); q) = C_{n,k}(\mathbf{x}; q) = \Delta'_{e_{k-1}} e_n \mid_{t=0} . \tag{3.11}$$

Thanks to Proposition 3.1, Equation (3.9), and the definition of $I_{n,k}$, Equation (3.11) follows from

$$I_{n,k} = \mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]}\Delta_n(\mathbf{a}) \tag{3.12}$$

when $\mathbf{a} = (k-1, \ldots, k-1) \in (\mathbb{Z}_{\geq 0})^r$. Our basic tool in proving (3.12) is the following lemma, which gives elements in $I_n(\mathbf{a})$ for any sequence $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$.

Lemma 3.2. *Let k, r be nonnegative integers with $k + r = n$ and let $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbb{Z}_{\geq 0})^r$ be an arbitrary length r sequence of nonnegative integers. The top k elementary symmetric polynomials $e_n, e_{n-1}, \ldots, e_{n-k+1} \in \mathbb{Q}[\mathbf{x}_n]$ lie in the annihilator* $\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]}\Delta_n(\mathbf{a})$.

Proof. We need to check that $e_d \cdot \Delta_n(\mathbf{a}) = \partial(e_d)\Delta_n(\mathbf{a}) = 0$ for $n-k < d \leq n$. We describe a combinatorial procedure for applying the differential operator $\partial(e_d)$ to $\Delta_n(\mathbf{a})$ for any $1 \leq d \leq n$.

Given $w \in \mathfrak{S}_n$ and $1 \leq i \leq n$, we have the following identity of operators on $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$:

$$\partial_{w(i)} \cdot w = w \cdot \partial_i \tag{3.13}$$

For $1 \leq d \leq n$, since $e_d \in \mathbb{Q}[\mathbf{x}_n]$ is a symmetric polynomial we have

$$\partial(e_d) \cdot w = w \cdot \partial(e_d) \tag{3.14}$$

which implies

$$\partial(e_d) \cdot \varepsilon_n = \varepsilon_n \cdot \partial(e_d) \tag{3.15}$$

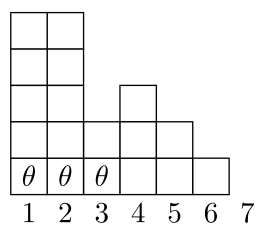
and therefore

$$\partial(e_d) \cdot \Delta_n(\mathbf{a}) = \partial(e_d) \cdot \varepsilon_n \cdot (x_1^{a_1} \cdots x_r^{a_r} x_{r+1}^{k-1} \cdots x_{n-1}^1 x_n^0 \theta_1 \cdots \theta_r) \tag{3.16}$$

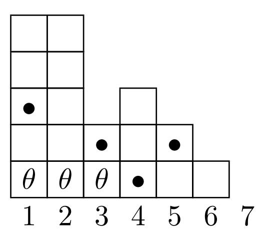
$$= \varepsilon_n \cdot \partial(e_d) \cdot (x_1^{a_1} \cdots x_r^{a_r} x_{r+1}^{k-1} \cdots x_{n-1}^1 x_n^0 \theta_1 \cdots \theta_r). \tag{3.17}$$

We describe the application of $\partial(e_d)$ to $(x_1^{a_1} \cdots x_r^{a_r} x_{r+1}^{k-1} \cdots x_{n-1}^1 x_n^0 \theta_1 \cdots \theta_r)$ combinatorially.

The supermonomial $x_1^{a_1} \cdots x_r^{a_r} x_{r+1}^{k-1} \cdots x_{n-1}^1 x_n^0 \theta_1 \cdots \theta_r$ is modeled by a diagram with n columns of boxes. For $1 \leq i \leq r$, the i^{th} column (from left to right) contains a box with a θ at the bottom, with a_i empty boxes on top. For $r+1 \leq i \leq n$, the i^{th} column consists of $n-i$ empty boxes. The case $n = 7, r = 3, \mathbf{a} = (4,4,1)$ is shown below. We refer to this diagram as the $\mathbf{a}$-*staircase.*



For $d \geq 0$, a d*-dotted* $\mathbf{a}$*-staircase* is an $\mathbf{a}$-staircase in which d of the boxes are marked with a $\bullet$, with no two marked boxes in the same column. An example with $d = 4$ is shown below.



Let σ be a d-dotted $\mathbf{a}$-staircase. The *weight* $\mathrm{wt}(\sigma) \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ of σ is the supermonomial $x_1^{b_1} \cdots x_n^{b_n} \theta_1 \cdots \theta_r$, where b_i is the number of empty boxes in column i. In the above example, we have $\mathrm{wt}(\sigma) = x_1^3 x_2^4 x_4^2 x_5 x_6 \theta_1 \theta_2 \theta_3$. It should be clear that

$$\partial(e_d) \cdot x_1^{a_1} \cdots x_r^{a_r} x_{r+1}^{k-1} \cdots x_{n-1}^1 x_n^0 \theta_1 \cdots \theta_r = \sum_{\sigma} \mathrm{wt}(\sigma), \tag{3.18}$$

where the sum is over all d-dotted $\mathbf{a}$-staircases σ.

In order to calculate $\partial(e_d) \cdot \Delta_n(\mathbf{a})$, we apply ε_n to both sides of Equation (3.18). By Equation (3.15), this yields

$$e_d \cdot \Delta_n(\mathbf{a}) = \partial(e_d) \cdot \Delta_n(\mathbf{a}) = \sum_{\sigma} \varepsilon_n \cdot \mathrm{wt}(\sigma). \tag{3.19}$$

Let σ be a d-dotted $\mathbf{a}$-staircase. If any column of σ contains a $\bullet$ but no θ, there will be two θ-free columns of σ with the same number of empty boxes so that $\varepsilon_n \cdot \mathrm{wt}(\sigma) = 0$. If $d > n-k$, any d-dotted $\mathbf{a}$-staircase σ has a $\bullet$ in a θ-free column so that $e_d \cdot \Delta_n(\mathbf{a}) = 0$. ☐

Remark 3.3. *If we let $X_{n,k}$ be the variety* (3.10) *of spanning configurations of n lines $(\ell_1, \ldots, \ell_n)$ in $\mathbb{C}^k$ and let $\ell_i^* \twoheadrightarrow X_{n,k}$ be the i^{th} tautological line bundle for $1 \leq i \leq n$, we can identify the variable x_i with the Chern class $x_i := c_1(\ell_i^*) \in H^2(X_{n,k})$. The Whitney Sum Formula can be used (see* [14]*) to deduce that the top k elementary symmetric polynomials $e_n, e_{n-1}, \ldots, e_{n-k+1}$ in the x_i vanish in $H^\bullet(X_{n,k}; \mathbb{Q})$. Since we have the identification $H^\bullet(X_{n,k}; \mathbb{Q}) = R_{n,k}$ (see* [14]*), this gives geometric intuition for why $e_n, e_{n-1}, \ldots, e_{n-k+1}$ 'should' lie in the ideal $I_{n,k}$.*

Assuming Equation (3.9)*, Lemma 3.2 gives algebraic and combinatorial intuition coming from superspace for why the elementary symmetric polynomials $e_n, e_{n-1}, \ldots, e_{n-k+1}$ 'should' lie in the ideal $I_{n,k}$ whose corresponding quotient models the Delta Conjecture at $t = 0$.*

3.3. **A Vandermonde model for $C_{n,k}$.** Our goal is the equality of ideals (3.12). Lemma 3.2 gives one of the containments right away.

Lemma 3.4. *Let k, r be nonnegative integers with $k + r = n$. Let $\mathbf{a} = (k-1, \ldots, k-1) \in (\mathbb{Z}_{\geq 0})^r$. Then $I_{n,k} \subseteq \mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})$.*

Proof. It suffices to show that the generators of the ideal $I_{n,k} \subseteq \mathbb{Q}[\mathbf{x}_n]$ annihilate the $\mathbf{a}$-superspace Vandermonde

$$\Delta_n(\mathbf{a}) = \varepsilon_n \cdot (x_1^{k-1} \cdots x_r^{k-1} x_{r+1}^{k-1} \cdots x_{n-1}^1 x_n^0 \cdot \theta_1 \cdots \theta_r). \tag{3.20}$$

Since no x-variable appearing in $\Delta_n(\mathbf{a})$ has exponent $\geq k$, we see immediately that

$$x_i^k \cdot \Delta_n(\mathbf{a}) = \partial_i^k \Delta_n(\mathbf{a}) = 0 \tag{3.21}$$

for all $1 \leq i \leq n$. The remaining generators of $I_{n,k}$ are handled by Lemma 3.2. ☐

Lemma 3.4 proves that the module $V_n(\mathbf{a})$ is not too large; it yields a surjection of vector spaces $R_{n,k} = \mathbb{Q}[\mathbf{x}_n]/I_{n,k} \twoheadrightarrow \mathbb{Q}[\mathbf{x}_n]/\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a}) \cong V_n(\mathbf{a})$. Our next task is to show that $V_n(\mathbf{a})$ is not too small. To do this, we need some facts about $R_{n,k}$.

Haglund, Rhoades, and Shimozono [11] proved that $\dim(R_{n,k}) = k! \cdot \mathrm{Stir}(n,k)$, the number of k-block ordered set partitions of $[n]$. There are a number of vector space bases of $R_{n,k}$ which are indexed by ordered set partitions [11, 14]. We describe the *substaircase monomial basis* here.

Recall that a *shuffle* of two sequences $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_s)$ is an interleaving $(c_1, \ldots, c_{r+s})$ which preserves the relative orders of the a's and the b's. If $k + r = n$, an (n,k)*-staircase* is a shuffle of the sequences $(k-1, k-1, \ldots, k-1)$ (r times) and $(k-1, k-2, \ldots, 1, 0)$. For example, the $(5,3)$-staircases are the shuffles of $(2,2)$ and $(2,1,0)$:

$$(2,2,2,1,0),\ (2,2,1,2,0),\ (2,2,1,0,2),\ (2,1,2,2,0),\ (2,1,2,0,2),\ \text{and}\ (2,1,0,2,2).$$

A sequence $(c_1, \ldots, c_n)$ of nonnegative integers is called (n,k)*-substaircase* if it is componentwise $\leq$ at lease one (n,k)-staircase. For example, the sequence $(2,0,2,1,0)$ is $(5,3)$-substaircase since we have the componentwise inequality $(2,0,2,1,0) \leq (2,2,1,0,2)$. It is shown in [11, Thm. 4.13] that

$$\{x_1^{c_1} \cdots x_n^{c_n} \,:\, (c_1, \ldots, c_n) \text{ is } (n,k)\text{-substaircase}\} \tag{3.22}$$

descends to a vector space basis [2] of $R_{n,k}$. In particular,

$$\text{number of } (n,k)\text{-substaircase sequences} = k! \cdot \mathrm{Stir}(n,k). \tag{3.23}$$

[2]Strictly speaking, [11, Thm. 4.13] states that the set $\{x_n^{c_1} \cdots x_1^{c_n} \,:\, (c_1, \ldots, c_n) \text{ is } (n,k)\text{-substaircase}\}$ of 'reversed' monomials descends to a basis of $R_{n,k}$, but since $R_{n,k}$ is an $\mathfrak{S}_n$-module this is equivalent.

The proof of (3.23) in [11] was recursive and rather involved. A bijective proof of (3.23) involving an extension of Lehmer code from permutations to ordered set partitions was given in [18]. We use substaircase monomials to bound $\dim V_n(\mathbf{a})$ from below.

Lemma 3.5. *Let $k, r \geq 0$ with $k + r = n$ and let $\mathbf{a} = (k-1, \ldots, k-1) \in (\mathbb{Z}_{\geq 0})^r$. We have $\dim V_n(\mathbf{a}) \geq k! \cdot \mathrm{Stir}(n,k)$.*

Proof. It is enough to exhibit $k! \cdot \mathrm{Stir}(n,k)$ linearly independent elements of the vector space $V_n(\mathbf{a})$. Thanks to (3.23) it is enough to show that

$$\{\partial_1^{c_1} \cdots \partial_n^{c_n} \Delta_n(\mathbf{a}) \,:\, (c_1, \ldots, c_n) \text{ is } (n,k)\text{-substaircase}\} \subseteq V_n(\mathbf{a}) \tag{3.24}$$

is linearly independent. We begin with the following seemingly weaker claim.
Claim: *The family of $\binom{n-1}{r}$ superpolynomials*

$$\left\{ \theta_{i_1}\theta_{i_2}\cdots\theta_{i_r} + \sum_{j=1}^{r} (-1)^j \theta_1 \theta_{i_1} \cdots \widehat{\theta_{i_j}} \cdots \theta_{i_r} \,:\, \begin{array}{c} \{i_1 < \cdots < i_r\} \textit{ is an} \\ r\textit{-element subset of } \{2,3,\ldots,n\} \end{array} \right\} \tag{3.25}$$

is linearly independent.

To see why the Claim is true, observe that $\theta_{i_1}\theta_{i_2}\cdots\theta_{i_r}$ only appears in the element corresponding to $\{i_1 < \cdots < i_r\}$. This completes the proof of the Claim.

We use our Claim to finish the proof. Suppose there were numbers $\gamma_{(c_1,\ldots,c_n)} \in \mathbb{Q}$ not all zero so that

$$\sum_{(c_1,\ldots,c_n)} \gamma_{(c_1,\ldots,c_n)} \partial_1^{c_1} \cdots \partial_n^{c_n} \Delta_n(\mathbf{a}) = 0, \tag{3.26}$$

where the sum is over all (n,k)-substaircases $(c_1, \ldots, c_n)$. Choose an (n,k)-substaircase $(d_1, \ldots, d_n)$ so that

(1) we have $\gamma_{(d_1,\ldots,d_n)} \neq 0$, and
(2) subject to (1) the number $d_1 + \cdots + d_n$ is minimal, and
(3) subject to (1) and (2) the sequence $(d_1, \ldots, d_n)$ is lexicographically least.

Since $(d_1, \ldots, d_n)$ is (n,k)-substaircase, there exists an (n,k)-staircase $(d_1', \ldots, d_n')$ with $d_i \leq d_i'$ for all i. Let $1 < i_1 < \cdots < i_r$ be the indices such that $d_1' = d_{i_1}' = \cdots = d_{i_r}' = k-1$. Write $p_i := d_i' - d_i$. We apply the operator $\partial_1^{p_1} \cdots \partial_n^{p_n}$ to both sides of Equation (3.26); this will force $\gamma_{(d_1,\ldots,d_n)} = 0$, contradicting Assumption (1) on $(d_1, \ldots, d_n)$.

The application of $\partial_1^{p_1} \cdots \partial_n^{p_n}$ to the term $\gamma_{(c_1,\ldots,c_n)} \partial_1^{c_1} \cdots \partial_n^{c_n} \Delta_n(\mathbf{a})$ in Equation (3.26) has the following effect.

- If $(c_1 + p_1, \ldots, c_n + p_n)$ is not a rearrangement of $(k-1, \ldots, k-1, k-2, \ldots, 1, 0)$ then $\gamma_{(c_1,\ldots,c_n)} \partial_1^{c_1+p_1} \cdots \partial_n^{c_n+p_n} \Delta_n(\mathbf{a}) = 0$ by Assumption (2) on $(d_1, \ldots, d_n)$.
- If $c_1 + p_1 < k-1$, Assumption (3) on $(d_1, \ldots, d_n)$ forces $\gamma_{(c_1,\ldots,c_n)} = 0$.
- Otherwise, let $1 < s_1 < \cdots < s_r$ be the unique indices such that

$$c_1 + p_1 = c_{s_1} + p_{s_1} = \cdots = c_{s_r} + p_{s_r} = k-1.$$

Directly computing the application of partial derivatives to $\Delta_n(\mathbf{a})$ yields

$$\partial_1^{c_1+p_1} \cdots \partial_n^{c_n+p_n} \gamma_{(c_1,\ldots,c_n)} \Delta_n(\mathbf{a}) \doteq \gamma_{(c_1,\ldots,c_n)} \left[\theta_{s_1}\theta_{s_2}\cdots\theta_{s_r} + \sum_{j=1}^{r} (-1)^j \theta_1 \theta_{s_1} \cdots \widehat{\theta_{s_j}} \cdots \theta_{s_r} \right]. \tag{3.27}$$

Observe that (3.27) is a superpolynomial appearing in the Claim. If $(s_1, \ldots, s_r) = (i_1, \ldots, i_r)$ in (3.27), Assumption (3) on $(d_1, \ldots, d_n)$ forces $(c_1, \ldots, c_n) = (d_1, \ldots, d_n)$.

Putting everything together, we apply $\partial_1^{p_1} \cdots \partial_n^{p_n}$ to both sides of (3.26), yielding

$$\sum_{(c_1,\ldots,c_n)} \gamma_{(c_1,\ldots,c_n)} \partial_1^{c_1+p_1} \cdots \partial_n^{c_n+p_n} \Delta_n(\mathbf{a}) = 0. \tag{3.28}$$

The term on the left-hand side corresponding to $(c_1, \ldots, c_n)$ is either zero, or a nonzero scalar multiple of some element appearing in the linearly independent set described in the Claim. Furthermore, the unique term yielding the element in the Claim corresponding to $(i_1, \ldots, i_r)$ occurs when $(c_1, \ldots, c_n) = (d_1, \ldots, d_n)$. In particular, the left-hand side of (3.28) has no cancellation involving the element in the Claim corresponding to $(i_1, \ldots, i_r)$. Linear independence forces the contradiction $\gamma_{(d_1,\ldots,d_n)} = 0$. □

By combining Lemmas 3.4 and 3.5, we prove Equation (3.9) and obtain our new model for the Delta coinvariants.

Theorem 3.6. *Let k, r be nonnegative integers with $k + r = n$. Let $\mathbf{a} = (k-1, k-1, \ldots, k-1)$ where there are r copies of $k-1$. We have*

$$\operatorname{grFrob}(V_n(\mathbf{a}); q) = C_{n,k}(\mathbf{x}; q) = \Delta'_{e_{k-1}} e_n \mid_{t=0} . \tag{3.29}$$

Proof. By Proposition 3.1, Equation (3.9), and Lemma 3.4, it is enough to show that

$$\dim V_n(\mathbf{a}) \geq \dim \mathbb{Q}[\mathbf{x}_n]/I_{n,k} = \dim R_{n,k} = k! \cdot \operatorname{Stir}(n, k). \tag{3.30}$$

This is a consequence of Lemma 3.5. □

Remark 3.7. *The irreducible representation $S^{(n-1,1)}$ corresponding to the partition $(n-1, 1) \vdash n$ is the $(n-1)$-dimensional* reflection representation *of $\mathfrak{S}_n$. This is the coordinate-permuting action of $\mathfrak{S}_n$ on the subspace of $\mathbb{Q}^n$ of vectors which sum to 0.*

The span of the $\binom{n-1}{r}$ polynomials in the θ-variables described in (3.25) *in the proof of Lemma 3.5 is closed under the action of $\mathfrak{S}_n$. This span is isomorphic to the exterior power $\wedge^r S^{(n-1,1)}$ as an $\mathfrak{S}_n$-module. If $\lambda = (n-r, 1, \ldots, 1) \vdash n$ (where there are r copies of 1), it is well-known that $\wedge^r S^{(n-1,1)} \cong S^\lambda$. Lemma 3.5 and Theorem 3.6 therefore explain the presence of hook-shaped Schur functions as the coefficient of $q^0 t^0$ in the Schur expansion of $\Delta'_{e_{k-1}} e_n$.*

The symmetric function $\operatorname{grFrob}(V_n(\mathbf{a}); q)$ of Theorem 3.6 can also be expressed in the Schur basis. Applying [21, Thm. 5.0.1] (with $m = 0$ after taking the coefficient of u^{n-k}) and [11, Cor. 6.13] we see that

$$\operatorname{grFrob}(V_n(\mathbf{a}); q) = \sum_{T \in \operatorname{SYT}(n)} q^{\operatorname{maj}(T) + \binom{n-k}{2} - (n-k) \cdot \operatorname{des}(T)} \begin{bmatrix} \operatorname{des}(T) \\ n-k \end{bmatrix}_q s_{\operatorname{shape}(T)}. \tag{3.31}$$

Here $\operatorname{SYT}(n)$ is the set of standard Young tableaux with n boxes, $\operatorname{maj}(T)$ is the major index of T, $\operatorname{des}(T)$ is the number of descents in T, and $\operatorname{shape}(T) \vdash n$ is the shape of T; see [11] or [21] for definitions of these terms. We can describe the Hilbert series of the module in Theorem 3.6 in terms of q-Stirling numbers.

Corollary 3.8. *Let k, r be nonnegative integers with $k + r = n$. Let $\mathbf{a} = (k-1, k-1, \ldots, k-1)$ where there are r copies of $k-1$. We have*

$$\operatorname{Hilb}(V_n(\mathbf{a}); q) = [k]!_q \cdot \operatorname{Stir}_q(n, k). \tag{3.32}$$

Proof. The asserted formula is the q-reversal of the formula for $\operatorname{Hilb}(R_{n,k}; q)$ given in [11]. □

4. Vandermondes and Other Graded Modules

In this section we extend Theorem 3.6 to calculate $\operatorname{grFrob}(V_n(\mathbf{a}); q)$ for other constant vectors $\mathbf{a}$. The first result involves uniformly increasing the entries of $\mathbf{a}$.

4.1. **Vandermondes and the quotient ring $R_{n,k,s}$.** Let k, s, and n be nonnegative integers with $k \leq s$. We define the ideal $I_{n,k,s} \subseteq \mathbb{Q}[\mathbf{x}_n]$ by

$$I_{n,k,s} := \langle x_1^s, x_2^s, \ldots, x_n^s, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle \tag{4.1}$$

and let $R_{n,k,s} := \mathbb{Q}[\mathbf{x}_n]/I_{n,k,s}$ be the corresponding quotient ring. When $k = s$ we have $I_{n,k,k} = I_{n,k}$ and $R_{n,k,k} = R_{n,k}$. The rings $R_{n,k,s}$ are graded $\mathfrak{S}_n$-modules which were used in [11] to inductively understand the rings $R_{n,k}$. Pawlowski and Rhoades proved [14] that $R_{n,k,s}$ presents the cohomology of a certain space $X_{n,k,s}$ of line configurations.

We extend (n,k)-staircases to include the parameter s as follows. An (n,k,s)-*staircase* is a shuffle of the sequences $(s-1, s-1, \ldots, s-1)$ (with $n-k$ copies of $s-1$) and $(k-1, \ldots, 1, 0)$. For example, the $(4,2,6)$-staircases are the shuffles of $(5,5)$ and $(1,0)$:

$$(5,5,1,0),\ (5,1,5,0),\ (5,1,0,5),\ (1,5,5,0),\ (1,5,0,5),\ \text{and } (1,0,5,5).$$

A sequence $(c_1, \ldots, c_n)$ is (n,k,s)-*substaircase* if it is componentwise $\leq$ at least one (n,k,s)-staircase. The (n,k,s)-substaircases parameterize a monomial basis of $R_{n,k,s}$.

Proposition 4.1. *Let k, s, and n be nonnegative integers with $k \leq s$. The set of monomials*

$$\{x_1^{c_1} \cdots x_n^{c_n} \,:\, (c_1, \ldots, c_n) \text{ is an } (n,k,s)\text{-substaircase}\}$$

descends to a basis of $R_{n,k,s}$.

Proof. In the case $k \leq s \leq n$, Haglund, Rhoades, and Shimozono computed [11, Sec. 6] the standard monomial basis of $R_{n,k,s}$ in terms of '(n,k,s)-nonskip monomials'. The arguments of [11, Sec. 6] go through without change to the case $n < s$. We show that this monomial basis coincides with the set of (n,k,s)-substaircase monomials.

Let $S = \{i_1 < \cdots < i_{n-k+1}\} \subseteq [n]$ have size $n-k+1$. The *skip sequence* $\gamma(S) = (\gamma_1, \ldots, \gamma_n)$ corresponding to S is defined by

$$\gamma_i = \begin{cases} i - j + 1 & \text{if } i = i_j \in S, \\ 0 & \text{if } i \notin S. \end{cases} \tag{4.2}$$

We also let $\gamma(S)^* := (\gamma_n, \ldots, \gamma_1)$ be the reverse of the sequence $\gamma(S)$. A sequence $(c_1, \ldots, c_n)$ of nonnegative integers is (n,k,s)-*nonskip* if

- $c_i < s$ for all $1 \leq i \leq n$ and
- the coordinatewise inequality $\gamma(S)^* \leq (c_1, \ldots, c_n)$ does *not* hold for any $S \subseteq [n]$ with $|S| = n-k+1$.

The arguments of [11, Sec. 6] show that the set $\{x_1^{c_1} \cdots x_n^{c_n} \,:\, (c_1, \ldots, c) \text{ is } (n,k,s)\text{-nonskip}\}$ descends to a basis of $R_{n,k,s}$. The proposition therefore reduces to the following Claim.

Claim: *Let $(c_1, \ldots, c_n)$ be a sequence of nonnegative integers. Then $(c_1, \ldots, c_n)$ is (n,k,s)-nonskip if and only if $(c_1, \ldots, c_n)$ is (n,k,s)-substaircase.*

When $k = s$ this claim is proven in [11], so we assume that $k < s$. The reverse implication reduces to showing that any (n,k,s)-staircase is (n,k,s)-nonskip, which we leave to the reader. For the forward implication, let $(c_1, \ldots, c_n)$ be an (n,k,s)-nonskip sequence. We produce an (n,k,s)-staircase $(b_1, \ldots, b_n)$ such that we have the componentwise inequality $(c_1, \ldots, c_n) \leq (b_1, \ldots, b_n)$.

Since we are assuming $k < s$, an (n,k,s)-staircase $(b_1, \ldots, b_n)$ is determined by the set

$$T := \{1 \leq i \leq n \,:\, b_i < k\} = \{t_1 < t_2 < \cdots < t_k\}$$

of positions of entries $< k$. We describe how to form T from $(c_1, \ldots, c_n)$.

We claim that there exists $1 \leq t_k \leq n$ such that $c_{t_k} < 1$. If not, we would have the componentwise inequality $(1, 1, \ldots, 1) \leq (c_1, c_2, \ldots, c_n)$. If $S = \{1, 2, \ldots, n-k+1\}$, we would have $\gamma(S)^* \leq (1, 1, \ldots, 1) \leq (c_1, c_2, \ldots, c_n)$, contradicting the assumption that $(c_1, c_2, \ldots, c_n)$ is (n,k,s)-nonskip. Let $1 \leq t_k \leq n$ be maximal such that $c_{t_k} < 1$.

With t_k as in the last paragraph, we claim that there exists $1 \leq t_{k-1} < t_k$ with $c_{t_{k-1}} < 2$. If not, we would have the componentwise inequality

$$(2, 2, \ldots, 2, 0, 1, 1, \ldots, 1) \leq (c_1, c_2, \ldots, c_n)$$

where the 0 is in position t_k. If we take $S \subseteq [n]$ to be

$$S = \{1, 2, \ldots, n - t_k, n - t_k + 2, n - t_k + 3, \ldots, n - k + 2\},$$

we would have $\gamma(S)^* \leq (c_1, c_2, \ldots, c_n)$, contradicting the assumption that $(c_1, c_2, \ldots, c_n)$ is (n, k, s)-nonskip. Let $1 \leq t_{k-1} < t_k$ be maximal such that $c_{t_{k-1}} < 2$.

Given $t_{k-1} < t_k$ as above, we claim that there exists $1 \leq t_{k-2} < t_{k-1}$ with $c_{t_{k-2}} < 3$. If not, the componentwise inequality

$$(3, 3, \ldots, 3, 0, 2, 2, \ldots, 2, 0, 1, 1, \ldots, 1) \leq (c_1, c_2, \ldots, c_n)$$

with the 0's in positions t_{k-1} and t_k would contradict $(c_1, \ldots, c_n)$ being (n, k, s)-nonskip. Choose $1 \leq t_{k-2} < t_{k-1}$ minimal such that $c_{t_{k-2}} < 3$.

Since $(c_1, \ldots, c_n)$ is (n, k, s)-nonskip, we can iterate this procedure to obtain a k-element subset $T = \{t_1 < \cdots < t_k\} \subseteq [n]$. Let $(b_1, \ldots, b_n)$ be the unique (n, k, s)-staircase whose entries which are $< k$ are in the positions indexed by T. By the construction of T we have $(c_1, \ldots, c_n) \leq (b_1, \ldots, b_n)$ so that $(c_1, \ldots, c_n)$ is (n, k, s)-substaircase. ☐

Proposition 4.1 can be used to extend Theorem 3.6 to constant vectors with larger entries.

Theorem 4.2. *Let $k, s,$ and n with $k \leq s$ be nonnegative integers and let $r = n - k$. Let $\mathbf{a} = (s - 1, s - 1, \ldots, s - 1)$ be the constant vector of length r with entries $s - 1$. We have*

$$\mathrm{grFrob}(V_n(\mathbf{a}); q) = (\mathrm{rev}_q \circ \omega)\mathrm{grFrob}(R_{n,k,s}; q). \tag{4.3}$$

Proof. When $k = s$, this is Theorem 3.6 so we assume $k < s$.

It is enough to demonstrate the equality of ideals $I_{n,k,s} = \mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})$. The containment $I_{n,k,s} \subseteq \mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})$ follows from Lemma 3.2 and the fact that no x-variable in $\Delta_n(\mathbf{a})$ has exponent $\geq s$. The desired equality of ideals will follow if we can show

$$\dim V_n(\mathbf{a}) = \dim(\mathbb{Q}[\mathbf{x}_n]/\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})) \geq \dim(\mathbb{Q}[\mathbf{x}_n]/I_{n,k,s}) = \dim R_{n,k,s}. \tag{4.4}$$

By Proposition 4.1, we know that $\dim(R_{n,k,s})$ equals the number of (n, k, s)-staircases. It is therefore enough to prove the following Claim.

Claim: *The subset*

$$\{\partial_1^{c_1} \cdots \partial_n^{c_n} \Delta_n(\mathbf{a}) \,:\, (c_1, \ldots, c_n) \text{ is } (n, k, s)\text{-substaircase}\} \subseteq V_n(\mathbf{a}) \tag{4.5}$$

is linearly independent.

Since $k < s$, the Claim is an easier version of Lemma 3.5. The set of $\binom{n}{r}$ supermonomials

$$\{\theta_{i_1} \cdots \theta_{i_r} \,:\, 1 \leq i_1 < \cdots < i_r \leq n\} \tag{4.6}$$

is certainly linearly independent. If the Claim were false, there would be scalars $\gamma_{(c_1, \ldots, c_n)}$ not all zero so that

$$\sum_{(c_1, \ldots, c_n)} \gamma_{(c_1, \ldots, c_n)} \partial_1^{c_1} \cdots \partial_n^{c_n} \Delta_n(\mathbf{a}) = 0, \tag{4.7}$$

where the sum is over all (n, k, s)-substaircase sequences $(c_1, \ldots, c_n)$.

Let $(d_1, \ldots, d_n)$ be the unique (n, k, s)-substaircase such that

1. we have $\gamma_{(d_1, \ldots, d_n)} \neq 0$,
2. subject to (1) the number $d_1 + \cdots + d_n$ is minimal, and
3. subject to (1) and (2) the sequence $(d_1, \ldots, d_n)$ is lexicographically least.

Let $(d'_1, \ldots, d'_n)$ be a (n, k, s)-staircase such that $d_i \leq d'_i$ for all i and set $p_i := d'_i - d_i$. Let $1 \leq i_1 < \cdots < i_r \leq n$ be the indices such that $d'_{i_1} = \cdots = d'_{i_r} = s - 1$.

Consider applying the operator $\partial_1^{p_1} \cdots \partial_n^{p_n}$ to both sides of Equation (4.7). This operator has the following effect on $\gamma_{(c_1,\ldots,c_n)} \partial_1^{c_1} \cdots \partial_n^{c_n} \Delta_n(\mathbf{a})$.

- If $(c_1 + p_1, \ldots, c_n + p_n)$ is not a rearrangement of $(s-1, \ldots, s-1, k-1, k-2, \ldots, 1, 0)$ then $\gamma_{(c_1,\ldots,c_n)} \partial_1^{c_1+p_1} \cdots \partial_n^{c_n+p_n} \Delta_n(\mathbf{a}) = 0$.
- If $(c_1 + p_1, \ldots, c_n + p_n)$ is a rearrangement of $(s-1, \ldots, s-1, k-1, k-2, \ldots, 1, 0)$ let $1 \leq t_1 < \cdots < t_r \leq n$ be the indices with $c_{t_1} + p_{t_1} = \cdots = c_{t_r} + p_{t_r} = s - 1$. We have
$$\partial_1^{c_1+p_1} \cdots \partial_n^{c_n+p_n} \Delta_n(\mathbf{a}) \doteq \theta_{t_1} \cdots \theta_{t_r}.$$

If $(c_1 + p_1, \ldots, c_n + p_n)$ is as in the second bullet point and we have $(t_1, \ldots, t_r) = (i_1, \ldots, i_r)$, the lexicographical minimality of $(d_1, \ldots, d_n)$ forces $(c_1, \ldots, c_n) = (d_1, \ldots, d_n)$. The linear independence of (4.6) implies that $\gamma_{(d_1,\ldots,d_n)} = 0$, which is a contradiction. $\square$

4.2. **Vandermondes and Tanisaki ideals.** The authors are unaware of a representation theoretic description of $V_n(\mathbf{a})$ when $0 < s < r$ and $\mathbf{a} = (s-1, \ldots, s-1)$ is a constant sequence of length r. However, we can describe $V_n(\mathbf{a})$ when $\mathbf{a} = (0, \ldots, 0)$ is the length r zero sequence. In order to state this result, we will need a couple more definitions.

For any subset $S \subseteq [n]$ and any $d \geq 0$, let $e_d(S) \in \mathbb{Q}[\mathbf{x}_n]$ be the degree d elementary symmetric polynomial in the variable set $\{x_i : i \in S\}$. We adopt the convention $e_d(S) = 0$ whenever $d > |S|$.

Let $\lambda \vdash n$. The *Tanisaki ideal* $I_\lambda \subseteq \mathbb{Q}[\mathbf{x}_n]$ is defined as follows. Write the *conjugate* partition to λ as $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n)$, where we will have trailing zeros unless $\lambda = (n)$. The ideal I_λ has generating set

$$\bigcup_{i=1}^{n} \{e_d(S) \,:\, |S| = i \text{ and } d > i - (\lambda'_{n-i+1} + \lambda'_{n-i+2} + \cdots + \lambda'_n)\}. \tag{4.8}$$

This ideal was used by Tanisaki [20] to present the cohomology of the *Springer fiber* $\mathcal{B}_\lambda$. We let $R_\lambda := \mathbb{Q}[\mathbf{x}_n]/I_\lambda$ be the corresponding quotient ring, which is a graded $\mathfrak{S}_n$-module.

Proposition 4.3. *Let $r < n$ be positive integers and let $\mathbf{a} = (0, \ldots, 0)$ be the length r zero sequence. Then*

$$\mathrm{grFrob}(V_n(\mathbf{a}); q) = (\mathrm{rev}_q \circ \omega) \mathrm{grFrob}(R_\lambda; q) \tag{4.9}$$

where λ is the hook-shaped partition $(r+1, 1, 1, \ldots, 1) \vdash n$.

Proof. Write $k = n - r$. We begin by showing that $I_\lambda \subseteq \mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})$.

The ideal I_λ is generated by:

(1) the elementary symmetric polynomials $e_1, e_2, \ldots, e_n$ in the full set of variables $\{x_1, \ldots, x_n\}$ and
(2) products of k distinct variables $x_{i_1} \cdots x_{i_k}$.

(The other generators of I_λ are redundant.) We show that each of these generators annihilates $\Delta_n(\mathbf{a})$. To do this, we adopt the notation in the proof of Lemma 3.2.

The generators of type (1) annihilate $\Delta_n(\mathbf{a})$ by an argument similar to that in the proof of Lemma 3.2. The key observation is that, since $\mathbf{a}$ is the zero sequence, all of the $\bullet$'s in any d-dotted $\mathbf{a}$-staircase must be in columns which do not contain a θ. The generators of type (2) annihilate $\Delta_n(\mathbf{a})$ because in any monomial appearing in $\Delta_n(\mathbf{a})$ there are only $k-1$ many x-variables with positive exponents. This completes the proof that $I_\lambda \subseteq \mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})$.

We must show that

$$\dim V_n(\mathbf{a}) = \dim(\mathbb{Q}[\mathbf{x}_n]/\mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})) \geq \dim(\mathbb{Q}[\mathbf{x}_n]/I_\lambda) = \dim R_\lambda. \tag{4.10}$$

The quantity $\dim R_\lambda$ has the following combinatorial description. An (n,k)-*hook staircase* is a shuffle of $(k-1, k-2, \ldots, 1, 0)$ and the length r zero sequence $(0, 0, \ldots, 0)$. For example, the $(5,3)$-hook staircases are the shuffles of $(2,1,0)$ and $(0,0)$:

$$(2,1,0,0,0),\ (2,0,1,0,0),\ (2,0,0,1,0),\ (0,2,1,0,0),\ (0,2,0,1,0),\ \text{and}\ (0,0,2,1,0).$$

A sequence $(c_1, \ldots, c_n)$ is (n,k)-*hook-substaircase* if it is componentwise $\leq$ some (n,k)-hook staircase.

It is known [9] that the set

$$\{x_1^{c_1} \cdots x_n^{c_n} \,:\, (c_1, \ldots, c_n) \text{ is } (n,k)\text{-hook-substaircase}\} \tag{4.11}$$

descends to a basis for R_λ. As in Theorems 3.6 and 4.2, we start with a family of linearly independent superpolynomials.

Observation: *The subset*

$$\left\{ \left(\sum_{j=1}^{r} (-1)^{j-1} \theta_{i_1} \cdots \widehat{\theta_{i_j}} \cdots \theta_{i_r} \theta_n \right) + (-1)^r \theta_{i_1} \cdots \theta_{i_r} \,:\, 1 \leq i_1 < \cdots < i_r \leq n-1 \right\} \tag{4.12}$$

is linearly independent.

We show that the set

$$\{\partial_1^{c_1} \cdots \partial_n^{c_n} \Delta_n(\mathbf{a}) \,:\, (c_1, \ldots, c_n) \text{ is } (n,k)\text{-hook-substaircase}\} \subseteq V_n(\mathbf{a}) \tag{4.13}$$

is linearly independent. Indeed, suppose we had a dependence relation

$$\sum_{(c_1, \ldots, c_n)} \gamma_{(c_1, \ldots, c_n)} \partial_1^{c_1} \cdots \partial_n^{c_n} \Delta_n(\mathbf{a}) = 0 \tag{4.14}$$

where the sum is over (n,k)-hook-substaircases $(c_1, \ldots, c_n)$ and the numbers $\gamma_{(c_1, \ldots, c_n)}$ are not all zero. As before, let $(d_1, \ldots, d_n)$ be the unique (n,k)-hook-substaircase such that

(1) we have $\gamma_{(d_1, \ldots, d_n)} \neq 0$,
(2) subject to (1) the number $d_1 + \cdots + d_n$ is minimal, and
(3) subject to (1) and (2) the sequence $(d_1, \ldots, d_n)$ is lexicographically least.

Let $(d'_1, \ldots, d'_n)$ be any (n,k)-hook staircase with $d_i \leq d'_i$ for all i and set $p_i := d'_i - d_i$. An argument similar to that of Lemma 3.5 yields the contradiction $\gamma_{(d_1, \ldots, d_n)} = 0$ upon application of $\partial_1^{p_1} \cdots \partial_n^{p_n}$ to both sides of Equation (4.14); we leave the details to the reader. □

4.3. **A positroid superspace quotient.** In recent work related to an operation on symmetric functions and vector bundles called 'Chern plethysm', Billey, Rhoades, and Tewari [5] defined a quotient of superspace which gives a bigraded refinement of an action of $\mathfrak{S}_n$ on size n positroids. In this subsection we use superpolynomials similar to **a**-superspace Vandermondes to give an alternative model for their module.

Following [5], we let $J_n \subseteq \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ be the bihomogeneous ideal

$$J_n := \langle x_1\theta_1, x_2\theta_2, \ldots, x_n\theta_n, e_1, e_2, \ldots, e_n \rangle, \tag{4.15}$$

where the elementary symmetric polynomials $e_d = e_d(\mathbf{x}_n)$ are in the x-variables. Let $S_n := \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]/J_n$ be the corresponding superspace quotient. The ring S_n is a bigraded $\mathfrak{S}_n$-module. By [5, Thm. 5.3], we have

$$\mathrm{grFrob}(S_n; q, z) = \sum_{r=0}^{n} z^r \cdot e_r(\mathbf{x}) \cdot \mathrm{rev}_q Q'_{(1^{n-r})}(\mathbf{x}; q), \tag{4.16}$$

where q tracks x-degree and z tracks θ-degree. Recall that $\mathrm{rev}_q Q'_{(1^{n-r})}(\mathbf{x}; q)$ is the graded Frobenius image of the coinvariant ring R_{n-r} attached to $\mathfrak{S}_{n-r}$.

The module S_n is related to positroids. A *positroid* [3] of size n is a length n sequence $p_1 \dots p_n$ of nonnegative integers which contains r copies of 0 (for some $0 \leq r \leq n$) and a single copy of $1, 2, \dots, n-r$. Let P_n be the family of positroids of size n. For example,

$$P_3 = \{123, 213, 132, 231, 312, 321, 120, 210, 102, 201, 012, 021, 001, 010, 100, 000\}.$$

By [5, Prop. 5.2, Thm. 5.3] the dimension of S_n counts size n positroids:

$$\dim(S_n) = |P_n| = \sum_{r=0}^{n} \frac{n!}{r!}. \tag{4.17}$$

The $\mathbb{Q}$-vector space $\mathbb{Q}[P_n]$ with basis P_n carries an action of $\mathfrak{S}_n$ defined on adjacent transpositions by

$$(i, i+1).p_1 \dots p_i p_{i+1} \dots p_n := \pm(p_1 \dots p_{i+1} p_i \dots p_n), \quad 1 \leq i \leq n-1 \tag{4.18}$$

where the sign is $-$ if $p_i = p_{i+1}$ and $+$ if $p_i \neq p_{i+1}$. By [5, Prop. 5.2, Thm. 5.3] we have

$$\mathrm{Frob}(S_n) = \mathrm{Frob}(\mathbb{Q}[P_n]) \tag{4.19}$$

so that S_n gives a bigraded refinement of this $\mathfrak{S}_n$-action on positroids.

We will give an alternative model for S_n as a bigraded subspace of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ rather than as a quotient ring. For $0 \leq k \leq n$ we define $\rho_{n,k} \in \mathbb{Q}[\mathbf{x}_n, \theta_n]$ to be the superpolynomial

$$\rho_{n,k} := \varepsilon_k \cdot (x_1^{k-1} \cdots x_{k-1}^1 x_k^0) \cdot \theta_{k+1} \cdots \theta_{n-1} \theta_n. \tag{4.20}$$

Here $\varepsilon_k = \sum_{w \in \mathfrak{S}_k} \mathrm{sign}(w) \cdot w$ acts on the subscripts of the first k variables. In particular, the element $\rho_{n,n} = \Delta_n$ is the classical Vandermonde.

Definition 4.4. *Let M_n be the smallest linear subspace of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ which*

- *contains each of the superpolynomials $\rho_{n,0}, \rho_{n,1}, \dots, \rho_{n,n}$,*
- *is closed under the action of $\mathfrak{S}_n$, and*
- *is closed under the partial derivatives $\partial_1, \partial_2, \dots, \partial_n$ acting on the x-variables.*

The vector space M_n is a bigraded $\mathfrak{S}_n$-module.

Proposition 4.5. *The bigraded $\mathfrak{S}_n$-modules S_n and M_n are isomorphic. Equivalently, we have*

$$\mathrm{grFrob}(M_n; q, z) = \sum_{r=0}^{n} z^r \cdot e_r(\mathbf{x}) \cdot \mathrm{rev}_q Q'_{(1^{n-r})}(\mathbf{x}; q). \tag{4.21}$$

Proof. For $0 \leq r \leq n$ let $M_{n,r}$ be the smallest subspace of M_n containing $\rho_{n,n-r}$ which is closed under the action of $\mathfrak{S}_n$ and the partial derivatives $\partial_1, \dots, \partial_n$. Then $M_{n,r}$ is the θ-homogeneous piece of M_n of degree r, so that

$$\mathrm{grFrob}(M_n; q, z) = \sum_{r=0}^{n} z^r \cdot \mathrm{grFrob}(M_{n-r}; q), \tag{4.22}$$

so it suffices to verify

$$\mathrm{grFrob}(M_{n,r}; q) = e_r(\mathbf{x}) \cdot \mathrm{rev}_q Q'_{(1^{n-r})}(\mathbf{x}; q), \tag{4.23}$$

where q tracks x-degree.

Equation (4.23) states that $M_{n,r}$ is the induction product of the sign representation of $\mathfrak{S}_r$ with the coinvariant algebra attached to $\mathfrak{S}_{n-r}$. Indeed, for any $I = \{i_1 < \cdots < i_r\} \subseteq [n]$ with $|I| = r$ and complement $J = [n] - I = \{j_1 < \cdots < j_{n-r}\}$, let $M_I \subseteq M_{n,r}$ be the smallest linear subspace such that

[3] A size n positroid is more typically defined as a permutation in $\mathfrak{S}_n$ whose fixed points are colored either black or white, but these objects are in bijection with P_n.

- we have $\left[\left(\sum_{w \in \mathfrak{S}_J} \mathrm{sign}(w) \cdot w\right) \cdot x_{j_1}^{k-1} \cdots x_{j_{n-r-1}}^1 x_{j_{n-r}}^0\right] \cdot \theta_{i_1} \cdots \theta_{i_r} \in M_I$, and
- M_I is closed under the partial derivatives $\partial_1, \ldots, \partial_n$.

We have the vector space direct sum decomposition

$$M_{n,r} = \bigoplus_{\substack{I \subseteq [n] \\ |I|=r}} M_I. \tag{4.24}$$

Taking $I = [r] = \{1, 2, \ldots, r\}$, the space $M_{[r]}$ is a graded $\mathfrak{S}_r \times \mathfrak{S}_{n-r}$-module isomorphic to $\mathrm{sign}_r \otimes R_{n-r}$, where sign_r is the 1-dimensional sign representation of $\mathfrak{S}_r$ and R_{n-r} is the coinvariant ring attached to $\mathfrak{S}_{n-r}$. Equation (4.24) leads to the identification of $M_{n,r}$ as the induction product

$$M_{n,r} \cong \mathrm{sign}_r \circ R_{n-r}, \tag{4.25}$$

which implies Equation (4.23). ☐

5. Antisymmetric differentiation

The singly graded modules $V_n(\mathbf{a})$ are based on an action of the partial derivative operators ∂_i acting on the x-variables in $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$. The goal of this section is to describe how new operators ∂_i^θ acting on the θ-variables can be used to build new doubly graded modules $W_n(\mathbf{a})$. The modules $W_n(\mathbf{a})$ will contain the $V_n(\mathbf{a})$ as their top antisymmetric components and will exhibit a new kind of duality which is invisible at the level of $V_n(\mathbf{a})$.

5.1. **The operators ∂_i^θ.** How can we differentiate with respect to a skew-commuting variable? Recall from Section 1 that $\partial_i^\theta : \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n] \to \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ is the $\mathbb{Q}[\mathbf{x}_n]$-linear operator determined on θ-monomials by the rule

$$\partial_i^\theta : \theta_{j_1} \cdots \theta_{j_r} \mapsto \begin{cases} (-1)^{k-1} \theta_{j_1} \cdots \widehat{\theta_{j_k}} \cdots \theta_{j_r} & \text{if } j_k = i, \\ 0 & \text{if } i \notin \{j_1 < \cdots < j_k\} \end{cases} \tag{5.1}$$

for all $1 \leq j_1 < \cdots < j_r \leq n$. The sign $(-1)^{k-1}$ ensures that ∂_i^θ is a well-defined $\mathbb{Q}[\mathbf{x}_n]$-endomorphism of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$. For another characterization of these operators, see Remark 5.2. We begin with some basic identities satisfied by the ∂_i^θ.

Lemma 5.1. *Let $1 \leq i, j \leq n$. We have the following identities of operators on $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$.*

$$\partial_i \partial_j = \partial_j \partial_i, \quad \partial_i \partial_j^\theta = \partial_j^\theta \partial_i, \quad \partial_i^\theta \partial_j^\theta = -\partial_j^\theta \partial_i^\theta. \tag{5.2}$$

Furthermore, if $w \in \mathfrak{S}_n$ we have the operator identities

$$w \cdot \partial_i \cdot w^{-1} = \partial_{w(i)}, \quad w \cdot \partial_i^\theta \cdot w^{-1} = \partial_{w(i)}^\theta. \tag{5.3}$$

Finally, if $f, g \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ and f is concentrated in θ-degree r we have

$$\partial_i^\theta(fg) = \partial_i^\theta(f) g + (-1)^r f \partial_i^\theta(g). \tag{5.4}$$

Proof. We begin with (5.2). The first assertion is the standard commutativity of mixed partials. The second follows because ∂_i acts on x-variables and ∂_i^θ acts on θ-variables. The third can be verified directly on any θ-monomial $\theta_{k_1} \cdots \theta_{k_r}$ for $1 \leq k_1 < \cdots < k_r \leq n$. There are two cases depending on whether $i, j \in \{k_1, \ldots, k_r\}$; we leave the details to the reader.

We turn our attention to (5.3). The first assertion of (5.3) is (3.13). For the second assertion, it suffices to consider the case $w = (p, p+1)$ is an adjacent transposition in $\mathfrak{S}_n$ for some $1 \leq p \leq n-1$. Given $1 \leq k_1 < \cdots < k_r \leq n$, it is enough to show that

$$w \cdot \partial_i^\theta \cdot w^{-1}(\theta_{k_1} \cdots \theta_{k_r}) = \partial_{w(i)}^\theta(\theta_{k_1} \cdots \theta_{k_r}), \quad \text{where } w = (p, p+1). \tag{5.5}$$

We have

$$w(i) \in \{k_1, \ldots, k_r\} \text{ if and only if } i \in \{w^{-1}(k_1), \ldots w^{-1}(k_r)\}. \tag{5.6}$$

If (5.6) does not hold, then both sides of (5.5) equal 0, so assume (5.6) does hold. If $i \notin \{p, p+1\}$ then $w(i) = i$ and both sides of (5.5) equal $(-1)^{s-1}\theta_{k_1} \cdots \widehat{\theta_{k_s}} \cdots \theta_{k_r}$ where $i = k_s$. If $i = p$ then $w(i) = i+1$; we leave it for the reader to check that both sides of (5.5) equal $(-1)^{s-1}\theta_{k_1} \cdots \widehat{\theta_{k_s}} \cdots \theta_{k_r}$ where $i + 1 = k_s$ (there are two cases depending on whether $i \in \{k_1, \ldots, k_r\}$). The case $i = p + 1$ is similar to the case $i = p$ and left to the reader.

Since ∂_i^θ is a map of $\mathbb{Q}[\mathbf{x}_n]$-modules, (5.4) need only be verified in the case where $f = \theta_{k_1} \cdots \theta_{k_r}$ and $g = \theta_{\ell_1} \cdots \theta_{\ell_s}$ are monomials in the θ-variables and the indices appearing in f and g are distinct. One observes that (5.1) still applies when the indices of the θ's are merely distinct (as opposed to increasing). The proof is casework depending on whether θ_i appears in f, g, both, or neither; the details are left for the reader. ☐

Remark 5.2. *We will not use the Leibniz relation* (5.4) *in this paper, but the operator ∂_i^θ can be characterized as the unique $\mathbb{Q}$-linear endomorphism of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ which satisfies*

$$\partial_i^\theta(x_j) = 0 \quad \text{and} \quad \partial_i^\theta(\theta_j) = \delta_{i,j} \tag{5.7}$$

(where $1 \leq j \leq n$ and $\delta_{i,j}$ is the Kronecker delta) together with (5.4).

By (5.2), superspace $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ acts on itself by the rule

$$f \cdot g := \partial(f)(g) \quad \text{for } f, g \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n], \tag{5.8}$$

where $\partial(f)$ is obtained from f by replacing every x_i by ∂_i and every θ_i by ∂_i^θ. This extends the action of $\mathbb{Q}[\mathbf{x}_n]$ on superspace discussed earlier. We repeat Definition 1.4, which introduces the bigraded modules of study.

Definition 1.4. *Suppose $n = k + r$ for $k, r \geq 0$ and let $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$. Let $W_n(\mathbf{a})$ to be the smallest $\mathbb{Q}$-linear subspace of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ containing $\Delta_n(\mathbf{a})$ which is closed under the n partial derivative operators $\partial_1, \ldots, \partial_n$ as well as the n operators $\partial_1^\theta, \ldots, \partial_n^\theta$.*

Definition 1.4 can also be interpreted as saying that $W_n(\mathbf{a})$ is the cyclic $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$-submodule of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ generated by $\Delta_n(\mathbf{a})$. Since $\Delta_n(\mathbf{a})$ is alternating, the relations (5.3) imply that $W_n(\mathbf{a})$ is closed under the action of $\mathfrak{S}_n$. The $\mathfrak{S}_n$-module $W_n(\mathbf{a})$ is bigraded. We have $V_n(\mathbf{a}) \subseteq W_n(\mathbf{a})$; in fact, $V_n(\mathbf{a})$ is the θ-homogeneous piece of $W_n(\mathbf{a})$ of θ-degree r.

The spaces $W_n(\mathbf{a})$ have nicer algebraic properties than the $V_n(\mathbf{a})$. For example, we will see that $W_n(\mathbf{a})$ may be presented as a bigraded quotient of superspace. We repeat the relevant Definition 1.5.

Definition 1.5. *Suppose $n = k + r$ for $k, r \geq 0$ and let $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbb{Z}_{\geq 0})^r$. Let $I_n(\mathbf{a}) \subseteq \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ be the ideal $I_n(\mathbf{a}) := \{f \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n] \,:\, f \cdot \Delta_n(\mathbf{a}) = 0\}$ and let $R_n(\mathbf{a}) := \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]/I_n(\mathbf{a})$ be the corresponding quotient ring.*

The bigraded vector space $W_n(\mathbf{a})$ and the bigraded ring $R_n(\mathbf{a})$ posses a duality which is invisible at the level of $V_n(\mathbf{a})$. Establishing this duality – as well as the equivalence of $W_n(\mathbf{a})$ and $R_n(\mathbf{a})$ as doubly graded $\mathfrak{S}_n$-modules – is our next goal.

5.2. **A duality of $W_n(\mathbf{a})$.** To better understand the duality enjoyed by the $W_n(\mathbf{a})$, let us look at some examples of their bigraded Frobenius images. The symmetric function $\mathrm{grFrob}(W_3(1); q, z)$ is displayed in a matrix below, where the entry in row i and column j gives the coefficient of $z^i q^j$ in the Schur basis.

$$\begin{pmatrix} s_3 & s_3 + s_{21} & s_{21} \\ s_{21} & s_{21} + s_{111} & s_{111} \end{pmatrix}$$

The symmetric function $\mathrm{grFrob}(W_4(1,1); q, z)$ is similarly displayed below.

$$\begin{pmatrix} s_4 & s_4+s_{31} & s_4+s_{31}+s_{22} & s_{31} \\ s_{31} & 2s_{31}+s_{22}+s_{211} & s_{31}+s_{22}+2s_{211} & s_{211} \\ s_{211} & s_{22}+s_{211}+s_{1111} & s_{211}+s_{1111} & s_{1111} \end{pmatrix}$$

Finally, we display the symmetric function $\mathrm{grFrob}(W_4(2,1);q,z)$.

$$\begin{pmatrix} s_4 & s_4+s_{31} & s_4+2s_{31}+s_{22} & s_4+2s_{31}+s_{22}+s_{211} & s_{31}+s_{211} \\ s_4+s_{31} & s_4+3s_{31}+s_{22}+s_{211} & 3s_{31}+3s_{22}+3s_{211} & s_{31}+s_{22}+3s_{211}+s_{1111} & s_{211}+s_{1111} \\ s_{31}+s_{211} & s_{31}+s_{22}+2s_{211}+s_{1111} & s_{22}+2s_{211}+s_{1111} & s_{211}+s_{1111} & s_{1111} \end{pmatrix}$$

These tables have the property that if they are rotated 180°, the effect is the same as if the ω involution were applied to each entry. The goal of this section is to prove (Corollary 5.6) that this is a general phenomenon.

The action $f \cdot g := \partial(f)(g)$ of superspace on itself yields a bilinear form on $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$. More precisely, given $f, g \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ we define a rational number $\langle f, g\rangle \in \mathbb{Q}$ by

$$\langle f, g\rangle := \text{constant term of } f \cdot g = \text{constant term of } \partial(f)(g). \tag{5.9}$$

In particular, if f and g are homogeneous superpolynomials we have $\langle f, g\rangle = 0$ unless f and g have the same x-degree and the same θ-degree.

As an example of the form $\langle -, -\rangle$ we calculate $\langle x_1^3 x_2^2 \theta_1\theta_2, x_1^3 x_2^2\theta_1\theta_2\rangle = -3!2!$, where the minus sign comes from the action of the operator ∂_2^θ. In particular, the form $\langle -, -\rangle$ is not positive definite. However, the form $\langle -, -\rangle$ enjoys a graded version of positive definiteness.

Lemma 5.3. *The bilinear form $\langle -, -\rangle$ defined above is symmetric. Furthermore we have:*

(1) *Let $f \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ be a θ-homogeneous superpolynomial of θ-degree r. Then*

$$\begin{cases} \langle f, f\rangle \geq 0 & \text{if } r \equiv 0, 1 \pmod 4, \\ \langle f, f\rangle \leq 0 & \text{if } r \equiv 2, 3 \pmod 4, \end{cases}$$

with equality if and only if $f = 0$.

(2) *For any θ-homogeneous $f, g, f', g' \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ we have*

$$\langle f, (f'\cdot g')\cdot g\rangle = \pm\langle f\cdot g, f'\cdot g'\rangle.$$

where the sign is determined by the θ-degrees of f, g, f', g'.

The proof of Lemma 5.3 (2) is notationally cumbersome, but worth it. This fact will be crucial for our duality result (Theorem 5.5) below.

Proof. Let $m = x_1^{a_1}\cdots x_n^{a_n}\theta_{i_1}\cdots\theta_{i_r}$ and $m' = x_1^{a_1'}\cdots x_n^{a_n'}\theta_{i_1'}\cdots\theta_{i_s'}$ be two supermonomials (so that $i_1 < \cdots < i_r$ and $i_1' < \cdots < i_s'$). A direct computation gives

$$\langle m, m'\rangle = \begin{cases} (-1)^{\binom{r}{2}} a_1!\cdots a_n! & \text{if } m = m', \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

This shows that $\langle -, -\rangle$ is symmetric. The supermonomials are orthogonal with respect to $\langle -, -\rangle$. Since $\binom{r}{2}$ is even when $r \equiv 0, 1 \pmod 4$ and odd when $r \equiv 2, 3 \pmod 4$, Assertion (1) of the lemma is true.

Assertion (2) can be verified by direct computation when f, g, f', g' are supermonomials and applying the bilinearity of $\langle -, -\rangle$. A faster proof can be given as follows. The authors thank Josh Swanson for pointing out this argument.

Let $\epsilon \in \{1, -1\}$ be the sign determined by the equation

$$f(f'\cdot g') = \epsilon(f'\cdot g')f. \tag{5.11}$$

The sign ϵ only depends on the θ-degrees of f, f', and g. We calculate

$$f\cdot((f'\cdot g')\cdot g) = (f\cdot(f'\cdot g'))\cdot g = \epsilon((f'\cdot g')\cdot f)\cdot g = \epsilon(f'\cdot g')\cdot(f\cdot g). \tag{5.12}$$

Now we have

$$\langle f, ((f' \cdot g') \cdot g\rangle = \text{constant coefficient of } f \cdot ((f' \cdot g') \cdot g \tag{5.13}$$
$$= \text{constant coefficient of } \epsilon(f' \cdot g') \cdot (f \cdot g) \tag{5.14}$$
$$= \epsilon\langle f' \cdot g', f \cdot g\rangle \tag{5.15}$$
$$= \epsilon\langle f \cdot g, f' \cdot g'\rangle, \tag{5.16}$$

where the last equality used the fact that $\langle -, - \rangle$ is symmetric. ☐

For $0 \leq j \leq n$, let $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j$ be the (infinite-dimensional) subspace of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ consisting of superpolynomials which are θ-homogeneous of θ-degree j. We have a direct sum decomposition

$$\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n] = \bigoplus_{j=0}^{n} \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j. \tag{5.17}$$

The bilinear form $\langle -, - \rangle$ on $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ may be restricted to $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j$ for any j. Lemma 5.3 (1) states that this restriction will be positive definite if $\binom{j}{2}$ is even and negative definite if $\binom{j}{2}$ is odd.

Lemma 5.4. *Suppose $V \subseteq \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j$ is a linear subspace (for some fixed $0 \leq j \leq n$) which is x-homogeneous (i.e. for any $f \in V$, all of the x-homogeneous components of f are contained in V). Define a new subspace $V^\perp \subseteq \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j$ by*

$$V^\perp = \{f \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j \,:\, \langle f, g \rangle = 0 \textit{ for all } g \in V\}. \tag{5.18}$$

We have the direct sum of vector spaces $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j = V \oplus V^\perp$.

Proof. For $j \geq 0$, let $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_{i,j}$ be the finite-dimensional subspace of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_i$ which is x-homogeneous of x-degree i and let $V_i := V \cap \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_{i,j}$. By the assumption on V we have $V = \bigoplus_{i \geq 0} V_i$. If we set

$$V_i^\perp = \{f \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_{i,j} \,:\, \langle f, g \rangle = 0 \text{ for all } g \in V_i\}, \tag{5.19}$$

Lemma 5.3 (1) implies $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_{i,j} = V_i \oplus V_i^\perp$. Taking a direct sum over $j \geq 0$ gives the result. ☐

Informally, Lemma 5.4 says that we can take orthogonal complements in superspace as long as the subspaces in question are concentrated in a single θ-degree. This will play a key role in proving the following result.

Theorem 5.5. *Let $0 \leq r \leq n$ and let $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{Z}_{\geq 0}^r$. Define a map*

$$\iota : W_n(\mathbf{a}) \to W_n(\mathbf{a}) \tag{5.20}$$

by the rule $\iota(f) := f \cdot \Delta_n(\mathbf{a}) = \partial(f)(\Delta_n(\mathbf{a}))$.

The map ι is a linear automorphism of $W_n(\mathbf{a})$ which complements both x-degree and θ-degree simultaneously. Furthermore, the map ι satisfies

$$\iota(w \cdot f) = \mathrm{sign}(w) w \cdot \iota(f) \quad \textit{for all } w \in \mathfrak{S}_n \textit{ and } f \in W_n(\mathbf{a}). \tag{5.21}$$

Proof. The remaining statements of the theorem will follow immediately if we can show that ι is bijective. Since ι is $\mathbb{Q}$-linear and $W_n(\mathbf{a})$ is finite-dimensional, it is enough to check that ι is surjective.

For $0 \leq j \leq r$, let $W_n(\mathbf{a})_j$ be the θ-homogeneous piece of $W_n(\mathbf{a})$ of θ-degree j. Similarly, let $I_n(\mathbf{a})_j$ for the θ-homogeneous piece of $I_n(\mathbf{a})$ of θ-degree j. We have the direct sum decompositions

$$W_n(\mathbf{a}) = \bigoplus_{j=0}^{r} W_n(\mathbf{a})_j \quad \text{and} \quad I_n(\mathbf{a}) = \bigoplus_{j=0}^{r} I_n(\mathbf{a})_j. \tag{5.22}$$

Fix $0 \leq j \leq r$. Let $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j$ be the space of superpolynomials which are θ-homogeneous of θ-degree j. Consider the orthogonal complement

$$W_n(\mathbf{a})_j^{\perp} = \{f \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j \,:\, \langle f, g \rangle = 0 \text{ for all } g \in W_n(\mathbf{a})_j\}. \tag{5.23}$$

Claim: *We have* $I_n(\mathbf{a})_j = W_n(\mathbf{a})_j^{\perp}$.

To prove the Claim, we consider both containments separately. For the containment $\subseteq$, let $f \in I_n(\mathbf{a})_j$ and $g \in W_n(\mathbf{a})_j$. There exists $h \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ such that $g = h \cdot \Delta_n(\mathbf{a})$. After discarding redundant terms if necessary, we may assume that h is θ-homogeneous. We have

$$f \cdot g = f \cdot (h \cdot \Delta_n(\mathbf{a})) = \partial(fh)(\Delta_n(\mathbf{a})) = \pm\partial(hf)(\Delta_n(\mathbf{a})) = \pm\partial(h)(f \cdot \Delta_n(\mathbf{a})) = 0, \tag{5.24}$$

where the third equality used the θ-homogeneity of f and h. Taking the constant term gives $\langle f, g \rangle = 0$ so that $f \in W_n(\mathbf{a})_j^{\perp}$.

Now we prove the containment $\supseteq$. Let $f \in W_n(\mathbf{a})_j^{\perp}$. We want to show that $f \cdot \Delta_n(\mathbf{a}) = 0$. Since both f and $\Delta_n(\mathbf{a})$ are θ-homogeneous, Lemma 5.3 (1) implies that

$$f \cdot \Delta_n(\mathbf{a}) = 0 \text{ if and only if } \langle f \cdot \Delta_n(\mathbf{a}), f \cdot \Delta_n(\mathbf{a}) \rangle = 0 \tag{5.25}$$

On the other hand, Lemma 5.3 (2) implies that

$$\langle f \cdot \Delta_n(\mathbf{a}), f \cdot \Delta_n(\mathbf{a}) \rangle = \pm\langle f, (f \cdot \Delta_n(\mathbf{a})) \cdot \Delta_n(\mathbf{a}) \rangle = 0, \tag{5.26}$$

where the second equality used $f \in W_n(\mathbf{a})_j^{\perp}$. This completes the proof of the Claim.

We proceed to prove that the map ι is surjective, which will prove the Theorem. Fix $0 \leq j \leq r$. We prove that $W_n(\mathbf{a})_{r-j}$ is contained in the image $\mathrm{Image}(\iota)$ of ι. By the definition of $W_n(\mathbf{a})$, we have

$$W_n(\mathbf{a})_{r-j} = \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j \cdot \Delta_n(\mathbf{a}) = \{f \cdot \Delta_n(\mathbf{a}) \,:\, f \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j\}. \tag{5.27}$$

The desired containment $W_n(\mathbf{a})_{r-j} \subseteq \mathrm{Image}(\iota)$ is equivalent to the seemingly stronger statement

$$W_n(\mathbf{a})_{r-j} = W_n(\mathbf{a})_j \cdot \Delta_n(\mathbf{a}) = \{f \cdot \Delta_n(\mathbf{a}) \,:\, f \in W_n(\mathbf{a})_j\}. \tag{5.28}$$

However, Lemma 5.4 and our Claim give the direct sum decomposition

$$\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_j = W_n(\mathbf{a})_j \oplus W_n(\mathbf{a})_j^{\perp} = W_n(\mathbf{a})_j \oplus I_n(\mathbf{a})_j. \tag{5.29}$$

Since $I_n(\mathbf{a})_j$ annihilates $\Delta_n(\mathbf{a})$, (5.27) and (5.28) are equivalent. □

The map ι gives our desired duality immediately.

Corollary 5.6. *Let* $0 \leq r \leq n$ *and let* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^r$. *The module* $W_n(\mathbf{a})$ *has the duality property*

$$\omega(\mathrm{grFrob}(W_n(\mathbf{a}); q, z)) = (\mathrm{rev}_q \circ \mathrm{rev}_z)\mathrm{grFrob}(W_n(\mathbf{a}); q, z). \tag{5.30}$$

Proof. The automorphism ι of Theorem 5.5 reverses both x-degree and θ-degree, as well as twisting by the sign representation. □

In Corollary 5.6 the operator rev_q acts on formal power series in $\mathbb{Q}[[q, z, x_1, x_2, \ldots]]$ with finite q-degree by regarding them as polynomials in $\mathbb{Q}[[z, x_1, x_2, \ldots]][q]$. A similar remark applies to rev_z. Corollary 5.6 immediately shows that $W_n(\mathbf{a})$ and $R_n(\mathbf{a})$ are isomorphic as bigraded $\mathfrak{S}_n$-modules.

Corollary 5.7. *For any* $0 \leq r \leq n$ *and any sequence* $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$ *we have*

$$\mathrm{grFrob}(R_n(\mathbf{a}); q, z) = \mathrm{grFrob}(W_n(\mathbf{a}); q, z) \tag{5.31}$$

where q *tracks* x*-degree and* z *tracks* θ*-degree.*

Proof. The same argument used to prove Proposition 3.1 shows that

$$\mathrm{grFrob}(\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]/I_n(\mathbf{a}); q, z) = (\mathrm{rev}_q \circ \mathrm{rev}_z \circ \omega)\mathrm{grFrob}(W_n(\mathbf{a}); q, z). \tag{5.32}$$

By Corollary 5.6 the operator $(\mathrm{rev}_q \circ \mathrm{rev}_z \circ \omega)$ leaves $\mathrm{grFrob}(W_n(\mathbf{a}); q, z)$ unchanged. □

Our duality gives us another model for the quotient rings $R_{n,k}$ of [11].

Corollary 5.8. *Let $r, k \geq 0$ with $n = k + r$. Let $\mathbf{a} = (k-1, \dots, k-1) \in (\mathbb{Z}_{\geq 0})^n$ be a length r sequence of $(k-1)$'s. Let $W_n(\mathbf{a})_0$ be the subspace of $W_n(\mathbf{a})$ of θ-degree zero. Then $W_n(\mathbf{a})_0 \cong R_{n,k}$ as singly graded $\mathfrak{S}_n$-modules.*

Proof. Combine Theorem 3.6 and Corollary 5.6. □

5.3. **Poincaré duality.** In this subsection we prove that the bigraded rings $R_n(\mathbf{a})$ exhibit an algebraic structure which is reminiscent of Poincaré duality and propose the problem of finding a geometric explanation for this fact.

Consider Corollaries 5.6 and 5.7 in the case $r = 0$, so that $\mathbf{a} = \varnothing$ is the empty sequence and $\Delta_n(\mathbf{a}) = \Delta_n$ is the classical Vandermonde. In this case $R_{n,k} = R_n$ is the classical coinvariant algebra and these corollaries give the classical result

$$\mathrm{grFrob}(R_n; q) = (\mathrm{rev}_q \circ \omega)\mathrm{grFrob}(R_n; q) \tag{5.33}$$

which implies the palindromicity of the Hilbert series

$$\mathrm{Hilb}(R_n; q) = [n]!_q = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}). \tag{5.34}$$

The palindromicity of $\mathrm{Hilb}(R_n; q) = [n]!_q$ has a geometric interpretation. Let $\mathcal{F}\ell_n$ be the variety of complete flags in $\mathbb{C}^n$. Borel [4] proved that the rational cohomology ring $H^\bullet(\mathcal{F}\ell_n)$ can be presented as the coinvariant ring R_n. Since $\mathcal{F}\ell_n$ is a smooth compact complex projective variety, the top cohomology

$$H^{\mathrm{top}}(\mathcal{F}\ell_n) = H^{n(n-1)}(\mathcal{F}\ell_n) \cong \mathbb{Q} \tag{5.35}$$

is a 1-dimensional vector space and for any $0 \leq d \leq n(n-1)$ the cup product

$$H^d(\mathcal{F}\ell_n) \otimes H^{n(n-1)-d}(\mathcal{F}\ell_n) \to \mathbb{Q} \tag{5.36}$$

is a perfect pairing by Poincaré duality.

The variety

$$X_{n,k} = \{(\ell_1, \dots, \ell_n) \,:\, \ell_i \subseteq \mathbb{C}^k,\, \dim(\ell_i) = 1,\, \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}$$

of spanning configurations of n lines in $\mathbb{C}^k$ introduced in [14] is not compact, and so does not satisfy the hypotheses of Poincaré duality. Indeed, the Hilbert series of its cohomology

$$\mathrm{Hilb}(H^\bullet(X_{n,k}); \sqrt{q}) = \mathrm{Hilb}(R_{n,k}; q) = \mathrm{rev}_q([k]! \cdot \mathrm{Stir}_q(n,k)) \tag{5.37}$$

is not palindromic. However, $H^\bullet(X_{n,k}) = R_{n,k}$ is a 1-dimensional slice of a 2-dimensional self-dual object.

Corollary 5.9. *Let $0 \leq r \leq n$ and let $\mathbf{a} = (a_1, \dots, a_r) \in (\mathbb{Z})^r_{\geq 0}$. The top x-degree component of $R_n(\mathbf{a})$ is $s := a_1 + \cdots + a_r + \binom{k}{2}$, and the top θ-degree component is r. Write $R_n(\mathbf{a})_{i,j}$ for the component of $R_n(\mathbf{a})$ of x-degree i and θ-degree j. The component $R_n(\mathbf{a})_{r,s} \cong \mathbb{Q}$ is a 1-dimensional vector space.*

For any $0 \leq i \leq s$ and any $0 \leq j \leq r$ the multiplication map

$$R_n(\mathbf{a})_{i,j} \otimes R_n(\mathbf{a})_{s-i,r-j} \to \mathbb{Q} \tag{5.38}$$

is a perfect pairing.

Proof. Let $\{f_1, \dots, f_m\}$ be a basis for $W_n(\mathbf{a})_{i,j}$ and let $\{g_1, \dots, g_m\}$ be a basis for $W_n(\mathbf{a})_{s-i,r-j}$ (by Theorem 5.5, these bases have the same size m). The direct sum decomposition (5.29) in the proof of Theorem 5.5 guarantees that $\{f_1, \dots, f_m\}$ descends to a basis of $R_n(\mathbf{a})_{i,j}$ and $\{g_1, \dots, g_m\}$ descends to a basis of $R_n(\mathbf{a})_{s-i,r-j}$.

Let $A = (a_{p,q})_{1 \leq p,q \leq m}$ be the $m \times m$ rational matrix whose entries are

$$a_{p,q} := (f_p g_q) \cdot \Delta_n(\mathbf{a}). \tag{5.39}$$

It is enough to verify that A is nonsingular. By Theorem 5.5, the set $\{g_1 \cdot \Delta_n(\mathbf{a}), \dots, g_m \cdot \Delta_n(\mathbf{a})\}$ descends to a basis of $R_n(\mathbf{a})_{i,j}$. The matrix element $a_{p,q}$ is equal to $\langle f_p, g_q \cdot \Delta_n(\mathbf{a}) \rangle$, so that A is the Gram matrix of a bilinear form on $R_n(\mathbf{a})_{i,j}$ which (by Lemma 5.3 (1)) is either positive definite or negative definite, and hence nonsingular. □

Problem 5.10. *Find a geometric enhancement of $X_{n,k}$ which explains Corollary 5.9.*

5.4. **Superspace coinvariants.** It is well-known that the ring $\mathbb{Q}[\mathbf{x}_n]^{\mathfrak{S}_n}$ of symmetric polynomials has algebraically independent generators given by the set $\{e_1, e_2, \dots, e_n\}$ of symmetric polynomials. In general, a set of n algebraically independent symmetric polynomials $\{f_1, f_2, \dots, f_n\}$ is a *fundamental system of invariants* if $\mathbb{Q}[\mathbf{x}_n]^{\mathfrak{S}_n} = \mathbb{Q}[f_1, f_2, \dots, f_n]$. A fundamental system of invariants other than $\{e_1, e_2, \dots, e_n\}$ is the set of *power sums* $p_1, p_2, \dots, p_n$ where $p_d = p_d(\mathbf{x}_n) := x_1^d + \cdots + x_n^d$.

Recall that $\mathfrak{S}_n$ acts diagonally on superspace $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$. The collection of invariant superpolynomials $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]^{\mathfrak{S}_n}$ forms a subalgebra of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$. Given a fundamental system of invariants $\{f_1, f_2, \dots, f_n\} \subseteq \mathbb{Q}[\mathbf{x}_n]$, Solomon described a generating set of $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]^{\mathfrak{S}_n}$. The *differential* map $d : \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n] \to \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ is defined by

$$df := \sum_{i=1}^{n} \partial_i f \times \theta_i, \quad \text{for } f \in \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]. \tag{5.40}$$

Solomon proved [19] that $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]^{\mathfrak{S}_n}$ is generated as a $\mathbb{Q}$-algebra by $\{f_1, f_2, \dots, f_n, df_1, df_2, \dots, df_n\}$. Extensions of various symmetric polynomial bases to superspace were studied in [7].

Let $\mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n}$ denote the space of $\mathfrak{S}_n$-invariant superpolynomials with vanishing constant term. The *superinvariant ideal* is the ideal

$$SI_n := \langle \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n} \rangle = \langle e_1, e_2, \dots, e_n, dp_1, dp_2, \dots, dp_n \rangle, \tag{5.41}$$

where the second equality is justified by Solomon's result [19] and the equality $\mathbb{Q}[e_1, \dots, e_n] = \mathbb{Q}[p_1, \dots, p_n]$. The *supercoinvariant algebra* is the quotient

$$SR_n := \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n] / \langle \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n} \rangle. \tag{5.42}$$

The ring SR_n is a bigraded $\mathfrak{S}_n$-module.

The following conjectural expression for the bigraded Frobenius image of SR_n was obtained in the algebraic combinatorics seminar at the Fields Institute; it is a special case of the conjecture of Mike Zabrocki in [22]:

$$\mathrm{grFrob}(SR_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n \mid_{t=0} \tag{5.43}$$

where q tracks x-degree and z tracks θ-degree. By Theorem 3.6, Equation (5.43) is equivalent to the statement that the θ-homogeneous piece of SR_n of θ-degree $n-k$ is isomorphic to $V_n(\mathbf{a})$ as a (singly) graded $\mathfrak{S}_n$-module, where $\mathbf{a} = (k-1, \dots, k-1)$ is a length $n-k$ sequence of $(k-1)$'s. By Theorem 3.6 we have the following potential road to proving Equation (5.43).

Conjecture 5.11. *Let $1 \leq k \leq n$ and let $\mathbf{a} = (k-1, \dots, k-1)$ be a length $n-k$ sequence of $(k-1)$'s. The canonical projection from $V_n(\mathbf{a}) \subseteq \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]$ to the θ-homogeneous piece of SR_n of θ-degree $n-k$ is an isomorphism of vector spaces.*

Proposition 5.12. *Conjecture 5.11 implies Equation* (5.43)*.*

We have been unable to use Vandermondes to analyze the supercoinvariant algebra SR_n directly, but we have the following result describing elements of the annihilator of $\Delta_n(\mathbf{a})$ in terms of the superinvariant ideal SI_n in the case where $\mathbf{a} = (k-1, \dots, k-1)$ is a constant sequence of $(k-1)$'s of length $n-k$.

Proposition 5.13. *Let $0 \leq r \leq n$ and let $k = n - r$. Let $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$ be the constant sequence $\mathbf{a} = (k-1, k-1, \dots, k-1)$ of length r. Each of the superpolynomials*

$$x_1^k, x_2^k, \dots, x_n^k, \quad e_n, e_{n-1}, \dots, e_{n-k+1}, \quad dp_1, dp_2, \dots, dp_n \tag{5.44}$$

annihilates $\Delta_n(\mathbf{a})$.

The ideal generated by the superpolynomials appearing in Proposition 5.13 has generators similar to the superinvariant ideal $SI_n = \langle e_1, \dots, e_n, dp_1, \dots, dp_n \rangle$, but without the low degree elementary symmetric polynomials $e_1, e_2, \dots, e_{n-k}$ and with the variable powers $x_1^k, x_2^k, \dots, x_n^k$. Indeed, these ideals are not equal. Despite this, we hope that the similarity between these ideals will assist in the proof of Zabrocki's conjecture (5.43).

Proof. Lemma 3.4 implies $x_1^k, \dots, x_n^k, e_n, e_{n-1}, \dots, e_{n-k+1} \in \mathrm{ann}_{\mathbb{Q}[\mathbf{x}_n]} \Delta_n(\mathbf{a})$.

Let $1 \leq j \leq n$. By (5.3) dp_j commutes with the action of $\mathfrak{S}_n$, and hence the action of ε_n. It follows that

$$dp_j \cdot \Delta_n(\mathbf{a}) = \varepsilon_n \cdot dp_j \cdot \left[x_1^{k-1} \cdots x_r^{k-1} x_{r+1}^{k-1} \cdots x_{n-1}^1 x_n^0 \cdot \theta_1 \theta_2 \cdots \theta_r \right]. \tag{5.45}$$

A direct computation gives $dp_j \cdot \left[x_1^{k-1} \cdots x_r^{k-1} x_{r+1}^{k-1} \cdots x_{n-1}^1 x_n^0 \cdot \theta_1 \theta_2 \cdots \theta_r \right] = 0$ if $j > k$ and

$$dp_j \cdot \left[x_1^{k-1} \cdots x_r^{k-1} x_{r+1}^{k-1} \cdots x_n^0 \cdot \theta_1 \cdots \theta_r \right] \doteq \sum_{i=1}^{r} (-1)^{i-1} x_1^{k-1} \cdots x_i^{k-j} \cdots x_r^{k-1} x_{r+1}^{k-1} \cdots x_n^0 \theta_1 \cdots \widehat{\theta_i} \cdots \theta_r \tag{5.46}$$

if $j \leq k$. In term i in the sum on the right-hand-side of Equation (5.46), the exponents of x_i and x_{n-k+j} coincide. Since neither θ_i nor θ_{n-k+j} appear in this term, this term is annihilated by the application of ε_n. We conclude that Equation (5.46) itself is annihilated by ε_n, so that Equation (5.45) equals 0. $\square$

6. Conclusion

6.1. A conjecture on Tanisaki quotients.

In this paper we defined a graded $\mathfrak{S}_n$-module $V_n(\mathbf{a})$ for any nonnegative integer sequence $\mathbf{a}$ of length $\leq n$. Theorem 3.6, Theorem 4.2, and Proposition 4.3 calculate the graded module structure of $V_n(\mathbf{a})$ for certain constant sequences $\mathbf{a}$. It is natural to ask what $V_n(\mathbf{a})$ looks like for general sequences $\mathbf{a}$. While we do not have a full conjecture in this direction, computational evidence suggests a relationship between $V_n(\mathbf{a})$ and the Tanisaki quotients R_λ.

More precisely, let $\leq$ be the componentwise partial order on length r sequences of nonnegative integers. Given a length r sequence $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$ and $n \geq r$, define the graded $\mathfrak{S}_n$-modules

$$V_n^{\leq}(\mathbf{a}) := \sum_{\mathbf{b} \leq \mathbf{a}} V_n(\mathbf{b}), \quad V_n^{<}(\mathbf{a}) := \sum_{\mathbf{b} < \mathbf{a}} V_n(\mathbf{b}), \quad \text{and } V_n^{=}(\mathbf{a}) := V_n^{\leq}(\mathbf{a}) / V_n^{<}(\mathbf{a}). \tag{6.1}$$

It can be checked that

$$\begin{cases} (\mathrm{rev}_q \circ \omega) \mathrm{grFrob}(V_4^{=}(0,0); q) = \mathrm{grFrob}(R_{(3,1)}; q), \\ (\mathrm{rev}_q \circ \omega) \mathrm{grFrob}(V_4^{=}(1,0); q) = \mathrm{grFrob}(R_{(3,1)}; q), \text{ and} \\ (\mathrm{rev}_q \circ \omega) \mathrm{grFrob}(V_4^{=}(1,1); q) = \mathrm{grFrob}(R_{(2,2)}; q). \end{cases}$$

Conjecture 6.1. *Let $r \leq n$ be nonnegative integers with $k = n - r$ and let $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$. There exists a partition $\lambda \vdash n$ with k parts such that*

$$(\mathrm{rev}_q \circ \omega) \mathrm{grFrob}(V_n^{=}(\mathbf{a}); q) = \mathrm{grFrob}(R_\lambda; q). \tag{6.2}$$

Equivalently, if $Q'_\lambda(X; q)$ is the Hall-Littlewood Q'-function, we have

$$\omega[\mathrm{grFrob}(V_n^{=}(\mathbf{a}); q)] \propto Q'_\lambda(X; q), \tag{6.3}$$

where $\propto$ denotes equality up to a power of q.

Proposition 4.3 proves Conjecture 6.1 when $\mathbf{a}$ is a zero sequence and $\lambda = (r+1, 1, \ldots, 1) \vdash n$. By Proposition 3.1 and [11, Thm. 6.14] if $\mathbf{a} = (k-1, \ldots, k-1)$ is a length r sequence of $(k-1)$'s we have

$$\operatorname{grFrob}(V_n(\mathbf{a}); q) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = k}} q^{\sum (i-1)(\lambda_i - 1)} \begin{bmatrix} k \\ m_1(\lambda), \ldots, m_n(\lambda) \end{bmatrix}_q \omega Q'_\lambda(X; q). \tag{6.4}$$

Conjecture 6.1 can be thought of as giving a filtration on $V_n(\mathbf{a})$ which is compatible with Equation (6.4). We do not have a conjecture for how to produce λ from $\mathbf{a}$ in general. It should be noted that N. Bergeron and A. Garsia obtained a model for R_λ as a space of harmonics inside $\mathbb{Q}[\mathbf{x}_n]$ [3].

The generalized coinvariant ring $R_{n,k}$ of [11] and the positroid quotient S_n of [5] have graded Frobenius images which are (up to q-reversal) positive in the Q'-basis of symmetric functions. In [17] the authors defined a quotient of the polynomial ring $\mathbb{Q}[\mathbf{x}_n]$ corresponding to hook Schur-delta operator images $\Delta_{s_{(r,1^{n-1})}} e_n \mid_{t=0}$ whose graded Frobenius image is also (up to q-reversal) Q'-positive. Haglund, Rhoades, and Shimozono [12] gave a manifestly positive Q'-expansion of $\Delta'_{s_\lambda} e_n \mid_{t=0}$, where s_λ is any Schur function (up to ω). It may be interesting to use superspace to build modules for the symmetric functions appearing in [12] and [17].

6.2. **Additional sets of variables, $\Delta'_{e_{k-1}} e_n$, and beyond.** Zabrocki [22] conjectured an extension of (5.43) to more sets of variables. Let $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n]$ be the $\mathbb{Q}$-algebra with $2n$ commuting variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ and n anticommuting variables $\theta_1, \ldots, \theta_n$ (where any two variables of different species commute). Zabrocki [22] verified that

$$\operatorname{grFrob}(\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n] / \langle \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n} \rangle; q, t, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n \tag{6.5}$$

for $n \leq 6$. Here q tracks x-degree, t tracks y-degree, and z tracks θ-degree. Thus, the quotient $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n] / \langle \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n} \rangle$ gives a conjectural representation theoretic model for $\Delta'_{e_{k-1}} e_n$ in antisymmetric degree $n-k$.

Superspace Vandermondes can be used to give another conjectural representation theoretic model for $\Delta'_{e_{k-1}} e_n$. To describe this, we need the *polarization operators* on $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n]$. For $j \geq 1$, the j^{th} *polarization operator (from the x-variables to the y-variables)* on $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n]$ is the operator

$$p^{(j)}_{x \to y} := y_1 \partial^j_{x_1} + y_2 \partial^j_{x_2} + \cdots + y_n \partial^j_{x_n}. \tag{6.6}$$

Definition 6.2. *Let $n = k + r$ and let $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbb{Z}_{\geq 0})^r$. Let $\mathcal{V}_n(\mathbf{a})$ be the smallest subspace of $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n]$ such that*

- *$\mathcal{V}_n(\mathbf{a})$ contains the $\mathbf{a}$-superspace Vandermonde*

$$\Delta_n(\mathbf{a}) = \varepsilon_n \cdot (x_1^{a_1} \cdots x_r^{a_r} x_{r+1}^{k-1} \cdots x_{n-1}^1 x_n^0 \theta_1 \cdots \theta_r)$$

  in the x-variables and θ-variables,
- *$\mathcal{V}_n(\mathbf{a})$ is closed under all partial derivatives ∂_{x_i} and ∂_{y_i} in commuting variables, and*
- *$\mathcal{V}_n(\mathbf{a})$ is closed under all polarization operators $p^{(j)}_{x \to y}$ for $j \geq 1$.*

The space $\mathcal{V}_n(\mathbf{a})$ has fixed θ-degree r. By considering the x-degree and y-degree separately, we view $\mathcal{V}_n(\mathbf{a})$ as a doubly graded $\mathfrak{S}_n$-module. The space $\mathcal{V}_n(\mathbf{a})$ specializes to $V_n(\mathbf{a})$ when the y-variables are set to zero.

Conjecture 6.3. *Let $n = k + r$ and let $\mathbf{a} = (k-1, k-1, \ldots, k-1)$ be a length r sequence of $(k-1)$'s. Then*

$$\operatorname{grFrob}(\mathcal{V}_n(\mathbf{a}); q, t) = \Delta'_{e_{k-1}} e_n. \tag{6.7}$$

Conjecture 6.3 has been checked by computer for $n \leq 4$. Theorem 3.6 proves Conjecture 6.3 in the case $t = 0$; Zabrocki's conjecture (6.5) is open even in the case $t = 0$. Just as we hope that $V_n(\mathbf{a})$ will lead to a better understanding of the supercoinvariant algebra $SR_n = \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]/\langle \mathbb{Q}[\mathbf{x}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n} \rangle$, we hope that $\mathcal{V}_n(\mathbf{a})$ will help in understanding the quotient $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n]/\langle \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n} \rangle$ appearing in (6.5).

Given any vector $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbb{Z}_{\geq 0})^r$, we have a symmetric function $\mathrm{grFrob}(\mathcal{V}_n(\mathbf{a}); q, t)$. It might be interesting to study the combinatorics of these symmetric functions when $\mathbf{a}$ is a vector other than $(k-1, \ldots, k-1)$.

There has been a significant amount of interest in extensions of the diagonal coinvariant ring to > 2 species of n commuting variables (see [2]). Let us remark that we may extend our modules $\mathcal{V}_n(\mathbf{a})$ to any number of species of commuting and skew-commuting variables.

Let $S(n, c, s)$ be the $\mathbb{Q}$-algebra generated by $c \times n$ **c**ommuting variables

$$x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}, \quad x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}, \quad \ldots \quad , \text{ and } x_1^{(c)}, x_2^{(c)}, \ldots, x_n^{(c)},$$

and $s \times n$ **s**kew-commuting variables

$$\theta_1^{(1)}, \theta_2^{(1)}, \ldots, \theta_n^{(1)}, \quad \theta_1^{(2)}, \theta_2^{(2)}, \ldots, \theta_n^{(2)}, \quad \ldots \quad , \text{ and } \theta_1^{(s)}, \theta_2^{(s)}, \ldots, \theta_n^{(s)},$$

where any $x_i^{(j)}$ and $\theta_{i'}^{(j')}$ commute. The ring $S(n, c, s)$ is a multigraded $\mathbb{Q}$-algebra with c kinds of commutative grading and s kinds of skew-commutative grading.

The ring $S(n, c, s)$ carries a 'diagonal' action of $\mathfrak{S}_n$ by simultaneous subscript permutation. We may also act on $S(n, c, s)$ by any partial derivative $\partial/\partial x_i^{(j)}$ or $\partial/\partial \theta_i^{(j)}$ with respect to any commuting or skew-commuting variable. We may also polarize between any two species of commutating variables $1 \leq i, i' \leq c$ (at polarization parameter j) by the operator

$$p_{i \to i'}^{(j)} := x_1^{(i')} (\partial/\partial x_1^{(i)})^j + x_2^{(i')} (\partial/\partial x_2^{(i)})^j + \cdots + x_n^{(i')} (\partial/\partial x_n^{(i)})^j \tag{6.8}$$

and between any two species of skew-commuting variables $1 \leq i, i' \leq s$ by the operator

$$q_{i \to i'}^{(j)} := \theta_1^{(i')} (\partial/\partial \theta_1^{(i)}) + \theta_2^{(i')} (\partial/\partial \theta_2^{(i)}) + \cdots + \theta_n^{(i')} (\partial/\partial \theta_n^{(i)}). \tag{6.9}$$

Given $r \leq n$ and sequence $\mathbf{a} = (a_1, \ldots, a_r) \in (\mathbb{Z}_{\geq 0})^r$, we may define $\mathbb{V}_n(\mathbf{a}, c, s)$ to be the smallest $\mathbb{Q}$-linear subspace of $S(n, c, s)$ containing the $\mathbf{a}$-superspace Vandermonde $\Delta_n(\mathbf{a})$ in the $x^{(1)}$ and $\theta^{(1)}$-variables which is closed under all possible partial differentiation and polarization operators. We have $\mathbb{V}_n(\mathbf{a}, 1, 1) = V_n(\mathbf{a})$ and $\mathbb{V}_n(\mathbf{a}, 2, 1) = \mathcal{V}_n(\mathbf{a})$.

The vector space $\mathbb{V}_n(\mathbf{a}, c, s)$ is a multigraded $\mathfrak{S}_n$-module. Its isomorphism type is encoded in a symmetric function

$$\mathrm{grFrob}(\mathbb{V}_n(\mathbf{a}, c, s); q_1, q_2, \ldots, q_c, z_1, z_2, \ldots, z_s) \tag{6.10}$$

Following the work of F. Bergeron [1], it may be interesting to study this symmetric function as $c, s \to \infty$.

6.3. **A conjectural Lefschetz property of $W_n(\mathbf{a})$.** For $r \leq n$ and $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$, we defined a doubly graded $\mathfrak{S}_n$-module $W_n(\mathbf{a})$.

Problem 6.4. *Find the doubly graded Frobenius image* $\mathrm{grFrob}(W_n(\mathbf{a}); q, z)$.

The z^0-coefficient of $\mathrm{grFrob}(W_n(\mathbf{a}); q, z)$ gives the graded isomorphism type of $R_{n,k}$. The z^r-coefficient of $\mathrm{grFrob}(W_n(\mathbf{a}); q, z)$ is the reversed and sign-twisted version of the z^0-coefficient. The authors do not have a conjecture for the intermediate powers of z. Indeed, we do not even know the vector space dimension $\dim W_n(\mathbf{a})$.

Let $r \leq n$ and $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$. We close with a conjecture on the bigraded Hilbert series

$$\mathrm{Hilb}(R_n(\mathbf{a}); q, z) := \sum_{i,j} \dim R_n(\mathbf{a})_{i,j} \cdot q^i z^j \tag{6.11}$$

of the doubly graded ring $R_n(\mathbf{a})$. Here $R_n(\mathbf{a})_{i,j}$ is the homogeneous piece of $R_n(\mathbf{a})$ of x-degree i and θ-degree j. By Proposition 5.7 the polynomial (6.11) is unchanged if we replace $R_n(\mathbf{a})$ by the doubly graded vector space $W_n(\mathbf{a})$.

We may display the bivariate polynomial (6.11) as a matrix of coefficients. The case $n = 5, \mathbf{a} = (2, 2)$ is shown below, with column indices recording x-degree and row indices recording θ-degree.

$$\begin{pmatrix} 1 & 5 & 15 & 29 & 39 & 35 & 20 & 6 \\ 4 & 19 & 50 & 77 & 77 & 50 & 19 & 4 \\ 6 & 20 & 35 & 39 & 29 & 15 & 5 & 1 \end{pmatrix}$$

As guaranteed by Theorem 5.5, this matrix is symmetric under 180° rotation. Recall that an integer sequence $(c_1, c_2, \dots, c_m)$ is *unimodal* if there is some i with $c_1 \leq \cdots \leq c_i \geq c_{i+1} \geq \cdots \geq c_m$.

Conjecture 6.5. *For any $r \leq n$ and $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^r$, the matrix of coefficients of* $\mathrm{Hilb}(R_n(\mathbf{a}); q, z)$ *has unimodal rows and columns.*

When $\mathbf{a} = \varnothing$, Conjecture 6.5 follows from the Hard Lefschetz Theorem. In this case, we have the geometric interpretation $R_n(\varnothing) = R_n = H^\bullet(\mathcal{F}\ell_n)$ of $R_n(\varnothing)$ as the (singly-graded) cohomology of the Kähler manifold $\mathcal{F}\ell_n$. Recall that $n(n-1)$ is the top degree of the cohomology ring $H^\bullet(\mathcal{F}\ell_n)$. The Hard Lefschetz Theorem states that there is an element $\ell \in H^2(\mathcal{F}\ell_n)$ such that for all $d \leq \binom{n}{2}/2$ the multiplication map

$$\ell^{\binom{n}{2}-2d} \times (-) : H^{2d}(\mathcal{F}\ell_n) \to H^{n(n-1)-2d}(\mathcal{F}\ell_n) \tag{6.12}$$

is an isomorphism of vector spaces. Any element $\ell \in H^2(\mathcal{F}\ell_n)$ with this property is called a *(strong) Lefschetz element*. In terms of the presentation $H^\bullet(\mathcal{F}\ell_n) = R_n = \mathbb{Q}[x_1, \dots, x_n]/\langle e_1, \dots, e_n \rangle$, we may represent any element $\ell \in H^2(\mathcal{F}\ell_n)$ as a $\mathbb{Q}$-linear combination $c_1x_1 + \cdots + c_nx_n$ of the variables $x_1, \dots, x_n$. The element ℓ is Lefschetz if and only if $c_i \neq c_j$ for all $i \neq j$ [13].

Conjecture 6.5 would be best proven by a doubly graded version of the Hard Lefschetz Theorem. For the symmetric grading, one could hope that multiplication by an appropriate linear form $\ell = c_1x_1 + \cdots + c_nx_n$ with $c_i \neq c_j$ for $i \neq j$ would be surjective or injective depending on the relative sizes of the entries in a row of $\mathrm{Hilb}(R_n(\mathbf{a}); q, z)$. On the other hand, if $\tau = c_1\theta_1 + \cdots + c_n\theta_n$ is *any* $\mathbb{Q}$-linear combination of the θ-variables, we have $\tau^2 = 0$, so we would need a new model for the antisymmetric part of a Lefschetz element.

Ideally, the unimodality of Conjecture 6.5 would be explained by the geometry of objects with algebraic invariants given by superspace quotients. We leave this project for future work.

7. Acknowledgements

B. Rhoades is grateful to the organizers and hosts of 'Representation Theory Connections to (q, t)-Combinatorics' at BIRS in January 2019 where this project was initiated. B. Rhoades was partially supported by NSF Grant DMS-1500838. The authors are grateful to Nantel Bergeron, Sara Billey, Jeff Rabin, Josh Swanson, and Mike Zabrocki for helpful discussions. In particular, the authors thank Josh Swanson for a slick proof of Lemma 5.3 and Mike Zabrocki for suggesting the use of polarization operators. The authors thank an anonymous referee for their careful reading of the paper.

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