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Higher Specht Bases for Generalizations of the Coinvariant Ring

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Abstract. The classical coinvariant ring R_n is defined as the quotient of a polynomial ring in n variables by the positive-degree S_n-invariants. It has a known basis that respects the decomposition of R_n into irreducible S_n-modules, consisting of the *higher Specht polynomials* due to Ariki, Terasoma, and Yamada (Hiroshima Math J 27(1):177–188, 1997). We provide an extension of the higher Specht basis to the generalized coinvariant rings $R_{n,k}$ introduced in Haglund et al. (Adv Math 329:851–915, 2018). We also give a conjectured higher Specht basis for the Garsia–Procesi modules R_μ, and we provide a proof of the conjecture in the case of two-row partition shapes μ. We then combine these results to give a higher Specht basis for an infinite subfamily of the modules $R_{n,k,\mu}$ recently defined by Griffin (Trans Amer Math Soc, to appear, 2020), which are a common generalization of $R_{n,k}$ and R_μ.

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1. Introduction and Background

The **Specht polynomials** provide one of the many ways of directly constructing the irreducible representations of the symmetric group S_n. To define them, recall that a *standard Young tableau* on a partition λ of n is a filling of the Young diagram of λ with the numbers $1, \dots, n$ that is increasing across rows and up columns (using the 'French' convention for tableaux; see Fig. 1). Given a standard Young tableau T, the Specht polynomial F_T is defined as

$$F_T = \prod_{C} \prod_{\substack{i,j \in C, \\ i<j}} (x_j - x_i),$$

where the outer product is over all columns of T. For example, if T is the tableau in Fig. 1, then $F_T = (x_1 - x_2)(x_1 - x_5)(x_2 - x_5)(x_3 - x_4)(x_6 - x_7)$.

$$\begin{array}{|c|c|c|}\cline{1-1} 5 & \multicolumn{2}{c}{} \\ \hline 2 & 4 & 7 \\ \hline 1 & 3 & 6 \\ \hline \end{array}$$

FIGURE 1. A standard Young tableau T of partition shape $\lambda = (3, 3, 1)$

Given a fixed partition λ of n, the set of Specht polynomials

$$\{F_T : T \text{ has shape } \lambda\}$$

spans a subspace of $\mathbb{Q}[x_1, \ldots, x_n]$ isomorphic to the irreducible representation V_λ of S_n (under the usual S_n-action on the variables x_i). Moreover, the polynomials F_T are linearly independent, forming a basis of this representation. (See [19] for proofs of these facts along with a general overview of symmetric group representation theory and symmetric function theory.)

1.1. Higher Specht Polynomials for the Coinvariant Ring

The Specht polynomial construction has been generalized in [1] to higher degree copies of V_λ appearing in polynomial rings. In particular, the S_n-module structure of the full polynomial ring is easily determined from that of the **coinvariant ring**

$$R_n = \mathbb{Q}[x_1, \ldots, x_n]/(e_1, \ldots, e_n).$$

Here $e_1, \ldots, e_n$ are the elementary symmetric functions in $x_1, \ldots, x_n$, defined by

$$e_d = e_d(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}.$$

It is known that R_n, as an ungraded S_n-module, is isomorphic to the regular representation. Thus, each irreducible S_n-module V_λ appears $\dim V_\lambda$ times, which is precisely the number of standard Young tableaux of shape λ. Hence, a basis of generalized Specht polynomials for R_n should be indexed by pairs of standard Young tableaux of the same shape.

To this end, in [1] (and more succinctly described in [2]), Ariki, Terasoma, and Yamada defined the **higher Specht polynomials** using the well-known *cocharge*[1] statistic. We first recall the definition of cocharge for permutations and tableaux here.

Definition 1. Let $\pi = \pi_1 \ldots \pi_n$ be a permutation in S_n. The **cocharge word** $\mathrm{cw}(\pi) = c_1 \ldots c_n$ is defined as follows. Label the 1 in π with the subscript 0. Assuming the letter i in π has been labeled j, assign the letter $i + 1$ in π the

[1] In [1], the terminology used is 'charge', but we use 'cocharge' to be consistent with the original notation of Lascoux and Schutzenberger [14].

$$\begin{array}{|c|cc}\hline 7 & & \\ \hline 3 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{6} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|cc}\hline 3 & & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline 0 & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

FIGURE 2. A standard Young tableau S at left, with its cocharge labels shown at right

label j if $\pi_i^{-1} < \pi_{i+1}^{-1}$ and $j+1$ if $\pi_i^{-1} > \pi_{i+1}^{-1}$. Then $\mathrm{cw}(\pi) = c_1 \dots c_n$ is the list of labels, read left-to-right.

Definition 2. If S is a standard tableau, then $\mathrm{cw}(S)$ is the cocharge word of the *reading word* of S, formed by concatenating the rows from top to bottom.

For example, if S is the tableau at left in Fig. 2, the reading word is 7346125 so that the cocharge labeling is

$$7_3\, 3_1\, 4_1\, 6_2\, 1_0\, 2_0\, 5_1$$

and $\mathrm{cw}(S) = 3112001$. We can also represent $\mathrm{cw}(S)$ as a tableau by replacing the entry i in S with its cocharge label, as shown at right in Fig. 2.

Definition 3. For any word w or standard tableau S, we define its **cocharge**, written $\mathrm{cc}(w)$ or $\mathrm{cc}(S)$, respectively, to be the sum of the labels in the cocharge word.

Now suppose we have two standard tableaux S and T with the same shape. Define the monomial

$$\mathbf{x}_T^{\mathrm{cw}(S)} = \prod_{i=1}^{n} x_i^{\mathrm{cw}(i)},$$

where $\mathrm{cw}(i)$ is the cocharge label in $\mathrm{cw}(S)$ in the same square as i in T. If T is the tableau in Fig. 1 and S is at left in Fig. 2, then

$$\mathbf{x}_T^{\mathrm{cw}(S)} = x_1^0 x_2^1 x_3^0 x_4^1 x_5^3 x_6^1 x_7^2 = x_2 x_4 x_5^3 x_6 x_7^2.$$

Finally, define the *higher Specht polynomial* F_T^S to be

$$F_T^S := \varepsilon_T \cdot \mathbf{x}_T^{\mathrm{cw}(S)}, \tag{1}$$

where $\varepsilon_T \in \mathbb{Q}[S_n]$ is the **Young idempotent** corresponding to T. That is,

$$\varepsilon_T = \sum_{\tau \in C(T)} \sum_{\sigma \in R(T)} \mathrm{sgn}(\tau)\tau\sigma,$$

where $C(T) \subseteq S_n$ is the group of *column permutations* of T (those that send every number to another number in its column in T), and $R(T) \subseteq S_n$ is the group generated by row permutations.

Example 1. Suppose S is an SYT of shape λ with the property that the numbers $1, \dots, \lambda_1$ are in the bottom row, the numbers $\lambda_1 + 1, \lambda_1 + 2, \dots, \lambda_1 + \lambda_2$ are in the second, and so on. Then its cocharge indices are $i - 1$ in the ith row for all i. In this case, if T is any SYT of shape λ, then we have $F_T^S = F_T$, where F_T is the ordinary Specht polynomial defined above.

If V is a finite-dimensional S_n-module, there are unique multiplicities c_λ such that $V \cong \bigoplus_{\lambda \vdash n} c_\lambda V_\lambda$. The *Frobenius character* of V is the symmetric function $\mathrm{Frob}(V) := \sum_\lambda c_\lambda s_\lambda$ obtained by replacing each copy of V_λ with the corresponding Schur function s_λ. More generally, if $V = \bigoplus_{d \geq 0} V_d$ is a graded S_n-module with each piece V_d finite-dimensional, the *graded Frobenius character* of V is $\mathrm{grFrob}(V; q) := \sum_{d \geq 0} \mathrm{Frob}(V) \cdot q^d$.

Let $\mathrm{SYT}(n)$ be the set of all standard Young tableaux with n boxes. In [1], Ariki, Terasoma, and Yamada proved that the set

$$\mathcal{B}_n := \{F_T^S \,:\, S, T \in \mathrm{SYT}(n) \text{ have the same shape}\}$$

descends to a basis for the classical coinvariant algebra R_n. Since F_T^S is obtained by the action of the idempotent ε_T, it follows that the subspace generated by those elements F_T^S with a fixed T is a copy of the irreducible representation V_λ where $\lambda = \mathrm{shape}(T) = \mathrm{shape}(S)$ is the partition shape of S and T. (See [6, page 46].) As an immediate corollary, one obtains the known fact that the graded Frobenius character of R_n is given by

$$\mathrm{grFrob}(R_n; q) = \sum_{S \in \mathrm{SYT}(n)} q^{\mathrm{cc}(S)} s_{\mathrm{shape}(S)} = \sum_{S \in \mathrm{SYT}(n)} q^{\mathrm{maj}(S)} s_{\mathrm{shape}(S)}.$$

Here maj is the major index (see Definition 6 below). The second equality follows from the equidistribution of cocharge and major index on standard tableaux of a given shape (see [13]).

Our goal is to extend this setup to several important generalizations of the coinvariant ring. To be precise, we define a *higher Specht basis* of an arbitrary S_n-module as follows.

Definition 4. Let R be an S_n-module with decomposition

$$R = \bigoplus_\lambda c_\lambda V_\lambda$$

into irreducible S_n-modules. Then a **higher Specht basis** of R is a set of elements $\mathcal{B}$ such that there exists a decomposition $\mathcal{B} = \bigcup_\lambda \bigcup_{i=1}^{c_\lambda} \mathcal{B}_{\lambda,i}$ such that the elements of $\mathcal{B}_{\lambda_i}$ are a basis of the ith copy of V_λ in the decomposition of R.

We now describe three important generalizations of the coinvariant ring in the following subsections, with the goal of constructing a higher Specht basis for each.

1.2. The Rings $R_{n,k}$

For positive integers $k \leq n$, Haglund, Rhoades, and Shimozono [11] defined a quotient ring

$$R_{n,k} := \mathbb{Q}[x_1, \ldots, x_n]/I_{n,k}, \tag{2}$$

where $I_{n,k} \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ is the ideal

$$I_{n,k} := \langle x_1^k, x_2^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle. \tag{3}$$

Since the ideal $I_{n,k}$ is homogeneous and S_n-stable, the ring $R_{n,k}$ is a graded S_n-module. When $k = n$, we recover the classical coinvariant ring, i.e. $R_{n,n} = R_n$. As an ungraded S_n-module, the ring $R_{n,k}$ is isomorphic [11] to the permutation action of S_n on k-block ordered set partitions of $\{1, 2, \ldots, n\}$.

The *Delta Conjecture* of Haglund, Remmel, and Wilson [10] depends on two positive integers $k \leq n$ and predicts the equality of three formal power series in an infinite set of variables $\mathbf{x} = (x_1, x_2, \ldots)$ and two additional parameters q and t:

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(\mathbf{x}; q, t) = \mathrm{Val}_{n,k}(\mathbf{x}; q, t). \tag{4}$$

Here $\Delta'_{e_{k-1}}$ is a Macdonald eigenoperator and Rise and Val are defined in terms of lattice path combinatorics; see [10] for details.

Although the Delta Conjecture is open in general, it is proven when one of the parameters q, t is set to zero. Combining the results of [5, 10–12, 17, 23], we have that $\Delta'_{e_{k-1}} e_n \mid_{t=0}$ is equal to

$$\begin{aligned} \mathrm{Rise}_{n,k}(\mathbf{x}; q, 0) &= \mathrm{Rise}_{n,k}(\mathbf{x}; 0, q) \\ &= \mathrm{Val}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Val}_{n,k}(\mathbf{x}; 0, q). \end{aligned} \tag{5}$$

If $C_{n,k}(\mathbf{x}; q)$ is the common symmetric function in Eq. (5), we have [11]

$$\mathrm{grFrob}(R_{n,k}; q) = (\mathrm{rev}_q \circ \omega) C_{n,k}(\mathbf{x}; q), \tag{6}$$

where rev_q reverses the coefficient sequences of polynomials in q and ω is the symmetric function involution which trades e_n and h_n, so that $R_{n,k}$ gives a representation-theoretic model for the Delta Conjecture at $t = 0$.

The rings $R_{n,k}$ also have a geometric interpretation. For $k \leq n$, Pawlowski and Rhoades [15] introduced the variety

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) : \ell_i \text{ a line in } \mathbb{C}^k \text{ and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\} \tag{7}$$

of n-tuples of 1-dimensional subspaces of $\mathbb{C}^k$ which have full span. They proved [15] that the rational cohomology of $X_{n,k}$ is presented by the ring $R_{n,k}$. Rhoades and Wilson [18] gave another interpretation of $R_{n,k}$ using an extension of the Vandermonde determinant to superspace.

1.3. The Rings R_μ

The **Garsia–Procesi modules** R_μ, indexed by partitions $\mu \vdash n$, are another generalization of the coinvariant ring defined by

$$R_\mu = \mathbb{Q}[x_1, \ldots, x_n]/I_\mu,$$

where we define I_μ using the notation of Garsia and Procesi [4] as follows. For a subset $S \subseteq \{x_1, \ldots, x_n\}$, define the *partial elementary symmetric functions*

$e_r(S)$ to be the elementary symmetric function of degree r in the restricted set of variables S. For instance, $e_2(x_1, x_4, x_5) = x_1x_4 + x_1x_5 + x_4x_5$.

Let μ' be the conjugate partition formed by reflecting μ about the diagonal, and define

$$c_t(\mu) = \mu'_1 + \cdots + \mu'_t - t \tag{8}$$

to be the number of squares in the first t columns that lie above the first row. Then we have[2]

$$I_\mu = \langle e_r(S) \,:\, c_{n-|S|}(\mu) < r \leq |S| \rangle. \tag{9}$$

Note that in the case $\mu = (1^n)$, we recover the coinvariant ring, that is, $R_{(1^n)} = R_n$. In general, the graded Frobenius character of R_μ is given by

$$\mathrm{grFrob}(R_\mu; q) = \widetilde{H}_\mu(\mathbf{x}; q),$$

where $\widetilde{H}_\mu(\mathbf{x}; q)$ are the classical Hall–Littlewood polynomials. These exhibit a combinatorial formula in terms of the following notions.

Definition 5. A **semistandard Young tableau** T of shape λ is a filling of the Young diagram of λ with positive integers such that the rows are weakly increasing left to right and the columns are strictly increasing bottom to top. The **content** of a tableau T (or word w) is the tuple $(m_1, m_2, \ldots)$, where m_i is the number of times i appears in T (or w).

Write $\mathrm{SSYT}(\lambda, \mu)$ for the set of all semistandard Young tableaux of shape λ and content μ. Then it was shown in [14] that

$$\widetilde{H}_\mu(\mathbf{x}; q) = \sum_\lambda \sum_{S \in \mathrm{SSYT}(\lambda,\mu)} q^{\mathrm{cc}(S)} s_\lambda,$$

where cc is a generalization of the cocharge statistic that we describe in detail in Sect. 3.

The rings R_μ also have a geometric interpretation in terms of *Springer fibers*. Define $\mathcal{B}_\mu$ to be the subvariety of the full flag variety

$$\mathrm{Fl}_n = \{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \,:\, \dim(V_i) = i \text{ for all } i\}$$

consisting of the flags fixed by the action of a fixed unipotent element u of $\mathrm{GL}_n(\mathbb{C})$ having Jordan blocks of size $\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}$. The space $\mathcal{B}_\mu$ is a fiber of the Springer resolution of the unipotent subvariety of GL_n, and its cohomology ring comes with a graded S_n-module structure whose top degree component is precisely the irreducible representation V_μ [21]. The work of [3] and [22] shows that R_μ is isomorphic to the cohomology ring of the Springer fiber $\mathcal{B}_\mu$, both as a graded ring and as a graded S_n-module.

[2] It is straightforward to verify that the inequality in (9) is equivalent to the one stated in [4], and we omit the proof for brevity.

1.4. The Rings $R_{n,k,\mu}$

In [8], Griffin introduced a common generalization of R_μ and $R_{n,k}$. While Griffin's notation for these generalized modules is $R_{n,\lambda,s}$, here we change the variable s to k and λ to μ and interchange their order to instead write $R_{n,k,\mu}$. This notation is more compatible with the way we denoted the two known modules above.

Griffin defines the ideal $I_{n,k,\mu}$ generated by

– the monomials $x_1^k, \ldots, x_n^k$, and
– the partial elementary symmetric functions $e_r(S)$ satisfying

$$c_{n-|S|}(\mu) + (n - |\mu|) < r \leq |S|,$$

where the notation $c_t(\mu)$ is the same as in Eq. (8).

Then we have

$$R_{n,k,\mu} = \mathbb{C}[x_1, \ldots, x_n]/I_{n,k,\mu}.$$

Notice that if $|\mu| = n$ and $k \geq \ell(\mu)$, then $R_{n,k,\mu} = R_\mu$, and if $\mu = (1^k)$ then $R_{n,k,\mu} = R_{n,k}$.

In [8], Griffin gives several combinatorial formulas for the graded Frobenius series of $R_{n,k,\mu}$. The most relevant of these to our purposes is an expansion in terms of Hall–Littlewood polynomials. In the following, we write

$$H_\lambda(x;q) := q^{n(\lambda)}\widetilde{H}(x;q^{-1}) = \mathrm{rev}_q(\widetilde{H}(x;q))$$

to denote the 'charge' Hall–Littlewood polynomials, where $n(\lambda) = \sum_i \binom{\lambda_i'}{2}$ for any partition λ. With this notation, we have

$$\mathrm{grFrob}(R_{n,k,\mu};q) = \mathrm{rev}_q\left(\sum_{\substack{\lambda\supseteq\mu\\ \ell(\lambda)\leq k\\ |\lambda|=n}} q^{n(\lambda,\mu)} \prod_{i\geq 0}\binom{\lambda_i' - \mu_{i+1}'}{\lambda_i' - \lambda_{i+1}'}_q H_\lambda(x;q)\right), \tag{10}$$

where $n(\lambda,\mu) = \sum_i \binom{\lambda_i'-\mu_i'}{2}$ and where the notation $\binom{a}{b}_q$ denotes the q-binomial coefficient $\prod_{i=0}^{b-1}\frac{1-q^{a-i}}{1-q^{b-i}}$. The notation $\lambda \supseteq \mu$ indicates that the Young diagram of μ is contained inside that of λ.

The modules $R_{n,k,\mu}$ have a geometric interpretation as well, in the limit as $k \to \infty$. The *Eisenbud–Saltman rank variety* $\overline{O}_{n,\mu}$ is the subvariety of $\mathfrak{gl}_n$ defined by

$$\overline{O}_{n,\mu} = \{X \in \mathfrak{gl}_n : \dim\ker X^d \geq \mu_1' + \cdots + \mu_d', d = 1, \ldots, n\}.$$

In the case that $|\mu| = n$, this coincides with the closure of the variety O_μ of nilpotent matrices with Jordan block type μ. Setting $R_{n,\mu}$ to be the limiting module of $R_{n,k,\mu}$ as $k \to \infty$, Griffin shows that $R_{n,\mu}$ is the coordinate ring of the scheme theoretic intersection

$$\overline{O}_{n,\mu'} \cap \mathfrak{t},$$

where $\mathfrak{t}$ is the Cartan subalgebra of diagonal matrices in $\mathfrak{gl}_n$. This is a strict generalization of the analogous result for R_μ and $\overline{O}_\mu$, which was an essential step in de Concini and Procesi's work [3] on the connections to the Springer fibers.

1.5. Main Results

Our main results are as follows.

Theorem 1. *Let $k \leq n$ be positive integers. Consider the set of polynomials*

$$\mathcal{B}_{n,k} := \{F_T^S \cdot e_1^{i_1} e_2^{i_2} \cdots e_{n-k}^{i_{n-k}}\},$$

where $T, S \in \mathrm{SYT}(n)$ have the same shape and $(i_1, i_2, \ldots, i_{n-k})$ is a tuple of $n-k$ nonnegative integers whose sum is $< k - \mathrm{des}(S)$. The set $\mathcal{B}_{n,k}$ descends to a higher Specht basis for $R_{n,k}$.

More details on the notation above, as well as the proof, can be found in Sect. 2. For now, note that since S_n acts trivially on any elementary symmetric polynomial, Theorem 1 immediately implies [11, Cor. 6.13]:

$$\mathrm{grFrob}(R_{n,k}; q) = \sum_{S \in \mathrm{SYT}(n)} q^{\mathrm{maj}(S)} \binom{n - \mathrm{des}(S) - 1}{n-k}_q s_{\mathrm{shape}(S)}.$$

For R_μ, we use a generalization of the cocharge statistic to define *semistandard higher Specht polynomials* F_T^S, where S is a semistandard tableau with content μ and T is a standard Young tableau of the same shape as S. (See Sect. 3.) The polynomial F_T^S is homogeneous of degree $\mathrm{cc}(S)$.

Conjecture 1. Let $\mu \vdash n$. Consider the set of semistandard higher Specht polynomials

$$\mathcal{B}_\mu = \{F_T^S\}$$

for which S has content μ and $T \in \mathrm{SYT}(n)$ has the same shape as S. Then $\mathcal{B}_\mu$ descends to a higher Specht basis of R_μ.

Numerically, the conjectured basis matches what we would expect based on the graded Frobenius character of R_μ (as computed in [14]), which is given by

$$\mathrm{grFrob}(R_\mu; q) = \sum_\lambda \sum_{S \in \mathrm{SSYT}(\lambda, \mu)} q^{\mathrm{cc}(S)} s_\lambda.$$

Our main progress towards proving this conjecture is the following.

Theorem 2. *Conjecture 1 holds when $\mu = (k, n-k)$ has two rows.*

Finally, we combine these two results to give a higher Specht basis for an infinite family of the modules $R_{n,k,\mu}$ as follows.

Theorem 3. *Suppose μ is the one-row partition $(n-1)$. Consider the set of polynomials*

$$\mathcal{B}_{n,k,(n-1)} = \{F_T^S \cdot e_1^i\},$$

where $F_T^S \in \mathcal{B}_{(n-1,1)}$ is a semistandard higher Specht polynomial for the shape $(n-1, 1)$, and $i < k - \mathrm{des}(S)$. Then $\mathcal{B}_{n,k,(n-1)}$ descends to a higher Specht basis of $R_{n,k,(n-1)}$.

We can see numerically that the basis of Theorem 3 matches what we would expect from the graded Frobenius character. In particular, setting $\mu = (n-1)$ in Eq. (10), the summation has two terms, with $\lambda = (n)$ and $\lambda = (n-1,1)$. In both cases, we have $n(\lambda,\mu) = 0$, and the only nontrivial q-binomial coefficient occurs at $i = 0$ with $\lambda = (n-1,1)$. Hence,

$$\begin{aligned}
\mathrm{grFrob}(R_{n,k,(n-1)};q) &= \mathrm{rev}_q \left[H_{(n)}(x;q) + \binom{k-1}{k-2}_q H_{(n-1,1)}(x;q) \right] \\
&= \mathrm{rev}_q \left[H_{(n)}(x;q) + (1+q+\cdots+q^{k-2}) H_{(n-1,1)}(x;q) \right] \\
&= q^{k-1} \Big[H_{(n)}(x;q^{-1}) + (1+q^{-1}+q^{-2}+\cdots \\
&\qquad + q^{2-k}) H_{(n-1,1)}(x;q^{-1}) \Big] \\
&= q^{k-1} \widetilde{H}_{(n)}(x;q) + (1+q+q^2+\cdots+q^{k-2}) \widetilde{H}_{(n-1,1)}(x;q),
\end{aligned}$$

where the third equality above follows from the fact that $H_{(n)}(x;q)$ has degree 0 in q and $H_{(n-1,1)}$ has degree 1, so that the entire polynomial has degree $k-1$. Finally, note that $\widetilde{H}_{(n)}(x;q)$ and $\widetilde{H}_{(n-1,1)}(x;q)$ are the Frobenius series of the Garsia–Procesi modules $R_{(n)}$ and $R_{(n-1,1)}$, respectively. It follows from Theorem 2 that the basis $\mathcal{B}_{n,k,(n-1)}$ of Theorem 3 gives the correct number of irreducible S_n representations in each degree.

It is our hope that these methods can be generalized to construct a higher Specht basis for $R_{n,k,\mu}$ of the form $\{F_T^S \cdot e_1^{i_1} \cdots e_{n-|\mu|}^{i_{n-|\mu|}}\}$, where the polynomials F_T^S are semistandard higher Specht polynomials for various partitions $\lambda \vdash n$ such that $\lambda \supset \mu$, and where there is an appropriate bound on the exponents i_j. As it is, one current limitation is that for any partition μ with $|\mu| < n$ and $\mu \neq (n-1)$, there exists a partition λ of n containing μ that has at least three rows. This exceeds the two-row condition of Theorem 2.

1.6. Outline

The remainder of the paper is organized as follows. In Sect. 2, we prove Theorem 1. In Sect. 3, we prove Theorem 2 and provide additional evidence and work towards Conjecture 1. Finally, we prove Theorem 3 in Sect. 4.

2. Higher Specht Bases for $R_{n,k}$

We will obtain our new basis for $R_{n,k}$ by multiplying the higher Specht polynomials F_T^S (for standard tableaux T and S of the same shape) by appropriate elementary symmetric polynomials. Before stating our basis, we recall some notions from commutative algebra.

A sequence of polynomials $f_1, f_2, \ldots, f_r$ in the ring $\mathbb{Q}[x_1, \ldots, x_n]$ is a **regular sequence** if for all $1 \le j \le r$, the endomorphism

$$\mathbb{Q}[x_1,\ldots,x_n]/(f_1,\ldots,f_{j-1}) \xrightarrow{\times f_j} \mathbb{Q}[x_1,\ldots,x_n]/(f_1,\ldots,f_{j-1})$$

on the quotient ring $\mathbb{Q}[x_1,\ldots,x_n]/(f_1,\ldots,f_{j-1})$ induced by multiplication by f_j is injective. The longest possible length of a regular sequence $f_1,\ldots,f_r$ in $\mathbb{Q}[x_1,\ldots,x_n]$ is $r=n$. The elementary symmetric polynomials $e_1,e_2,\ldots,e_n$ constitute one such length n regular sequence.

Let $f_1,\ldots,f_n$ be any length n regular sequence in $\mathbb{Q}[x_1,\ldots x_n]$ such that the f_j are homogeneous. Then the quotient $\mathbb{Q}[x_1,\ldots,x_n]/(f_1,\ldots,f_n)$ is graded and if $\mathcal{B}$ is a family of homogeneous polynomials which descends to a $\mathbb{Q}$-basis of $\mathbb{Q}[x_1,\ldots,x_n]/(f_1,\ldots,f_n)$, then the infinite set of polynomials

$$\left\{g\cdot f_1^{i_1}f_2^{i_2}\cdots f_n^{i_n}\ :\ g\in\mathcal{B}\right\}$$

is a basis of the full polynomial ring $\mathbb{Q}[x_1,\ldots,x_n]$. To describe our basis of $R_{n,k}$, we need one more definition.

Definition 6. A **descent** of a standard Young tableau S is an entry i which occurs in a lower row than $i+1$ (written in French notation). The **major index** of S, written $\mathrm{maj}(S)$, is the sum of the descents of S, and we write $\mathrm{des}(S)$ for the number of descents.

For instance, if S is the tableau at left in Fig. 1, then $\mathrm{maj}(S)=1+3+4+6=14$ and $\mathrm{des}(S)=4$.

We now restate Theorem 1 here for the reader's convenience.

Theorem 1. *Let $k\le n$ be positive integers. Consider the set*

$$\mathcal{B}_{n,k}:=\{F_T^S\cdot e_1^{i_1}e_2^{i_2}\cdots e_{n-k}^{i_{n-k}}\}$$

consisting of all polynomials of the form $F_T^S\cdot e_1^{i_1}e_2^{i_2}\cdots e_{n-k}^{i_{n-k}}$, where $S,T\in\mathrm{SYT}(n)$ have the same shape and $(i_1,i_2,\ldots,i_{n-k})$ is a tuple of $n-k$ nonnegative integers whose sum is $<k-\mathrm{des}(S)$. The set $\mathcal{B}_{n,k}$ descends to a higher Specht basis for $R_{n,k}$.

We first prove an enumerative lemma which will help us in the proof of Theorem 1.

Lemma 1. *Let n and k be positive integers. There are exactly k^n tuples of the form $(S,T,i_1,\ldots,i_n)$, where S and T are standard Young tableaux of the same shape with n boxes and $i_1,\ldots,i_n$ are nonnegative integers with $i_1+\cdots+i_n<k-des(S)$.*

Proof. The *Robinson–Schensted–Knuth correspondence* gives a bijection between words $w\in\{1,2,\ldots,k\}^n$ and pairs (R,T) of Young tableaux with n boxes having the same shape, such that R is semistandard with entries in $\{1,2,\ldots,k\}$ and T is standard.

Since there are k^n words $w\in\{1,2,\ldots,k\}^n$, it suffices to give a bijection between the tuples $(S,T,i_1,\ldots,i_n)$ in question and the pairs (R,T) described above. Given a standard Young tableau S, define the *destandardization* of S, denoted S' as follows. If $d_1<d_2<\cdots<d_n$ are the descents of S, replace $1,\ldots,d_1$ with 1, replace $d_1+1,\ldots,d_2$ with 2, etc. See Fig. 3 for an example.

Note that S' is semistandard by the definition of a descent, and S can be uniquely reconstructed from S'. Notice also that the largest entry of S' is $\mathrm{des}(S)+1$.

$$\begin{array}{|c|}\hline 6 \\ \hline \end{array}\quad\begin{array}{|c|}\hline 3 \\ \hline \end{array}\quad\begin{array}{|c|}\hline 6 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 6 & & & \\ \hline 3 & 7 & 8 & \\ \hline 1 & 2 & 4 & 5 \\ \hline \end{array}\qquad \begin{array}{|c|c|c|c|}\hline 3 & & & \\ \hline 2 & 3 & 3 & \\ \hline 1 & 1 & 2 & 2 \\ \hline \end{array}\qquad \begin{array}{|c|c|c|c|}\hline 6 & & & \\ \hline 3 & 7 & 7 & \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array}$$

FIGURE 3. A standard Young tableau S (at left) and its destandardization S' (middle). The tableau R defined from S' in the proof of Lemma 1 arising from the tuple $(i_1, \ldots, i_8) = (0, 1, 0, 0, 2, 0, 1, 0)$ is shown at right

Now let $i_1, \ldots, i_n$ be such that $i_1 + \cdots + i_n < k - \mathrm{des}(S)$. Let $a_1 \leq \cdots \leq a_n$ be the entries of S' in order (breaking ties by the corresponding ordering in S). Then define R by increasing each of the numbers $a_1, \ldots, a_n$ by i_1, then increasing $a_2, \ldots, a_n$ by i_2, then increasing $a_3, \ldots, a_n$ by i_3, and so on. Because $i_1 + \cdots + i_n < k - \mathrm{des}(S)$, the result R has largest entry at most k. This process is reversible, and so the proof is complete.

We now prove Theorem 1.

Proof of Theorem 1. It will be convenient to consider a broader family of quotients $R_{n,k,s} = \mathbb{Q}[x_1, \ldots, x_n]/I_{n,k,s}$ defined for $s \leq k \leq n$. Here

$$I_{n,k,s} := \langle x_1^k, x_2^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-s+1} \rangle. \tag{11}$$

In particular, we have $I_{n,k,k} = I_{n,k}$ and $R_{n,k,k} = R_{n,k}$. We allow s to be zero, in which case, no es appear in our ideal at all. However, we assume that n, k are positive.

Consider the following extended set

$$\mathcal{B}_{n,k,s} := \{F_T^S \cdot e_1^{i_1} e_2^{i_2} \cdots e_{n-s}^{i_{n-s}}\} \tag{12}$$

consisting of all polynomials $F_T^S \cdot e_1^{i_1} e_2^{i_2} \cdots e_{n-s}^{i_{n-s}}$ for which $S, T \in \mathrm{SYT}(n)$ have the same shape and $(i_1, i_2, \ldots, i_{n-s})$ is a list of $n - s$ nonnegative integers whose sum is $< k - \mathrm{des}(S)$.

We claim that $\mathcal{B}_{n,k,s}$ descends to a basis for $R_{n,k,s}$ for all n, k, s. This is stronger than the statement of the Theorem. When $n = k = s$, the quotient $R_{n,n,n}$ is the classical coinvariant algebra and the fact that $\mathcal{B}_{n,n,n}$ descends to a basis for R_n is precisely the result of [1].

To begin, the proof of [11, Lem. 6.9] gives a short exact sequence

$$0 \to R_{n,k-1,s} \to R_{n,k,s} \to R_{n,k,s+1} \to 0, \tag{13}$$

where the first map is induced by multiplication by e_{n-s} and the second map is the canonical projection. (In fact, the [11, Lem. 6.9] is only proven in the case where $s > 0$; the case $s = 0$ has the same proof after observing that $\dim(R_{n,k,0}) = k^n$, so that the dimensions of the rings on either end add up to the dimension of the ring in the middle in this case.)

By exactness, if $\mathcal{B}$ descends to a basis for $R_{n,k-1,s}$ and if $\mathcal{C}$ descends to a basis for $R_{n,k,s+1}$, the disjoint union

$$\{e_{n-s} \cdot f \in \mathcal{B}\} \sqcup \{g \, : \, g \in \mathcal{C}\} \tag{14}$$

descends to a basis for $R_{n,k,s}$. Using this property and the fact that

$$\{e_{n-s} \cdot f \in \mathcal{B}_{n,k-1,s}\} \sqcup \{g \, : \, g \in \mathcal{B}_{n,k,s+1}\} = \mathcal{B}_{n,k,s}, \tag{15}$$

we are inductively reduced to proving the result when $s = 0$. That is, it remains to show that $\mathcal{B}_{n,k,0}$ descends to a basis for $R_{n,k,0}$.

By definition, we have

$$\mathcal{B}_{n,k,0} = \{F_T^S \cdot e_1^{i_1} e_2^{i_2} \cdots e_n^{i_n}\}, \tag{16}$$

where $S, T \in \mathrm{SYT}(n)$ have the same shape and $(i_1, \ldots, i_n)$ is a sequence of nonnegative integers whose sum is $< k - \mathrm{des}(S)$. By the definition of the cocharge word $\mathrm{cw}(S)$, the largest possible exponent appearing in the monomial $\mathbf{x}_T^{\mathrm{cw}(S)}$ or the polynomial $F_T^S = \varepsilon_T \cdot \mathbf{x}_T^{\mathrm{cw}(S)}$ is $\mathrm{des}(S)$. Since elementary symmetric polynomials are sums of square-free monomials, we see that the largest possible exponent appearing in a polynomial in $\mathcal{B}_{n,k,0}$ is $k - 1$. Since

$$R_{n,k,0} = \mathbb{Q}[x_1, \ldots, x_n]/\langle x_1^k, \ldots, x_n^k \rangle \tag{17}$$

and $|\mathcal{B}_{n,k,0}| = k^n = \dim(R_{n,k,0})$, (where the first equality uses Lemma 1) we conclude that $\mathcal{B}_{n,k,0}$ descends to a basis for $R_{n,k,0}$ if and only if $\mathcal{B}_{n,k,0}$ is linearly independent in the full polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$.

We finish the proof by showing that $\mathcal{B}_{n,k,0}$ is linearly independent in $\mathbb{Q}[x_1, \ldots, x_n]$. To do this, we apply the main result of [1] that the set

$$\mathcal{B}_n = \{F_T^S \, : \, S, T \in \mathrm{SYT}(n) \text{have the same shape}\}$$

descends to a basis for the coinvariant ring R_n. Since the ideal defining R_n is cut out by the regular sequence $e_1, \ldots, e_n$, we know that the set

$$\{F_T^S \cdot e_1^{i_1} \cdots e_n^{i_n} \, : \, S, T \in \mathrm{SYT}(n) \text{ have the same shape and} i_1, \ldots, i_n \geq 0\} \tag{18}$$

is a basis of the full polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. Since it is a subset of this basis, the set $\mathcal{B}_{n,k,0}$ is linearly independent in $\mathbb{Q}[x_1, \ldots, x_n]$, as desired.

3. Higher Specht Bases for R_μ

We now give a conjectured generalization of the higher Specht basis to the Garsia–Procesi modules R_μ, and prove it in the case that μ has at most two rows. We first recall the generalization of cocharge, defined in [14], to words whose *content* (Definition 5) is a partition. Throughout this section, we assume w is a word with partition content μ.

For an entry w_j of w and a positive integer k, define the **cyclically previous** k before w_j, denoted $\mathrm{cprev}(k, w_j)$, to be the rightmost k cyclically to the left of w_j in w. That is, it is the rightmost k to the left of w_j if such a k exists, or the rightmost k in w otherwise.

Definition 7. Let $w_{i_1} = 1$ be the rightmost 1 in w, and recursively define $i_2, \ldots, i_{\ell(\mu)}$ by

$$w_{i_{j+1}} = \mathrm{cprev}(j+1, w_{i_j}).$$

We call the subword $w^{(1)}$ consisting of the entries w_{i_j} the **first standard subword** of w.

Definition 8. The **standard subword decomposition** of w is obtained by setting $w^{(1)}$ to be the first standard subword of w, and recursively defining $w^{(i)}$, for $i > 1$, to be the first standard subword of the entries of w not in $w^{(1)}, \ldots, w^{(i-1)}$.

Definition 9. The **cocharge** of w is

$$\mathrm{cc}(w) = \sum_i \mathrm{cc}(w^{(i)}),$$

where $w^{(1)}, w^{(2)}, \ldots, w^{(\mu_1)}$ is its standard subword decomposition. The **cocharge word** $\mathrm{cw}(w)$ is defined as the labeling on w given by labeling the letters of $w^{(i)}$ with its cocharge word $\mathrm{cw}(w^{(i)})$ for each i.

For a semistandard Young tableau S having reading word w, we define

$$\mathrm{cc}(S) = \mathrm{cc}(w).$$

For a square s in the diagram of S, we write $\mathrm{cw}_S(s)$ for the cocharge word label of the corresponding letter of w.

Example 2. The semistandard Young tableau

$$S = \begin{array}{|c|c|c|c|c|} \cline{1-1} 4 \\ \hline 2 & 2 & 3 & 3 & 4 \\ \hline 1 & 1 & 1 & 2 & 3 \\ \hline \end{array}$$

has reading word $w = 42233411123$. If we label the first standard subword $w^{(1)}$ (shown in boldface below) with its cocharge labeling as subscripts, we get

$$4_{2\mathbf{2}_1}3_{3\mathbf{4}_2}1_{1\mathbf{1}_0}2_{\mathbf{3}1}.$$

Then we label $w^{(2)}$ to obtain:

$$\mathbf{4}_2\mathbf{2}_1 2_1 3\mathbf{3}_1 4_2 1\mathbf{1}_0 1_0 2 3_1.$$

We finally label $w^{(3)}$ to obtain

$$4_2 2_1 2_1 \mathbf{3}_1 3_1 4_2 \mathbf{1}_0 1_0 1_0 \mathbf{2}_0 3_1.$$

It follows that $\mathrm{cw}(w) = 2111200001$ and $\mathrm{cc}(S) = \mathrm{cc}(w) = 8$.

We can now define the conjectured basis for R_μ.

Definition 10. Let (S, T) be a pair of Young tableaux of the same shape $\lambda \vdash n$ where S is semistandard and has content μ and T has content (1^n) (but is not necessarily standard). Then we define

$$\mathbf{x}_T^S = \prod_{s \in D(\lambda)} x_{T(s)}^{\mathrm{cw}_S(s)},$$

where $D(\lambda)$ is the set of squares in the diagram of λ. Finally, define the **semistandard higher Specht polynomial**

$$F_T^S = \varepsilon_T \cdot \mathbf{x}_T^S.$$

Recall that $\mathrm{SSYT}(\lambda,\mu)$ is the set of all semistandard Young tableaux of shape λ and content μ. We also write $\mathrm{SYT}(\lambda) = \mathrm{SSYT}(\lambda,(1^n))$ for the set of standard Young tableaux of shape λ. Then we can restate Conjecture 1 as follows.

Conjecture 2. (Conjecture 1 restated) The set of polynomials

$$\mathcal{B}_\mu = \left\{ F_T^S \,:\, (S,T) \in \bigcup_{\lambda \vdash n} \mathrm{SSYT}(\lambda,\mu) \times \mathrm{SYT}(\lambda) \right\}$$

is a basis of R_μ.

3.1. Semistandard Higher Specht Modules in $\mathbb{Q}[\mathbf{x}_n]$

As a step towards proving Conjecture 1, we consider the modules generated by the semistandard higher Specht polynomials as submodules of the full polynomial ring $\mathbb{Q}[\mathbf{x}_n]$, before descending to the quotient R_μ. In particular, we show that these give copies of the ordinary polynomial Specht modules in higher degrees.

Definition 11. Write $\mathrm{Tab}(\lambda)$ to denote the set of all (not necessarily standard) tableaux of shape λ and of content (1^n). In other words, $\mathrm{Tab}(\lambda)$ is the set of all $n!$ ways of filling the boxes of λ with the numbers $1, 2, 3, \ldots, n$ in any manner.

Note that if $\lambda \vdash n$ then S_n naturally acts on $\mathrm{Tab}(\lambda)$ by permuting entries in a tableau.

Definition 12. For a fixed $S \in \mathrm{SSYT}(\lambda,\mu)$, define

$$V^S := \mathrm{span}\{F_T^S : T \in \mathrm{Tab}(\lambda)\}$$

to be the span of the higher Specht polynomials associated with S, considered as a subspace of $R = \mathbb{C}[x_1, \ldots, x_n]$, where $n = |\lambda|$. Similarly, define $\overline{V^S}$ to be its image in the quotient R_μ.

We first show that V^S is an irreducible S_n-module isomorphic to the standard Specht module V^λ. We begin with several technical lemmas. Throughout, we fix a choice of semistandard Young tableau $S \in \mathrm{SSYT}(\lambda,\mu)$.

Proposition 1. *Let* $\omega \in S_n$ *and* $T \in \mathrm{Tab}(\lambda)$. *Then*

$$\omega F_T^S = F_{\omega T}^S.$$

Proof. First note that if $\tau \in C(T)$ then $\tau' := \omega\tau\omega^{-1} \in C(\omega T)$, and similarly, if $\sigma \in R(T)$ then $\sigma' := \omega\sigma\omega^{-1} \in R(\omega T)$. Notice also that $\omega\mathbf{x}_T^S = \mathbf{x}_{\omega T}^S$. We, therefore, have

$$\omega F_T^S = \omega\varepsilon_T \mathbf{x}_T^S = \sum_{\tau \in C(T)} \sum_{\sigma \in R(T)} \mathrm{sgn}(\tau)\omega\tau\sigma\mathbf{x}_T^S$$

$$\begin{aligned}
&= \sum_{\tau \in C(T)} \sum_{\sigma \in R(T)} \operatorname{sgn}(\omega\tau\omega^{-1})(\omega\tau\omega^{-1})(\omega\sigma\omega^{-1})\omega \mathbf{x}_T^S \\
&= \sum_{\tau' \in C(\omega T)} \sum_{\sigma' \in R(\omega T)} \operatorname{sgn}(\tau')\tau'\sigma' \mathbf{x}_{\omega T}^S \\
&= F_{\omega T}^S
\end{aligned}$$

as desired.

Corollary 1. *The space V^S is a cyclic S_n-submodule of R.*

We now show that, assuming the polynomials F_T^S are independent for T standard, the submodule V^S is a copy of the irreducible S_n-module V^λ. We recall (see, for instance [16]) the Garnir relations that govern the S_n-module structure of V^λ with respect to the standard Specht basis.

Definition 13. Let $T \in \mathrm{Tab}(\lambda)$. Let a and b, with $a < b$, be the indices of two distinct columns of T, and let $t \leq \lambda_b'$ be a row index of one of the entries of column b. Then we write $S_t^{a,b}$ to be the subgroup of S_n consisting of all permutations of the set of elements of T residing either in column a weakly above t, or in column b weakly below t.

The **Garnir element** $G_t^{a,b}$ is the partial antisymmetrizer

$$G_t^{a,b} := \sum_{\omega \in S_t^{a,b}} \operatorname{sgn}(\omega)\omega.$$

Proposition 2. *The element F_T^S, for any $T \in \mathrm{Tab}(\lambda)$, satisfies the Garnir relation $G_t^{a,b}(F_T^S) = 0$.*

To prove this proposition, we first show that the analog of Lemma 3.3 in [16] holds here. To state it, we introduce the Young (anti)symmetrizers α and β defined as follows. For any subgroup $U \subseteq S_n$, define

$$\alpha(U) = \sum_{\tau \in U} \operatorname{sgn}(\tau)\tau \qquad \text{and} \qquad \beta(U) = \sum_{\sigma \in U} \sigma.$$

In this notation, the Young symmetrizer ε_T can be written as

$$\varepsilon_T = \alpha(C(T))\beta(R(T)).$$

Lemma 2. *Let U be any subgroup of S_n and let $C = C(T)$, where $T \in \mathrm{Tab}(\lambda)$. Suppose there is an involution $\sigma \mapsto \sigma'$ on UC such that for each $\sigma \in UC$, there exists $\rho_\sigma \in R(T)$ for which $\rho_\sigma^2 = 1$, $\operatorname{sgn}(\rho_\sigma) = -1$, and $\sigma' = \sigma\rho_\sigma$. Then*

$$\alpha(U)F_T^S = 0.$$

Proof. We have $\alpha(U)\alpha(C) = |U \cap C|\alpha(UC)$ (see Lemma 3.2 in [16]). Therefore,

$$\begin{aligned}
\alpha(U)F_T^S &= \alpha(U)\varepsilon_T \mathbf{x}_T^S \\
&= \alpha(U)\alpha(C)\beta(R)\mathbf{x}_T^S \\
&= |U \cap C|\alpha(UC)\beta(R)\mathbf{x}_T^S,
\end{aligned}$$

where $R = R(T)$ is the group of row permutations. Due to the involution $\sigma \mapsto \sigma'$, which has no fixed points because $\mathrm{sgn}(\rho_\sigma) = -1$ for each ρ_σ, we have that the terms in $\alpha(UC)$ can be partitioned into pairs of terms $\mathrm{sgn}(\sigma)\sigma + \mathrm{sgn}(\sigma')\sigma'$. We claim that each of these two-term sums kills $\beta(R)x_T^S$. Indeed, we have

$$\begin{aligned}
(\mathrm{sgn}(\sigma)\sigma + \mathrm{sgn}(\sigma')\sigma')\beta(R)\mathbf{x}_T^S &= (\mathrm{sgn}(\sigma)\sigma + \mathrm{sgn}(\sigma)\mathrm{sgn}(\rho_\sigma)\sigma\rho_\sigma)\beta(R)\mathbf{x}_T^S \\
&= (\mathrm{sgn}(\sigma)\sigma - \mathrm{sgn}(\sigma)\sigma\rho_\sigma)\beta(R)\mathbf{x}_T^S \\
&= (\mathrm{sgn}(\sigma)\sigma\beta(R) - \mathrm{sgn}(\sigma)\sigma\rho_\sigma\beta(R))\mathbf{x}_T^S \\
&= (\mathrm{sgn}(\sigma)\sigma\beta(R) - \mathrm{sgn}(\sigma)\sigma\beta(R))\mathbf{x}_T^S \\
&= 0,
\end{aligned}$$

where the last computation follows because $\rho_\sigma \in R$ and therefore ρ_σ permutes the terms of $\beta(R)$. It follows that $\alpha(U)F_T^S = 0$, as desired.

The above lemma is the exact analog to Lemma 3.3 in [16]. Using this lemma, the proof of Proposition 2 now exactly follows that of Theorem 3.1 in [16] for the ordinary Specht polynomials, since the remaining steps of Peel's proof only depend on the operators $\alpha(U)$ and not on the specific polynomials they are applied to. We, therefore, omit the rest of the details and refer to [16].

It now follows that the elements F_T^S, for T a standard tableau of shape $\mathrm{shape}(S)$, span the space V^S. Finally, we show the polynomials F_T^S for $T \in \mathrm{SYT}(n)$ are linearly independent in $\mathbb{C}[x_1, \ldots, x_n]$. In fact, their images are independent in the coinvariant ring R_n.

To prove this, we use the **last letter order** $\lessdot$ on standard Young tableaux defined in [1]. In particular, for any two standard tableaux T_1, T_2 of the same shape, let $m(T_1, T_2)$ be the largest letter that is not in the same square in T_1 as in T_2. Then we say $T_1 \lessdot T_2$ if $m(T_1, T_2)$ is in row ℓ in T_1 and row k in T_2 with $\ell < k$.

Example 3. The last letter order on the shape $(2, 4)$ puts the standard tableaux in the following order from least to greatest:

$$\begin{ytableau} 2 & 4 \\ 1 & 3 & 5 & 6 \end{ytableau}, \quad \begin{ytableau} 3 & 4 \\ 1 & 2 & 5 & 6 \end{ytableau}, \quad \begin{ytableau} 2 & 5 \\ 1 & 3 & 4 & 6 \end{ytableau}, \quad \begin{ytableau} 3 & 5 \\ 1 & 2 & 4 & 6 \end{ytableau}, \quad \begin{ytableau} 4 & 5 \\ 1 & 2 & 3 & 6 \end{ytableau},$$
$$\begin{ytableau} 2 & 6 \\ 1 & 3 & 4 & 5 \end{ytableau}, \quad \begin{ytableau} 3 & 6 \\ 1 & 2 & 4 & 5 \end{ytableau}, \quad \begin{ytableau} 4 & 6 \\ 1 & 2 & 3 & 5 \end{ytableau}, \quad \begin{ytableau} 5 & 6 \\ 1 & 2 & 3 & 4 \end{ytableau}.$$

We also require the following elementary linear algebra fact, whose proof we omit.

Lemma 3. *Let V and W be vector spaces over a field k of characteristic 0, with a nondegenerate bilinear form $\langle , \rangle : V \times W \to k$. Let $v = \{v_1, \ldots, v_r\} \subseteq V$ and $w = \{w_1, \ldots, w_r\} \subseteq W$, and suppose $\langle v_i, w_j \rangle = 0$ whenever $i < j$ and further that $\langle v_i, w_i \rangle \neq 0$ for all i. Then v and w are both independent sets of vectors in V and W, respectively.*

Proposition 3. *For a fixed $S \in \mathrm{SSYT}(\lambda)$, the polynomials F_T^S, for $T \in \mathrm{SSYT}(\lambda, (1^n))$, are independent in the coinvariant ring R_n.*

Proof. We make use of the bilinear form and ordering on tableaux defined in [1]. In particular, for $f, g \in R_n$ define

$$\langle f, g\rangle = \frac{1}{\Delta}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\sigma(\tilde{f}\tilde{g})\mid_{x_1=x_2=\cdots=x_n=0}$$

where $\tilde{f}$ and $\tilde{g}$ are lifts of f and g in $\mathbb{C}[x_1,\ldots,x_n]$ and $\Delta = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant.

Define $G_T^S = \varepsilon_{T'}\mathbf{x}_T^{T-1-S}$, where T' is the transpose of the standard Young tableau T, and $T-1-S$ denotes the tableau formed by reducing all entries of T by 1 and then subtracting, element-wise, the entries in S. Thus, in particular, $\mathbf{x}_T^{T-1-S}\cdot\mathbf{x}_T^S = x_1^0x_2^1x_3^2\cdots x_n^{n-1}$.

Then by an identical computation as in [1] (in the last paragraph of the proof of Proposition 1 part (2)), we have that

$$\langle F_{T_1}^S, G_{T_2}^S\rangle$$

is nonzero if $T_1 = T_2$, since the only surviving term in the computation is the product $x_1^0x_2^1x_3^2\cdots x_n^{n-1}$, whose antisymmetrization is the Vandermonde determinant Δ itself. Moreover, as in the proof of Proposition 2 of [1], we see that $\langle F_{T_1}^S, G_{T_2}^S\rangle$ is equal to 0 if $T_1 > T_2$ in the last letter order, as in this case $\varepsilon_{T_1}\varepsilon_{T_2} = 0$ as operators (see [1]), and so it does not matter that we are applying the operators to different monomials than in [1].

Thus, we have an upper triangular transition matrix between the F and G polynomials, and so the polynomials F_T^S for $T \in \mathrm{Tab}(\lambda)$ are independent in R_n.

Corollary 2. *The space V^S is a copy of the irreducible S_n-module V^λ.*

3.2. Independence in R_μ for Two-Row Shapes

We now show that, for two-row shapes μ, the set of semistandard higher Specht polynomials for R_μ is independent in R_μ, by induction on the size of μ. Our main tool is a recursion developed by Garsia and Procesi [4]. We recall their notation as follows.

Definition 14. Let $\mu = (\mu_1,\ldots,\mu_r)$ be a partition and let $i \le r$. Then $\mu^{(i)}$ is the partition whose parts are $\mu_1,\ldots,\mu_{i-1},\mu_i - 1,\mu_{i+1},\ldots,\mu_r$ (not necessarily in nonincreasing order).

Example 4. If $\mu = (3,3,2)$, then $\mu^{(1)} = \mu^{(2)} = (3,2,2)$, and $\mu^{(3)} = (3,3,1)$.

Garsia and Procesi [4] show that

$$R_\mu = \bigoplus_{i=1}^{\mu_1'} x_n^{i-1}R_\mu/x_n^iR_\mu$$

as vector spaces. Moreover, considered as S_{n-1}-modules, we have

$$R_{\mu^{(i)}} \cong x_n^{i-1}R_\mu/x_n^iR_\mu$$

via the map $p \mapsto x_n^{i-1}p$. It follows that there is an S_{n-1}-module decomposition

$$R_\mu = \bigoplus_{i=1}^{\mu_1'} R_{\mu^{(i)}}.$$

We, therefore, can conclude the following.

Lemma 4. *Let $\mu \vdash n$ and suppose $\mathcal{C}(\mu^{(i)})$ is a basis of $R_{\mu^{(i)}}$ for each $i = 1, 2, \ldots, \mu_1'$. Then $\bigcup x_n^{i-1}\mathcal{C}(\mu^{(i)})$ is a basis of R_μ.*

With this in mind, we outline the following general strategy for proving that $\mathcal{B}_\mu$ is a basis for R_μ. We assume for induction that $\mathcal{B}_\lambda$ is a basis for R_λ for all smaller shapes λ contained in μ. Then we define $\mathcal{C}_\mu = \bigcup_i x_n^{i-1}\mathcal{B}_{\mu^{(i)}}$, which is a basis of R_μ by the induction hypothesis and Lemma 4. Finally, if we can show that the transition matrix between $\mathcal{B}_\mu$ and $\mathcal{C}_\mu$ is invertible, then the induction is complete.

We can further simplify this process by noting that we can restrict to basis elements of a given degree.

Definition 15. For any set of homogeneous polynomials $\mathcal{B}$, we write $\mathcal{B}^{(d)}$ to denote the subset of degree d polynomials in $\mathcal{B}$.

Note that it suffices to show that the transition matrix between $\mathcal{B}_\mu^{(d)}$ and $\mathcal{C}_\mu^{(d)}$ is invertible for every d, since both sets consist of homogeneous polynomials and R_μ is degree graded. We will implement this inductive approach for two-row shapes below, by showing that the transition matrix is in fact lower triangular in this case. We begin by illustrating this phenomenon with an example.

Example 5. Consider the case $\mu = (3,3)$ and $d = 2$. Figure 4 shows the transition matrix from $\mathcal{B}_\mu^{(2)}$ to $\mathcal{C}_\mu^{(2)}$. Here, the elements of $\mathcal{B}_\mu^{(2)}$ are written in last letter order down the left hand side of the table. The elements of $\mathcal{C}_\mu^{(2)} = \mathcal{B}_{(3,2)}^{(2)} \cup x_6\mathcal{B}_{(3,2)}^{(1)}$ are written across the top, with the elements from $\mathcal{B}_{(3,2)}^{(2)}$ coming before those of $x_6\mathcal{B}_{(3,2)}^{(1)}$, with ties broken in last letter order. If a coefficient is 0 we leave that entry blank.

Here,

$$S = \begin{array}{|c|c|cc}\hline 2 & 2 \\ \hline 1 & 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \qquad S' = \begin{array}{|c|c|c|}\hline 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline\end{array}, \qquad S'' = \begin{array}{|c|ccc}\hline 2 \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline\end{array}.$$

We will show in the proof of Theorem 2 that, if the largest number n is in the bottom row of T, then $F_T^S = cF_{T'}^{S'}$ for some constant c, where T' is formed by removing the largest entry n from T. On the other hand, if the largest number n is in the top row of T, then we will show that

$$F_T^S = \alpha x_n F_{T'}^{S''} + \beta \sum_{j=b_{d+1}}^{b_{n-d}} F_{T_j'}^{S'},$$

where $\alpha = \frac{d}{n-2d+1} + d$ and $\beta = \frac{n-d}{n-2d+1}$, and where T_j' is the tableau formed by removing j from the bottom row of T' and inserting it in the top row. Here

	$F^{S'}_{\begin{smallmatrix}2&4\\1&3&5\end{smallmatrix}}$	$F^{S'}_{\begin{smallmatrix}3&4\\1&2&5\end{smallmatrix}}$	$F^{S'}_{\begin{smallmatrix}2&5\\1&3&4\end{smallmatrix}}$	$F^{S'}_{\begin{smallmatrix}3&5\\1&2&4\end{smallmatrix}}$	$F^{S'}_{\begin{smallmatrix}4&5\\1&2&3\end{smallmatrix}}$	$x_6F^{S''}_{\begin{smallmatrix}2\\1&3&4&5\end{smallmatrix}}$	$x_6F^{S''}_{\begin{smallmatrix}3\\1&2&4&5\end{smallmatrix}}$	$x_6F^{S''}_{\begin{smallmatrix}4\\1&2&3&5\end{smallmatrix}}$	$x_6F^{S''}_{\begin{smallmatrix}5\\1&2&3&4\end{smallmatrix}}$
$F^{S}_{\begin{smallmatrix}2&4\\1&3&5&6\end{smallmatrix}}$	4								
$F^{S}_{\begin{smallmatrix}3&4\\1&2&5&6\end{smallmatrix}}$		4							
$F^{S}_{\begin{smallmatrix}2&5\\1&3&4&6\end{smallmatrix}}$			4						
$F^{S}_{\begin{smallmatrix}3&5\\1&2&4&6\end{smallmatrix}}$				4					
$F^{S}_{\begin{smallmatrix}4&5\\1&2&3&6\end{smallmatrix}}$					4				
$F^{S}_{\begin{smallmatrix}2&6\\1&3&4&5\end{smallmatrix}}$	4/3		4/3			8/3			
$F^{S}_{\begin{smallmatrix}3&6\\1&2&4&5\end{smallmatrix}}$		4/3		4/3			8/3		
$F^{S}_{\begin{smallmatrix}4&6\\1&2&3&5\end{smallmatrix}}$	−4/3	4/3			4/3			8/3	
$F^{S}_{\begin{smallmatrix}5&6\\1&2&3&5\end{smallmatrix}}$	4/3		−8/3	4/3	4/3				8/3

FIGURE 4. The transition matrix that expresses the elements of $\mathcal{B}^{(2)}_{(3,3)}$ (the row labels) in terms of those of $\mathcal{C}^{(2)}_{(3,3)} = \mathcal{B}^{(2)}_{(3,2)} \cup x_6\mathcal{B}^{(1)}_{(3,2)}$ (the column labels)

$n = 6$ and $d = 2$, so $\alpha = 8/3$ and $\beta = 4/3$. Thus, for instance,

$$F^{S}_{\begin{smallmatrix}2&6\\1&3&4&5\end{smallmatrix}} = \frac{8}{3}x_6F^{S''}_{\begin{smallmatrix}2\\1&3&4&5\end{smallmatrix}} + \frac{4}{3}F^{S'}_{\begin{smallmatrix}2&4\\1&3&5\end{smallmatrix}} + \frac{4}{3}F^{S'}_{\begin{smallmatrix}2&5\\1&3&4\end{smallmatrix}}.$$

Indeed, subtracting the right-hand side from the left-hand side of the above equation yields the polynomial

$$\begin{aligned}-\frac{8}{3}(x_2 - x_1)(x_3 + x_4 + x_5 + x_6) &= -\frac{8}{3}(e_2(x_2, x_3, x_4, x_5, x_6) \\ &\quad - e_2(x_1, x_3, x_4, x_5, x_6)) \\ &\in I_\mu.\end{aligned}$$

We similarly have

$$F^{S}_{\begin{smallmatrix}4&6\\1&3&4&5\end{smallmatrix}} = \frac{8}{3}x_6F^{S''}_{\begin{smallmatrix}4\\1&2&3&5\end{smallmatrix}} + \frac{4}{3}F^{S'}_{\begin{smallmatrix}4&3\\1&2&5\end{smallmatrix}} + \frac{4}{3}F^{S'}_{\begin{smallmatrix}4&5\\1&2&3\end{smallmatrix}}.$$

The second summand is not a basis element, but we can straighten it using the Garnir relations to express it in terms of $F^{S'}_{T'}$ elements where T' is standard, to obtain the second to last row of the matrix above.

The following lemma will be used repeatedly in the proof of Theorem 2.

Lemma 5. *Suppose μ is a two-row shape and S has content μ. Then S has at most two rows, say of lengths d and $n-d$. If T is the tableau of the same shape as S with entries $t_1, \ldots, t_d$ in the top row and $b_1, \ldots, b_{n-d}$ in the bottom, we have*

$$F_T^S = d!(n-d)! \prod_{i=1}^{d} (x_{t_i} - x_{b_i}). \tag{19}$$

Proof. The tableau S looks like

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \cline{1-3} 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ \hline \end{array},$$

where there are μ_1 1's, μ_2 2's, and exactly d of the 2's are in the top row. Thus, the cocharge indices are all 0 in the first row and 1 in the second. The equation follows.

We now prove Theorem 2, which we restate here for the reader's convenience.

Theorem 2. *If $\mu = (n-k, k)$ for some $k \geq 0$, the set*

$$\mathcal{B}_\mu = \{F_T^S : (S,T) \in \bigcup_{\lambda \vdash n} \mathrm{SSYT}(\lambda, \mu) \times \mathrm{SYT}(\lambda)\}$$

descends to a higher Specht basis of R_μ. In other words, Conjecture 1 holds for one- and two-row shapes.

Proof. The base case, $n = 1$, holds trivially for the unique partition $\mu = (1)$.

Let $\mu = (n-k, k)$ and assume for induction that the claim holds for all smaller two-row (or one-row) shapes fitting inside μ. In particular, it holds for $\mu^{(1)}$ and $\mu^{(2)}$. Then by Lemma 4, the set

$$\mathcal{C}_\mu := \mathcal{B}_{\mu^{(1)}} \cup x_n \mathcal{B}_{\mu^{(2)}}$$

is a basis for R_μ.

Let $t = |\mathcal{B}_\mu| = |\mathcal{C}_\mu| = \binom{n}{\mu}$. We will show there are total orderings $b_1, \ldots, b_t$ and $c_1, \ldots, c_t$ on $\mathcal{B}_\mu$ and $\mathcal{C}_\mu$, respectively, for which

$$b_i = \sum_{j \leq i} \alpha_{i,j} c_j$$

for some constants $\alpha_{i,j}$ with $\alpha_{i,i} \neq 0$. Since the transition matrix $[\alpha_{i,j}]$ is lower triangular with a nonzero diagonal, it will follow that $\mathcal{B}_\mu$ is a basis of R_μ.

To define these orderings, first note that the sets $\mathcal{B}_\mu$ and $\mathcal{C}_\mu$ both consist of homogeneous polynomials, and R_μ is graded by degree. We, therefore, can define $b_i < b_j$ if $\deg(b_i) < \deg(b_j)$ and similarly $c_i < c_j$ if $\deg(c_i) < \deg(c_j)$. With respect to this partial ordering, we have $\alpha_{i,j} = 0$ if $i < j$. Thus, it suffices to choose a fixed degree d and consider just the basis elements b_i and c_i of degree d, and choose an appropriate total ordering on the corresponding subsets $\mathcal{B}_\mu^{(d)}$ and $\mathcal{C}_\mu^{(d)}$ to show that the corresponding sub-matrix $M^{(d)}$ is lower triangular.

Since the cocharge of a tableau with only 1's and 2's is equal to the size of the top row, the elements in $\mathcal{B}_\mu^{(d)}$ are precisely those of the form F_T^S, where S is the unique tableau of shape $\lambda = (n-d, d)$ and content μ, and $T \in \mathrm{SYT}(\lambda)$. Since S is fixed, we define our ordering based on T. In particular, we define $F_{T_1}^S < F_{T_2}^S$ if and only if $T_1 \lessdot T_2$ in the last letter order. (See the row ordering of the matrix $M^{(d)}$ in Fig. 4 for an example.)

To order the elements of $\mathcal{C}_\mu^{(d)}$, let S' be the unique tableau of content $\mu^{(1)}$ and shape $\lambda^{(1)} = (n-d-1, d)$ (where if $n-d = d$ then S' is undefined), and let S'' be the unique tableau of content $\mu^{(2)}$ and shape $\lambda^{(2)} = (n-d, d-1)$. Then we have

$$\begin{aligned}\mathcal{C}_\mu^{(d)} &= \mathcal{B}_{\mu^{(1)}}^{(d)} \cup x_n \mathcal{B}_{\mu^{(2)}}^{(d-1)} \\ &= \{F_{T'}^{S'}\} \cup \{x_n F_{T''}^{S''}\},\end{aligned}$$

where in the first set above $T' \in \mathrm{SYT}(\lambda^{(1)})$ and in the second, $T'' \in \mathrm{SYT}(\lambda^{(2)})$. We enforce that the elements $F_{T'}^{S'}$ come before those of the form $x_n F_{T''}^{S''}$ in our ordering, and then we break ties by the last letter order on the subscripts T' and T'' respectively. (See the column ordering of the matrix $M^{(d)}$ in Fig. 4.)

Now, consider the set $\mathcal{B}_0$ of elements $F_T^S \in \mathcal{B}_\mu^{(d)}$ for which n is in the bottom row of T (so necessarily $d \neq n-d$). Note that $\mathcal{B}_0$ forms an initial sequence of the total ordering on $\mathcal{B}_\mu^{(d)}$. Removing n from the bottom row of such a tableau T forms a standard tableau T' of shape $\lambda^{(1)} = (n-d-1, d)$. We claim that

$$F_T^S = c F_{T'}^{S'}$$

for some constant c. Indeed, let $t_1, \ldots, t_d$ be the entries in the top row of T, and let $b_1, \ldots, b_d$ be the first d entries in the bottom row; then by Eq. (19), both polynomials are nonzero constant multiples of

$$(x_{t_1} - x_{b_1})(x_{t_2} - x_{b_2}) \cdots (x_{t_k} - x_{b_k}).$$

Thus, the sets $\mathcal{B}_0$ and $\mathcal{B}_{\mu^{(1)}}^{(d)}$, which are both initial sequences of their respective orderings, are scalar multiples of one another, and so the transition matrix $M^{(d)}$ is block lower triangular, of the form

$$\begin{pmatrix} cI & 0 \\ X & Y \end{pmatrix}.$$

It remains to show that Y is lower triangular with nonzero diagonal entries. We in fact will show that $Y = \alpha I$ for some constant α as well.

Indeed, let $\mathcal{B}_1$ be the set of elements of $\mathcal{B}_\mu^{(d)}$ of the form F_T^S where n is in the top row of T. Let T be such a tableau, with top row having entries $t_1, \ldots, t_d = n$ and bottom row having entries $b_1, \ldots, b_{n-d}$. Define T'' to be the tableau formed by deleting n from T, and define the tableau T_j' for $j \in \{b_{d+1}, \ldots, b_{n-d}\}$ to be the tableau formed by deleting j from the bottom row of T'' and placing it at the end of the top row. Note that T_j' may not be standard. However, since the Garnir relations are satisfied, $F_{T_j'}^{S'}$ is a linear combination of the polynomials $F_{T'}^{S'}$, where T'' is standard, which come before elements of the form $x_n F_{T''}^{S''}$ in the ordering on $\mathcal{C}_\mu^{(d)}$.

We will show that

$$F_T^S = \alpha x_n F_{T''}^{S''} + \beta \sum_{j=b_{d+1}}^{b_{n-d}} F_{T'_j}^{S'} \tag{20}$$

for some nonzero constants α and β. In light of the Garnir relations and the ordering, it will follow that $Y = \alpha I$ as claimed.

To show (20), set

$$\alpha = \frac{d}{n-2d+1} + d$$

and

$$\beta = \frac{n-d}{n-2d+1}.$$

Then we have, using (19) repeatedly,

$$\begin{aligned} F_T^S - \alpha x_n F_{T''}^{S''} - \beta \sum_{j=b_{d+1}}^{b_{n-d}} F_{T'_j}^{S'} &= d!(n-d)! \prod_{i=1}^{d} (x_{t_i} - x_{b_i}) \\ &\quad - \alpha x_n (d-1)!(n-d)! \prod_{i=1}^{d-1} (x_{t_i} - x_{b_i}) \\ &\quad - \beta \sum_{j=b_{d+1}}^{b_{n-d}} d!(n-d-1)!(x_j - x_{b_d}) \prod_{i=1}^{d-1} (x_{t_i} - x_{b_i}). \end{aligned}$$

We wish to show that the right-hand side is equal to 0 in R_μ. Thus, we may divide the right-hand side by $(d-1)!(n-d-1)!$, and as a shorthand define $P = \prod_{i=1}^{d-1}(x_{t_i} - x_{b_i})$, so that we wish to show that the simpler expression

$$P \cdot \left(d(n-d)(x_n - x_{b_d}) - \alpha(n-d)x_n - d\beta \sum_{j=b_{d+1}}^{b_{n-d}} (x_j - x_{b_d}) \right)$$

is 0 in R_μ, that is, it lies in the ideal I_μ. In the parenthetical above, substituting α and β in for the expressions, it is easily verified that the coefficients of x_n, x_{b_d}, and each x_j for $j = b_{d+1}, \ldots, b_{n-d}$ are all equal to $-d(n-d)/(n-2d+1)$. Thus, the entire expression is a constant multiple of

$$P \cdot (x_{b_d} + x_{b_{d+1}} \cdots + x_{b_{n-d}} + x_n). \tag{21}$$

Finally, we show that this expression is in I_μ. Note that $e_d(X) \in I_\mu$ for any set X of $n-d+1$ variables by the definition of the Tanisaki generators (Eq. (9)) and the fact that μ has two rows, the second of which is at least d. Thus

$$e_d(\{x_{r_1}, \ldots, x_{r_{d-1}}\} \cup \{x_{b_d}, x_{b_{d+1}}, \ldots, x_{b_{n-d}}, x_n\}) \in I_\mu$$

for any choice of subscripts in which r_i is either equal to t_i or b_i for each $i = 1, \ldots, d-1$. We assign this partial elementary symmetric function a sign of -1 if there are an odd number of r_i subscripts equal to b_i, and a sign of 1 otherwise. Summing these signed functions yields the expression (21).

3.3. Beyond Two-Row Shapes

In this section, we provide computer evidence that our inductive approach above may be able to be extended to all partition shapes.

First, Conjecture 1 has been verified using Sage [20] for all partition shapes μ of size at most 7. We have also verified it for the three-row shape $(3, 3, 2)$ of size 8, which is often the smallest shape in which conjectures related to cocharge start to break down (see, for instance, [7], in which a property of cocharge is proven combinatorially for all shapes of the form $(a, b, 1^k)$, but the method does not extend to any other three row shapes).

Second, while the transition matrix expressing $\mathcal{B}_\mu^{(d)}$ in terms of $\mathcal{C}_\mu^{(d)}$ is not always lower triangular for partition shapes μ having more than two rows, it is very nearly so, in the following sense.

Definition 16. We say an $n \times n$ matrix M is **almost lower triangular** if there is an upper triangular $n \times n$ matrix A for which MA is lower triangular with nonzero diagonal entries.

Clearly every invertible lower triangular matrix is almost lower triangular, and every almost lower triangular matrix is invertible. Computer evidence indicates that there always exist orderings on the sets $\mathcal{B}_\mu^{(d)}$ and $\mathcal{C}_\mu^{(d)}$ such that the transition matrix between them in R_μ is almost lower triangular.

For example, the transition matrix for $\mu = (3, 1, 1)$ and $d = 2$ is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 1/2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 1/2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1/2 & -1/2 & 1/2 & 0 & 0 & 1 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 & -5/4 \end{pmatrix}$$

which is almost lower triangular. Indeed, multiplying M on the right by the upper triangular matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

yields a lower triangular matrix with nonzero diagonal entries.

4. A Higher Specht Basis for $R_{n,k,(n-1)}$

We now combine the methods of the previous two sections to prove Theorem 3, which we restate here for the reader's convenience.

Theorem 3. *Consider the set of polynomials*

$$\mathcal{B}_{n,k,(n-1)} = \{F_T^S \cdot e_1^i\},$$

where $F_T^S \in \mathcal{B}_{(n-1,1)}$ is a semistandard higher Specht polynomial for the shape $(n-1,1)$, and $i < k - \mathrm{des}(S)$. Then $\mathcal{B}_{n,k,(n-1)}$ descends to a higher Specht basis for $R_{n,k,(n-1)}$.

Proof. Since S_n acts trivially on the elementary symmetric function e_1, if $\mathcal{B}_{n,k,(n-1)}$ is a basis then it is indeed a higher Specht basis. In particular, the polynomials $F_T^S \cdot e_1^i$ for a fixed i and for a fixed tableau S of shape λ span a copy of the irreducible representation V^λ of S_n.

To show that $\mathcal{B}_{n,k,(n-1)}$ is a basis, we make use of a short exact sequence for the modules $R_{n,k,\mu}$ that is analogous to the sequence (13) for $R_{n,k,s}$ used in Sect. 2. Griffin shows [8, Lem. 4.12] that there is a short exact sequence of S_n-modules

$$0 \to R_{n,k,\mu} \to R_{n,k+1,\mu} \to R_{n,k+1,\mu+(1)} \to 0$$

for any $k < n$ and μ for which $R_{n,k,\mu}$ is defined. Here the notation $\mu + (1)$ indicates that we simply add one part of size 1 to the partition μ. In the sequence above, the first nontrivial map is multiplication by $e_{n-|\mu|}$ and the second is given by setting $e_{n-|\mu|} = 0$.

Setting $\mu = (n-1)$, we have the exact sequences

$$0 \to R_{n,k,(n-1)} \to R_{n,k+1,(n-1)} \to R_{n,k+1,(n-1,1)} \to 0$$

for any $k \geq 1$.

We now prove the claim by induction on k. For the base case $k = 1$, note that we have $R_{n,1,(n-1)} = \mathbb{C}[x_1, \ldots, x_n]/I_{n,1,(n-1)}$, where the ideal $I_{n,1,(n-1)}$ includes all the variables $x_1, \ldots, x_n$ as generators, since $k = 1$. Hence we simply have $R_{n,1,(n-1)} \cong \mathbb{C}$, generated by the single basis element 1. The set $\mathcal{B}_{n,1,(n-1)}$ consists of all polynomials $F_T^S \cdot e_1^i$ for which S has content $(n-1,1)$

and $i < 1 - \mathrm{des}(S)$, which forces $\mathrm{des}(S) = 0$ and $i = 0$. The only such tableau S is

$$S = \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{\cdots}\boxed{1}\boxed{1}\boxed{2}$$

which forces

$$T = \boxed{1}\boxed{2}\boxed{3}\boxed{\cdots}\boxed{n},$$

and these give rise to the unique basis element $F_T^S = 1$.

For the induction step, let $k \geq 2$ and assume the claim holds for all smaller k. Note that since $(n-1,1)$ is a partition of n, the right-hand module $R_{n,k+1,(n-1,1)}$ of the exact sequence is simply the Garsia–Procesi module $R_{(n-1,1)}$ for any $k \geq 1$. Hence, by Theorem 2, a higher Specht basis for this module is given by $\mathcal{B}_{(n-1,1)}$.

By the induction hypothesis, the left-hand term of the exact sequence has $\mathcal{B}_{n,k,(n-1)}$ as a basis. It follows that the middle term $R_{n,2,(n-1)}$ has basis $e_1\mathcal{B}_{n,k,(n-1)} \cup \mathcal{B}_{(n-1,1)}$. By the definition of the bases, this is simply equal to $\mathcal{B}_{(n,k+1,(n-1))}$, and the proof is complete.

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