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Vandermondes, Superspace, and Delta Conjecture modules

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Abstract. Superspace is an algebra Ω_n with n commuting generators $x_1, \ldots, x_n$ and n anticommuting generators $\theta_1, \ldots, \theta_n$. We present an extension $\delta_{n,k}$ of the Vandermonde determinant to Ω_n which depends on positive integers $k \leq n$. We use superspace Vandermondes to build representations of the symmetric group S_n. In particular, we construct a doubly graded S_n-module $\mathbb{V}_{n,k}$ whose bigraded Frobenius image $\mathrm{grFrob}(\mathbb{V}_{n,k}; q, t)$ conjecturally equals the symmetric function $\Delta'_{e_{k-1}} e_n$ appearing in the Haglund-Remmel-Wilson Delta Conjecture. We prove the specialization of our conjecture at $t = 0$. We use a differentiation action of Ω_n on itself to build bigraded quotients $\mathbb{W}_{n,k}$ of Ω_n which extend the Delta Conjecture coinvariant rings $R_{n,k}$ defined by Haglund-Rhoades-Shimozono and studied geometrically by Pawlowski-Rhoades. Despite the fact that the Hilbert polynomials of the $R_{n,k}$ are not palindromic, we show that $\mathbb{W}_{n,k}$ exhibits a superspace version of Poincaré Duality.

Keywords: Vandermonde, superspace, S_n-module

1 Introduction

The symmetric group S_n acts on the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ by subscript permutation. Polynomials in the invariant subring

$$\mathbb{Q}[x_1, \ldots, x_n]^{S_n} := \{ f \in \mathbb{Q}[x_1, \ldots, x_n] \, : \, w.f = f \text{ for all } w \in S_n \} \tag{1.1}$$

are called *symmetric polynomials*. The $\mathbb{Q}$-algebra $\mathbb{Q}[x_1, \ldots, x_n]^{S_n}$ is generated by the n elementary symmetric polynomials $e_1, e_2, \ldots, e_n$.

Let $\mathbb{Q}[x_1, \ldots, x_n]^{S_n}_+$ be the space of symmetric polynomials with vanishing constant term. The *invariant ideal* $I_n \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ is given by

$$I_n := \langle \mathbb{Q}[x_1, \ldots, x_n]^{S_n}_+ \rangle = \langle e_1, e_2, \ldots, e_n \rangle, \tag{1.2}$$

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and the *coinvariant ring* is the corresponding quotient

$$R_n := \mathbb{Q}[x_1, \dots, x_n]/I_n. \tag{1.3}$$

The quotient R_n is simultaneously a graded ring and a graded S_n-module. The module R_n is among the most important in algebraic combinatorics, with representation theory tied to permutation combinatorics and a geometric realization as the cohomology of the flag variety [1, 3].

The symmetric group S_n acts diagonally on the polynomial ring $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ in $2n$ variables, viz. $w.x_i = x_{w(i)}$ and $w.y_i := y_{w(i)}$ for all $w \in S_n$ and $1 \leq i \leq n$. Garsia and Haiman [4] initiated the study of the the *diagonal coinvariant ring* DR_n defined by modding out by those S_n-invariants with vanishing constant term:

$$DR_n := \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]/\langle \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]_+^{S_n} \rangle. \tag{1.4}$$

Considering x-degree and y-degree separately, the ring DR_n is a doubly graded S_n-module which specializes to R_n when the y-variables are set to zero.

Haiman proved [8] that as ungraded S_n-modules we have $DR_n \cong \mathbb{Q}[\mathrm{Park}_n] \otimes \mathrm{sign}$ where Park_n is the permutation action of S_n on size n parking functions and sign is the 1-dimensional sign representation of S_n. Haiman also proved more refined results on the bigraded S_n-module structure of DR_n; to state these we recall some facts about S_n-modules.

The irreducible representations of S_n over $\mathbb{Q}$ are indexed by partitions of n; if $\lambda \vdash n$ is a partition, let S^λ be the corresponding S_n-irreducible. If V is any finite-dimensional S_n-module, there exist unique multiplicities $c_\lambda \geq 0$ so that $V \cong \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$. Let Λ denote the ring of symmetric functions over the ground field $\mathbb{Q}(q,t)$ in the infinite variable set $\mathbf{x} = (x_1, x_2, \dots)$. The *Frobenius image* of V is the symmetric function $\mathrm{Frob}(V) \in \Lambda$ given by $\mathrm{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda s_\lambda$, where s_λ is the Schur function.

In this extended abstract we will consider (multi)graded S_n-modules. If $V = \bigoplus_{i \geq 0} V_i$ is a graded S_n-module with each graded piece V_i finite-dimensional, the *graded Frobenius image* of V is $\mathrm{grFrob}(V;q) := \sum_{i \geq 0} q^i \cdot \mathrm{Frob}(V_i)$. Even more generally, if $V = \bigoplus_{i,j \geq 0} V_{i,j}$ or $V = \bigoplus_{i,j,k \geq 0} V_{i,j,k}$ is a doubly or triply graded S_n-module, we have the associated bigraded and trigraded Frobenius images

$$\mathrm{grFrob}(V;q,t) := \sum_{i,j \geq 0} q^i t^j \cdot \mathrm{Frob}(V_{i,j}) \quad \text{or} \quad \mathrm{grFrob}(V;q,t,z) := \sum_{i,j,k \geq 0} q^i t^j z^k \cdot \mathrm{Frob}(V_{i,j,k}),$$

respectively.

Haiman proved [8] that $\mathrm{grFrob}(DR_n;q,t) = \nabla e_n$, where e_n is the degree n elementary symmetric function and ∇ is the Bergeron-Garsia *nabla operator*. Therefore, describing the bigraded S_n-isomorphism type of DR_n is equivalent to finding the Schur expansion of ∇e_n, but there is not even a conjecture in this direction. The monomial expansion of ∇e_n is given by the *Shuffle Theorem* [2].

The *Delta Conjecture* is a conjectural extension of the Shuffle Theorem due to Haglund, Remmel, and Wilson [6]. It depends on two positive integers $k \leq n$ and reads

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(\mathbf{x}; q, t) = \mathrm{Val}_{n,k}(\mathbf{x}; q, t). \tag{1.5}$$

Here $\Delta'_{e_{k-1}}$ is a certain symmetric function operator and Rise and Val are formal power series defined using the combinatorics of lattice paths; see [6] for details. When $k = n$, the Delta Conjecture reduces to the Shuffle Theorem.

The Delta Conjecture is open as of this writing, but combining the work of [5, 7, 11, 14] it is known at $q = 0$. More precisely, we have

$$\Delta'_{e_{k-1}} e_n \mid_{t=0} = \mathrm{Rise}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Rise}_{n,k}(\mathbf{x}; 0, q) = \mathrm{Val}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Val}_{n,k}(\mathbf{x}; 0, q). \tag{1.6}$$

In this paper we define a doubly graded S_n-module $\mathbb{V}_{n,k}$ for any positive integers $k \leq n$ and conjecture that $\mathrm{grFrob}(\mathbb{V}_{n,k}; q, t) = \Delta'_{e_{k-1}} e_n$ (see Conjecture 1). That is, we conjecture that $\mathbb{V}_{n,k}$ is a module for the Delta Conjecture. We prove this conjecture at $t = 0$. In order to describe $\mathbb{V}_{n,k}$, we introduce new combinatorial objects called *superspace Vandermondes*.

Superspace of rank n is the unital associative $\mathbb{Q}$-algebra Ω_n generated by $2n$ symbols $x_1, \ldots, x_n, \theta_1, \ldots, \theta_n$ subject to the relations

$$x_i x_j = x_j x_i \qquad x_i \theta_j = \theta_j x_i \qquad \theta_i \theta_j = -\theta_j \theta_i$$

for all $1 \leq i, j \leq n$.[1] Setting the θ-variables to zero recovers the classical polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. By considering x-degree and θ-degree separately, Ω_n is a doubly graded algebra. The ring Ω_n carries a diagonal action of S_n given by $w.x_i := x_{w(i)}$ and $w.\theta_i := \theta_{w(i)}$ for $w \in S_n$ and $1 \leq i \leq n$.

Defintion 1. *Let $k \leq n$ be positive integers. The* superspace Vandermonde $\delta_{n,k}$ *is the following element of Ω_n:*

$$\delta_{n,k} := \varepsilon_n.(x_1^{k-1} x_2^{k-1} \cdots x_{n-k+1}^{k-1} x_{n-k+2}^{k-2} \cdots x_{n-1}^{1} x_n^{0} \theta_1 \theta_2 \cdots \theta_{n-k}). \tag{1.7}$$

Here $\varepsilon_n := \sum_{w \in S_n} \mathrm{sign}(w) \cdot w \in \mathbb{Q}[S_n]$ is the antisymmetrizing element in the symmetric group algebra.

For example, when $n = 3$ and $k = 2$ we have

$$\delta_{3,2} = \varepsilon_3.(x_1 x_2 \theta_1) = x_1 x_2 \theta_1 - x_1 x_2 \theta_2 - x_1 x_3 \theta_1 + x_1 x_3 \theta_3 + x_2 x_3 \theta_2 - x_2 x_3 \theta_3.$$

[1] The 'super' in superspace comes from supersymmetry in physics: the x-variables index bosons and the θ-variables index fermions. Extending coefficients to the reals, Ω_n is the ring of polynomial-valued differential forms on Euclidean n-space – this is why we write Ω.

The superpolynomial $\delta_{n,k}$ is always a nonzero element of Ω_n, thanks to the θ-variables. When $k = n$, the superspace Vandermonde $\delta_{n,k}$ reduces to the classical Vandermonde determinant $\varepsilon_n.(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1 x_n^0)$.

The $\delta_{n,k}$ are seeds we use to grow modules. By starting with $\delta_{n,k}$ and closing under various differential operators and linearity we will construct:

- A singly graded subspace $V_{n,k}$ of Ω_n which satisfies $\mathrm{grFrob}(V_{n,k};q) = \Delta'_{e_{k-1}} e_n \mid_{t=0}$ (see **Section 2**).
- A doubly graded extension $\mathbb{V}_{n,k}$ of $V_{n,k}$ with $\mathrm{grFrob}(\mathbb{V}_{n,k};q,t)$ conjecturally given by $\Delta'_{e_{k-1}} e_n$ (see **Section 2**).
- A doubly graded S_n-stable quotient $\mathbb{W}_{n,k}$ of Ω_n which extends $V_{n,k}$ and exhibits a number of symmetries including a superspace variant of Poincaré Duality (see **Section 4**). $\mathbb{W}_{n,k}$ extends the cohomology of the space of spanning line configurations studied by Pawlowski and Rhoades [10].

This paper is not the first to propose connections between the Delta Conjecture and superspace. The Fields Institute Combinatorics Group in general, and Mike Zabrocki in particular, conjectured [15] that representation-theoretic models for the Delta Conjecture can be obtained by looking at coinvariant-type quotients defined using superspace Ω_n and an extension $\Omega_n[y_1,\ldots,y_n]$ of superspace involving n new commuting variables $y_1,\ldots,y_n$. We discuss the connection between our work and their conjectures in **Section 3**. In a nutshell, we are able to prove that our proposed Delta model $\mathbb{V}_{n,k}$ is valid at $t = 0$, but the corresponding case of their conjecture remains open.

2 The S_n-modules $V_{n,k}$ and $\mathbb{V}_{n,k}$ and the Delta Conjecture

For $1 \leq i \leq n$, the partial derivative operator $\partial/\partial x_i$ acts naturally on the polynomial ring $\mathbb{Q}[x_1,\ldots,x_n]$. Superspace admits the tensor product decomposition

$$\Omega_n = \mathbb{Q}[x_1,\ldots,x_n] \otimes \wedge\{\theta_1,\ldots,\theta_n\} \tag{2.1}$$

where $\wedge\{\theta_1,\ldots,\theta_n\}$ is the exterior algebra on the generators $\theta_1,\ldots,\theta_n$. The operator $\partial/\partial x_i$ therefore acts on Ω_n by acting on the first tensor factor.

Our first new S_n-module is defined as follows. Starting with the superspace Vandermonde $\delta_{n,k}$, we close under the operators $\partial/\partial x_1,\ldots,\partial/\partial x_n$ and linearity.

Defintion 2. *Let $k \leq n$ be positive integers. The vector space $V_{n,k}$ is the smallest $\mathbb{Q}$-linear subspace of Ω_n which*

- *contains the superspace Vandermonde $\delta_{n,k}$, and*

- *is closed under the n partial derivatives* $\partial/\partial x_1, \ldots, \partial/\partial x_n$.

The subspace $V_{n,k} \subseteq \Omega_n$ is closed under the action of S_n. Furthermore, $V_{n,k}$ a doubly graded subspace of Ω_n. If we ignore the θ-grading (which is constant of degree $n-k$) and focus on the x-grading, we see that $V_{n,k}$ is a singly-graded S_n-module.

To describe the Schur expansion of $\mathrm{grFrob}(V_{n,k};q)$, we need some notation. Let T be a standard Young tableau with n boxes. A number $1 \leq i \leq n-1$ is a *descent* of T if i appears in a row above $i+1$. The *descent number* $\mathrm{des}(T)$ is the number of descents and the *major index* $\mathrm{maj}(T)$ is the sum of the descents in T. We write $\mathrm{shape}(T) \vdash n$ for the partition of n obtained by erasing the numbers in T. We also use the standard q-numbers, q-factorials, and q-binomials:

$$[n]_q := 1 + q + \cdots + q^{n-1} \quad [n]!_q := [n]_q [n-1]_q \cdots [1]_q \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}. \tag{2.2}$$

Theorem 1. *Let $k \leq n$ be positive integers. The graded Frobenius image of $V_{n,k}$ is given by either of the expressions*

$$\mathrm{grFrob}(V_{n,k};q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T) + \binom{n-k}{2} - (n-k)\cdot \mathrm{des}(T)} \begin{bmatrix} \mathrm{des}(T) \\ n-k \end{bmatrix}_q s_{\mathrm{shape}(T)} \tag{2.3}$$

$$= \Delta'_{e_{k-1}} e_n \mid_{t=0} \tag{2.4}$$

where the sum is over all standard Young tableaux T with n boxes.

Equation (1.6) allows us to replace the $\Delta'_{e_{k-1}} e_n \mid_{t=0}$ in Theorem 1 with any of the symmetric functions $\mathrm{Rise}_{n,k}(\mathbf{x};q,0)$, $\mathrm{Rise}_{n,k}(\mathbf{x};0,q)$, $\mathrm{Val}_{n,k}(\mathbf{x};q,0)$, or $\mathrm{Val}_{n,k}(\mathbf{x};0,q)$. Thanks to Theorem 1, it is easy to describe the ungraded S_n-isomorphism type of $V_{n,k}$.

Corollary 1. *Let $k \leq n$ be positive integers and consider the permutation action of S_n on the family $\mathcal{OP}_{n,k}$ of k-block ordered set partitions $(B_1, B_2, \ldots, B_k)$ of $\{1, 2, \ldots, n\}$. As ungraded S_n-modules we have*

$$V_{n,k} \cong \mathbb{Q}[\mathcal{OP}_{n,k}] \otimes \mathrm{sign} \tag{2.5}$$

where sign *is the 1-dimensional sign representation of S_n.*

The *(signless) Stirling number of the second kind* $\mathrm{Stir}(n,k)$ counts (unordered) k-block set partitions of $\{1, 2, \ldots, n\}$. Corollary 1 implies $\dim V_{n,k} = k! \cdot \mathrm{Stir}(n,k)$. The graded dimension of $V_{n,k}$ is given by a suitable q-analog of this formula.

Recall that the *Hilbert series* of a graded vector space $V = \bigoplus_{i \geq 0} V_i$ is the formal power series $\mathrm{Hilb}(V;q) := \sum_{i \geq 0} q^i \cdot \dim V_i$. The *q-Stirling number* $\mathrm{Stir}_q(n,k)$ is defined by the recursion

$$\mathrm{Stir}_q(n,k) = \mathrm{Stir}_q(n-1,k-1) + [k]_q \cdot \mathrm{Stir}_q(n-1,k) \tag{2.6}$$

together with the initial conditions $\mathrm{Stir}_q(0,0) = 1$ and $\mathrm{Stir}_q(0,k) = 0$ for any $k > 0$.

Corollary 2. *The Hilbert series of $V_{n,k}$ is* $\mathrm{Hilb}(V_{n,k};q) = [k]!_q \cdot \mathrm{Stir}_q(n,k)$.

In order to describe our proposed model for the Delta Conjecture, we need more variables. Let $y_1, \ldots, y_n$ be n new commuting variables and consider the extension $\Omega_n[y_1, \ldots, y_n]$ of superspace defined formally by the tensor product

$$\Omega_n[y_1, \ldots, y_n] := \mathbb{Q}[x_1, \ldots, x_n] \otimes \mathbb{Q}[y_1, \ldots, y_n] \otimes \wedge\{\theta_1, \ldots, \theta_n\}. \tag{2.7}$$

This is a *triply* graded S_n-module with action $w.x_i := x_{w(i)}, w.y_i := y_{w(i)}, w.\theta_i := \theta_{w(i)}$. This ring admits an action of partial derivatives $\partial/\partial x_i$ and $\partial/\partial y_i$ in both the x-variables and y-variables.

Defintion 3. *For $k \leq n$, let $\mathbb{V}_{n,k}$ be the smallest $\mathbb{Q}$-linear subspace of $\Omega_n[y_1, \ldots, y_n]$ which*

- *contains the superspace Vandermonde $\delta_{n,k}$ (in the x-variables and θ-variables alone),*
- *is closed under the* polarization operator $\sum_{s=1}^{n} y_s (\partial/\partial x_s)^j$ *for each $j \geq 1$, and*
- *is closed under the $2n$ partial derivatives $\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial y_1, \ldots, \partial/\partial y_n$.*

The S_n-module $\mathbb{V}_{n,k}$ is concentrated in θ-degree $n-k$. By considering x-degree and y-degree, the space $\mathbb{V}_{n,k}$ attains the structure of a doubly graded S_n-module.

Conjecture 1. *Let $k \leq n$ be positive integers. The doubly graded Frobenius image of $\mathbb{V}_{n,k}$ is given by*

$$\mathrm{grFrob}(\mathbb{V}_{n,k};q,t) = \Delta'_{e_{k-1}} e_n. \tag{2.8}$$

Conjecture 1 is true at $t = 0$ by Theorem 1. Conjecture 1 is true when $k = n$ by the work of Haiman [8]. Conjecture 1 has been checked on computer for $n \leq 4$ (and at $n = 5$ on the level of bigraded Hilbert series). Since every increase $n \to n+1$ adds two new commuting variables and one new anticommuting variable, studying Conjecture 1 involves considerable computational challenges as n grows.

3 The Fields and Zabrocki Conjectures

In this section we describe alternative conjectural representation-theoretic models for the Delta Conjecture arising from quotients of Ω_n and $\Omega_n[y_1, \ldots, y_n]$. Recall that the symmetric group S_n acts diagonally on superspace Ω_n. Solomon proved [12] that the ring $(\Omega_n)^{S_n} \subseteq \Omega_n$ of S_n-invariants is a free $\mathbb{Q}[x_1, \ldots, x_n]^{S_n}$-module on the generating set $\{de_{i_1} \cdots de_{i_r} : 1 \leq i_1 < \cdots < i_r \leq n\}$ where $d := \sum_{j=1}^{n} \theta_j \cdot (\partial/\partial x_j)$ is the total derivative operator.

Let $\langle (\Omega_n)_+^{S_n} \rangle \subseteq \Omega_n$ be the two-sided ideal of Ω_n generated by S_n-invariants with vanishing constant term. By considering x-degree and θ-degree, the quotient $\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle$

is a doubly graded S_n-module. We view this quotient as a 'superspace coinvariant ring'. The following conjecture about its doubly graded Frobenius image was made by the Combinatorics Group at the Fields Institute.

Fields Conjecture. *(see [15]) Let n be a positive integer. The doubly graded Frobenius image of $\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle$ is given by*

$$\mathrm{grFrob}(\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n \mid_{t=0}, \tag{3.1}$$

where q tracks x-degree and z tracks θ-degree.

If the Fields Conjecture is true, the bigraded Hilbert series of $\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle$ would be given by

$$\mathrm{Hilb}(\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot [k]!_q \cdot \mathrm{Stir}_q(n,k) \tag{3.2}$$

where q tracks x-degree and z tracks θ-degree. The Fields Combinatorics Group proved (personal communication) the inequality

$$\mathrm{Hilb}(\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle; q, z) \geq \sum_{k=1}^{n} z^{n-k} \cdot [k]!_q \cdot \mathrm{Stir}_q(n,k) \tag{3.3}$$

where $f(q,z) \geq g(q,z)$ means that the difference $f(q,z) - g(q,z)$ is a polynomial in q, z with nonnegative coefficients.

Recall that the *alternating subspace* of an S_n-module V is given by

$$\{v \in V \,:\, w.v = \mathrm{sign}(w) \cdot v \text{ for all } w \in S_n\}.$$

Let A_n be the alternating subspace of $\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle$. The alternant space A_n is a doubly graded vector space. The Fields Conjecture would imply that

$$\mathrm{Hilb}(A_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot q^{\binom{k}{2}} \cdot \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{3.4}$$

Equation (3.4) has been verified by Swanson and Wallach [13], giving further evidence for the Fields Conjecture.

If the Fields Conjecture is true, we would have an isomorphism of ungraded S_n-modules $\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle \cong \bigoplus_{k=1}^{n} (\mathbb{Q}[\mathcal{OP}_{n,k}] \otimes \mathrm{sign})$. At present, it is unknown whether either of these S_n-modules injects into the other.

The symmetric functions appearing in the Fields Conjecture and Theorem 1 are closely related. We propose the following 'bridge conjecture' whose truth would yield the Fields Conjecture. Let φ be the composite linear map

$$\varphi : V_{n,1} \oplus \cdots \oplus V_{n,n} \hookrightarrow \Omega_n \twoheadrightarrow \Omega_n / \langle (\Omega_n)_+^{S_n} \rangle \tag{3.5}$$

obtained by including the direct sum $V_{n,1} \oplus \cdots \oplus V_{n,n}$ into superspace and then projecting onto the superspace coinvariant ring.

Conjecture 2. *The linear map φ is bijective.*

Mike Zabrocki studied the triply diagonal action of S_n on the ring $\Omega_n[y_1, \ldots, y_n]$ and the associated space $\Omega_n[y_1, \ldots, y_n]_+^{S_n}$ of S_n-invariants with vanishing constant term. He checked the following conjecture by computer for $n \leq 6$.

Zabrocki Conjecture. *([15]) Let n be a positive integer. We have*

$$\mathrm{grFrob}(\Omega_n[y_1, \ldots, y_n] / \langle \Omega_n[y_1, \ldots, y_n]_+^{S_n} \rangle; q, t, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n \tag{3.6}$$

where q tracks x-degree, t tracks y-degree, and z tracks θ-degree.

The Zabrocki Conjecture is related to Conjecture 1 in the same way as the Fields Conjecture is related to Theorem 1. Since Theorem 1 is proven whereas the Fields Conjecture remains open, superspace Vandermondes might prove an easier road to Delta Conjecture modules than quotient rings.

4 The ring $\mathbb{W}_{n,k}$ and Super Poincaré Duality

So far we have built S_n-modules $V_{n,k}$ and $\mathbb{V}_{n,k}$ by starting with the superspace Vandermonde $\delta_{n,k}$ and closing under partial derivatives in the commuting variables x_i, y_i (and potentially polarization operators). The modules $V_{n,k}$ and $\mathbb{V}_{n,k}$ have the defect of not being closed under multiplication and not admitting a natural ring structure. In this section we build a new bigraded S_n-module $\mathbb{W}_{n,k}$ from $\delta_{n,k}$. The module $\mathbb{W}_{n,k}$ is naturally a bigraded quotient of Ω_n. The module $\mathbb{W}_{n,k}$ turns out to extend both $V_{n,k}$ and the cohomology ring $H^\bullet(X_{n,k}; \mathbb{Q})$ of a variety $X_{n,k}$ of line configurations studied by Pawlowski and Rhoades. In order to define $\mathbb{W}_{n,k}$, we need operators $\partial/\partial\theta_i$ on Ω_n which differentiate with respect to anticommuting variables.

For $1 \leq i \leq n$, let $\partial/\partial\theta_i : \Omega_n \to \Omega_n$ be the $\mathbb{Q}[x_1, \ldots, x_n]$-module endomorphism characterized by

$$\partial/\partial\theta_i : \theta_{j_1} \cdots \theta_{j_r} \mapsto \begin{cases} (-1)^{s-1} \theta_{j_1} \cdots \widehat{\theta_{j_s}} \cdots \theta_{j_r} & \text{if } j_s = i \\ 0 & \text{if } i \neq j_1, \ldots, j_r \end{cases} \tag{4.1}$$

where $1 \leq j_1, \ldots, j_r \leq n$ are distinct indices and $\widehat{\cdot}$ means omission. The sign $(-1)^{s-1}$ is necessary to ensure that $\partial/\partial\theta_i$ is well-defined.

Defintion 4. *For positive integers $k \leq n$, let $\mathbb{W}_{n,k}$ be the smallest linear subspace of Ω_n which*

- *contains the superspace Vandermonde $\delta_{n,k}$, and*
- *is closed under the $2n$ operators $\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial\theta_1, \ldots, \partial/\partial\theta_n$.*

The vector space $\mathbb{W}_{n,k}$ is a bigraded S_n-module. We use an action of superspace on itself to show that $\mathbb{W}_{n,k}$ is naturally a bigraded quotient of Ω_n.

The operators $\partial/\partial\theta_i$ and $\partial/\partial x_i$ on Ω_n satisfy the relations

$$(\partial/\partial x_i)(\partial/\partial x_j) = (\partial/\partial x_j)(\partial/\partial x_i) \quad (\partial/\partial x_i)(\partial/\partial\theta_j) = (\partial/\partial\theta_j)(\partial/\partial x_i)$$

$$(\partial/\partial\theta_i)(\partial/\partial\theta_j) = -(\partial/\partial\theta_j)(\partial/\partial\theta_i)$$

for all $1 \leq i, j \leq n$. These are the defining relations of Ω_n, so for any superpolynomial $f = f(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n)$ we have an unambiguous operator ∂f on Ω_n obtained by replacing each x_i in f with $\partial/\partial x_i$ and each θ_i in f by $\partial/\partial\theta_i$. This gives rise to an action $\odot : \Omega_n \times \Omega_n \to \Omega_n$ of superspace on itself by the rule

$$f \odot g := \partial f(g). \tag{4.2}$$

Proposition 1. *Let $\mathrm{ann}(\delta_{n,k}) := \{f \in \Omega_n : f \odot \delta_{n,k} = 0\}$ be the annihilator in Ω_n of the superspace Vandermonde $\delta_{n,k}$. Then $\mathrm{ann}(\delta_{n,k})$ is a two-sided ideal in Ω_n which is S_n-stable and bigraded. The canonical composition*

$$\mathbb{W}_{n,k} \hookrightarrow \Omega_n \twoheadrightarrow \Omega_n/\mathrm{ann}(\delta_{n,k}) \tag{4.3}$$

is an isomorphism of bigraded S_n-modules.

Thanks to Proposition 1, there is a natural multiplication operation on $\mathbb{W}_{n,k}$, so that the anticommuting differentiation operators $\partial/\partial\theta_i$ give rise to a ring structure which $V_{n,k}$ and $\mathbb{V}_{n,k}$ lack.

What do the bigraded S_n-modules $\mathbb{W}_{n,k}$ look like? We display $\mathrm{grFrob}(\mathbb{W}_{4,2}; q, z)$ in matrix format, with rows labeling θ-degree and columns labeling x-degree.

$$\mathrm{grFrob}(\mathbb{W}_{4,2}; q, z) = \begin{pmatrix} s_4 & s_4 + s_{31} & s_4 + s_{31} + s_{22} & s_{31} \\ s_{31} & 2s_{31} + s_{22} + s_{211} & s_{31} + s_{22} + 2s_{211} & s_{211} \\ s_{211} & s_{22} + s_{211} + s_{1111} & s_{211} + s_{1111} & s_{1111} \end{pmatrix} \tag{4.4}$$

The matrices $\mathrm{grFrob}(\mathbb{W}_{n,k}; q, z)$ enjoy the following properties. Let $U_n = S^{(n-1,1)}$ be the $(n-1)$-dimensional reflection representation of S_n.

Theorem 2. *There hold the following facts about the bigraded S_n-module $\mathbb{W}_{n,k}$.*

1. (Special k) *We have $\mathbb{W}_{n,n} \cong R_n$ (coinvariant ring) and $\mathbb{W}_{n,1} \cong \wedge U_n$ (exterior algebra).*
2. (Bottom x-degree) *The x-degree 0 piece of $\mathbb{W}_{n,k}$ is isomorphic to $\bigoplus_{j=0}^{n-k} \wedge^j U_n$.*

3. (Top x-degree) *The top x-degree of $\mathbb{W}_{n,k}$ is $\binom{k}{2} + (n-k)\cdot(k-1)$; this piece of $\mathbb{W}_{n,k}$ is isomorphic to $\bigoplus_{j=0}^{n-k} \wedge^j U_n \otimes \mathrm{sign}$.*

4. (Top θ-degree) *The top ($= n-k$) θ-degree piece of $\mathbb{W}_{n,k}$ is isomorphic to $V_{n,k}$.*

5. (Bottom θ-degree) *Let $I_{n,k} \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ be $I_{n,k} := \langle x_1^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle$ and let*
$$R_{n,k} := \mathbb{Q}[x_1, \ldots, x_n]/I_{n,k}. \tag{4.5}$$
The θ-degree 0 piece of $\mathbb{W}_{n,k}$ is isomorphic to $R_{n,k}$.

The quotient rings $R_{n,k}$ displayed in Item 5 of Theorem 2 were introduced by Haglund, Rhoades, and Shimozono [7]. They proved that

$$\mathrm{grFrob}(R_{n,k}; q) = (\mathrm{rev}_q \circ \omega)\Delta'_{e_{k-1}} e_n \mid_{t=0}, \tag{4.6}$$

where ω is the symmetric function involution which trades s_λ and $s_{\lambda'}$ and rev_q reverses the coefficient sequences of polynomials in q. The ring $R_{n,k}$ was the first model for a coinvariant ring attached to the Delta Conjecture.

The rings $R_{n,k}$ have a geometric interpretation. A *line* in the k-dimensional complex vector space $\mathbb{C}^k$ is a 1-dimensional linear subspace. Pawlowski and Rhoades defined [10] the variety $X_{n,k}$ of spanning configurations of n lines in $\mathbb{C}^k$:

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) : \ell_i \subseteq \mathbb{C}^k \text{ a line and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}. \tag{4.7}$$

The space $X_{n,k}$ and its cohomology ring $H^\bullet(X_{n,k}; \mathbb{Q})$ admit S_n-actions by line permutation. Pawlowski and Rhoades presented [10] the cohomology $H^\bullet(X_{n,k}; \mathbb{Q})$ as

$$H^\bullet(X_{n,k}; \mathbb{Q}) = \mathbb{Q}[x_1, \ldots, x_n]/I_{n,k} = R_{n,k}. \tag{4.8}$$

We may therefore interpret the θ-degree 0 piece of $\mathbb{W}_{n,k}$ as the cohomology of $X_{n,k}$.

The 'twist' $(\mathrm{rev}_q \circ \omega)$ involved in Equation (4.6) can be visualized in the matrix representing $\mathrm{grFrob}(\mathbb{W}_{4,2}; q, z)$ in (4.4). Namely, the top row can be obtained from the bottom row by reversal together with applying the operator ω. The reader may notice that the middle row of $\mathrm{grFrob}(\mathbb{W}_{4,2}; q, z)$ is invariant under reversal followed by ω. This observation generalizes as follows.

Theorem 3. *The matrix representing* $\mathrm{grFrob}(\mathbb{W}_{n,k}; q, z)$ *is invariant under* 180° *rotation followed by the application of ω to each entry.*

Recall that a sequence of numbers $(a_0, a_1, \ldots, a_d)$ is *palindromic* if $a_i = a_{d-i}$ for all i and *unimodal* if $a_0 \leq a_1 \leq \cdots \leq a_r \geq a_{r+1} \geq \cdots \geq a_d$ for some r. A famous example of a polynomial in $\mathbb{Q}[q]$ with a palindromic and unimodal coefficient sequence is the q-factorial $[n]!_q = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$. While these facts about

$[n]!_q$ follow from showing that if $f(q)$ and $g(q)$ have palindromic unimodal coefficient sequences, so does $f(q) \cdot g(q)$, there is a deeper derivation coming from geometry.

A finite-dimensional graded $\mathbb{Q}$-algebra $A = \bigoplus_{i=0}^{d} A_i$ is a *Poincaré Duality Algebra* if $A_d \cong \mathbb{Q}$ is 1-dimensional and if for all $0 \leq i \leq d$ the map $A_i \otimes A_{d-i} \to A_d \cong \mathbb{Q}$ is a perfect pairing. This forces $\dim A_i = \dim A_{d-i}$.

Let $\mathcal{F}\ell_n$ be the variety of complete flags in $\mathbb{C}^n$. Borel proved [1] that the cohomology of $\mathcal{F}\ell_n$ has presentation $H^\bullet(\mathcal{F}\ell_n; \mathbb{Q}) = R_n$ given by the coinvariant ring. Since $\mathcal{F}\ell_n$ is a compact complex manifold, the ring $H^\bullet(\mathcal{F}\ell_n; \mathbb{Q})$ is a Poincaré Duality Algebra and the palindromicity of its Hilbert polynomial $[n]!_q$ follows.

The complex variety $X_{n,k}$ is smooth, but usually not compact. Indeed, the cohomology ring $H^\bullet(X_{n,k}; \mathbb{Q}) = R_{n,k}$ does not usually have a palindromic Hilbert series, e.g. $\mathrm{Hilb}(R_{3,2}; q) = 1 + 3q + 2q^2$. However, the extension $\mathbb{W}_{n,k} \supseteq R_{n,k}$ exhibits a superspace version of Poincaré Duality.

Let $A = \bigoplus_{i=0}^{d} \bigoplus_{j=0}^{e} A_{i,j}$ be a finite-dimensional bigraded $\mathbb{Q}$-algebra. We say that A is a *Super Poincaré Duality Algebra* if $A_{d,e} \cong \mathbb{Q}$ and $A_{i,j} \otimes A_{d-i,e-j} \to A_{d,e}$ is a perfect pairing for all $0 \leq i \leq d$ and $0 \leq j \leq e$.

Theorem 4. *The bigraded algebra $\mathbb{W}_{n,k}$ is a Super Poincaré Duality Algebra.*

Does Theorem 4 have geometric meaning? Is there a 'superspace version' of cohomology which yields $\mathbb{W}_{n,k}$ when applied to $X_{n,k}$?

The unimodality of $[n]!_q$ also has geometric meaning. A Poincaré Duality Algebra $A = \bigoplus_{i=0}^{d} A_i$ satisfies the *Hard Lefschetz Property* if there exists an element $\ell \in A_1$ (called a *Lefschetz element*) such that for any $i \leq d/2$ the map $A_i \xrightarrow{\times \ell^{d-2i}} A_{d-i}$ of multiplication by ℓ^{d-2i} is bijective.

Since $\mathcal{F}\ell_n$ is a compact complex manifold and $H^\bullet(\mathcal{F}\ell_n; \mathbb{Q}) = R_n$, the ring R_n satisfies the Hard Lefschetz Property. Maneo, Numata, and Wachi proved [9] that a linear form $\ell = c_1 x_1 + \cdots + c_n x_n$ is a Lefschetz element if and only if $c_1, \ldots, c_n \in \mathbb{Q}$ are distinct.

As a closing example, we display the bigraded Hilbert series $\mathrm{Hilb}(\mathbb{W}_{4,2}; q, z)$ as a matrix where rows index θ-degree and columns index x-degree.

$$\mathrm{Hilb}(\mathbb{W}_{4,2}; q, z) = \begin{pmatrix} 1 & 4 & 6 & 3 \\ 3 & 11 & 11 & 3 \\ 3 & 6 & 4 & 1 \end{pmatrix} \tag{4.9}$$

Either Theorem 3 or Theorem 4 imply that the matrix $\mathrm{Hilb}(\mathbb{W}_{n,k}; q, z)$ is always invariant under 180° rotation.

Conjecture 3. *Each row and column in the matrix representing* $\mathrm{Hilb}(\mathbb{W}_{n,k}; q, z)$ *is unimodal.*

Conjecture 3 would be best proven by showing that $\mathbb{W}_{n,k}$ satisfies an as-yet-undefined 'Super Hard Lefschetz Property'.

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