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Coarse and Lipschitz universality

by

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Abstract. We provide several *metric universality* results. For certain classes $\mathcal{C}$ of metric spaces we exhibit families of metric spaces $(M_i, d_i)_{i\in I}$ which have the property that a metric space (X, d_X) in $\mathcal{C}$ is coarsely, resp. Lipschitzly, universal for all spaces in $\mathcal{C}$ if $(M_i, d_i)_{i\in I}$ equi-coarsely, respectively equi-Lipschitzly, embeds into (X, d_X). Such families are built as certain Schreier-type metric subsets of c_0. We deduce a metric analogue of Bourgain's theorem, which generalized Szlenk's theorem, and prove that a space which is coarsely universal for all separable reflexive asymptotic-c_0 Banach spaces is coarsely universal for all separable metric spaces. One of our coarse universality results is valid under Martin's Axiom and the negation of the Continuum Hypothesis. We discuss the strength of the universality statements that can be obtained without these additional set-theoretic assumptions. In the second part of the paper, we study universality properties of Kalton's interlacing graphs. In particular, we prove that every finite metric space embeds almost isometrically into some interlacing graph of large enough diameter.

Contents

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1. Introduction. A metric space Y_{cu} is said to be *coarsely universal* for a class $\mathcal{M}$ of metric spaces if every metric space in $\mathcal{M}$ coarsely embeds into Y_{cu}. By modifying the definition accordingly we can obviously consider universality in various categories: [Banach spaces $\sim$ isomorphic embeddings], [metric spaces $\sim$ bi-Lipschitz embeddings], etc. A natural question is thus the following: Given a class of metric spaces can we find a metric space that is universal for this class with respect to a given type of metric embedding?

There are numerous embedding results that provide satisfactory answers to this broad question. That ℓ_∞ is isometrically universal for the class of separable metric spaces is a reformulation of the (elementary but fundamental) Fréchet–Kuratowski embedding theorem [Fré10, Kur35]. Note that ℓ_∞ is not separable and thus does not belong to the class it is a universal space for. This leads us to refine the question to, say: is there a member of the class that is universal for the class itself? Urysohn's space [Ury25] answers positively this question for the class of separable metric spaces and isometric embeddings.

However, it is not always possible to find a universal space within the class considered. A (relatively) simple example is the class of separable super-reflexive Banach spaces when universality refers to isomorphic embeddings. A much more difficult result of Szlenk [Szl68] states that there is no separable reflexive Banach space that is isomorphically universal for the class of separable reflexive Banach spaces. Szlenk's theorem was improved by Bourgain [Bou80] who showed that a separable Banach space that is isomorphically universal for the class of separable reflexive spaces is also isomorphically universal for *all* separable Banach spaces. So if we want to show that a separable Banach space contains an isomorphic copy of every separable Banach space we only need to show that it contains an isomorphic copy of every separable reflexive Banach space. To prove this remarkable rigidity result in the context of isomorphic universality, Bourgain ingeniously incorporated techniques from descriptive set theory. Bourgain's descriptive set-theoretic approach for universality problems was further extended by Bossard [Bos02] to show that a class of Banach spaces which is analytic, in the Effros–Borel structure of subspaces of $C[0,1]$, and contains all separable reflexive Banach spaces, must contain a universal space.

We will not discuss the numerous variants of the universality problem but instead we will focus on the following rigidity phenomenon in the context

of universality. We voluntarily refrain from specifying a particular type of embeddings.

PROBLEM 1.1. *For what classes $\mathcal{C}$ and $\mathcal{D}$ of metric spaces such that $\mathcal{C} \subset \mathcal{D}$, a universal space for $\mathcal{C}$ is also a universal space for $\mathcal{D}$?*

The first part of the article revolves around Problem 1.1 in the Lipschitz and coarse categories. Our first theorem says that a metric space is Lipschitzly universal for the class of all separable metric spaces, if it is universal for the uncountable collection $\mathcal{C} := \{(\mathrm{S}_\alpha(\mathbb{Q}), d_\infty) \colon \alpha < \omega_1\}$, which we will refer to as the collection of *rational-valued smooth Schreier metric spaces*. Each metric space $(\mathrm{S}_\alpha(\mathbb{Q}), d_\infty)$, in a certain sense developed in Section 3, exhibits c_0 behavior of complexity ω^α. Consequently, the entire hierarchy captures enough structure of c_0, and thus confers its good universality properties. None of the metric spaces in $\mathcal{C}$ is coarsely universal. The most natural Banach space analogue of a metric space $(\mathrm{S}_\alpha(\mathbb{Q}), d_\infty)$ is S_α^*, the dual of the classical Schreier Banach space, into which $(\mathrm{S}_\alpha(\mathbb{Q}), d_\infty)$ Lipschitz embeds. It it worth pointing out that S_α^* is a (non-reflexive) separable dual space, thus it fails to contain c_0 and it has the RNP (respectively by the classical theorems of Bessaga–Pełczyński from [BP58] and Uhl from [Uhl72]). By another classical theorem of Heinrich–Mankiewicz from [HM82], no S_α^* (and thus also $(\mathrm{S}_\alpha(\mathbb{Q}), d_\infty)$) contains a Lipschitz copy of c_0 (or of a dense subset of it).

THEOREM A. *If a complete separable metric space contains bi-Lipschitz copies of $(\mathrm{S}_\alpha(\mathbb{Q}), d_\infty)$ for every countable ordinal α, then it is Lipschiztly universal for the class of all separable metric spaces.*

Theorem A should be thought of as a purely Lipschitz analogue of the linear universality result that states that if a Banach space X is isomorphically universal for the class of separable reflexive asymptotic-c_0 Banach spaces then X contains an isomorphic copy of c_0. This linear universality can be found in [OSZ07], as it is explained at the end of Section 1. Similarly to the linear setting we use an ordinal index à la Bourgain.

In the context of coarse universality, technical difficulties arise and we need some additional set-theoretic axioms (Martin's Axiom and the negation of the Continuum Hypothesis) to prove a coarse analogue of Theorem A. Note that here we consider integer-valued Schreier metric spaces $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$, $\alpha < \omega_1$. Each space $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$ is a subset of $(\mathrm{S}_\alpha(\mathbb{Q}), d_\infty)$ and therefore it Lipschitz (i.e., also coarsely) embeds into the Banach space S_α^*. To deduce that each $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$ fails to be coarsely universal we need to argue slightly differently since we do not know if S_α^* has this property or not. The space $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$ Lipschitz (and thus coarsely) embeds into the dual Tsirelson space T_α^*, a reflexive Banach space. By Kalton's result from [Kal07], c_0 does

not coarsely embed in a reflexive space and therefore $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$ cannot be coarsely universal.

THEOREM B. (MA + ¬CH) *If a separable metric space contains coarse copies of* $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$ *for every countable ordinal* α*, then it is coarsely universal for the class of all separable metric spaces.*

We end the first part with several results which have statements somewhat weaker than Theorem B, but can be shown without any further axioms. In particular, we show the following.

THEOREM C. *If a separable metric space* (M, d) *contains coarse copies of* $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$ *for every countable ordinal* α*, then the class of all separable bounded metric spaces equi-coarsely embeds into* (M, d)*.*

With the help of a deep result of Dodos [Dod09], we prove Theorem D below. Note that the assumption is formally stronger than that of Theorem B or Theorem C.

THEOREM D. *If a separable metric space is coarsely universal for the class of all reflexive asymptotic-*c_0 *Banach spaces then it is coarsely universal for the class of all separable metric spaces.*

The second part of the article discusses some universality properties of the sequence of interlacing graphs $([\mathbb{N}]^k, d_{\mathrm{I}})_k$ and their applications to universality problems. The geometry of these graphs is intimately connected with the geometry of c_0 via the summing norm, and we prove the following universality property.

THEOREM E. *For every finite metric space* X *and every* $\varepsilon > 0$*, there exists* $k := k(X, \varepsilon) \in \mathbb{N}$ *such that* X *admits a bi-Lipschitz embedding into* $([\mathbb{N}]^k, d_{\mathrm{I}})$ *with distortion at most* $1 + \varepsilon$*.*

Note that it follows from this almost isometric universality property of the interlacing graphs and the work of Eskenazis, Mendel and Naor [EMN19] that the sequence of interlacing graphs $([\mathbb{N}]^k, d_{\mathrm{I}})_k$ does not equi-coarsely embed into any Alexandrov space of non-positive curvature.

Then, we discuss the connection between metric universality, the geometry of the interlacing graphs, and a non-linear version of Johnson–Odell elasticity.

In [Kal07], Kalton showed that a separable Banach space X that is coarsely universal for all separable metric spaces cannot have all its iterated duals separable. The argument is based on the existence of uncountably many well separated copies of the interlacing graphs in c_0. We conclude the paper by showing that it can be generalized to prove the following.

THEOREM F. *Let* X *be a separable Banach space with non-separable bidual* X^{**} *and such that no spreading model generated by a normalized weakly*

null sequence in X is equivalent to the ℓ_1-unit vector basis. Assume that X coarsely embeds into a Banach space Y. Then there exists $k \in \mathbb{N}$ such that $Y^{(2k)}$ is non-separable.

In connection with this last result, it is important to note that ℓ_1 is known to coarsely embed into ℓ_2.

2. Preliminaries

2.1. Coarse and Lipschitz geometry. If X and Y are two metric spaces, the *Y-distortion* of X, denoted $c_Y(X)$, is defined as the infimum of those $D \in [1, \infty)$ such that there exist $s \in (0, \infty)$ and a map $f\colon X \to Y$ such that for all $x, y \in X$ we have

$$s \cdot d_X(x, y) \leq d_Y(f(x), f(y)) \leq s \cdot D \cdot d_X(x, y). \tag{2.1}$$

When (2.1) holds, we say that X *bi-Lipschitzly embeds into Y with distortion D*. We introduce some convenient terminology and notation that will allow us to treat all at once various embedding notions.

Definition 2.1. Let X and Y be metric spaces. Let $\rho, \omega\colon [0, \infty) \to [0, \infty)$. We say that X *(ρ, ω)-embeds into Y* if there exists $f\colon X \to Y$ such that for all $x, y \in X$ we have

$$\rho(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \omega(d_X(x, y)). \tag{2.2}$$

If $\{X_i\}_{i \in I}$ is a collection of metric spaces, then we say that $\{X_i\}_{i \in I}$ *(ρ, ω)-embeds into Y* if for every $i \in I$, X_i (ρ, ω)-embeds into Y.

We say that $\{X_i\}_{i \in I}$ *equi-coarsely embeds into Y* if there exist non-decreasing functions $\rho, \omega\colon [0, \infty) \to [0, \infty)$ such that $\lim_{t \to \infty} \rho(t) = \infty$ and $\{X_i\}_{i \in I}$ (ρ, ω)-embeds into Y. We say that $\{X_i\}_{i \in I}$ *equi-bi-Lipschitzly embeds into Y* if $\{X_i\}_{i \in I}$ (ρ, ω)-embeds into Y, where ρ and ω are increasing and linear on $[0, \infty)$.

Note that equi-bi-Lipschitz embeddability is a stronger condition than merely assuming that $\sup_{i \in I} c_Y(X_i) < \infty$ since it does not allow for arbitrarily large or arbitrarily small scaling factors in (2.1). However if Y is a Banach space, rescaling is possible, and the two notions coincide.

Aharoni's embedding theorem [Aha74] states that there exists a universal constant $K \in [1, \infty)$ such that every separable metric space bi-Lipschitzly embeds into c_0 with distortion at most K. The optimal distortion in Aharoni's embedding theorem is $K = 2$ as shown in [KL08]. A consequence of Aharoni's embedding theorem, which will be used repeatedly, is that a metric space is Lipschitzly (resp. coarsely) universal for the class of separable metric spaces if and only if it contains a bi-Lipschitz (resp. coarse) copy of c_0.

2.2. Trees, derivations, and Bourgain's index theory. A *tree* T over a set X is a collection of finite sequences $(x_1, \dots, x_n)$ of elements of a set X with the property that whenever $(x_1, \dots, x_n)$ is in T then $(x_1, \dots, x_{n-1})$ is in T as well. A tree is *well founded* if it has no infinite branch, i.e., there is no sequence $(x_k)_{k=1}^{\infty}$ in X such that for all $n \in \mathbb{N}$, $(x_1, \dots, x_n) \in T$. There is a classical ordinal derivation on trees which is defined transfinitely as follows:

- $T^0 = T$,
- $T^{\alpha+1} = \{(x_1, \dots, x_n) \colon (x_1, \dots, x_n, x_{n+1}) \in T^\alpha\}$ for any ordinal α,
- $T^\beta = \bigcap_{\alpha<\beta} T^\alpha$ for any limit ordinal β.

We define $o(T)$, the *order* of a tree T, to be the least ordinal number such that $T^{o(T)} = \emptyset$, and by convention we set $o(T) = \infty$ if such an ordinal does not exist. Note that if T is well founded then the derivation produces a strictly decreasing sequence of trees and thus $o(T) < \infty$. For every ordinal α it is easy to construct a tree T_α such that $o(T_\alpha) = \alpha$.

In Section 2 we will need to strengthen a crucial result about trees on Polish spaces, which are complete, separable and metrizable spaces. A tree T on a topological space X is *closed* if for every $n \in \mathbb{N}$, $\mathsf{T} \cap X^n$ is closed in X^n equipped with the product topology. The following result, which follows from [Kec95, Theorem 31.1], was observed by Bourgain [Bou80, Proposition 3].

PROPOSITION 2.2. *If T is a closed and well founded tree on a Polish space, then $o(T) < \omega_1$, where ω_1 denotes the first uncountable ordinal.*

In order to facilitate the reading of Section 2, we recall Bourgain's ordinal index "measuring" the presence of a given basic sequence in a Banach space. This idea was introduced in [Bou80] for a basis of $C[0,1]$, but can be (and has been extensively) applied for other basic sequences (see for instance [AJO05, Definitions 3.1 and 3.6] or [Ode04]). In this article we will be mostly interested in the canonical basis of c_0.

Let $(e_i)_i$ be a normalized basic sequence, X be a Banach space, and $K \geq 1$. Denote by $T(X, (e_i)_i, K)$ the set of finite sequences $(x_1, \dots, x_n)$ of elements in X such that

$$\frac{1}{K}\Big\|\sum_{k=1}^{n} a_k x_k\Big\| \leq \Big\|\sum_{k=1}^{n} a_k e_k\Big\| \leq K\Big\|\sum_{k=1}^{n} a_k x_k\Big\|. \tag{2.3}$$

It is clear that $T(X, (e_i)_i, K)$ is a closed tree on X. It is also straightforward that X contains a K^2-isomorphic copy of $Y = \overline{\operatorname{span}(e_i)}$ if and only if $T(X, (e_i)_i, K)$ is not well founded (or in other words, has an infinite branch). Moreover, if X is separable (and thus Polish), it follows from Proposition 2.2 that X contains a K^2-isomorphic copy of $Y = \overline{\operatorname{span}(e_i)}$ if and only if $o(T(X, (e_i)_i, K)) = \omega_1$. At the technical level, Bourgain constructed, for every ordinal α, a separable reflexive Banach space X_α such that for

some universal constant $K > 0$, $T(X_\alpha, (e_i)_i, K) \geq \alpha$, where $(e_i)_i$ is a basis of $C[0,1]$.

If a separable Banach space Z is isomorphically universal for all separable reflexive Banach spaces, it is easy to see that it must be C-isomorphically universal for all separable reflexive Banach spaces for some $C \geq 1$. Indeed, if there exists a sequence (X_n) of reflexive separable Banach spaces such that their embedding constants escape to infinity, then the reflexive separable space $(\sum_n X_n)_2$ will not embed into Z. Thus Z will contain a C-isomorphic copy of all the X_α's and thus $T(Z, (e_i)_i, D) = \omega_1$ for some $D \geq 1$, and based on the above discussion it follows that Z contains an isomorphic copy of $C[0,1]$ (which is well known to be linearly isometrically universal for all separable Banach spaces thanks to Banach's embedding theorem [Ban32]).

Bourgain's (e_i)-*index* of X is defined as follows:

$$\mathrm{I}(X, (e_i)) = \sup\{o(T(X, (e_i)_i, K)) \colon K \geq 1\}.$$

We collect the key properties of Bourgain's index of the canonical basis of c_0, simply denoted by I_{c_0}, that we will need later on.

PROPOSITION 2.3. *Let X, Y be separable Banach spaces.*

(1) *If X is a subspace of Y then $I_{\mathrm{c}_0}(X) \leq I_{\mathrm{c}_0}(Y)$.*
(2) *If X is isomorphically equivalent to Y then $I_{\mathrm{c}_0}(X) = I_{\mathrm{c}_0}(Y)$.*
(3) *c_0 isomorphically embeds into X if and only if $I_{\mathrm{c}_0}(X) \geq \omega_1$.*

2.3. Schreier sets and higher order Tsirelson spaces. Schreier sets proved to be very useful to measure indices as well as to construct Banach spaces having certain indices. We will also use them in the more general metric context. We denote by $[\mathbb{N}]^{<\omega}$ the set of finite subsets of $\mathbb{N}$. An element $\bar{n} = \{n_1, \ldots, n_k\} \in [\mathbb{N}]^{<\omega}$ will always be written in strictly increasing order, i.e., $n_1 < \cdots < n_k$. If A and B are finite subsets of $\mathbb{N}$, we write $n \leq A < B$ if $n \leq \min(A) \leq \max(A) < \min(B)$.

For a countable ordinal α we denote by $\mathrm{S}_\alpha \subset [\mathbb{N}]^{<\omega}$ the *Schreier family of order α*, which is defined recursively as follows:

- $\mathrm{S}_0 = \{\{n\} \colon n \in \mathbb{N}\}$,
- $\mathrm{S}_{\alpha+1} = \{\bigcup_{j=1}^n E_j \colon E_j \in \mathrm{S}_\alpha \text{ for } j = 1, \ldots, n \text{ and } n \leq E_1 < \cdots < E_n\}$,
- $\mathrm{S}_\beta = \{A \in [\mathbb{N}]^{<\omega} \colon \exists n \in \mathbb{N} \text{ such that } n \leq A \text{ and } A \in \mathrm{S}_{\alpha_n}\}$ if β is a limit ordinal and $(\alpha_n) \subset [0, \alpha)$ is a (fixed) sequence which increases to β.

The above definition of S_β, for β a limit ordinal, is dependent on the choice of the sequence (α_n), but for our purposes the specific choice of (α_n) will be irrelevant. The Schreier sets $(\mathrm{S}_\alpha)_{\alpha<\omega_1}$ are collections of finite subsets of $\mathbb{N}$ with increasing complexity which naturally generate trees $T(\mathrm{S}_\alpha) :=$

$\{(n_1,\dots,n_k)\colon \{n_i\}_{i=1}^k \in \mathrm{S}_\alpha\}$ on $\mathbb{N}$. It is not difficult to prove by transfinite induction that $o(T(\mathrm{S}_\alpha)) = \omega^\alpha + 1$.

We now describe a procedure to generate metric spaces using Schreier sets. Let $\mathcal{G}$ be a family of finite subsets of $\mathbb{N}$ and let $\mathbb{E}$ be a non-empty (finite or infinite) countable subset of $\mathbb{R}$. We define the following subset of $\mathrm{c}_{00}(\mathbb{N})$:

$$X_{\mathcal{G},\mathbb{E}} = \Big\{\sum_{i\in G} c_i e_i \colon G \in \mathcal{G},\ c_i \in \mathbb{E} \text{ for } i \in G\Big\}$$

where (e_i) is the canonical basis of c_{00}. We will endow $X_{\mathcal{G},\mathbb{E}}$ with the metric d_∞ induced by the standard c_0-norm $\|\cdot\|_\infty$. When $\mathcal{G} = \mathrm{S}_\alpha$, we will simply denote by $(\mathrm{S}_\alpha(\mathbb{E}), d_\infty)$ the metric space obtained. These metric spaces naturally embed into the *higher order Tsirelson spaces* T^*_α, which are reflexive Banach spaces whose duals T_α have norms which are implicitly defined based on an admissibility condition that involves the Schreier sets. Although the original space constructed by Tsirelson [Tsi74] was T^*_α for $\alpha = 1$, nowadays their duals T_α are usually referred to as *Tsirelson spaces*, and it is easier to define T^*_α by first defining T_α.

We recall the crucial properties of the Banach space T^*_α (cf. [OSZ07]) that are needed in this article. The separable reflexive Banach space T^*_α is asymptotic-c_0 and has a 1-unconditional basis $(u_i)_i$ with the property that for any $G \in \mathrm{S}_\alpha$ the sequence $(u_i)_{i\in G}$ is 2-equivalent to the unit vector basis of $\ell_\infty^{|G|}$. From the latter property it follows that the natural embedding of $(\mathrm{S}_\alpha(\mathbb{E}), d_\infty)$ into T^*_α (mapping $\sum_{i\in G} c_i e_i$ to $\sum_{i\in G} c_i u_i$, for $G \in \mathrm{S}_\alpha$) is a 4-Lipschitz isomorphism. Moreover, it follows from [OSZ07] that Bourgain's c_0-index of T^*_α tends to ω_1 as α tends to ω_1.

3. Metric universality via descriptive set theory. This section is deeply inspired by the profound ideas introduced by Bourgain and Bossard in connection with isomorphic universality, and the unification of these approaches initiated by Argyros and Dodos [AD07]. The most natural approach to prove Theorem A (resp. Theorem B), is to mimic Bourgain's strategy and construct an ordinal index that will detect the presence of a bi-Lipschitz (resp. coarse) copy of c_0, and which behaves similarly to Bourgain c_0-index. We can indeed (though non-trivially) adjust Bourgain's approach to prove the Lipschitz universality result in Section 3.1.

Unfortunately some difficulties arise in the coarse setting. On the one hand, in Section 3.2, we use additional set-theoretic axioms to prove Theorem B. On the other hand, we need to resort to the delicate theory of strongly bounded classes of Banach spaces to prove Theorem D. This is carried out in Section 3.3 where we will use a deep theorem of Dodos. With this organization, we hope it will be clear what is the scope of application of Bourgain's strategy and why it partially fails to work in the coarse framework.

3.1. Lipschitz universality via a Lipschitz c_0-index. To detect the presence of a linear isomorphic copy of $C[0,1]$ Bourgain used a tree ordinal index where the trees are defined by a fixed basis of $C[0,1]$. By completeness, we only need to find a dense subset of c_0 in order to detect a Lipschitz copy of c_0, while to detect a coarse copy of c_0 we only need to find a 1-net of c_0. Note that $X_{[\mathbb{N}]^{<\omega},\mathbb{Q}}$ is a dense subset of c_0 and that $X_{[\mathbb{N}]^{<\omega},\mathbb{Z}}$ is a 1-net in c_0. It will be very useful to understand $X_{\mathcal{G},\mathbb{E}}$ as the collection of all $f\colon \mathbb{N}\to\mathbb{E}$ for which there is $G\in\mathcal{G}$ such that $\operatorname{supp}(f)\subset G$.

To handle the non-linearity of our universality problem we will introduce combinatorial objects called *vines* which will be a substitute for trees. The elements of a vine $\mathcal{V}$ will also be collections of elements of X, but they will be indexed over collections of finitely supported functions $f\colon\mathbb{N}\to\mathbb{E}$, where $\mathbb{E}$ is a fixed countable subset of $\mathbb{R}$, with $0\in\mathbb{E}$. Such elements will be called *bunches*. For a collection $\mathcal{V}$ of bunches to be called a vine it must also be closed under a certain restriction operation.

Formally, for a (finite or infinite) countable subset $\mathbb{E}$ of $\mathbb{R}$, with $0\in\mathbb{E}$, and finite subset G of $\mathbb{N}$ we call the set

$$[\mathbb{E},G]=\{f\colon\mathbb{N}\to\mathbb{E}\text{ with }\operatorname{supp}(f)\subset G\}$$

an $\mathbb{E}$-*bunch*. Note that if $G=\emptyset$, then $[\mathbb{E},G]=\{0\}$, where $0\colon\mathbb{N}\to\mathbb{E}$ is the constant zero map. We put

$$c_{00}^{\mathbb{E}}=\bigcup_{G\in[\mathbb{N}]^{<\omega}}[G,\mathbb{E}]=\{(\xi_j)\subset\mathbb{E}\colon\{j\in\mathbb{N}\colon\xi_j\neq0\}\text{ is finite}\}$$

which is dense in c_0 if $\mathbb{E}$ is dense in $\mathbb{R}$. Given a set X and a countable subset $\mathbb{E}$ of $\mathbb{R}$, every element of the form $\chi=(x_f)_{f\in[\mathbb{E},G]}$ in $X^{[\mathbb{E},G]}$ will be called an $\mathbb{E}$-*bunch over* X. We define a partial order on the set of $\mathbb{E}$-bunches over X as follows. If $\chi=(x_f)_{f\in[\mathbb{E},F]}$, $\psi=(y_f)_{f\in[\mathbb{E},G]}$, we will write $\chi\preceq\psi$ if F is an initial segment of G and for every $f\in[\mathbb{E},F]$ we have $y_f=x_f$. This makes sense because $[\mathbb{E},F]\subset[\mathbb{E},G]$. If $G=\emptyset$ then $X^{[\mathbb{E},G]}$ will be in an obvious way identified with X, and we note that for $G\in[\mathbb{N}]^{<\omega}$ and $(x_f)_{f\in[\mathbb{E},G]}$, $x_0\equiv(x_f)_{f\in[\mathbb{E},\emptyset]}\preceq(x_f)_{f\in[\mathbb{E},G]}$, or more generally $(x_f)_{f\in[\mathbb{E},F]}\preceq(x_f)_{f\in[\mathbb{E},G]}$ for all initial segments F of G.

A set $\mathcal{V}$ of $\mathbb{E}$-bunches over X is called an $\mathbb{E}$-*vine over* X if for all $\chi\in\mathcal{V}$ the set $[\psi\preceq\chi]$ is a subset of $\mathcal{V}$. Note that $[\psi\preceq\chi]$ is finite and totally ordered and hence $(\mathcal{V},\preceq)$ is a tree in the abstract classical sense. We will say that the $\mathbb{E}$-vine $\mathcal{V}$ is *well founded* if the tree $(\mathcal{V},\preceq)$ is well founded, i.e., it contains no infinite totally ordered subsets. We define the derivatives of vines: for a vine $\mathcal{V}$ we put

$$\mathcal{V}^{(1)}=\mathcal{V}\setminus\{\chi\in\mathcal{V}\colon\chi\text{ is }\preceq\text{-maximal}\},$$

and recursively for any ordinal α,

$$\mathcal{V}^{(\alpha+1)}=(\mathcal{V}^{(\alpha)})^{(1)},$$

and for a limit ordinal α,

$$\mathcal{V}^{(\alpha)} = \bigcap_{\beta<\alpha} \mathcal{V}^{(\beta)}.$$

Then the ordinal index of $\mathcal{V}$ is $o(\mathcal{V}) = \min\{\alpha\colon \mathcal{V}^{(\alpha)} = \emptyset\}$. This is well defined if $\mathcal{V}$ is well founded. As for trees, under appropriate assumptions, being well founded is equivalent to having countable ordinal index. This will be proven in Proposition 3.2.

For $n \in \mathbb{N} \cup \{0\}$ we define

$$\mathcal{V}_{(n)} = \{\chi = (x_f)_{f\in[\mathbb{E},G]}\colon |G| = n\} = \mathcal{V} \cap \bigcup_{G\in[\mathbb{N}]^n} X^{[\mathbb{E},G]}.$$

If X is a topological space then for each $G \in [\mathbb{N}]^n$ the set $X^{[\mathbb{E},G]}$ can be equipped with the product topology. Then the disjoint union $\bigcup_{G\in[\mathbb{N}]^n} X^{[\mathbb{E},G]}$ can be endowed with the induced topology. In particular, $\mathcal{V}_{(n)}$ is a topological space. We will call $\mathcal{V}$ a *closed* $\mathbb{E}$*-vine* if $\mathcal{V}_{(n)}$ is a closed subset of $\bigcup_{G\in[\mathbb{N}]^n} X^{[\mathbb{E},G]}$ for all $n \in \mathbb{N}$. This is equivalent to saying that for all $G \in [\mathbb{N}]^{<\omega}$ the set $\mathcal{V} \cap X^{[\mathbb{E},G]}$ is closed. Note that $\mathcal{V}$ being closed does *not* imply that the set $\bigcup_{\chi=(x_f)_{f\in[\mathbb{E},G]}\in\mathcal{V}}\{x_f\colon f \in [\mathbb{E}, G]\}$ is a closed subset of X.

We can define $\pi_n\colon \mathcal{V}_{(n+1)} \to \mathcal{V}_{(n)}$ as follows. If $G \in [\mathbb{N}]^{n+1}$, we set $G' = G\backslash\{\max(G)\}$. Given $\chi = (x_f)_{f\in[\mathbb{E},G]}$ in $\mathcal{V}_{(n+1)}$ we define $\pi_n(\chi) = (x_f)_{f\in[\mathbb{E},G']}$, which is in $\mathcal{V}_{(n)}$. Note that a collection $\mathcal{V}$ of $\mathbb{E}$-bunches over X is an $\mathbb{E}$-vine if and only if for all $n \in \mathbb{N}$ we have $\pi_n[\mathcal{V}_{(n+1)}] \subset \mathcal{V}_{(n)}$. Also, if X is a topological space then π_n is a continuous function.

The following is an analogue for vines of [Bou80, Lemma 2] and the proof is nearly identical.

Lemma 3.1. *Let $\mathbb{E}$ be a countable subset of $\mathbb{R}$ and $\mathcal{V}$ be a closed $\mathbb{E}$-vine over a complete metric space (X, d). Assume that for all $n \in \mathbb{N}$ we have $\mathcal{V}_{(n)} = \overline{\pi_n[\mathcal{V}_{(n+1)}]}$. Then either $\mathcal{V} = \emptyset$ or $\mathcal{V}$ is not well founded.*

Proof. We fix an enumeration $\{\varepsilon_i\colon i \in \mathbb{N}\}$ of $\mathbb{E}$ and for all $n \in \mathbb{N}$ we set $\mathbb{E}_n = \{\varepsilon_1, \dots, \varepsilon_n\}$. Assuming $\mathcal{V} \neq \emptyset$, we can find an $x_0 \equiv (x_f)_{f\in[\mathbb{E},\emptyset]} \in \mathcal{V} \cap X = \mathcal{V}_{(0)}$. Since $V_{(0)} = \overline{\pi_0(V_{(1)})}$, we find $\chi_1 = (x_f^{(1)})_{f\in[\mathbb{E},\{k_1\}]} \in \mathcal{V}_{(1)}$ such that $\|\pi_0(\chi_1) - x_0\| < 1$. By assumption there exists $k_2 > k_1$ and $\chi_2 = (x_f^{(2)})_{f\in[\mathbb{E},\{k_1,k_2\}]} \in \mathcal{V}_{(2)}$ such that for $f \in [\mathbb{E}_1, \{k_1\}]$ we have $d(x_f^{(1)}, x_f^{(2)}) \leq 1/2$. Proceed inductively to find an increasing sequence of integers $(k_m)_{m=1}^\infty$ and a sequence $(\chi_m)_{m=1}^\infty$ such that $\chi_m = (x_f^{(m)})_{f\in[\mathbb{E},\{k_1,\dots,k_m\}]} \in \mathcal{V}_{(m)}$ and for all $m \in \mathbb{N}$ and $f \in [\mathbb{E}_m, \{k_1, \dots, k_m\}]$ we have $d(x_f^{(m)}, x_f^{(m+1)}) \leq 1/2^m$. We conclude that for any $m_0 \in \mathbb{N}$ and $f \in [\mathbb{E}_{m_0}, \{k_1, \dots, k_{m_0}\}]$ the sequence $(x_f^{(m)})_{m\geq m_0}$ is Cauchy and we denote its limit by y_f. Because $\mathcal{V}$ is an $\mathbb{E}$-

vine, it is closed under taking projections π_n, and because $\mathcal{V}$ is assumed to be closed, we deduce that $\psi_m = (y_f)_{f\in[\mathbb{E},\{k_1,\dots,k_m\}]}$ is in $\mathcal{V}$ for all $m \in \mathbb{N}$. Because $(\psi_m)_m$ is an infinite chain, the $\mathbb{E}$-vine $\mathcal{V}$ must be ill founded. ∎

The following is the analogue of Proposition 2.2 for vines.

PROPOSITION 3.2. *Let $\mathbb{E}$ be a countable subset of $\mathbb{R}$ and $\mathcal{V}$ be a closed $\mathbb{E}$-vine on a Polish space. If $\mathcal{V}$ is well founded then $o(\mathcal{V}) < \omega_1$.*

Proof. We will show that there is $\eta < \omega_1$ such that $\mathcal{V}^{(\eta)} = \emptyset$. It is easily observed that for any $n \in \mathbb{N}$ and ordinal α we have

$$(\mathcal{V}^{(\alpha+1)})_{(n)} = \pi_n[(\mathcal{V}^{(\alpha)})_{(n+1)}], \tag{3.1}$$

i.e., a χ of length n is in $\mathcal{V}^{(\alpha+1)}$ if an only if it is the direct predecessor of a ψ of length $n+1$ in $\mathcal{V}^{(\alpha)}$. For $n \in \mathbb{N}$, consider the decreasing hierarchy of closed sets $\overline{(\mathcal{V}^{(\alpha)})_{(n)}}$, $\alpha < \omega_1$, of $\bigcup_{G\in[\mathbb{N}]^n} X^{[\mathbb{E},G]}$. Because X is Polish, so is $\bigcup_{G\in[\mathbb{N}]^n} X^{[\mathbb{E},G]}$ and therefore there must exist an $\alpha_n < \omega_1$ such that for all $\beta > \alpha_n$ we have $\overline{(\mathcal{V}^{(\alpha_n)})_{(n)}} = \overline{(\mathcal{V}^{(\beta)})_{(n)}}$. This is because in a Polish space there can be no strictly increasing transfinite hierarchy of open sets of length ω_1. Take $\eta = \sup_n \alpha_n$ and define $\mathcal{W} = \bigcup_{n=1}^{\infty} \overline{(\mathcal{V}^{(\eta)})_{(n)}}$. We observe that $\mathcal{W}$ is an $\mathbb{E}$-vine over X. We show that $\mathcal{W}$ satisfies the assumption of Lemma 3.1. Indeed, for $n \in \mathbb{N}$ we have

$$\begin{aligned}
\mathcal{W}_{(n)} &= \overline{(\mathcal{V}^{(\eta)})_{(n)}} = \overline{(\mathcal{V}^{(\eta+1)})_{(n)}} && \text{(by the choice of } \eta\text{)}\\
&= \overline{\pi_n[(\mathcal{V}^{(\eta)})_{(n+1)}]} && \text{(by (3.1))}\\
&= \overline{\pi_n\Big[\overline{(\mathcal{V}^{(\eta)})_{(n+1)}}\Big]} && \text{(by continuity of } \pi_n\text{)}\\
&= \overline{\pi_n[\mathcal{W}_{(n+1)}]}.
\end{aligned}$$

This means that either $\mathcal{W} = \emptyset$ or $\mathcal{W}$ is ill founded. Because $\mathcal{V}$ is closed, $\mathcal{W} \subset \mathcal{V}$, and because $\mathcal{V}$ is well founded, so is $\mathcal{W}$ and hence $\mathcal{W} = \emptyset$. It follows that $\mathcal{V}^{(\eta)} = \bigcup_{n\in\mathbb{N}}(\mathcal{V}^{(\eta)})_{(n)} \subset \mathcal{W} = \emptyset$. Therefore, $o(\mathcal{V}) \le \eta$. ∎

We can now introduce an ordinal index that will capture the presence of a bi-Lipschitz copy of c_0 in a metric space. For any $C > 0$, any metric space (M, d), and any countable subset $\mathbb{E}$ of $\mathbb{R}$, it is easy to verify that the set (think of $[\mathbb{E}, G]$ being a subset of c_0)

$$\mathcal{V}(M,\mathbb{E},C) = \left\{ (x_f)_{f\in[\mathbb{E},G]} : \begin{array}{r} G \in [\mathbb{N}]^{<\omega},\ x_f \in M \text{ for } f \in [\mathbb{E},G],\\ \text{and } \forall f, g \in [\mathbb{E},G]\\ \frac{1}{C}\|f-g\|_\infty \le d(x_f,x_g) \le C\|f-g\|_\infty \end{array} \right\}$$

is a closed $\mathbb{E}$-vine on M. We define the *Lipschitz* c_0*-index* of M as

$$\mathrm{I}^{\mathrm{Lip}}_{\mathrm{c}_0}(M) = \sup\{o(\mathcal{V}(M,\mathbb{Q},C)) : C > 0\}.$$

PROPOSITION 3.3. *Let M be a Polish space. Then*

$$\mathrm{c}_0 \text{ bi-Lipschitzly embeds into } M \quad \text{if and only if} \quad \mathrm{I}_{\mathrm{c}_0}^{\mathrm{Lip}}(M) \geq \omega_1.$$

Proof. The necessary implication is easy. Indeed, if ψ is a Lipschitz embedding from c_0 into M, then for $G \in [\mathbb{N}]^{<\omega}$ and $f \in [\mathbb{Q}, G]$ define $x_f = \psi(\sum_{i\in G} f(i)e_i)$. Thus, for some $C \geq 1$, the set $\{(x_f)_{f\in[\mathbb{Q},G]} \colon G \in [\mathbb{N}]^{<\omega}\}$ is included in $\mathcal{V}(M, \mathbb{Q}, C)$, which is therefore ill founded.

Assume now that $\mathrm{I}_{\mathrm{c}_0}^{\mathrm{Lip}}(M) = \omega_1$. Then for every countable ordinal α there exists $C_\alpha > 0$ such that $o(\mathcal{V}(M, \mathbb{Q}, C_\alpha)) \geq \alpha$. Using a simple pigeonhole argument we can find $C \geq 1$ and an uncountable subcollection U of $[1, \omega_1)$, such that for all $\alpha \in U$ we have $C_\alpha \leq C$. Since obviously $o(\mathcal{V}(M, \mathbb{Q}, C)) \geq o(\mathcal{V}(M, \mathbb{Q}, C_\alpha)) \geq \alpha$ for every $\alpha \in U$, it follows from Proposition 3.2 that $\mathcal{V}(M, \mathbb{Q}, C)$ is not well founded, i.e., there exists a strictly increasing sequence $(k_m)_m$ of integers and for $m \in \mathbb{N} \cup \{0\}$ an M-bunch $\chi_m = (x_f^{(m)} \colon f \in [\{k_1, \dots, k_m\}, \mathbb{E}]) \in \mathcal{V}(M, \mathbb{Q}, C)$ such that $\chi_0 \preceq \chi_1 \preceq \chi_2$. But this means that for every finitely supported $f \colon \{k_1, k_2, \dots\} \to \mathbb{Q}$ there is an $x_f \in M$ such that $\chi_m = (x_f \colon f \in [\{k_1, \dots, k_m\}, \mathbb{E}])$ for $m \in \mathbb{N}$. We define

$$\psi \colon \mathrm{c}_{00}^{\mathbb{Q}} \to M \quad \text{by } \psi((q_j)_j) = x_f$$

where

$$f \colon \{k_1, k_2, \dots\} \to \mathbb{Q} \quad \text{is defined by } f(k_j) = q_j.$$

It follows that ψ is a bi-Lipschitz embedding from $\mathrm{c}_{00}^{\mathbb{Q}}$ (with the c_0-norm) into M. Since $\mathrm{c}_{00}^{\mathbb{Q}}$ is dense in c_0 and M is complete, ψ can be extended to a bi-Lipschitz embedding from c_0 into M. ∎

To complete the proof of Theorem A it remains to show that if a complete separable metric space M is Lipschitzly universal for the collection of rational, valued Schreier metrics then $\mathrm{I}_{\mathrm{c}_0}^{\mathrm{Lip}}(M) \geq \omega_1$.

Proof of Theorem A. Assume that for every ordinal α, (M, d) admits bi-Lipschitz embeddings of $(\mathrm{S}_\alpha(\mathbb{Q}), d_\infty)$. Thus, after a possible extraction argument, there exist a constant $C > 0$, an uncountable $A \subset [0, \omega_1)$, and maps $F_\alpha \colon (\mathrm{S}_\alpha(\mathbb{Q}), d_\infty) \to (M, d)$, $\alpha \in A$, such that for all $f, g \in \mathrm{S}_\alpha(\mathbb{Q})$ and $\alpha \in A$,

$$\frac{1}{C}\|f - g\|_\infty \leq d(F_\alpha(f), F_\alpha(g)) \leq C\|f - g\|_\infty. \tag{3.2}$$

It follows that $\mathcal{V}(M, \mathbb{Q}, C)$ has ordinal index at least $o(\mathrm{S}_\alpha) = \omega^\alpha + 1$ for all $\alpha \in A$. To see this, define for every f in $\mathrm{S}_\alpha(\mathbb{Q})$ the vector $x_f = F_\alpha(f)$ and let $\mathcal{W} = \{(x_f)_{f\in[\mathbb{Q},G]} \colon G \in \mathrm{S}_\alpha\}$, which is thanks to (3.2) a subvine of $\mathcal{V}(M, \mathbb{Q}, C)$ that has the same tree index as S_α. ∎

3.2. Coarse universality via a coarse c_0-index in MA + ¬CH. The technique from Section 3.1 does not seem to be robust enough to prove

the statement of Theorem B without any further set-theoretic assumptions. The main roadblock is that the simple extraction argument that provides equi-bi-Lipschitz embeddings from an uncountable collection of bi-Lipschitz embeddings does not hold in the coarse setting. Under some additional set-theoretic axioms, MA + ¬CH, we can prove Theorem B. The advantage of assuming that Martin's Axiom holds, but the Continuum Hypothesis fails, lies in the fact that the following *diagonalization property* of infinite subsets of $\mathbb{N}$ (cf. [Fre84, pp. 3 ff]) will be valid.

LEMMA 3.4. (MA + ¬CH) *Let $(N_\alpha)_{\alpha<\omega_1} \subset [\mathbb{N}]^\omega$ have the property that $N_\beta \setminus N_\alpha$ is finite whenever $\alpha < \beta$ (in which case we say that N_β is* almost contained *in N_α and write $N_\beta \subset^a N_\alpha$). Then there exists N in $[\mathbb{N}]^\omega$ such that $N \subset^a N_\alpha$ for all $\alpha < \omega_1$.*

This diagonalization property will now be used to prove a principle of "equi-regularization" for expansion and compression moduli. Let us detail the case of the compression modulus. We first need some preparation. Denote by $\mathcal{I}$ the class of all non-decreasing maps $f\colon \mathbb{N} \to \mathbb{N} \cup \{0\}$ satisfying $f(1) = 0$, $\lim_{n\to\infty} f(n) = \infty$ and $f(n+1) \leq f(n)+1$ for all $n \in \mathbb{N}$. It will be useful to note that the map $j\colon \mathcal{I} \to [\mathbb{N}]^\omega$, defined by $j(f) = \{n \in \mathbb{N}\colon f(n+1) > f(n)\}$ is a bijection, and that its inverse is given by $j^{-1}(A)(n) = \sum_{i<n} \mathbb{1}_A(i)$ for $A \in [\mathbb{N}]^\omega$ and $n \in \mathbb{N}$. We will also use the following easy fact. If $j(f) = \{m_1 < m_2 < \cdots\}$ and $j(g) = \{n_1 < n_2 < \cdots\}$ with $n_i \leq m_i$ for all $i \in \mathbb{N}$, then $f \leq g$. In particular, if $j(f) \subset j(g)$, then $f \leq g$. We start with an easy lemma.

LEMMA 3.5. *Let $(g_\alpha)_{\alpha<\omega_1} \subset \mathcal{I}$. Then there exists $(f_\alpha)_{\alpha<\omega_1} \subset \mathcal{I}$ such that*

(1) *for all $\alpha < \omega_1$ and all $n \in \mathbb{N}$, $f_\alpha(n) \leq g_\alpha(n)$;*
(2) *for all $\alpha < \beta < \omega_1$, $j(f_\beta) \subset^a j(f_\alpha)$.*

Proof. We will build $(f_\alpha)_{\alpha<\omega_1}$ by transfinite induction. So, set $f_1 = g_1$ and assume that $\beta_0 < \omega_1$ is such that we have found $(f_\alpha)_{\alpha<\beta_0}$ satisfying (1) and (2). Since $\{\alpha < \beta_0\}$ is countable, a classical diagonal argument yields the existence of $M \in [\mathbb{N}]^\omega$ such that $M \subset^a j(f_\alpha)$ for all $\alpha < \beta_0$. Let $j(g_{\beta_0}) = \{n_1 < n_2 < \cdots\}$. Then pick $m_1 < m_2 < \cdots \in M$ so that $n_i \leq m_i$ for all $i \in \mathbb{N}$ and set $f_{\beta_0} = j^{-1}(\{m_1, m_2, \ldots\})$. We see that $f_{\beta_0} \leq g_{\beta_0}$ and $j(f_{\beta_0}) \subset M \subset^a j(f_\alpha)$ for all $\alpha < \beta_0$. This concludes our induction. ∎

Armed with Lemma 3.4 we can now prove our "equi-regularization" principle below for compression moduli.

PROPOSITION 3.6. (MA+¬CH) *Let $(\rho_\alpha)_{\alpha<\omega_1}$ be a family of non-decreasing maps from $[0,\infty)$ to $[0,\infty)$ and such that $\lim_{t\to\infty} \rho_\alpha(t) = \infty$ for all $\alpha < \omega_1$. Then there exist an uncountable subset C of ω_1 and $\rho\colon [0,\infty) \to [0,\infty)$ such that $\rho \leq \rho_\alpha$ for all $\alpha \in C$ and $\lim_{t\to\infty} \rho(t) = \infty$.*

Proof. First, note that for all $\alpha < \omega_1$, we can find $g_\alpha \in \mathcal{I}$ such that $g_\alpha(n) \leq \rho_\alpha(n)$ for all $n \in \mathbb{N}$. Then we consider the family $(f_\alpha)_{\alpha<\omega_1}$ associated to $(g_\alpha)_{\alpha<\omega_1}$ through Lemma 3.5. Next, we apply Lemma 3.4 to get $M \in [\mathbb{N}]^\omega$ such that $M \subset^a j(f_\alpha)$ for all $\alpha < \omega_1$. For $n \in \mathbb{N}$, define

$$C_n = \{\alpha < \omega_1 \colon M \cap \{n, n+1, \ldots\} \subset j(f_\alpha)\}.$$

Clearly, there exists $n_0 \in \mathbb{N}$ such that C_{n_0} is uncountable. We set $C = C_{n_0}$ and define $f = j^{-1}(M \cap \{n_0, n_0+1, \ldots\}) \in \mathcal{I}$. Then, for all $\alpha \in C$, we have $j(f) \subset j(f_\alpha)$ and therefore $f \leq f_\alpha$. Finally, ρ defined by $\rho = 0$ on $[0,1)$ and $\rho = f(n)$ on $[n, n+1)$, for $n \in \mathbb{N}$, is the desired map. ∎

Similarly, for expansion moduli, we have:

PROPOSITION 3.7. (MA+¬CH) *Let $(\omega_\alpha)_{\alpha<\omega_1}$ be a family of non-decreasing maps from $[0,\infty)$ to $[0,\infty)$ and such that $\omega_\alpha(0) = 0$. Then there exist an uncountable subset C of ω_1 and $\omega \colon [0,\infty) \to [0,\infty)$ such that $\omega \geq \omega_\alpha$ for all $\alpha \in C$.*

Proof. The argument is very similar. Let us just describe the few adjustments. We now consider the class $\mathcal{J}$ of all functions $f \colon \mathbb{N}\cup\{0\} \to \mathbb{N}\cup\{0\}$ such that $f(0) = 0$ and $f(n+1) \geq f(n) + 1$ for all $n \geq 0$. The map $k \colon \mathcal{J} \to [\mathbb{N}]^\omega$ defined by $k(f) = k(\mathbb{N})$ is a bijection. Then, for every $\alpha < \omega_1$, there exists $g_\alpha \in \mathcal{J}$ such that $g_\alpha(n) \geq \omega_\alpha(n)$ for all $n \in \mathbb{N}$. Playing the same game as before, but with the sets $k(g_\alpha)$ instead of $j(g_\alpha)$, we obtain (under MA + ¬CH) the existence of an uncountable subset C of ω_1 and of $g \in \mathcal{J}$ such that $g \geq g_\alpha$ for all $\alpha \in C$. The proof is then concluded by setting $\omega(0) = 0$ and $\omega = g(n)$ on $(n-1, n]$ for $n \in \mathbb{N}$. ∎

From Propositions 3.6 and 3.7, we deduce immediately:

PROPOSITION 3.8. (MA+¬CH) *If $(X_\alpha, d_\alpha)_{\alpha<\omega_1}$ is a collection of metric spaces such that for all $\alpha < \omega_1$, X_α coarsely embeds into a metric space (M,d), then there exists an uncountable subset C of ω_1 such that $(X_\alpha, d_\alpha)_{\alpha\in C}$ equi-coarsely embeds into (M,d).*

Proof of Theorem B. The argument goes along essentially the same lines as the proof of Theorem A, modulo the fact that we have to work with vines defined in terms of the compression and expansion moduli. Let us outline the main steps and the place where MA + ¬CH is used.

Let ρ, ω be two elements of the class $\mathcal{F}$ of all non-decreasing functions from $[0,\infty)$ to $[0,\infty)$ that are vanishing at 0 and tending to ∞ at ∞. Let also (M,d) be a complete separable metric space. Then we define

$$\mathcal{V}(M,\mathbb{Z},\rho,\omega) = \left\{ \begin{array}{c} G \in [\mathbb{N}]^{<\omega},\ x_f \in M \text{ for } f \in [\mathbb{E},G], \\ (x_f)_{f\in[\mathbb{Z},G]} \colon \text{and for } f,g \in [\mathbb{E},G] \text{ we have} \\ \rho(\|f-g\|_\infty) \leq d(x_f,x_g) \leq \omega(\|f-g\|_\infty) \end{array} \right\},$$

and the coarse c_0-index of M as

$$\mathrm{I}_{c_0}^{\mathrm{coarse}}(M) = \sup\{o(\mathcal{V}(M,\mathbb{Z},\rho,\omega))\colon \rho,\omega\in\mathcal{F}\}.$$

The next step is to prove the analogue of Proposition 3.3: c_0 coarsely embeds into M if and only if $\mathrm{I}_{c_0}^{\mathrm{coarse}}(M)\geq\omega_1$. For the non-trivial implication, the pigeonhole argument yielding a uniform constant C is replaced by Propositions 3.6 and 3.7 to prove the existence of $\rho,\omega\in\mathcal{F}$ such that $\mathcal{V}(M,\mathbb{Z},\rho,\omega)$ is not well founded (this is where (MA + ¬CH) is used). Then it implies the existence of a coarse embedding of the integer grid of c_0 (and therefore of c_0) into M.

Finally, assume that a separable metric space (M,d), that we may assume to be complete, contains a coarse copy of all spaces $(\mathrm{S}_\alpha(\mathbb{Z}),d_\infty)$ for $\alpha<\omega_1$. As in the proof of Theorem A, this implies that $\mathrm{I}_{c_0}^{\mathrm{coarse}}(M)\geq\omega_1$. ∎

Remark 3.9. We recall that a metric space X *coarse-Lipschitz* embeds into a metric space Y if X (ρ,ω)-embeds into Y where, for all $t\geq 0$, $\rho(t)=At-B$ and $\omega(t)=Ct+D$ for some constants $A,B,C,D>0$. It follows clearly from the tools and arguments developed in the last two subsections that we have, without assuming any further set-theoretic axioms, the following statement: a separable metric space containing coarse-Lipschitz all the spaces $(\mathrm{S}_\alpha(\mathbb{Z}),d_\infty)$, for $\alpha<\omega_1$, must contain a coarse-Lipschitz copy of c_0.

It is natural to wonder if Theorem B holds without MA + ¬CH.

Problem 3.10. *If a separable metric space contains a coarse copy of the metric space $(\mathrm{S}_\alpha(\mathbb{Z}),d_\infty)$ for every countable ordinal α, is it coarsely universal for the class of all separable metric spaces?*

We discuss some positive partial results in Section 4.

3.3. Coarse universality via strong boundedness. While we do not know how to prove Theorem B without further set axioms, we can prove Theorem D. Recall that the canonical embedding of $(\mathrm{S}_\alpha(\mathbb{Z}),d_\infty)$ in T_α^* is a 4-Lipschitz isomorphism onto its image, and thus the stronger assumption that the metric space contains every separable reflexive asymptotic-c_0 space, rather than merely the collection $(\mathrm{S}_\alpha(\mathbb{Z}))_{\alpha<\omega_1}$ of metric spaces, allows us to take advantage of the deep theory of strongly bounded classes of Banach spaces introduced by Argyros and Dodos [AD07]. A class $\mathcal{C}$ of separable Banach spaces is said to be *strongly bounded* if for every analytic subset A of $\mathcal{C}$, there exists $Y\in\mathcal{C}$ that contains isomorphic copies of every $X\in A$. Recall also that an infinite-dimensional Banach space X is said to be *minimal* if X isomorphically embeds into every infinite-dimensional subspace of itself (e.g. the classical sequence space c_0 is minimal). We will need the following deep result of Dodos [Dod09, Theorem 7].

THEOREM 3.11. *For any infinite-dimensional minimal Banach space Z not containing ℓ_1, the class*

$$\mathsf{NC}_Z := \{Y \in \mathsf{SB} \colon Z \text{ does not linearly embed into } Y\}$$

is strongly bounded.

Proof of Theorem D. Denote by R the set of all reflexive elements of SB and by As_{c_0} the set of all elements of SB that are asymptotic-c_0. Let now M be a separable metric space such that every space in $\mathsf{R} \cap \mathsf{As}_{c_0}$ coarsely embeds into M. If we denote $\mathsf{CE}_M = \{Y \in \mathsf{SB} \colon Y \text{ coarsely embeds into } M\}$, we have $\mathsf{R} \cap \mathsf{As}_{c_0} \subset \mathsf{CE}_M$. It is easily checked that CE_M is analytic (see [Bra19, proof of Theorem 1.7 in Section 7.1]). Recall that we denoted by NC_{c_0} the set of all $Y \in \mathsf{SB}$ such that c_0 does not linearly embed into Y. If we assume, aiming for a contradiction, that $\mathsf{CE}_M \subset \mathsf{NC}_{c_0}$, then since CE_M is an analytic subset of NC_{c_0}, which is strongly bounded by Theorem 3.11, there would exist $X \in \mathsf{NC}_{c_0}$ such that any element of CE_M, and therefore any element of $\mathsf{R} \cap \mathsf{As}_{c_0}$, linearly embeds into X. This is actually impossible since Bourgain's c_0-index of the separable, reflexive and asymptotic-c_0 space T^*_α tends to ω_1 as α tends to ω_1 (see [OSZ07]). Therefore Bourgain's c_0-index of X would be uncountable and X would contain an isomorphic copy of c_0; a contradiction with $X \in \mathsf{NC}_{c_0}$. So we can now deduce the existence of $Y \in \mathsf{CE}_{\mathsf{M}}$ such that c_0 linearly embeds into Y, and hence by composition c_0 coarsely embeds into M. Since by a theorem of Aharoni [Aha74], every separable metric space bi-Lipschitzly embeds into c_0, every separable metric space coarsely embeds into M. ∎

REMARKS 3.12. The same technique was used by B. de Mendonça Braga [Bra19] to prove that a Banach space which is coarsely universal for all reflexive separable Banach spaces is coarsely universal for all separable metric spaces.

The reader will easily adapt the above proof to show that a Banach space that is Lipschitz universal for $\mathsf{R} \cap \mathsf{As}_{c_0}$ is Lipschitz universal for all separable metric spaces. But this was also a consequence of Theorem A.

4. Coarse universality and barycentric gluing. The motivation for this section is to provide a somewhat weaker statement than Theorem B, that does not require MA+¬CH. We will show that containing coarse copies of the Schreier metric spaces $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$ for every $\alpha < \omega_1$ is a sufficient condition to equi-coarsely contain every separable *bounded* metric space. The reason for the boundedness restriction is because without MA + ¬CH we only have the following equi-regularization principle (which is weaker than the equi-regularization principle obtained under MA + ¬CH).

LEMMA 4.1. *Let C_0 be an uncountable subset of ω_1. Assume that for each $\alpha \in C_0$, we have increasing functions $\rho_\alpha, \omega_\alpha \colon [0,\infty) \to [0,\infty)$ such that $\rho_\alpha(t) \le \omega_\alpha(t)$ for all $t \in [0,\infty)$ and $\lim_{t\to\infty} \rho_\alpha(t) = \infty$. Then there exist increasing functions $\rho, \omega \colon [0,\infty) \to [0,\infty)$ and a decreasing nested sequence $(C_k)_{k\in\mathbb{N}}$ of uncountable subsets of ω_1 such that*

(i) *$\rho(t) \le \omega(t)$ for all $t \in [0,\infty)$, and $\lim_{t\to\infty} \rho(t) = \infty$;*
(ii) *for all $k \in \mathbb{N}$, $\alpha \in C_k$, $0 \le t \le k$ we have $\rho(t) \le \rho_\alpha(t)$ and $\omega_\alpha(t) \le \omega(t)$.*

Proof. For all $k \in \mathbb{N}$ and $\alpha \in C_0$ define

$$s(\alpha,k) = \min\{t \in \mathbb{N} \colon \rho_\alpha(t) \ge k\}, \qquad t(\alpha,k) = \max\{t \in \mathbb{N} \colon \omega_\alpha(t) \le k\}$$

and also define

$$M(\alpha,k) = \min\{n \in \mathbb{N} \colon s(\alpha,k) < s(\alpha,n)\},$$
$$N(\alpha,k) = \min\{n \in \mathbb{N} \colon t(\alpha,k) < t(\alpha,n)\}.$$

As, for each fixed $k \in \mathbb{N}$, the sets $\{s(\alpha,k) \colon \alpha \in C_0\}$, $\{t(\alpha,k) \colon \alpha \in C_0\}$, $\{M(\alpha,k) \colon \alpha \in C_0\}$, $\{N(\alpha,k) \colon \alpha \in C_0\}$ are all countable we may find uncountable sets $C_0 \supset C_1 \supset C_2 \supset \cdots$ such that for each $k \in \mathbb{N}$ and $\alpha, \beta \in C_k$ we have $s(\alpha,k) = s(\beta,k) = s_k$, $t(\alpha,k) = t(\beta,k) = t_k$, $M(\alpha,k) = M(\beta,k) = M_k$, and $N(\alpha,k) = N(\beta_k) = N_k$. Clearly, $s_k \le s_{k+1}$ and $t_k \le t_{k+1}$ for all $k \in \mathbb{N}$. We also observe that $\lim_k s_k = \lim_k t_k = \infty$. Indeed, it is easy to see that $s_k < s_{M_k}$ and $t_k < t_{N_k}$. Pick $k_1 < k_2 < \cdots$ so that $(s_{k_j})_j$ and $(t_{k_j})_j$ are both strictly increasing.

We now define $\rho, \tilde{\omega} \colon [0,\infty) \to [0,\infty)$ as follows:

$$\rho(t) = \begin{cases} 0 & \text{if } 0 \le t < s_{k_1}, \\ k_j & \text{if } s_{k_j} \le t < s_{k_{j+1}},\ j \in \mathbb{N}, \end{cases}$$

$$\tilde{\omega}(t) = \begin{cases} k_1 & \text{if } 0 \le t \le t_{k_1}, \\ k_j & \text{if } t_{k_{j-1}} < t \le t_{k_j},\ j \ge 2 \end{cases}$$

and $\omega(t) = \rho(t) \vee \tilde{\omega}(t)$. The conclusion follows straightforwardly after observing that $s_{k_j}, t_{k_j} \ge j$ for all $j \in \mathbb{N}$. ∎

Using the concept of vines introduced in Section 3.2 in the coarse context we now deduce the following.

THEOREM 4.2. *Let (M,d) be a separable metric space and assume that for every $\alpha < \omega_1$ the metric space $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$ coarsely embeds into (M,d). Then the class of all separable bounded metric spaces equi-coarsely embeds into (M,d). More precisely, there exist $m_0 \in M$ and equi-coarse embeddings $F_n \colon B_n = \{x \in c_0 \colon \|x\|_\infty \le n\} \to (M,d)$ such that for all $n \in \mathbb{N}$ we have $F_n(0) = m_0$.*

Proof. Let $\tilde{M}$ be a countable dense subset of M. Since each $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$ is a uniformly discrete metric space, it follows from a straightforward per-

turbation argument that every $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$ coarsely embeds into $(\tilde{M}, d)$ via a map f_α with compression and expansion moduli $\rho_\alpha, \omega_\alpha$. By passing to an uncountable set $C_0 \subset \omega_1$ we can assume that there exists $m_0 \in \tilde{M}$ such that for all $\alpha \in C_0$ we have $f_\alpha(0) = m_0$. For each $\alpha \in C_0$ and $n \in \mathbb{N}$ denote $B_{\alpha,n} = \{x \in \mathrm{S}_\alpha(\mathbb{Z})\colon \|x\|_\infty \leq n\}$ and $F_{\alpha,n} = f_\alpha|_{B_{\alpha,n}}\colon (B_{\alpha,n}, d_\infty) \to \tilde{M}$.

Take the functions ρ, ω and the sets $C_0 \supset C_1 \supset C_2 \supset \cdots$ given by Lemma 4.1. By the conclusion of that lemma, it follows that for every $\beta \in C_{2n}$ the function $f_\beta\colon (\mathrm{S}_\beta(\mathbb{Z}), d_\infty) \to \tilde{M}$ is a (ρ, ω)-coarse embedding on every subset of $(\mathrm{S}_\beta(\mathbb{Z}), d_\infty)$ with diameter at most $2n$, and also because $\beta \in C_0$ we have $f_\beta(0) = m_0$. In particular, for all $n \in \mathbb{N}$ and $\alpha \in C_{2n}$, $F_{\alpha,n}\colon (B_{\alpha,n}, d_\infty) \to \tilde{M}$ is a (ρ, ω)-embedding with $F_{\alpha,n}(0) = m_0$.

Next, we will use Proposition 3.2 to show that for all $N \in \mathbb{N}$ there exists a function $F_N\colon B_N \to M$ that is a (ρ, ω)-coarse embedding with the additional property that $F_N(0) = m_0$. More precisely, we will define this F_N on the subset $B(N, \mathbb{Z})$ of B_N consisting of all integer-valued sequences in the set B_N. Because this is a 1-net of B_N and N is arbitrary, we may then deduce the desired conclusion. We denote $\mathbb{I}_N = [-N, N] \cap \mathbb{Z}$ and consider the closed $\mathbb{I}_N$-vine defined by

$$\mathcal{V} := \left\{ (x_f)_{f\in[\mathbb{I}_N,G]}\colon \begin{array}{c} G \in [\mathbb{N}]^{<\omega},\ x_f \in M \text{ for } f \in [\mathbb{I}_N, G], \\ x_0 = m_0, \text{ and for } f, g \in [\mathbb{I}_N, G] \text{ we have} \\ \rho(\|f-g\|_\infty) \leq d(x_f, x_g) \leq \omega(\|f-g\|_\infty) \end{array} \right\}.$$

Because for each $\alpha \in C_{2N}$ the space $(B_{\alpha,N}, d_\infty)$ (ρ, ω)-embeds into (M, d) via $F_{\alpha,N}$, which maps 0 to m_0, it follows that $o(\mathcal{V}) \geq \omega_1$. Because we are only considering coarse embeddings, we may assume that (M, d) is complete. By Proposition 3.2 the $\mathbb{I}_N$-vine $\mathcal{V}$ must be ill founded, i.e., there exists a strictly increasing sequence $(k_m)_m$ of integers and for every finitely supported $f\colon \{k_i\colon i \in \mathbb{N}\} \to \mathbb{I}_N$ there exists $x_f \in M$ such that $\chi_m = (x_f)_{f\in[\mathbb{I}_N,\{k_1,\ldots,k_m\}]}$ is in $\mathcal{V}$ for all $m \in \mathbb{N}$. By the definition of $\mathcal{V}$ it follows that the map from $B(N, \mathbb{Z})$ to (M, d) given by $\sum_{i=1}^\infty m_i e_i \mapsto x_f$, where $f\colon \{k_i\colon i \in \mathbb{N}\} \to I_n$ is the function with $f(k_i) = m_i$, is a (ρ, ω)-embedding. ∎

The last result of this section is a variation of the barycentric gluing technique, which has an interest on its own. With this gluing technique we can show that if we can equi-coarsely embed the bounded subsets of c_0 (or equivalently every separable bounded metric spaces) into a metric space M then M^4 is coarsely universal. In particular, an immediate consequence of Theorem 4.2 and of Theorem 4.4, below, is

Corollary 4.3. *Let (M, d) be a separable metric space. If for every $\alpha < \omega_1$ the metric space $(\mathrm{S}_\alpha(\mathbb{Z}), d_\infty)$ coarsely embeds into (M, d), then c_0 coarsely embeds into M^4.*

The original barycentric gluing technique (see [Bau07]) creates a coherent embedding of a metric space into a Banach space, by pasting embeddings of balls of growing radii together. Here, the process is reversed in the sense that we will paste balls of Banach spaces into metric spaces, but our proof has the caveat that it requires the gluing into M^4, rather than into M. Our general result is the following.

THEOREM 4.4. *Let $(X, \|\cdot\|)$ be a Banach space and (M, d) be a metric space. Assume that there exist increasing functions $\rho, \omega\colon [0,\infty) \to [0,\infty)$ that are tending to ∞ at ∞, $m_0 \in M$, and for all $n \in \mathbb{N}$, maps $h_n\colon nB_X \to M$ such that $h_n(0) = m_0$, and for all $x, y \in nB_X$,*

$$\rho(\|x-y\|) \le d(h_n(x), h_n(y)) \le \omega(\|x-y\|). \tag{4.1}$$

We equip M^4 with the ℓ_∞-metric associated with d, which we still denote d. Then there is a $(\tilde\rho, \tilde\omega)$-embedding of X into M^4 where $\tilde\rho(t) = \frac{1}{2}\rho(t/2)$ and $\tilde\omega(t) = 8\omega(3t)$ for all $0 < t < \infty$.

Proof. We choose inductively $r_0 = 0 < r_1 < r_2 < \cdots$ in $\mathbb{N}$ so that

$$\rho(r_{n+1}) > 2\omega(r_n) \quad \text{and} \quad r_{n+1} \ge 2r_n. \tag{4.2}$$

For $n \in \mathbb{N}$ we define the following map $\alpha_n\colon [0,\infty) \to [0,1]$ (set $r_{-4} = r_{-3} = r_{-2} = r_{-1} = r_0 = 0$):

$$\alpha_n(t) = \begin{cases} 0 & \text{if } t < r_{n-4} \text{ or } t > r_n, \\ \dfrac{t - r_{n-4}}{r_{n-3} - r_{n-4}} & \text{if } r_{n-4} \le t < r_{n-3}, \\ 1 & \text{if } r_{n-3} \le t \le r_{n-1}, \\ \dfrac{r_n - t}{r_n - r_{n-1}} & \text{if } r_{n-1} < t \le r_n. \end{cases}$$

The support of α_n is (r_{n-4}, r_n), and $\{t\colon \alpha_n(t) = 1\} = [r_{n-3}, r_{n-1}]$.

For $i \in \{0,1,2,3\}$ we define $F^{(i)}\colon X \to M$ as follows: For $x \in X$ we choose $l \in \mathbb{Z}^+$ so that $r_{4(l-1)+i} \le \|x\| < r_{4l+i}$, and put

$$F^{(i)}(x) = h_{r_{4l+i}}(\alpha_{r_{4l+i}}(\|x\|)x).$$

Then we define the map

$$F\colon X \to M^4, \qquad x \mapsto (F^{(0)}(x), F^{(1)}(x), F^{(2)}(x), F^{(3)}(x)).$$

We will show that F satisfies

$$\frac{1}{2}\rho\left(\frac{\|x-y\|}{2}\right) \le d(F(x), F(y)) \le 3\omega(3\|x-y\|). \tag{4.3}$$

Firstly, we estimate the compression function. Let $x, y \in X$, and assume without loss of generality that $\|x\| \le \|y\|$. Choose $l \in \mathbb{N}_0$ and $i \in \{0,1,2,3\}$ so that $r_{4l+i-2} \le \|y\| \le r_{4l+i-1}$. It is sufficient to show that

$d(F^{(i)}(y), F^{(i)}(x)) \geq \tilde{\rho}(\|x-y\|)$. We first note that $\alpha_{4l+i}(y) = 1$ and thus $F^{(i)}(y) = h_{r_{4l+i}}(y)$. We consider two cases.

Case 1: $r_{4l+i-3} \leq \|x\|$, thus $\alpha_{4l+i}(x) = 1$ and $F^{(i)}(x) = h_{r_{4l+i}}(x)$. It follows that

$$d(F^{(i)}(x), F^{(i)}(y)) = d((h_{r_{4l+i}}(x), h_{r_{4l+i}}(y)) \geq \rho(\|x-y\|).$$

Case 2: $\|x\| < r_{4l+i-3}$. Thus, for some $m \leq l$,

$$\begin{aligned} d(F^{(i)}(x), F^{(i)}(y)) &\geq d(F^{(i)}(y), m_0) - d(F^{(i)}(x), m_0) \\ &= d(h_{r_{4l+i}}(y), h_{r_{4l+i}}(0)) \\ &\quad - d\big(h_{r_{4m+i}}(\alpha_{4m+i}(\|x\|)x), h_{r_{4m+i}}(0)\big) \\ &\geq \rho(\|y\|) - \omega(\|x\|) \\ &\geq \frac{1}{2}\rho(\|y\|) + \frac{1}{2}\rho(r_{4l+i-2}) - \omega(r_{4l+i-3}) \\ &\geq \frac{1}{2}\rho(\|y\|) \geq \frac{1}{2}\rho\bigg(\frac{\|x-y\|}{2}\bigg). \end{aligned}$$

Secondly, we estimate the expansion function. We fix $i \in \{0,1,2,3\}$ and consider three cases.

Case 1: For some $n \in \mathbb{N}$ we have $r_{n-1} \leq \|x\| \leq \|y\| \leq r_n$.

If $n = 4l+i-1$ or $n = 4l+i-2$, then $\alpha_{4l+i}(\|y\|) = \alpha_{4l+i}(\|x\|) = 1$, and therefore

$$d(F^{(i)}(x), F^{(i)}(y)) = d(h_{r_{4l+i}}(x), h_{r_{4l+i}}(y)) \leq \omega(\|x-y\|).$$

If $n = 4l+i-3$ or $n = 4l+i$, then

$$|\alpha_{4l+i}(\|x\|) - \alpha_{4l+i}(\|y\|)| = \left|\frac{\|x\| - \|y\|}{r_n - r_{n-1}}\right| \leq \frac{2}{r_n}\|x-y\|,$$

and therefore

$$\begin{aligned} \|\alpha_{4l+i}(\|x\|)x - \alpha_{4l+i}(\|y\|)y\| &\leq \alpha_{4l+i}(\|x\|)\|x-y\| \\ &\quad + \|y\|\,|\alpha_{4l+i}(\|x\|) - \alpha_{4l+i}(\|y\|)| \\ &\leq 3\|x-y\|, \end{aligned}$$

which implies that

$$\begin{aligned} d(F^{(i)}(x), F^{(i)}(y)) &= d(h_{r_{4l+i}}(\alpha_{4l+i}(\|x\|)x), h_{r_{4l+i}}(\alpha_{4l+i}(\|y\|)y)) \\ &\leq \omega(3\|x-y\|). \end{aligned}$$

From now on we assume that there are $m, n \in \mathbb{N}$, $m < n$, such that $r_m \leq \|x\| \leq r_{m+1} \leq r_n \leq \|y\| \leq r_{n+1}$. For $j = 1, \ldots, n-m$, let z_j be the element on the segment $[x,y]$ (i.e., points of the form $x + t(y-x)$ with $0 \leq t \leq 1$) such that $\|z_j\| = r_{m+j}$, and put $z_0 = x$ and $z_{n-m+1} = y$.

CASE 2: $n - m \le 3$. We have

$$d(F^{(i)}(x), F^{(i)}(y)) \le \sum_{i=1}^{n-m+1} d(F^{(i)}(z_{i-1}), F^{(i)}(z_i)) \le \sum_{i=1}^{n-m+1} \omega(3\|z_{i-1} - z_i\|) \le 4\omega(3\|x - y\|).$$

CASE 3: $n - m \ge 4$. It then follows from Case 2 that

$$\begin{aligned} d(F^{(i)}(x), F^{(i)}(y)) &\le d(F^{(i)}(x), m_0) + d(F^{(i)}(y), m_0) \\ &= d(F^{(i)}(x), F^{(i)}(z_{j_1})) + d(F^{(i)}(y), F^{(i)}(z_{j_2})) \\ &\le 4\omega(3\|x - z_1\|) + 4\omega(3\|y - z_2\|) \le 8\omega(3\|x - y\|). \ \blacksquare \end{aligned}$$

5. Universality properties of interlacing graphs. In [Kal07], Kalton showed that a Banach space X that is coarsely universal for the class of all separable metric spaces, or equivalently that coarsely contains c_0, cannot have separable iterated duals, i.e., $X^{(r)}$ is non-separable from some $r \ge 2$ on. Kalton's argument is based on the metric properties of the interlacing graphs. As we will see in the next section, these graphs introduced by Kalton have some remarkable universality properties.

5.1. Almost isometric universality of the interlacing graphs. We define a slightly larger class of interlacing graphs than the ones introduced by Kalton. The set of vertices is $[\mathbb{N}]^{<\omega}$, the set of finite subsets of $\mathbb{N}$, and we declare that two vertices $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ in $[\mathbb{N}]^{<\omega}$ are adjacent if and only if $a \ne b$ and one of the following interlacing relations holds:

(i) $n = m + 1$ and $a_i \le b_i \le a_{i+1}$ for $1 \le i \le m$,
(ii) $m = n + 1$ and $b_i \le a_i \le b_{i+1}$ for $1 \le i \le n$,
(iii) $n = m$, $a_i \le b_i \le a_{i+1}$ for $1 \le i < n$, and $a_n \le b_n$, or
(iv) $n = m$, $b_i \le a_i \le b_{i+1}$ for $1 \le i < n$, and $b_n \le a_n$.

We also connect the empty set with all singletons. We refer to this graph as the *universal interlacing graph*, and we denote by $([\mathbb{N}]^{<\omega}, d_{\mathrm{I}})$ the universal interlacing graph equipped with its canonical graph metric. Kalton's interlacing graphs are defined in the same way except that only vertices with the same length were considered. More precisely, Kalton's interlacing graph of diameter k is the space $([\mathbb{N}]^k, d_{\mathrm{I}}^{(k)})$, where the graph metric only refers to the interlacing relations (iii) or (iv) in this case. For $A, B \in [\mathbb{N}]^k$, although it is obvious that $d_{\mathrm{I}}^{(k)}(A, B) = 1$ if and only if $d_{\mathrm{I}}(A, B) = 1$, it is not immediately clear that on $[\mathbb{N}]^k$ the metrics $d_{\mathrm{I}}^{(k)}$ and d_{I} coincide. As we will see later, this is indeed the case.

The universality properties of the interlacing graphs stem from the fact that the interlacing metric admits an interpretation in terms of the summing norm on c_0. For A, B in $[\mathbb{N}]^{<\omega}$ define the *summing distance*

$$d_{\mathrm{sum}}(A,B) = \Big\| \sum_{i\in A} s_i - \sum_{i\in B} s_i \Big\|_{\mathrm{sum}}$$

where $(s_i)_i$ denotes the summing basis of c_0, endowed with the usual bimonotone version of the summing norm, i.e.

$$\Big\| \sum_i a_i s_i \Big\|_{\mathrm{sum}} = \sup\Big\{ \Big| \sum_{i=k}^{m} a_i \Big| : k, m \in \mathbb{N},\ k \le m \Big\}. \tag{5.1}$$

In (5.1) one only needs to consider intervals at whose boundaries there are sign-changes of the a_i's. More precisely for a sequence (a_i) in c_{00} let $0 = m_0 < m_1 < \cdots < m_s$ be chosen in $\mathbb{N}$ so that for all $i \le s$ the signs of a_j on $j \in [m_{i-1}+1, m_i]$ are the same (i.e., all non-negative or all non-positive). Then

$$\Big\| \sum_i a_i s_i \Big\|_{\mathrm{sum}} = \sup\Big\{ \Big| \sum_{i=k}^{l} \sum_{m_{i-1}+1}^{m_i} a_j \Big| : 1 \le k \le l \le s \Big\}. \tag{5.2}$$

Thus for $A, B \subset [\mathbb{N}]^{<\omega}$ we write $A \triangle B$ in increasing order as $A \triangle B = \{x_1, \ldots, x_n\}$ and note that

$$\begin{aligned} d_{\mathrm{sum}}(A,B) &= \max\{|\#(A\cap E) - \#(B\cap E)| : E \text{ is an interval of } \mathbb{N}\} \\ &= \max\{|\#(A\cap [x_i, x_j]) - \#(B\cap [x_i, x_j])| : 1 \le i \le j \le n\}. \end{aligned} \tag{5.3}$$

The above forms of the metric d_{sum} will be used more often. We first show that the interlacing metric and the summing distance coincide. For fixed k, the coincidence of $d_{\mathrm{I}}^{(k)}(A,B)$ with

$$\max\{|\#(A\cap E) - \#(B\cap E)| : E \text{ is an interval of } \mathbb{N}\}$$

was already shown in [LPP20], where it was afterwards used in connection with the canonical norm of c_0 instead of $\|\cdot\|_{\mathrm{sum}}$.

For $n \in \mathbb{N}$, $A = \{a_1, \ldots, a_n\} \in \mathbb{N}^{<\omega_1}$, with $a_1 < \cdots < a_n$, we call $A' = \{a'_1, \ldots, a'_n\}$, with $a'_1 < \ldots < a'_n$ a *shift to the left of* A if $A' \neq A$ and $a_1 \le a'_1 \le a_2 \le a'_2 \le \cdots \le a_n \le a'_n$. We note that in this case

$$d_{\mathrm{I}}(A,A') = d_{\mathrm{I}}(A, A'\setminus\{a'_n\}) = d_{\mathrm{sum}}(A,A') = d_{\mathrm{sum}}(A, A'\setminus\{a'_n\}). \tag{5.4}$$

For another set $B \in [\mathbb{N}]^{<\omega}$ we say that a left shift A' of A is a *shift towards* B if $A'\setminus A \subset B \setminus A$.

THEOREM 5.1. *For $A, B \in [\mathbb{N}]^{<\omega}$ we have $d_{\mathrm{sum}}(A,B) = d_{\mathrm{I}}(A,B)$. Moreover if $k = \#A = \#B$ then there is a path of length $d_{\mathrm{I}}(A,B)$ from A to B in the interlacing graph which stays in $[\mathbb{N}]^k$. Thus the restriction of d_{I} to $[\mathbb{N}]^k$ is $d_{\mathrm{I}}^{(k)}$.*

Proof. We prove our statement by induction for all $m \in \mathbb{N} \cup \{0\}$, and all $A, B \in [\mathbb{N}]^{<\omega}$ with $m = d_{\mathrm{sum}}(A, B)$.

If $m = 0$ and $d_{\mathrm{sum}}(A, B) = 0$ and thus $A = B$, our claim is trivial. If $m = 1$ and $d_{\mathrm{sum}}(A, B) = 1$, we will show that $d_{\mathrm{I}}(A, B) = 1$. Write $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_m\}$ and assume, without loss of generality, that $\min(A \triangle B) = a_{i_0} \in A$. Note that $|n - m| = |\#A - \#B| \leq d_{\mathrm{sum}}(A, B) = 1$ and by the assumption $\min(A \triangle B) = a_{i_0} \in A$ we have $1 \leq \#A \cap [a_{i_0}, \max(A \cup B)] - \#B \cap [a_{i_0}, \max(A \cup B)] = (n - i_0 + 1) - (m - i_0)$, i.e., $m \leq n \leq m + 1$. Next, observe that for $1 \leq i \leq \min(m, n)$ we have $a_i \leq b_i$. Otherwise, set $j_0 = \min\{1 \leq i \leq \min(m, n) \colon a_i > b_i\}$ and note that $a_{i_0} < b_{i_0}$ and thus $i_0 < j_0$. If we set $E = [b_{i_0}, b_{j_0}]$ then $d_{\mathrm{sum}}(A, B) = 1 \geq \#B \cap E - \#A \cap E = (j_0 - i_0 + 1) - (j_0 - i_0 - 1) = 2$. We also observe that for $1 \leq i \leq \min(m, n - 1)$ we have $b_i \leq a_{i+1}$. If this is not the case, set $s_0 = \min\{1 \leq i \leq \min(m, n - 1) \colon b_i > a_{i+1}\}$ and observe that if $E = [a_1, a_{s_0+1}]$ then $\#A \cap E = s_0 + 1$ whereas $\#B \cap E = s_0 - 1$, which is absurd. Finally we distinguish the cases $n = m$ and $n = m + 1$. If $n = m$ then we have demonstrated that (iii) of the definition of adjacency holds. If $n = m + 1$ then we have demonstrated that (i) holds.

Assume now that for some $m \geq 2 \in \mathbb{N}$ and all $A, B \in [\mathbb{N}]^{<\omega}$ with $d_{\mathrm{sum}}(A, B) < m$ it follows that $d_{\mathrm{sum}}(A, B) = d_{\mathrm{I}}(A, B)$, and that $d_{\mathrm{I}}(A, B) = d_{\mathrm{I}}^{(k)}(A, B)$ if $k = \#A = \#B$.

Let $A, B \in [\mathbb{N}]^{<\omega}$ with $d_{\mathrm{sum}}(A, B) = m$. If $A \subset B$, or $B \subset A$, and we assume without loss of generality that $B \subsetneq A$, we put $A' = A \setminus \{a\}$, where $a \in A \setminus B$. Then $d_{\mathrm{sum}}(A, A') = d_I(A, A') = 1$ and $d_{\mathrm{sum}}(A', B) = m - 1$, and we deduce our claim from the induction hypothesis.

Assuming that $A \not\subset B$ and $B \not\subset A$ we write $A \triangle B$ in increasing order as $A \triangle B = \{x_1, \dots, x_l\}$. It follows that

$$m = d_{\mathrm{sum}}(A, B) = \max_{i \leq j} |\#(A \cap [x_i, x_j]) - \#(B \cap [x_i, x_j])|.$$

Without loss of generality we can assume that $x_1 \in A \setminus B$. There is a $t \in \mathbb{N}$ and numbers $1 \leq i_1 < i_2 < \cdots < i_t < l$ such that

$$\{i_s \colon s = 1, \dots, t\} = \big\{i \in \{1, \dots, l - 1\} \colon x_i \in A \text{ and } x_{i+1} \in B\big\}.$$

We now define $A' \in [\mathbb{N}]^{\omega}$ for which $d_{\mathrm{I}}(A, A') = d_{\mathrm{sum}}(A, A') = 1$ and $d_{\mathrm{sum}}(A', B) \leq m - 1$, and consider the following two cases:

Case 1: For all $1 \leq j \leq l$ we have

$$\#(A \cap [x_j, x_l]) - \#(B \cap [x_j, x_l]) < m.$$

Then we put

$$A' = (A \setminus \{x_{i_s} \colon s \leq t\}) \cup \{x_{i_s+1} \colon s \leq t\},$$

which is a left shift of A towards B.

Case 2: There is a $j \le l$ such that $\#(A \cap [x_j, x_l]) - \#(B \cap [x_j, x_l]) = m$. It follows that $x_l \in A$ and we put

$$A' = ((A \setminus \{x_{i_s} \colon s \le t\}) \cup \{x_{i_s+1} \colon s \le t\}) \setminus \{x_l\}.$$

We observe that if $\#A = \#B$, the second case cannot happen. Indeed, assume that there is a $j \le l$ such that $\#(A \cap [x_j, x_l]) - \#(B \cap [x_j, x_l]) = m$; then $j > 1$ and it follows that

$$\begin{aligned}
&\#(B \cap [x_2, x_{j-1}]) - \#(A \cap [x_2, x_{j-1}]) \\
&\quad = \#(B \setminus A) - \#(B \cap (\{x_1\} \cup [x_j, x_l])) - \#(A \setminus B) \\
&\qquad - \#(A \cap (\{x_1\} \cup [x_j, x_l])) \\
&\quad = \#(A \cap (\{x_1\} \cup [x_j, x_l])) - \#(B \cap (\{x_1\} \cup [x_j, x_l])) = m + 1,
\end{aligned}$$

which is a contradiction.

Thus, $\#A' = \#A$ if $\#A = \#B$.

From (5.4) it follows that $d_{\mathrm{I}}(A, A') = d_{\mathrm{sum}}(A, A') = 1$. We need to show that $d_{\mathrm{sum}}(A', B) \le m - 1$, and thus, by the triangle inequality, $d_{\mathrm{sum}}(A', B) = m - 1$.

First let $i \in \{1, \dots, l\}$ and define, for $i \le j \le l$,

$$f(j) = \#(A' \cap [x_i, x_j]) - \#(B \cap [x_i, x_j]).$$

Observe that $f(i) \le 1 \le m - 1$. We claim that for all $i < j \le l$,

$$f(j) \le \min(\#(A \cap [x_i, x_j]) - \#(B \cap [x_i, x_j]), m - 1). \tag{5.5}$$

As $A' \setminus B \subset A \setminus B$, we have $f(j) \le \#(A \cap [x_i, x_j]) - \#(B \cap [x_i, x_j])$ for $i \le j$.

Assume that our claim is not true, and let k be the minimum of all $j > i$ such that

$$f(j) = 1 + \min(\#(A \cap [x_i, x_j]) - \#(B \cap [x_i, x_j]), m - 1).$$

Since $f(k) \le f(k-1) + 1$, it follows that $\#(A' \cap [x_i, x_{k-1}]) - \#(B \cap [x_i, x_{k-1}]) = m - 1$, and since $A' \triangle B \subset A \triangle B$, it follows that $x_k \in A$ and thus $\#(A \cap [x_i, x_{k-1}]) - \#(B \cap [x_i, x_{k-1}]) = m - 1$ (otherwise $\#(A \cap [x_i, x_{k-1}]) - \#(B \cap [x_i, x_{k-1}]) = m$ and so $\#(A \cap [x_i, x_k]) - \#(B \cap [x_i, x_k]) = m + 1$). Therefore, $\#(A \cap [x_i, x_k]) - \#(B \cap [x_i, x_k]) = m$.

Either $k < l$, then $x_{k+1} \in B$, and since $x_k \in A$, it follows from the definition of A' that $x_k \notin A'$ and thus $f(k) = f(k-1) = m - 1$, which contradicts our assumption. Or $k = l$, and therefore $\#(A \cap [x_i, x_l]) - \#(B \cap [x_i, x_l]) = m$, which implies that $x_k = x_l \notin A'$, since the second case in the definition of A' occurs; it would again follow that $f(k) = m - 1$, which is once more a contradiction.

Next we let $j = 1, \dots, l$, and for $i = 1, \dots, j$ put

$$g(i) = \#(B \cap [x_i, x_j]) - \#(A' \cap [x_i, x_j]),$$

and claim that $g(i) \leq \min(\#(B \cap [x_i, x_j]) - \#(A \cap [x_i, x_j]), m-1)$ for all $i \in \{1, \dots, l\}$. Then, since $B \triangle A' \subset B \triangle A$, it follows that $g(i) \leq \#(B \cap [x_i, x_j]) - \#(A \cap [x_i, x_j])$ for all $i \in \{1, \dots, j\}$.

Assume our claim is not true and let k be the maximal $k < j$ such that $g(k) = m$. So it follows that $\#(B \cap [x_k, x_j]) - \#(A \cap [x_k, x_j]) = m$, and thus $x_k \in B$, and $x_{k-1} \in A$ (note that $k \neq 1$ since $x_1 \in A$). But this means that $x_k \in A'$ and thus $\#(B \cap [x_k, x_j]) - \#(A' \cap [x_k, x_j]) = m - 1$, which is again a contradiction.

We have therefore shown that for all $1 \leq i < j \leq l$, $|\#(A' \cap [x_i, x_j]) - \#(B \cap [x_i, x_j])| \leq m - 1$, which finishes our proof of Theorem 5.1. ∎

The following corollary could of course be also proven directly very easily.

COROLLARY 5.2. *For all $k, m \in \mathbb{N}$ with $k < m$, $([\mathbb{N}]^k, d_{\mathrm{I}}^{(k)})$ embeds isometrically into $([\mathbb{N}]^m, d_{\mathrm{I}}^{(m)})$.*

The following quantitative embedding result implies Theorem E.

THEOREM 5.3. *Let (X, d) be an n-point metric space, and $\alpha := \alpha(X) = \frac{\operatorname{diam}(X)}{\operatorname{sep}(X)}$ be its aspect ratio, where $\operatorname{diam}(X) = \sup\{d_X(x, y) \colon x, y \in X\}$ and $\operatorname{sep}(X) = \inf\{d_X(x, y) \colon x, y \in X\}$. Then for every $0 < \varepsilon < 1$ and every integer $k \geq (n + 3/2)(\alpha/\varepsilon + \operatorname{diam}(X) + 1)$, (X, d) embeds with distortion $(1 - \varepsilon)^{-1}$ into $([\mathbb{N}]^k, d_{\mathrm{I}}^{(k)})$. In particular, for all $\varepsilon > 0$, (X, d) embeds with distortion at most $1 + \varepsilon$ into $([\mathbb{N}]^{<\omega}, d_{\mathrm{I}})$.*

Proof. The proof of the general situation can be reduced to the special case where (X, d) is a finite metric space with even distances. Assuming that we have proven the following claim, we can finish the proof of the general case.

CLAIM 5.4. *Assume that for all $x, y \in X$, $d(x, y)$ is an even integer and that k is an integer number such that $k \geq \frac{1}{2}(n + 3/2)(\operatorname{diam}(X) + 2)$. Then the space (X, d) embeds isometrically into $([\mathbb{N}]^k, d_{\mathrm{I}}^{(k)})$.*

Indeed, let $\varepsilon > 0$ and choose an integer q such that $(\operatorname{sep}(X)\varepsilon)^{-1} \leq q \leq (\operatorname{sep}(X)\varepsilon)^{-1} + 1$, and for each $x, y \in X$ define $k_{x,y} = \max\{k \in \mathbb{N} \cup \{0\} \colon k/q \leq d(x, y)\}$. Define a metric $\tilde{d}$ on X by

$$\tilde{d}(x, y) = \min\left\{ \sum_{i=1}^{\ell} \frac{k_{x_i, x_{i-1}}}{q} \colon x_i \in X \text{ for } 0 \leq i \leq \ell \text{ and } x = x_0,\ y = y_\ell \right\}.$$

One can check that $\tilde{d}$ is indeed a metric and for all $x, y \in X$ we have

$$(1 - \varepsilon) d(x, y) \leq \tilde{d}(x, y) \leq d(x, y),$$

hence, it suffices to embed the space $(X, \tilde{d})$ with distortion 1 into $([\mathbb{N}]^k, d_{\mathrm{I}}^{(k)})$ for an appropriate k. Note that if we denote $\tilde{X} = (X, \tilde{d})$, then $\operatorname{diam}(\tilde{X}) \leq$

$\operatorname{diam}(X)$. By Claim 5.4, the space $(X, 2q\tilde{d})$ embeds isometrically into $([\mathbb{N}]^k, d_1^{(k)})$ for $k \geq \frac{1}{2}(n+3/2)(2q\operatorname{diam}(X)+2)$. Recall $q \leq (\operatorname{sep}(X)\varepsilon)^{-1}+1$, which implies $\frac{1}{2}(n+3/2)(2q\operatorname{diam}(X)+2) \leq (n+3/2)(\alpha/\varepsilon+\operatorname{diam}(X)+1)$. ∎

Proof of Claim 5.4. We will find an embedding Φ from X into the linear span of $(s_i)_i$, endowed with the norm (5.1), such that for each $x \in X$ the vector $\Phi(x)$ is of the form $\sum_{i\in A(x)} s_i$ with $\#A(x) \leq \frac{1}{2}(n+3/2)(\operatorname{diam}(X)+2)$.

We enumerate $X = \{x_2, \dots, x_{n+1}\}$. We add one more point x_1 to obtain the set $\tilde{X} = \{x_1, x_2, \dots, x_{n+1}\}$. We extend d onto $\tilde{X}$ by setting for $d(x_1, x) = d(x, x_1) = D$, $x \in X$, where D is the minimal even integer with $2D \geq \operatorname{diam}(X)$. As the diameter of X is an even integer, we deduce $D \leq \operatorname{diam}(X)/2+1$. It is straightforward to verify that the triangle inequality is still satisfied on $\tilde{X}$. For notational reasons, we add a "ghost" point x_{n+1} with the property $d(x, x_{n+1}) = 0$ for all $x \in X$. We first define a map $\Phi_0 \colon X \to \langle \{s_i \colon i \in \mathbb{N}\}\rangle$, the linear span of the s_i's, by

$$\Phi_0(x) = \frac{1}{2}\sum_{i=1}^{n+1}(d(x, x_i) - d(x, x_{i+1}))s_i.$$

If we denote by $(s_i^*)_i$ the sequence of coordinate functionals associated to $(s_i)_i$, we observe that for all $i \in \mathbb{N}$ and $x \in X$, the number $s_i^*(\Phi_0(x))$ is an integer. We start by showing that $\Phi_0(x)$ is an isometric embedding. Let $1 \leq k \leq m \leq n+1$. Then

$$\begin{aligned}
\|\Phi_0(x) - \Phi_0(y)\| &= \frac{1}{2}\Big|\sum_{i=k}^{m}(d(x, x_i) - d(x, x_{i+1} - d(y, x_i) + d(y, x_{i+1}))\Big| \\
&= \frac{1}{2}|d(x, x_{m+1}) - d(y, x_{m+1}) - d(x, x_k) + d(y, x_k)| \\
&\leq \frac{1}{2}|d(x, x_{m+1}) - d(y, x_{m+1})| + \frac{1}{2}|d(x, x_k) - d(y, x_k)| \\
&\leq d(x, y).
\end{aligned}$$

For the inverse inequality, let $x = x_j$, $y = x_{j'}$ and assume without loss of generality $j < j'$. Define $k = j$ and $m = j' - 1$. Then

$$\begin{aligned}
\|\Phi_0(x) - \Phi_0(y)\| &\geq \frac{1}{2}\Big|\sum_{i=k}^{m}(d(x, x_i) - d(x, x_{i+1}) - d(y, x_i) + d(y, x_{i+1}))\Big| \\
&= \frac{1}{2}|d(x, x_{m+1}) - d(y, x_{m+1}) - d(x, x_k) + d(y, x_k)| \\
&= \frac{1}{2}|d(x, y) - d(y, y) - d(x, x) + d(y, x)| = d(x, y).
\end{aligned}$$

Define

$$\Phi_1(x) = \Phi_0(x) + D\sum_{i=1}^{n+1} s_i.$$

Then Φ_1 is an isometric embedding of X into $\langle\{s_i\colon i\in\mathbb{N}\}\rangle$ such that for all $i\in\mathbb{N}$ and $x\in X$ the number $s_i^*(\Phi_1(x))$ is a non-negative integer. For $k=1,\dots,n+1$ define

$$N_k=\max\{s_k^*(\Phi_1(x))\colon x\in X\}\quad\text{and}\quad M_k=\sum_{j=1}^{k}N_j.$$

Also let $M_0=0$. We are ready to define the desired embedding. For $x\in X$ set

$$\Phi(x)=\sum_{k=1}^{n+1}\sum_{\{\substack{M_{k-1}<i\\ \le M_{k-1}+s_k^*(\Phi_1(x))}\}}s_i.$$

We deduce that $\Phi(x)$ is of the form $\sum_{i\in A(x)}s_i$ with

$$\begin{aligned}\#A(x)&=\sum_{k=1}^{n+1}s_k^*(\Phi_1(x))=\frac{1}{2}\sum_{k=1}^{n+1}(d(x,x_i)-d(x,x_{i+1}))+D(n+1)\\&=\frac{1}{2}(d(x,x_1)-d(x,x_{n+1}))+D(n+1)=\frac{1}{2}D+D(n+1)\\&=\left(n+\frac{3}{2}\right)D\le\frac{1}{2}\left(n+\frac{3}{2}\right)(\operatorname{diam}(X)+2).\end{aligned}$$

Applying (5.2) to $m_i=M_i$, $i=1,\dots,n$, we deduce for $x,y\in X$ that

$$\begin{aligned}\|\Phi(x)-\Phi(y)\|&=\max\Big\{\Big|\sum_{i=p}^{q}\sum_{j=M_{i-1}+1}^{M_i}s_j^*(\Phi(x)-\Phi(y))\Big|\colon 1\le p\le q\le n\Big\}\\&=\max\Big\{\sum_{i=p}^{q}s_i^*(\Phi_0(x)-\Phi_0(y))\Big\}=d(x,y).\end{aligned}$$

Our conclusion follows therefore from Theorem 5.1 and Corollary 5.2. ∎

5.2. Metric universality and metric elasticity. It is a well known and long standing open problem whether c_0 isomorphically embeds into a Banach space whenever it bi-Lipschitzly embeds into it. Due to Aharoni's theorem this fundamental rigidity problem in non-linear Banach space geometry can be reformulated as the following universality question.

Problem 5.5. *Let X be a Banach space. If X is Lipschitz universal for the class of separable metric spaces, does X contain an isomorphic copy of c_0?*

It is possible to answer Problem 5.5 positively for Banach lattices using Kalton's work on the interlacing graphs. This fact seems to have been overlooked and we describe the argument in the ensuing discussion. Recall that a Banach space Y has Kalton's *property* $\mathcal{Q}$ if there exists $C\in(0,\infty)$ such that for all $k\in\mathbb{N}$ and every Lipschitz map f from $([\mathbb{N}]^k,d_1^{(k)})$ to Y, there

exists $\mathbb{M} \in [\mathbb{N}]^\omega$ such that for all $\bar{m}, \bar{n} \in [\mathbb{M}]^k$ we have

$$\|f(\bar{m}) - f(\bar{n})\|_Y \leq C \operatorname{Lip}(f). \tag{5.6}$$

Kalton showed that reflexive Banach spaces [Kal07, Theorem 4.1] and, more generally, Banach spaces whose unit ball uniformly embeds into a reflexive Banach space [Kal07, Corollary 4.3] have property $\mathcal{Q}$. It follows from (5.6) (and the fact that coarse embeddings f whose domains are graphs must be $\omega(1)$-Lipschitz) that the sequence of interlacing graphs cannot equi-coarsely embed into a Banach space with property $\mathcal{Q}$. Therefore if a Banach space X equi-coarsely contains the interlacing graphs, it must fail property $\mathcal{Q}$. By [Kal07, Corollary 4.3], the unit ball of X does not uniformly embed into a reflexive Banach space. But Kalton also proved [Kal07, Theorem 3.8] that for a separable Banach lattice X, B_X uniformly embeds into a reflexive Banach space if and only if X does not contain any subspace isomorphic to c_0. Thus, it follows from the above discussion that:

THEOREM 5.6 ([Kal07]). *If X is a separable Banach lattice and if $([\mathbb{N}]^k, d_{\mathrm{I}})_k$ equi-coarsely embeds into X, then X contains an isomorphic copy of c_0.*

The following statement is an immediate consequence of Theorem 5.6.

COROLLARY 5.7. *If a separable Banach lattice X is coarsely universal for the class of separable metric spaces, then X contains an isomorphic copy of c_0.*

Thus, Problem 5.5 (as well as its coarse analogue) has a positive solution for Banach lattices. It is worth pointing out that the coarse (resp. uniform) analogue of Problem 5.5 does not hold in general since, using the theory of Hölder free spaces, it is proven in [Kal04] that c_0 coarsely (resp. uniformly) embeds into a Schur space. Recall that a Banach space has the *Schur property* if every weakly null sequence converges to 0 in the norm topology, and hence a Schur space cannot contain any isomorphic copy of c_0.

REMARK 5.8. The Lipschitz version of Corollary 5.7 can be proven for a Banach space with an unconditional basis using classical linear and non-linear Banach space theory. Indeed, for Banach spaces with an unconditional basis the following dichotomy holds: either the unconditional basis is not boundedly complete, and by a result of James [Jam50], X will contain an isomorphic copy of c_0, or the unconditional basis is boundedly complete and hence X will be isomorphic to a dual space and thus X will have the Radon–Nikodým property. Note that the two possibilities are mutually exclusive. In the first situation the conclusion of Corollary 5.7 already holds, and in the second situation we can use classical differentiability theory and obtain a contradiction. A similar dichotomy argument fails for Banach lattices since L_1 is a Banach lattice that does not linearly contain c_0 and yet L_1 does not have the Radon–Nikodým property.

Theorem 5.6 also has an application to metric analogues of the linear notion of elasticity. In 1976 Schäffer raised the problem whether the isomorphism class of every infinite-dimensional Banach space X is unbounded in the sense that $D(X) := \sup\{d_{\mathrm{BM}}(Y,Z)\colon Y,Z \text{ are isomorphic to } X\} = \infty$ where d_{BM} denotes the Banach–Mazur distance ([1]). Johnson and Odell introduced the notion of elasticity for their solution of Schäffer's problem for *separable* Banach spaces.

Definition 5.9 ([JO05]). Let $K \in [1,\infty)$. A Banach space Y is *K-elastic* provided that if a Banach space X isomorphically embeds into Y then X must be K-isomorphically embeddable into Y, and Y is *elastic* if it is K-elastic for some K.

The connection with Schäffer's problem comes from the observation that if $D(X) < \infty$ then X as well as all isomorphic copies of X are $D(X)$-elastic. Elasticity is intimately connected with universality. First of all, it is immediate that every Banach space is crudely finitely representable into any elastic Banach space, in particular every elastic Banach space has trivial cotype. Second of all, a consequence of Banach–Mazur embedding theorem is that $C[0,1]$ is 1-elastic. Moreover, a key step in [JO05] is the following theorem.

Theorem 5.10 ([JO05, Theorem 7]). *Let X be a separable infinite-dimensional Banach space. If X is elastic then c_0 isomorphically embeds into X.*

The conjecture from [JO05] that a separable elastic Banach space must contain an isomorphic copy of $C[0,1]$ was recently solved positively by Alspach and Sarı [AS16].

We now discuss a metric analogue of Theorem 5.10. According to Johnson and Odell a Banach space Y is said to be *Lipschitz K-elastic* provided that if a Banach space is isomorphic to Y then X must bi-Lipschitzly embed into Y with distortion at most K. The definition by Johnson and Odell of Lipschitz elasticity is motivated by the fact that a Banach space X is K-elastic if and only if every isomorphic copy of X is K-isomorphic to a subspace of X (the proof uses a Hahn–Banach extension argument that goes back to Pełczyński [Peł60]). It was observed in [JO05] that it follows from Aharoni's embedding theorem and James' distortion theorem that there exists $K \geq 1$ such that every Banach space that contains an isomorphic copy of c_0 must be Lipschitz K-elastic. The constant K is related to the optimal distortion in Aharoni's embedding and can be taken to be $2+\varepsilon$ for every $\varepsilon > 0$ due to [KL08]. Motivated by Definition 5.9 the following definition is another metric analogue of elasticity.

([1]) This widely used terminology can be misleading since $\log(d_{\mathrm{BM}})$ (and not d_{BM}) is a semimetric.

DEFINITION 5.11. Let $K \in [1,\infty)$. A metric space Y is *metric K-elastic* provided that if a metric space X bi-Lipschitzly embeds into Y then X must be bi-Lipschitzly embeddable into Y with distortion at most K, and Y is *metric elastic* if it is metric K-elastic for some K.

It is immediate that a Banach space that is metric K-elastic (as a metric space) is Lipschitz K-elastic. With this stronger non-linear notion of elasticity we obtain the following theorem, which contains a strong non-linear analogue of Theorem 5.10 in the context of Banach lattices.

THEOREM 5.12. *Let X be a separable infinite-dimensional Banach lattice. The following assertions are equivalent:*

(1) c_0 *isomorphically embeds into X.*
(2) c_0 *bi-Lipschitzly embeds into X.*
(3) c_0 *coarsely embeds into X.*
(4) *X is metric elastic.*
(5) $([\mathbb{N}]^k, d_{\mathrm{I}})_{k\in\mathbb{N}}$ *equi-bi-Lipschitzly embeds into X.*
(6) $([\mathbb{N}]^k, d_{\mathrm{I}})_{k\in\mathbb{N}}$ *equi-coarsely embeds into X.*

Proof. (1)⇒(2)⇒(3) is trivial. (3)⇒(1) is Corollary 5.7. (2)⇒(4) follows from Aharoni's embedding theorem and the fact that separability is a Lipschitz invariant. For (4)⇒(5), observe that an infinite-dimensional Banach space X has a 1-separated sequence of unit vectors, and thus for all $k \in \mathbb{N}$, the k-dimensional interlacing graph $([\mathbb{N}]^k, d_{\mathrm{I}})$ (which is countable, 1-separated, and has diameter k) bi-Lipschitzly embeds into X with distortion at most k. Since X is metric K-elastic for some $K \geq 1$, it follows that $\sup_{k\in\mathbb{N}} c_X(([\mathbb{N}]^k, d_{\mathrm{I}})) \leq K$. For Banach spaces, (5)⇒(6) always holds. An appeal to Corollary 5.6 gives the remaining implication. ∎

5.3. Separating interlacing graphs in Banach spaces with non-separable biduals. The following concentration result for interlacing graphs was shown by Kalton [Kal07].

THEOREM 5.13 ([Kal07, Theorem 3.5]). *Let $k \in \mathbb{N}$ and Y be a Banach space such that $Y^{(2k)}$, the iterated dual of order $2k$ of Y, is separable. Assume that $(g_i)_{i\in I}$ is an uncountable family of 1-Lipschitz maps from $([\mathbb{N}]^k, d_{\mathrm{I}}^{(k)})$ to Y. Then there exist $i \neq j \in I$ and $\mathbb{M} \in [\mathbb{N}]^\omega$ such that for all $\bar{n} \in [\mathbb{M}]^k$ we have*

$$\|g_i(\bar{n}) - g_j(\bar{n})\| \leq 3.$$

Vaguely speaking, it follows from Theorem 5.13 that if a Banach space X contains uncountably many well separated 1-Lipschitz images of the interlacing graphs and if X coarsely embeds into a Banach space Y, then Y cannot have all its iterated duals separable. This idea was devised by Kalton in [Kal07] to show that if c_0 coarsely embeds into a Banach space Y, then

one of the iterated duals of Y is non-separable (in particular, Y cannot be reflexive). It was adapted in [LPP20] to show that the same conclusion holds if the James tree space JT or its predual coarsely embeds into Y. In these proofs the non-separability of the bidual of the embedded space plays an important role. However, since ℓ_1 coarsely embeds into ℓ_2, this is not a sufficient condition. We will prove that a certain presence of ℓ_1 in the embedded space is essentially the only obstruction.

THEOREM 5.14 (Theorem F). *Let X be a separable Banach space with non-separable bidual X^{**}, $\ell_1 \not\subset X$, and such that no spreading model generated by a normalized weakly null sequence in X is equivalent to the ℓ_1-unit vector basis. Assume that X coarsely embeds into a Banach space Y. Then there exists $k \in \mathbb{N}$ such that $Y^{(2k)}$ is non-separable.*

Proof. We start with the construction of our well separated 1-Lipschitz maps from $([\mathbb{N}]^k, d_1^{(k)})$ to X. Since X is separable and X^{**} is not, using the Riesz lemma and an easy transfinite induction, we can build $(x^{**}_\alpha)_{\alpha<\omega_1}$ in $S_{X^{**}}$ such that

$$\forall \alpha < \omega_1, \qquad d(x^{**}_\alpha, \overline{\mathrm{sp}}(X \cup \{x^{**}_\beta : \beta < \alpha\})) > 3/4.$$

Fix now $\alpha < \omega_1$. Since $\ell_1 \not\subset X$, it follows from a result by Odell and Rosenthal [OR75, equivalence of (1)–(5) on p. 376] that for each $\alpha < \omega_1$ there is a sequence $(x_{\alpha,n})_{n=1}^\infty$ in S_X which converges weak* in X^{**} to x^{**}_α. In particular, the sequence $(x_{\alpha,n})_{n=1}^\infty$ is weakly Cauchy. Since $d(x^{**}_\alpha, X) > 3/4$, we may as well assume, after extracting a subsequence, that $\|x_{\alpha,n} - x_{\alpha,m}\| > 3/4$, for all $n \neq m$.

After passing to a further subsequence we can also assume that $(x_{\alpha,n})_{n=1}^\infty$ has a spreading model. But this means that all the sequences of the form $(x_{\alpha,n_{2j}} - x_{\alpha,n_{2j-1}})_{j=1}^\infty \subset 2B_X$, with $(n_j)_{j=1}^\infty$ an increasing sequence in $\mathbb{N}$, have the same spreading model $(e^\alpha_j)_{j=1}^\infty$. Define now $\lambda^\alpha_k = \|\sum_{j=1}^k e^\alpha_j\|$. Since spreading models generated by weakly null sequences are 1-suppression unconditional, we see that for any $\alpha < \omega_1$, the sequence $(\lambda^\alpha_k)_k$ is non-decreasing. It also follows from our assumptions on X and from the fact that $(x_{\alpha,m} - x_{\alpha,n})_{n\neq m}$ is seminormalized that

$$\forall \alpha < \omega_1, \quad \lim_{k\to\infty} \frac{k}{\lambda^\alpha_k} = \infty.$$

For fixed $\alpha < \omega_1$ and $k \in \mathbb{N}$, we can apply Ramsey's theorem and, after passing to a further subsequence, we can assume that for all $\bar{m}, \bar{n} \in [\mathbb{N}]^k$, with $k \le m_1 < n_1 < m_2 < n_2 < \cdots < m_k < n_k$, we have

$$\Big\| \sum_{j=1}^k x_{\alpha,n_j} - x_{\alpha,m_j} \Big\| \le \frac{3}{2}\lambda^\alpha_k.$$

Applying then the usual diagonalization argument we can assume that for

all $k \in \mathbb{N}$ and all $\bar{m}, \bar{n} \in [\mathbb{N}]^k$ with $m_1 < n_1 < m_2 < n_2 < \cdots < m_k < n_k$,

$$\Big\| \sum_{j=1}^{k} (x_{\alpha,n_j} - x_{\alpha,m_j}) \Big\| \leq \frac{3}{2} \lambda_k^\alpha. \tag{5.7}$$

Then for $\alpha < \omega_1$ and $k \in \mathbb{N}$ we define

$$f_\alpha^{(k)} \colon [\mathbb{N}]^k \to X, \quad \bar{n} \mapsto \frac{2}{3\lambda_k^\alpha} \sum_{i=1}^{k} x_{\alpha,n_i}.$$

It follows from the definition of the interlaced distance, (5.7) and the monotonicity of $(\lambda_k^\alpha)_k$ that $f_\alpha^{(k)}$ is 1-Lipschitz.

Consider now $\alpha < \beta \in [1, \omega_1)$. Since $\operatorname{dist}(x_\beta^{**}, \operatorname{sp}(x_\alpha^{**})) > 3/4$, by Hahn–Banach, there exists an $x_{\alpha,\beta}^{***} \in S_{X^{***}}$ with $x_{\alpha,\beta}^{***}(x_\alpha^{**}) = 0$ and $x_{\alpha,\beta}^{***}(x_\beta^{**}) = \operatorname{dist}(x_\beta^{**}, \operatorname{sp}(x_\alpha^{**})) > 3/4$. By the principle of local reflexivity (applied to the space X^*) there exists $x_{\alpha,\beta}^{*} \in S_{X^*}$ with $x_\alpha^{**}(x_{\alpha,\beta}^{*}) = 0$ and $x_\beta^{**}(x_{\alpha,\beta}^{*}) > 3/4$. It therefore follows that for any $\mathbb{M} \in [\mathbb{N}]^\omega$,

$$\begin{aligned} &\sup_{\bar{n} \in [\mathbb{M}]^k} \|f_\alpha^{(k)}(\bar{n}) - f_\beta^{(k)}(\bar{n})\| \\ &\quad \geq \limsup_{n_1 \in \mathbb{M},\, n_1 \to \infty} \cdots \limsup_{n_k \in \mathbb{M}, n_k \to \infty} x_{\alpha,\beta}^{*} \Big(\frac{2}{3\lambda_k^\beta} \sum_{i=1}^{k} x_{\beta,n_i} - \frac{2}{3\lambda_k^\alpha} \sum_{i=1}^{k} x_{\alpha,n_i} \Big) \\ &\quad \geq \frac{2}{3\lambda_k^\beta} \frac{3k}{4} = \frac{k}{2\lambda_k^\beta}. \end{aligned} \tag{5.8}$$

This finishes our construction of uncountably many well separated X-valued Lipschitz maps.

Assume now that X coarsely embeds into a Banach space Y such that all the iterated duals of Y are separable and let $g \colon X \to Y$ be such a coarse embedding. Of course, we may assume that $\omega_g(1) \leq 1$. Then, for any $\alpha < \omega_1$ and $k \in \mathbb{N}$, we define $g_\alpha^{(k)} = g \circ f_\alpha^{(k)}$. We observe that $g_\alpha^{(k)}$ is 1-Lipschitz from $([\mathbb{N}]^k, d_1^{(k)})$ to Y. For a fixed $k \in \mathbb{N}$, we can therefore apply Theorem 5.13 to any uncountable subfamily of $(g_\alpha^{(k)})_{\alpha<\omega_1}$. We then deduce from (5.8) that for any $k \in \mathbb{N}$, $\{\alpha < \omega_1 \colon \rho_g(k/(3\lambda_k^\alpha)) > 3\}$ is countable. This implies that the set $\{\alpha < \omega_1 \colon \exists k \in \mathbb{N},\, \rho_g(k/(3\lambda_k^\alpha)) > 3\}$ is also countable and, since $[1, \omega_1)$ is uncountable, there exists $\alpha < \omega_1$ such that for all $k \in \mathbb{N}$, $\rho_g(k/(3\lambda_k^\alpha)) \leq 3$. This is in contradiction with the fact that for this given $\alpha < \omega_1$, $k/(3\lambda_k^\alpha) \nearrow \infty$ if $k \nearrow \infty$ and $\lim_{t\to\infty} \rho_g(t) = \infty$. ∎

Understanding quantitatively what is the order of the non-separable iterated dual in Theorem F is a very interesting problem.

PROBLEM 5.15. *Assume that X is c_0, or any separable Banach space with non-separable bidual X^{**} and $\ell_1 \not\subset X$ such that no spreading model generated*

by a normalized weakly null sequence is equivalent to the ℓ_1-unit vector basis. If X coarsely embeds into a Banach space Y, does this imply that Y^{**} is non-separable?

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References

[Aha74] I. Aharoni, *Every separable metric space is Lipschitz equivalent to a subset of c_0^+*, Israel J. Math. 19 (1974), 284–291.

[AJO05] D. Alspach, R. Judd, and E. Odell, *The Szlenk index and local ℓ_1-indices*, Positivity 9 (2005), 1–44.

[AS16] D. E. Alspach and B. Sarı, *Separable elastic Banach spaces are universal*, J. Funct. Anal. 270 (2016), 177–200..

[AD07] S. A. Argyros and P. Dodos, *Genericity and amalgamation of classes of Banach spaces*, Adv. Math. 209 (2007), 666–748.

[Ban32] S. Banach, *Théorie des opérations linéaires*, Warszawa, 1932.

[Bau07] F. Baudier, *Metrical characterization of super-reflexivity and linear type of Banach spaces*, Arch. Math. (Basel) 89 (2007), 419–429.

[BP58] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), 151–164.

[Bos02] B. Bossard, *A coding of separable Banach spaces. Analytic and coanalytic families of Banach spaces*, Fund. Math. 172 (2002), 117–152.

[Bou80] J. Bourgain, *On separable Banach spaces, universal for all separable reflexive spaces*, Proc. Amer. Math. Soc. 79 (1980), 241–246.

[Bra19] B. M. Braga, *On asymptotic uniform smoothness and nonlinear geometry of Banach spaces*, J. Inst. Math. Jussieu, online, 2019; arXiv:1808.03254.

[Dod09] P. Dodos, *On classes of Banach spaces admitting "small" universal spaces*, Trans. Amer. Math. Soc. 361 (2009), 6407–6428.

[EMN19] A. Eskenazis, M. Mendel, and A. Naor, *Nonpositive curvature is not coarsely universal*, Invent. Math. 217 (2019), 833–886.

[Fré10] M. Fréchet, *Les dimensions d'un ensemble abstrait*, Math. Ann. 68 (1910), 145–168.

[Fre84] D. H. Fremlin, *Consequences of Martin's Axiom*, Cambridge Tracts in Math. 84, Cambridge Univ. Press, Cambridge, 1984.

[HM82] S. Heinrich and P. Mankiewicz, *Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces*, Studia Math. 73 (1982), 225–251.

[Jam50] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. (2) 52 (1950), 518–527.

[JO05] W. B. Johnson and E. Odell, *The diameter of the isomorphism class of a Banach space*, Ann. of Math. (2) 162 (2005), 423–437.

[Kal04] N. J. Kalton, *Spaces of Lipschitz and Hölder functions and their applications*, Collect. Math. 55 (2004), 171–217.

[Kal07] N. J. Kalton, *Coarse and uniform embeddings into reflexive spaces*, Quart. J. Math. (Oxford) 58 (2007), 393–414.

[KL08] N. J. Kalton and G. Lancien, *Best constants for Lipschitz embeddings of metric spaces into* c_0, Fund. Math. 199 (2008), 249–272.

[Kec95] A. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Math. 156, Springer, New York, 1995.

[Kur35] C. Kuratowski, *Quelques problèmes concernant les espaces métriques non-séparables*, Fund. Math 25 (1935), 534–545.

[LPP20] G. Lancien, C. Petitjean, and A. Procházka, *On the coarse geometry of James spaces*, Canadian Math. Bull. 63 (2020), 77–93.

[Ode04] E. Odell, *Ordinal indices in Banach spaces*, Extracta Math. 19 (2004), 93–125.

[OR75] E. Odell and H. P. Rosenthal, *A double-dual characterization of separable Banach spaces containing* l^1, Israel J. Math. 20 (1975), 375–384.

[OSZ07] E. Odell, Th. Schlumprecht and A. Zsák, *Banach spaces of bounded Szlenk index*, Studia Math. 183 (2007), 63–97.

[Peł60] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. 19 (1960), 209–228.

[Uhl72] J. J. Uhl Jr., *A note on the Radon–Nikodym property for Banach spaces*, Rev. Roumaine Math. Pures Appl. 17 (1972), 113–115.

[Ury25] P. S. Urysohn, *Sur un espace métrique universel*, C. R. Acad. Sci. Paris 180 (1925), 803–806.

[Szl68] W. Szlenk, *The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces*, Studia Math. 30 (1968), 53–61.

[Tsi74] B. S. Tsirel'son, *Not every Banach space contains an imbedding of* l_p *or* c_0, Funct. Anal. Appl. 8 (1974), 138–141.

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