SURPRISING CIRCLES IN MORSE BOUNDARIES OF RIGHT-ANGLED COXETER GROUPS

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Given a graph Λ the associated *right-angled Coxeter group* is given by

$$W_\Lambda = \left\langle V(\Lambda) \mid \{v^2\}_{v\in V(\Lambda)} \cup \{[v,w]\}_{\{v,w\}\in E(\Lambda)} \right\rangle.$$

See [7, 4] for background.

A finitely generated group Γ admits a quasi-isometry invariant *Morse boundary* defined by Cordes [3] and denoted $\partial_M\Gamma$. The Morse boundary of a CAT(0) group, such as a right-angled Coxeter group, is equal to the contracting boundary of the group, defined by Charney–Sultan [2]. Moreover, the Morse boundary of a hyperbolic group is equal to the visual boundary of the group.

The topology of a boundary of a finitely generated group often captures algebraic information. For example, the visual boundary of a hyperbolic group is totally disconnected if and only if the group is virtually free. In contrast, the Morse boundary of every right-angled Artin group is totally disconnected [1, Theorem 1.1]. The classification of the right-angled Coxeter groups that have totally disconnected Morse boundary is open. An induced cycle in a graph is *burst* if it contains a pair of non-adjacent vertices that are contained in an induced 4-cycle.

Conjecture (Tran)**.** [12, Conjecture 1.14] *Let Λ be a graph. The Morse boundary $\partial_M W_\Lambda$ is totally disconnected if and only if every induced cycle of length at least four in Λ is burst.*

We give a negative answer to the above conjecture.

Theorem. *There exists a graph in which every induced cycle of length at least four is burst and the associated right-angled Coxeter group contains an embedded circle $\mathbb{S}^1$ in its Morse boundary.*

Proof. Let Λ and Λ' be the graphs in Figure 1. Every cycle in Λ is burst. The graph Λ' is obtained by doubling Λ over the star of x then deleting the vertex x. Thus, the group $W_{\Lambda'}$ is an index-2 subgroup of W_Λ by [5, Lemma 2.3][1]. Therefore, $\partial_M W_\Lambda = \partial_M W_{\Lambda'}$. The graph Λ' contains a non-burst cycle, drawn in red. Thus, $\partial_M W_{\Lambda'}$, and hence $\partial_M W_\Lambda$, contains $\mathbb{S}^1$ by [12, Corollary 1.12]. This also follows from [12, Theorem 1.4] together with [8, Proposition 4.9] or [11, Theorem 7.5]. ☐

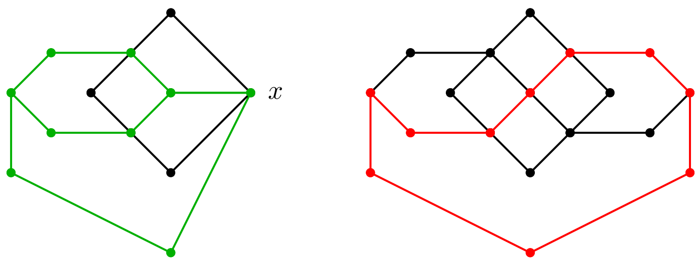

FIGURE 1. The graphs Λ on the left and Λ' on the right.

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[1]The lemma does not appear in the published version [6]

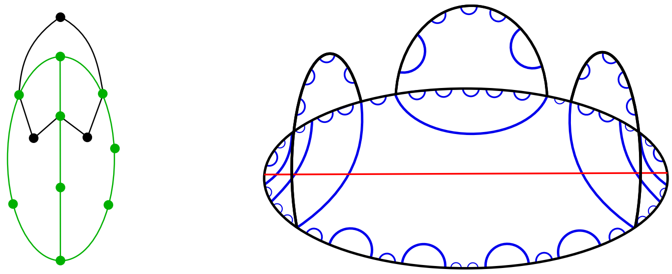

FIGURE 2. The graph Λ and hyperbolic planes branching along lines.

Geometric intuition. As shown above, the Morse boundary of the right-angled Coxeter group W_Λ contains an embedded circle. Indeed, the group W_Λ has an index-two subgroup whose defining graph yields a cocompact Fuchsian reflection subgroup that intersects any Euclidean reflection subgroup in only a bounded set. In particular, there is a quasi-isometrically embedded hyperbolic plane in the Davis complex for W_Λ that intersects any Euclidean plane in only a bounded subset; we sketch an alternative geometric description of such a subspace.

The graph Λ contains a subdivided Θ-graph Θ as an induced subgraph, as shown in green in the figures. The subgraph Θ contains three embedded induced cycles of length greater than four. These three cycles yield quasi-isometrically embedded hyperbolic planes in the Davis complex for W_Λ. However, the Morse boundary does not contain the visual boundary of these hyperbolic planes. Indeed, each such cycle is burst, so each such hyperbolic plane in the Davis complex intersects a Euclidean plane in a line.

Nonetheless, a hyperbolic plane in the Davis complex for W_Λ that meets any Euclidean plane in only a bounded subset can be constructed by piecing together subsets of these hyperbolic planes and their W_Θ-translates. The hyperbolic planes arising from the three cycles in Θ and their W_Θ-translates intersect to form $\mathcal{X}$, a "tree" of hyperbolic planes branching along lines. See Figure 2 and [9, Section 3] for additional details. The cycles of length four in Λ yield Euclidean planes whose intersections with $\mathcal{X}$ are quasi-isometric to lines. These lines are contained in one hyperbolic plane and intersect branching lines transversely, as shown in red in Figure 2. Thus, a hyperbolic plane meeting each Euclidean plane in a bounded subset can be constructed by making a choice at each branching line in $\mathcal{X}$.

Additional examples will appear in [10, Section 5.5].

Remark. A *finite-index reflection subgroup* of a right-angled Coxeter group W_Λ is a subgroup generated by reflections about the set of hyperplanes bounding a compact, convex subcomplex of the Davis complex for W_Λ. The conjecture above still fails if one is allowed to pass to a finite-index reflection subgroup. Indeed, after an initial preprint of this paper, Hung Cong Tran explained if Λ is the 1-skeleton of a 3-cube, then W_Λ also provides a counterexample to the conjecture above, using [1]. One can show there does not exist a finite-index reflection subgroup of W_Λ whose defining graph contains an induced cycle of length at least four which is not burst.

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