MODIFIED ARCSINE LAW FOR ONE-BIT SAMPLED STATIONARY SIGNALS WITH TIME-VARYING THRESHOLDS

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ABSTRACT

One-bit quantization has attracted considerable attention in signal processing for communications and sensing. The *arcsine law* is a useful relation often used to estimate the normalized covariance matrix of zero-mean stationary input signals when they are sampled by one-bit analog-to-digital converters (ADCs)—practically comparing the signals with a given threshold level. This relation, however, only considers a zero threshold which can cause a remarkable information loss. For the first time in the literature, this paper introduces an approach to extending the arcsine law to the case where one-bit ADCs apply time-varying thresholds. In particular, the proposed method is shown to accurately recover the variance and autocorrelation of the stationary signals of interest.

Index Terms**—** Arcsine law, covariance matrix, one-bit quantization, time-varying thresholds.

1. INTRODUCTION

Digital signal processing typically requires the quantization of the signals of interest through analog-to-digital converters (ADCs). In high resolution settings, a very large number of quantization levels is required in order to represent the original continuous signal. However, this leads to some difficulties in modern applications where the signals of interest have large bandwidths, and may pass through several RF chains that require using a plethora of ADCs. Moreover, the overall power consumption and manufacturing cost of ADCs, and chip area grows exponentially with the number of quantization bits. Such drawbacks lend support to the idea of utilizing fewer bits for sampling. The most extreme version of this idea would be to use *one-bit quantization*, in which ADCs are merely comparing the signals with given threshold levels, producing sign (± 1) outputs. This allows for sampling at a very high rate, with a significantly lower cost and energy consumption compared to conventional ADCs [1–4].

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In the context of one-bit sampling, until recently, most researchers approached the problem of estimating signal parameters by comparing the signal with a fixed threshold, usually zero. This introduces difficulties in the recovery of the signal amplitude. On the other hand, recent works have shown enhanced estimation performance for the signal parameters by employing time-varying thresholds [5–10].

The arcsine law is a fundamental statistical property of one-bit sampling [11–14], which connects the covariance of an unquantized signal with that of its quantized counterpart [3, 15]. An important disadvantage of the arcsine law is, however, that the one-bit quantization threshold is considered to be zero, which leads to a considerable loss of information. In this paper, we present a new approach to extending the arcsine law in the time-varying sampling thresholds, which can recover the covariance values of the input unquantized signal with high accuracy. To this end, we present a correlation recovery formula in Section 2 which does not seem to be amenable to analytical manipulation. In Section 3, we propose a one-point piece-wise Padé approximation (PA) approach to recast our integrands as rational expressions which are readily integrable. Next, we formulate an estimation criterion to recover our desired parameters. Section 4 is dedicated to the numerical results, while Section 5 concludes the paper.

2. PROBLEM FORMULATION

We consider a similar setting as in [11], with a zero-mean stationary Gaussian input signal $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{R}_{\mathbf{x}})$, where $\boldsymbol{R}_{\mathbf{x}}$ is a Toeplitz matrix associated with the autocorrelation function of $\mathbf{x}$ denoted as $R_{\mathbf{x}}$. Suppose x_i and x_j are the ith and jth entries of $\mathbf{x}$, and $\mathbf{y} = f(\mathbf{x})$ is the output process, where $f(x)$ is the sign function. The autocorrelation function of the output, denoted by $R_{\mathbf{y}}(l)$, with $l = |i-j|$, is connected to that of $\mathbf{x}$ via the arcsine law [11–13]:

$$R_{\mathbf{y}}(i,j) = R_{\mathbf{y}}(l) = \mathbb{E}\left\{y_i y_j\right\} = \frac{2}{\pi}\sin^{-1}\left(\frac{R_{\mathbf{x}}(l)}{R_{\mathbf{x}}(0)}\right), \quad (1)$$

where y_i and y_j are the ith and jth entries of $\mathbf{y}$, and $R_{\mathbf{x}}(l)$ denotes the input signal autocorrelation for the lag l.

We consider a non-zero time-varying Gaussian threshold $\boldsymbol{\tau}$ that is independent of the input signal with the distribution $\boldsymbol{\tau} \sim \mathcal{N}(\mathbf{d} = \mathbf{1}d, \boldsymbol{\Sigma})$. We define a new random process $\mathbf{w}$ such that $\mathbf{w} = \mathbf{x} - \boldsymbol{\tau}$. Clearly, $\mathbf{w}$ is a Gaussian stochastic process with $\mathbf{w} \sim \mathcal{N}(-\mathbf{d}, \boldsymbol{R}_{\mathbf{x}} + \boldsymbol{\Sigma} = \boldsymbol{P})$. Suppose w_i and w_j are the ith and jth entries of $\mathbf{w}$ ($i \neq j$), and that p_l and p_0 denote the autocorrelation term for lag $l = |i - j|$ and variance of $\mathbf{w}$, respectively. The autocorrelation of $f(w_i)$ and $f(w_j)$ is given by [11]:

$$R_{\mathbf{y}}(i,j) = \frac{1}{2\pi\sqrt{p_0^2 - p_l^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(w_i) f(w_j) e^{\lambda(d)} dw_i dw_j \tag{2}$$

where $\lambda(d)$ is defined as

$$\lambda(d) = \frac{(w_i + d)^2 p_0 + (w_j + d)^2 p_0 - 2p_l(w_i + d)(w_j + d)}{-2(p_0^2 - p_l^2)}. \tag{3}$$

The autocorrelation function in (2) can be rewritten as

$$\begin{aligned} R_{\mathbf{y}}(i,j) = \frac{1}{2\pi\sqrt{p_0^2 - p_l^2}} \Bigg(&\int_0^{\infty}\int_0^{\infty} e^{\lambda(d)}\, dw_i\, dw_j \\ &+ \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\lambda(d)}\, dw_i\, dw_j \\ &- \int_0^{\infty}\int_{-\infty}^{0} e^{\lambda(d)}\, dw_i\, dw_j \\ &- \int_{-\infty}^{0}\int_0^{\infty} e^{\lambda(d)}\, dw_i\, dw_j \Bigg). \end{aligned} \tag{4}$$

We can simplify (4) using the relation

$$\frac{1}{2\pi\sqrt{p_0^2 - p_l^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\lambda(d)}\, dw_i\, dw_j = 1. \tag{5}$$

In fact, using (5) one can verify that

$$R_{\mathbf{y}}(i,j) = \frac{1}{\pi\sqrt{p_0^2 - p_l^2}} \int_0^{\infty}\int_0^{\infty} \left(e^{\lambda(d)} + e^{\lambda(-d)}\right) dw_i\, dw_j - 1. \tag{6}$$

By employing polar coordinates $w_i = \rho\cos\theta$, $w_j = \rho\sin\theta$, we can recast the integral in (6) as

$$R_{\mathbf{y}}(i,j) = \frac{e^{\frac{-d^2}{p_0 + p_l}}}{\pi\sqrt{p_0^2 - p_l^2}} \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-\beta\rho^2}\left(e^{-\alpha\rho} + e^{\alpha\rho}\right) \rho\, d\rho\, d\theta - 1, \tag{7}$$

where

$$\alpha = \frac{d(\sin\theta + \cos\theta)}{p_0 + p_l}, \quad \beta = \frac{p_0 - p_l \sin 2\theta}{2(p_0^2 - p_l^2)}. \tag{8}$$

Integrating (7) with respect to ρ leads to

$$R_{\mathbf{y}}(i,j) = R_{\mathbf{y}}(l) = \frac{e^{\frac{-d^2}{p_0 + p_l}}}{\pi\sqrt{(p_0^2 - p_l^2)}} \left\{ \int_0^{\frac{\pi}{2}} \frac{1}{\beta} + \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{2\beta} e^{\frac{\alpha^2}{4\beta}} - \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{\beta} Q\left(\frac{\alpha}{\sqrt{2\beta}}\right) e^{\frac{\alpha^2}{4\beta}} d\theta \right\} - 1. \tag{9}$$

It remains to evaluate the integral in (9) in terms of p_0 and $\{p_l\}$ which have to be estimated—a task that is central to our efforts in the rest of this paper. Finding p_0 and $\{p_l\}$ results in input variance and autocorrelation recovery, which can be achieved by considering the relation:

$$\boldsymbol{R}_{\mathbf{x}}(i,j) = \boldsymbol{P}(i,j) - \boldsymbol{\Sigma}(i,j). \tag{10}$$

For $i = j$, the input variance is thus given by $\boldsymbol{R}_{\mathbf{x}}(i,i) = r_0 = p_0 - \boldsymbol{\Sigma}(i,i)$, while for $i \neq j$, we have the input autocorrelation for lag $l = |i - j|$ as $\boldsymbol{R}_{\mathbf{x}}(i,j) = \boldsymbol{R}_{\mathbf{x}}(l) = r_l = p_l - \boldsymbol{\Sigma}(i,j)$.

3. COVARIANCE RECOVERY METHOD

Since evaluating the integral in (9) appears to be difficult, we resort to rational approximations to facilitate its evaluation. In Subsection 3.1, a Padé approximation (PA) [16–19] is utilized to evaluate (9). This lays the ground for the recovery of p_0 and $\{p_l\}$ in Subsection 3.2.

3.1. Proposed Rational Approximations

According to [20], the Q-function is well-approximated with the sum of exponentials,

$$Q(x) \approx \frac{1}{12} e^{\frac{-x^2}{2}} + \frac{1}{4} e^{\frac{-2x^2}{3}}, \quad x > 0. \tag{11}$$

We further note that the integral in (9) may be evaluated by substituting $D_1(\theta; p_0, p_l, d) = \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{\beta} Q\left(\frac{\alpha}{\sqrt{2\beta}}\right) e^{\frac{\alpha^2}{4\beta}}$ and $D_2(\theta; p_0, p_l, d) = \sqrt{\frac{\pi}{\beta}}\frac{\alpha}{2\beta} e^{\frac{\alpha^2}{4\beta}}$ with Padé approximants, that yield the best approximation of a function by a rational function of given order through the *moment matching* technique.

For the sake of completeness, herein we present a brief introduction of the PA method. Suppose $I(t)$ is an *analytic function* at point $t = 0$ with the Taylor series:

$$I(t) = \sum_{n=0}^{\infty} c_n t^n, \quad c_n \in \mathbb{R}. \tag{12}$$

The PA of order $[L/M]$ for $I(t)$, denoted by $P^{[L/M]}(t)$, is defined as a rational function in the form [19]:

$$P^{[L/M]}(t) \triangleq \frac{\sum_{n=0}^{L} a_n t^n}{\sum_{n=0}^{M} b_n t^n} \tag{13}$$

where the coefficients $\{a_n\}$ and $\{b_n\}$ are defined so that

$$\lim_{t\to 0} \frac{\sum_{n=0}^{L} a_n t^n}{\sum_{n=0}^{M} b_n t^n} = \sum_{n=0}^{L+M} c_n t^n + O(t^{L+M+1}) \tag{14}$$

with $b_0 = 1$. The moment matching technique is a method widely used to obtain the coefficients of PA. The coefficients $\{b_n\}$ are obtained through the linear system of equations [19]:

$$\begin{bmatrix} c_{L-M+1} & c_{L+M+2} & \cdots & c_L \\ \vdots & \vdots & \vdots & \vdots \\ c_{L-M+k} & c_{L-M+k+1} & \cdots & c_{L+k-1} \\ \vdots & \vdots & \vdots & \vdots \\ c_L & c_{L+1} & \cdots & c_{L+M-1} \end{bmatrix} \begin{bmatrix} b_M \\ \vdots \\ b_k \\ \vdots \\ b_1 \end{bmatrix} = -\begin{bmatrix} c_{L+1} & \cdots & c_{L+k+1} & \cdots & c_{L+M} \end{bmatrix}^T \tag{15}$$

where the matrix in the left-hand side of (15) is a Hankel matrix. The coefficients $\{a_n\}$ are obtained by backsubstitution [19]:

$$a_j = c_j + \sum_{i=1}^{\min(M,j)} b_i c_{j-i}, \quad 0 \le j \le L. \tag{16}$$

The selection of the PA order is an important task in approximation; see [19] for a related study. Note that the integration in (9) occurs in the interval $\theta \in \left[0, \frac{\pi}{2}\right]$. To have a better fitness, we use the idea of piece-wise PA. Owing to the fact that the functions $D_1(\theta; p_0, p_l, d)$ and $D_2(\theta; p_0, p_l, d)$ have their extremum at $\theta = \frac{\pi}{4}$, the selection of three distinct intervals $\left[0, \frac{\pi}{8}\right]$, $\left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$, and $\left[\frac{3\pi}{8}, \frac{\pi}{2}\right]$ with the expansion points $\theta = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$ paves the way for a convenient approximation, with extra boundary points $\frac{\pi}{8}$ and $\frac{3\pi}{8}$ making the chosen intervals symmetric. Thus, the function $D_2(\theta; p_0, p_l, d)$ can be approximated as,

$$\begin{aligned} \theta \in \left[0, \frac{\pi}{8}\right] \cup \left[\frac{3\pi}{8}, \frac{\pi}{2}\right] &: \sqrt{\frac{\pi}{\beta}} \frac{\alpha}{2\beta} e^{\frac{\alpha^2}{4\beta}} \approx \frac{e + s\theta}{k + g\theta + h\theta^2}, \\ \theta \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] &: \sqrt{\frac{\pi}{\beta}} \frac{\alpha}{2\beta} e^{\frac{\alpha^2}{4\beta}} \approx \frac{z + u\theta + v\theta^2}{k' + g'\theta + h'\theta^2}. \end{aligned} \tag{17}$$

A similar approximation with same orders can be proposed for $D_1(\theta; p_0, p_l, d)$. As mentioned earlier, the two functions $D_1(\theta; p_0, p_l, d)$ and $D_2(\theta; p_0, p_l, d)$ should be analytic at the expansion points (which can be easily verified). The first part of the integration in (9) can be analytically evaluated as

$$\int_0^{\frac{\pi}{2}} \frac{1}{\beta} \, d\theta = \sqrt{p_0^2 - p_l^2} \left(\pi + 2\tan^{-1}\left[\frac{p_l}{\sqrt{p_0^2 - p_l^2}} \right] \right). \tag{18}$$

Substituting $D_2(\theta; p_0, p_l, d)$ with its approximation and evaluating the integration the associated parts of (9) results in:

$$\begin{aligned} \int_0^{\frac{\pi}{8}} \sqrt{\frac{\pi}{\beta}} \frac{\alpha}{2\beta} e^{\frac{\alpha^2}{4\beta}} \, d\theta \approx \frac{s}{2h} \ln\left(\frac{\left| k + \frac{\pi g}{8} + \frac{\pi^2 h}{64} \right|}{|k|} \right) + \\ \frac{2eh - sg}{h\sqrt{4hk - g^2}} \tan^{-1}\left(\frac{\pi h \sqrt{4hk - g^2}}{16hk + \pi gh} \right), \end{aligned} \tag{19}$$

$$\begin{aligned} \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sqrt{\frac{\pi}{\beta}} \frac{\alpha}{2\beta} e^{\frac{\alpha^2}{4\beta}} \, d\theta \approx \frac{\pi v}{4h'} + \\ \frac{uh' - vg'}{2h'^2} \ln\left(\frac{|64k' + 9\pi^2 h' + 24\pi g'|}{|64k' + \pi^2 h' + 8\pi g'|} \right) + \\ \frac{2vh'k' - 2zh'^2 + ug'h' - vg'^2}{h'^2 \sqrt{4k'h' - g'^2}} \\ \tan^{-1}\left(\frac{-8\pi h' \sqrt{4h'k' - g'^2}}{64h'k' + 3\pi^2 h'^2 + 16\pi h'g'} \right), \end{aligned} \tag{20}$$

$$\begin{aligned} \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \sqrt{\frac{\pi}{\beta}} \frac{\alpha}{2\beta} e^{\frac{\alpha^2}{4\beta}} \, d\theta \approx \frac{s}{2h} \ln\left(\frac{\left| k + \frac{\pi g}{2} + \frac{\pi^2 h}{4} \right|}{\left| k + \frac{3\pi g}{8} + \frac{9\pi^2 h}{64} \right|} \right) + \\ \frac{2eh - sg}{h\sqrt{4kh - g^2}} \tan^{-1}\left(\frac{\pi h \sqrt{4hk - g^2}}{16kh + 3\pi^2 h^2 + 7\pi hg} \right). \end{aligned} \tag{21}$$

Similar approximations can be obtained for terms associated with the function $D_1(\theta; p_0, p_l, d)$ which are not presented here due to the lack of space.

3.2. Recovery Criterion

In this subsection, p_0 and $\{p_l\}$ are estimated by formulating a minimization problem. For this purpose, one may consider the following criterion:

$$\begin{aligned} \bar{C}(p_0, p_l) \triangleq \log\Bigg(\Bigg| R_{\mathbf{y}}(l) - \frac{e^{\frac{-d^2}{p_0 + p_l}}}{\pi\sqrt{(p_0^2 - p_l^2)}} \Bigg\{ \int_0^{\frac{\pi}{2}} \frac{1}{\beta} \\ + \sqrt{\frac{\pi}{\beta}} \frac{\alpha}{2\beta} e^{\frac{\alpha^2}{4\beta}} - \sqrt{\frac{\pi}{\beta}} \frac{\alpha}{\beta} Q\left(\frac{\alpha}{\sqrt{2\beta}} \right) e^{\frac{a^2}{4\beta}} d\theta \Bigg\} + 1 \Bigg|^2 \Bigg) \end{aligned} \tag{22}$$

where the autocorrelation of output signal ($R_{\mathbf{y}}$) can be estimated with the given sign vector ($\mathbf{y}$) via the sample covariance matrix

$$\boldsymbol{R}_{\mathbf{y}} \approx \frac{1}{N} \sum_{k=1}^{N} \mathbf{y}(k)\mathbf{y}(k)^H. \tag{23}$$

Note that by now we have derived an approximated version of (9). Let $H(p_0, p_l)$ denote this approximation. Therefore, we can alternatively consider the criterion:

$$C(p_0, p_l) \triangleq \log\left(|R_{\mathbf{y}}(l) - H(p_0, p_l)|^2 \right). \tag{24}$$

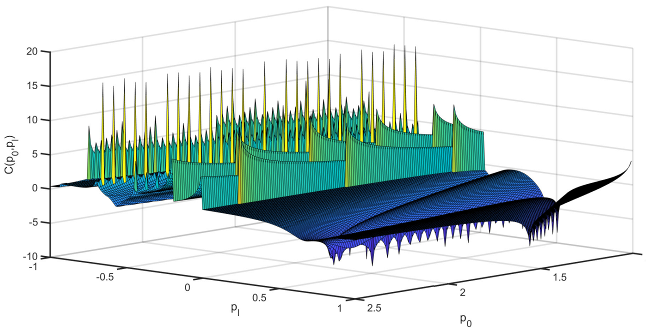

Fig. 1: Example plot of the estimation criterion $C(p_0, p_l)$ with respect to p_0 and p_l showing its multi-modality, i.e. having multiple local optima.

A numerical investigation of (24) reveals that it is highly multi-modal, with many local minima—see Fig. 1 for an example of the optimization landscape of $C(p_0, p_l)$. To filter out the undesired local minima, we resort to constraints re-enforcing the behaviour of an autocorrelation function. More precisely, we will consider the minimization problem:

$$\mathcal{P}_\ell : \quad \min_{p_0, p_l} \quad C(p_0, p_l), \quad \text{s.t.} \quad p_0^2 \geq p_l^2, \quad p_0 \geq 0, \tag{25}$$

where the first inequality constraint in (25) is imposed to ensure that the magnitude of the diagonal elements of the covariance matrix of $\mathbf{w}$ is greater than the magnitude of the off-diagonal elements. The non-convex problem in (25) may be solved via the gradient descent numerical optimization approach by employing multiple random initial points. Once p_0 and $\{p_l\}$ are obtained, one can estimate the autocorrelation values of $\mathbf{x}$ via (10). The optimum recovery results will be presented in the following.

4. NUMERICAL RESULTS

In this section, we will examine the proposed method by comparing its recovery results with the true input signal autocorrelation values. In all experiments, the input signals were generated as zero-mean Gaussian sequences with unit variance. Accordingly, we made use of the time-varying thresholds with $d = 0.7$ and diagonal $\mathbf{\Sigma}$ whose diagonal entries are equal to 0.3. Note that the values of d and $\mathbf{\Sigma}$ are best chosen based on the application, considering the magnitude of the input signal.

To show the effectiveness of the proposed approach, we present an example of autocorrelation sequence recovery. The true input signal autocorrelation and the estimated autocorrelation values by our approach are shown in Fig. 2 for a random sequence of length 31. Fig. 2 appears to confirm the possibility of recovering the autocorrelation values from one-bit sampled data with time-varying thresholds.

Next, we investigate the impact of a growing sample size in autocorrelation recovery, and in particular, the variance.

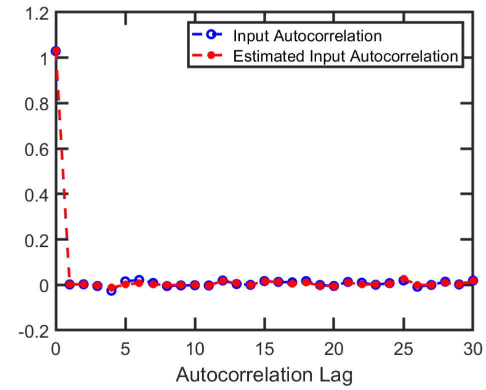

Fig. 2: Recovery of the input signal autocorrelation for a sequence of length 31 from one-bit sampled data, with the true values plotted along the estimates.

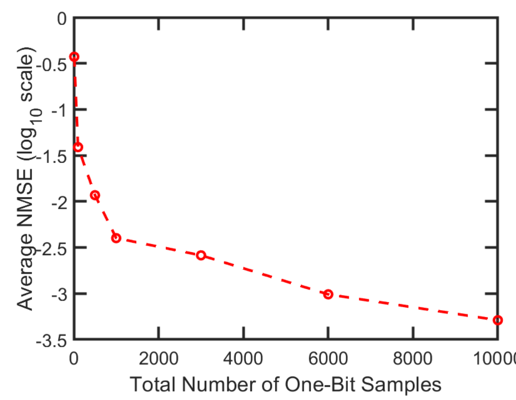

Fig. 3: Average NMSE for signal variance recovery for different one-bit sample sizes.

We define the normalized mean square error (NMSE) of an estimate $\hat{r}_0$ of a variance r_0 as

$$\text{NMSE} \triangleq \frac{|r_0 - \hat{r}_0|^2}{|r_0|^2}. \tag{26}$$

Each data point presented is averaged over 15 experiments. As can be seen in Fig. 3, the proposed method can estimate the variance elements of an input signal accurately. The results are obtained for lengths $10, 100, 500, 1000, 3000, 6000$, and 10000, with fixed d and $\mathbf{\Sigma}$ for each experiment. As expected, the accuracy of variance recovery will significantly enhance as the number of one-bit samples grows large.

5. CONCLUSION

We proposed a modified arcsine law through Padé approximations that can make use of non-zero time-varying thresholds in one-bit sampling. The numerical results showcase the effectiveness of the proposed approach in recovering the autocorrelation values of one-bit sampled stationary signals.

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