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GREEN FUNCTION OF ORR–SOMMERFELD EQUATIONS AWAY FROM CRITICAL LAYERS*

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Abstract. The classical Orr–Sommerfeld equations are the resolvent equations of the linearized Navier–Stokes equations around a stationary shear layer profile in the half plane. In this paper, we derive pointwise bounds on the Green function of the Orr–Sommerfeld problem away from its critical layers.

Key words. Green function, Orr–Sommerfeld equations, linearized Navier–Stokes, boundary layers, instability, semigroup bounds

AMS subject classifications. 35Q30, 35Q31, 35B25, 35B40

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1. Introduction. In this paper, we are interested in the study of linearized Navier–Stokes equations around a given fixed profile $U_s = (U(z), 0)$ as the viscosity goes to 0. Namely, we consider the following set of equations:

$$\partial_t v + U_s \cdot \nabla v + v \cdot \nabla U_s + \nabla p - \nu \Delta v = F, \tag{1.1}$$

$$\nabla \cdot v = 0 \tag{1.2}$$

on the half plane $x \in \mathbb{T}$, $z \geq 0$, with Dirichlet boundary condition

$$v = 0 \quad \text{on} \quad z = 0. \tag{1.3}$$

We focus on the periodic case $x \in \mathbb{T}$, the whole line case $x \in \mathbb{R}$ being similar. Throughout this paper, the background profile $U(z)$ is assumed to be sufficiently smooth, to satisfy $U(0) = 0$ and

$$|\partial_z^k (U(z) - U_+)| \leq C_k e^{-\eta_0 z} \qquad \forall\ z \geq 0, \quad k \geq 0, \tag{1.4}$$

for some finite constant U_+ and some positive constants C_k and η_0.

The inviscid limit problem (1.1)–(1.4) is a very classical problem that has led to a huge physical and mathematical literature, focusing in particular on the linear stability, the dispersion relation, the study of eigenvalues and eigenmodes, and the onset of nonlinear instabilities and turbulence (see [1] for an introduction on these topics and the classical achievements of Rayleigh, Orr, Sommerfeld, Heisenberg, Tollmien, C. C. Lin, and Schlichting).

Two cases arise. Either the profile U is linearly stable for the corresponding linearized Euler equations (the case when $\nu = 0$) or it is linearly unstable for these

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limiting equations. In this paper, we consider the unstable case, leaving the stable case to be treated in [10], which turns out to be much more delicate. In the unstable case, it is well known [6] that the profile U is linearly unstable for the linearized Navier–Stokes equations provided ν is sufficiently small or, equivalently, the Reynolds number $R = \nu^{-1}$ is sufficiently large. However, in order to go from linear to nonlinear instability, more precise information on solutions to the linearized problem is required. Let us mention several efforts in treating the stability and instability of nonlinear boundary layers in the small viscosity limit [2, 3, 4, 5, 6, 11, 13, 14].

A natural and traditional approach to studying linearized Navier–Stokes equations is to take the Fourier Laplace transform of these equations. For this, in order to take advantage of the incompressibility relation (1.2), we introduce the stream function ψ of v, defined by

$$v = \nabla^\perp \psi,$$

and take its Fourier transform in the x variable, with wave number α, and its Laplace transform in time, with Laplace variable

$$\lambda = -i\alpha c,$$

following historical notation. Equivalently, we focus on solutions v of linearized Navier–Stokes equations of the form

$$v = \nabla^\perp \Big(e^{i\alpha(x-ct)} \phi(z) \Big),$$

with the source term of the same form. This leads to the classical Orr–Sommerfeld equation

$$(U-c)(\partial_z^2 - \alpha^2)\phi - U''\phi = \epsilon(\partial_z^2 - \alpha^2)^2\phi - i\alpha^{-1} f, \qquad \epsilon = \frac{\nu}{i\alpha}, \tag{1.5}$$

on the half line $z \geq 0$, together with the boundary conditions

$$\phi_{|_{z=0}} = \phi'_{|_{z=0}} = 0, \qquad \lim_{z\to\infty} \phi(z) = 0. \tag{1.6}$$

Here, $\alpha \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ denotes the tangential wave number and $c \in \mathbb{C}$ is the complex phase velocity.

For the mathematical analysis, it is more convenient to multiply (1.5) by $i\alpha$, which leads to

$$(\lambda + i\alpha U)(\partial_z^2 - \alpha^2)\phi - i\alpha U''\phi - \nu(\partial_z^2 - \alpha^2)^2\phi = f. \tag{1.7}$$

Such a spectral formulation of the linearized Navier–Stokes equations near a boundary layer shear profile has been intensively studied in the physical literature. We in particular refer the reader to [1, 12, 15] for the major works of Heisenberg, Tollmien, C. C. Lin, and Schlichting on the subject. We also refer the reader to [7, 8, 9] for the rigorous spectral analysis of the Orr–Sommerfeld equations.

In this paper, we shall derive pointwise bounds on the Green function of the Orr–Sommerfeld problem (1.6)–(1.7). For convenience, let us denote

$$\Delta_\alpha := \partial_z^2 - \alpha^2$$

and

$$\mathrm{OS}_{\alpha,\lambda}(\phi) := (\lambda + i\alpha U)\Delta_\alpha \phi - i\alpha U''\phi - \nu \Delta_\alpha^2 \phi. \tag{1.8}$$

For each fixed $\alpha \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}$, we denote by $G_{\alpha,\lambda}(x,z)$ the corresponding Green kernel of the Orr–Sommerfeld problem. By definition, for each $x \in \mathbb{R}_+$, $G_{\alpha,\lambda}(x,z)$ solves

$$\mathrm{OS}_{\alpha,\lambda}(G_{\alpha,\lambda}(x,\cdot)) = \delta_x(\cdot)$$

on $z \geq 0$, together with the boundary conditions

$$G_{\alpha,\lambda}(x,0) = \partial_z G_{\alpha,\lambda}(x,0) = 0, \qquad \lim_{z\to\infty} G_{\alpha,\lambda}(x,z) = 0. \tag{1.9}$$

The Green kernel allows us to solve the inhomogeneous Orr–Sommerfeld problem

$$\mathrm{OS}_{\alpha,\lambda}(\phi) = f, \tag{1.10}$$

or, equivalently, the resolvent equations of the linearized Navier–Stokes operator, through the following explicit expression for the solution ϕ:

$$\phi(z) = \int_0^\infty G_{\alpha,\lambda}(x,z) f(x)\, dx.$$

To construct the Green function, let us first note that as $z \to +\infty$ the homogeneous Orr–Sommerfeld equation "converges" to the following constant-coefficient equation:

$$\mathrm{OS}_+(\phi) = (\lambda + i\alpha U_+)\Delta_\alpha \phi - \nu \Delta_\alpha^2 \phi = 0, \tag{1.11}$$

where $U_+ = \lim_{z\to\infty} U(z)$. This constant-coefficient equation has four independent solutions $e^{\pm\mu_s z}$ and $e^{\pm\mu_f^+ z}$, with

$$\mu_s = |\alpha|, \quad \mu_f(z) = \nu^{-1/2}\sqrt{\lambda + \nu\alpha^2 + i\alpha U(z)}, \quad \mu_f^+ = \lim_{z\to\infty} \mu_f(z), \tag{1.12}$$

in which we take the positive real part of the square root.

As will be proved later, there exist four solutions to the homogeneous Orr–Sommerfeld equation $\mathrm{OS}_{\alpha,\lambda}(\phi) = 0$ which have either a "slow behavior" $e^{\pm\mu_s z}$ or a "fast behavior" $e^{\pm\mu_f^+ z}$ as $z \to +\infty$. The two slow modes appear to be perturbations of solutions of the Rayleigh equation

$$Ray_{\alpha,\lambda}(\phi) = (\lambda + i\alpha U)\Delta_\alpha \phi - i\alpha U''\phi = 0,$$

whereas the two fast modes are linked to the Airy-type equation

$$(\lambda + i\alpha U - \nu\Delta_\alpha)\Delta_\alpha \phi = 0$$

or, recalling μ_f introduced in (1.12),

$$\nu(\partial_z^2 - \mu_f^2)\Delta_\alpha \phi = 0. \tag{1.13}$$

Let us first consider the Rayleigh equation $Ray_{\alpha,\lambda}(\phi) = 0$. As z goes to $+\infty$, this equation "converges" to $\Delta_\alpha \phi = 0$, and hence $Ray_{\alpha,\lambda}(\phi) = 0$ has two solutions $\phi_{\alpha,\pm}$, with respective behaviors $e^{\pm|\alpha|z}$ at infinity. We define the Evans function $E(\alpha,\lambda)$ by

$$E(\alpha,\lambda) = \phi_{\alpha,-}(0). \tag{1.14}$$

Note that the Rayleigh equation degenerates at points where $\lambda + i\alpha U(z)$ vanishes. In this paper, we restrict ourselves to the case when λ is away from the range of $-i\alpha U$.

Precisely, throughout the paper, letting ϵ_0 be an arbitrarily small, but fixed, positive constant, we shall consider the range of (α,λ) in $\mathbb{R}\setminus\{0\}\times\mathbb{C}$ so that

$$d(\alpha,\lambda)=\inf_{z\in\mathbb{R}_+}|\lambda+i\alpha U(z)|\geq\epsilon_0. \tag{1.15}$$

Note that $d(\alpha,\lambda)=\Re\lambda$ if $\Im\lambda\in-\alpha\mathrm{Range}(U)$. In any case,

$$d(\alpha,\lambda)\geq|\Re\lambda|. \tag{1.16}$$

It turns out that two independent "slow" solutions of Orr–Sommerfeld equations can be constructed as perturbations of these two solutions of the Rayleigh equation.

The two "fast" solutions come from the Airy equation (1.13). This equation degenerates when $\lambda+\alpha^2\nu+i\alpha U$ gets small. Points z_c such that $\alpha U(z_c)=-\Im\lambda$ are called "critical layers." The behavior of the Airy equation changes as we approach these points, and in this paper we only study this equation away from these critical layers. Let us quantify this notion. The Airy equation has a typical length scale:

$$\delta(z)=\frac{1}{\mu_f(z)}=\sqrt{\frac{\nu}{\lambda+\nu\alpha^2+i\alpha U(z)}}.$$

If $\delta(z)$ varies within a length $\delta(z)$, namely if $\delta'(z)\delta(z)\sim\delta(z)$ or, equivalently, if $\alpha\sim\nu^{-1/2}$, then the nature of the construction changes (see (2.1) for more details). In this paper, we restrict ourselves to the case $|\alpha|\ll\nu^{-1/2}$ or, more precisely, on $|\alpha|\leq\nu^{-\zeta}$ for some $\zeta<1/2$.

We are mainly interested in getting bounds on the Green function when λ has a small positive real part. In this case, the condition (1.15) implies

$$\Re\mu_f(z)=\nu^{-1/2}\Re\sqrt{\lambda+\nu\alpha^2+i\alpha U(z)}\geq\nu^{-1/2}\sqrt{\epsilon_0/2}\gg\mu_s \tag{1.17}$$

for sufficiently small ν and for $|\alpha|\leq\nu^{-\zeta}$ for some $\zeta<1/2$. We may also use these Green function bounds in order to obtain bounds on the solutions of linearized Navier–Stokes equations, through contour integrations. It turns out that these contours may be chosen such that $\mu_s\leq\Re\mu_f$. Therefore, we focus on this case in this paper, leaving aside the case when $\mu_s/\mu_f\geq1$.

Our main result is the following.

Theorem 1.1. *Let $U(z)$ be a boundary layer profile which satisfies* (1.4)*. For each α,λ, let by $G_{\alpha,\lambda}(x,z)$ be the Green kernel of the Orr–Sommerfeld equation, with the source term in x, and let*

$$\mu_s=|\alpha|,\qquad\mu_f(z)=\nu^{-1/2}\sqrt{\lambda+\nu\alpha^2+i\alpha U(z)}, \tag{1.18}$$

where we take the square root with a positive real part. Let $0<\theta_0<1$ and $\zeta<1/2$. Let $\sigma_0>0$ be arbitrarily small. Then there exists $C_0>0$ so that

$$|G_{\alpha,\lambda}(x,z)|\leq\frac{C_0}{\mu_s d(\alpha,\lambda)}e^{-\theta_0\mu_s|x-z|}+\frac{C_0}{|\mu_f(x)|d(\alpha,\lambda)}e^{-\theta_0|\int_x^z\Re\mu_f\,dy|} \tag{1.19}$$

uniformly for all $x,z\geq0$ and $0<\nu\leq1$, and uniformly in $(\alpha,\lambda)\in\mathbb{R}\setminus\{0\}\times\mathbb{C}$ so that $|\alpha|\leq\nu^{-\zeta}$, (1.15) *holds, and*

$$|E(\alpha,\lambda)|>\sigma_0.$$

In particular, we have

$$|G_{\alpha,\lambda}(x,z)| \le \frac{C_0}{\mu_s|\Re\lambda|}e^{-\theta_0\mu_s|x-z|} + \frac{C_0}{|\mu_f(x)||\Re\lambda|}e^{-\theta_0|\int_x^z \Re\mu_f\, dy|}. \tag{1.20}$$

In addition, the following derivative bounds hold:

$$|\partial_x^k\partial_z^\ell G_{\alpha,\lambda}(x,z)| \le \frac{C_0\mu_s^{k+\ell}}{\mu_s d(\alpha,\lambda)}e^{-\theta_0\mu_s|x-z|} + \frac{C_0|\mu_f(y)|^{k+\ell}}{|\mu_f(x)|d(\alpha,\lambda)}e^{-\theta_0|\int_x^z \Re\mu_f\, dy|} \tag{1.21}$$

for all $x,z\ge 0$ and $k,\ell\ge 0$, in which $M_f=\sup_z \Re\mu_f(z)$. Moreover,

$$|\Delta_\alpha G_{\alpha,\lambda}(x,z)| \le \frac{C_0}{d(\alpha,\lambda)}e^{-\theta_0\mu_s|x-z|} + \frac{C_0|\mu_f(y)|^2}{|\mu_f(x)|d(\alpha,\lambda)}e^{-\theta_0|\int_x^z \Re\mu_f\, dy|}, \tag{1.22}$$

where we "gain" a factor μ_s in the first term on the right-hand side.

We believe that the θ_0 factor is purely technical and that this theorem holds true for $\theta_0=1$. In addition, we note that $\Delta_\alpha G_{\alpha,\lambda}$ enjoys better bounds since $\Delta_\alpha e^{\pm|\alpha|z}=0$.

To prove this theorem, we first construct approximate solutions to the Orr–Sommerfeld equation and then construct an approximate Green function. An iteration argument yields the exact Green function together with the stated bounds. Our construction of the Green function for the Orr–Sommerfeld problem was inspired by the pointwise Green function approach introduced by Zumbrun and Howard [18] and Zumbrun [16, 17].

We are also interested in the construction of a pseudoinverse of the Orr–Sommerfeld operator near a simple eigenvalue, a construction which is detailed in section 5.

2. Approximate solutions of Orr–Sommerfeld. In this section, we construct four independent approximate solutions to the Orr–Sommerfeld equations $\mathrm{OS}_{\alpha,\lambda}(\phi)=0$, two with a "fast" behavior and two with a "slow" one. The fast modes are constructed using geometrical optics methods, namely following the BKW method. For the slow modes we will distinguish between three regimes:

- Bounded $|\alpha|$. In this case, the slow modes are perturbations of the eigenmodes of Rayleigh equations.
- $1\ll|\alpha|\le\nu^{-1/4}$ (or any small negative power of ν). We use the fact that the Rayleigh equation is a perturbation of Δ_α. The slow modes are perturbations of the eigenmodes of $\partial_z^2-\alpha^2$, namely $e^{\pm|\alpha|z}$.
- $\nu^{-1/4}\le|\alpha|\le\nu^{-\zeta}$ for $\zeta<1/2$. In this case, $e^{\pm|\alpha|z}$ is a sufficient approximation.

Solutions will be constructed in function spaces L^∞_η, for $\eta>0$, that consist of smooth functions f so that the norm

$$\|f\|_\eta := \sup_{z\ge 0} e^{\eta|z|}|f(z)|$$

is finite.

2.1. Fast modes. In this section, we shall construct two independent approximate solutions, which asymptotically behave like $e^{\pm\mu_f^+ z}$, of the Orr–Sommerfeld equation $\mathrm{OS}_{\alpha,\lambda}(\phi)=0$. We will use the BKW method. Let us first discuss its validity. Note that locally the characteristic length scale of the oscillations is

$$\delta(z)=\frac{1}{\mu_f(z)}=\sqrt{\frac{\nu}{\lambda+\nu\alpha^2+i\alpha U(z)}}.$$

The BKW method is valid provided δ has a small change during a period, namely provided $\delta'\delta \ll \delta$, or, equivalently,

$$\delta'(z) = \frac{-i\sqrt{\nu}\alpha U'(z)}{2(\lambda+\nu\alpha^2+i\alpha U(z))^{3/2}} \ll 1. \tag{2.1}$$

Note that it may happen that for some particular z_c, $\Im\lambda + \alpha U(z_c) = 0$. Such z_c are called critical layers, or turning points. If $\alpha \sim \nu^{-1/2}$, then the denominator and numerator of (2.1) are of order $O(1)$ at such points, and hence the condition (2.1) is not satisfied and $\delta'(z_c) \sim 1$.

On the contrary, if $\alpha \lesssim \nu^{-\zeta}$ with $\zeta < 1/2$, then near critical points, the denominator is of order $O(1)$ but the numerator is of order $O(\nu^{1/2-\zeta})$. Therefore, the condition (2.1) is satisfied provided ν is small enough. Similarly, $\mu_f^{-1-j}\partial_z^j\mu_f(z)$ is of order $O(\nu^{1/2-\zeta})$ or smaller for $j \ge 1$.

Proposition 2.1. *Let $N > 0$ be arbitrarily large. Then, for sufficiently small ν and for $|\alpha| \le \nu^{-\zeta}$ with $\zeta < 1/2$, there exist two approximate solutions $\phi^{app}_{f,\pm}(z)$ which solve Orr–Sommerfeld equations up to a small error term*

$$\mathrm{OS}_{\alpha,\lambda}(\phi^{app}_{f,\pm}) = O(\nu^N|\phi^{app}_{f,\pm}|),$$

with $\phi^{app}_{f,\pm}(0) = 1$ and

$$\phi^{app}_{f,\pm}(z) = e^{\pm\int_0^z \mu_f(y)\,dy}\Big(1+\phi_\pm(z)\Big), \tag{2.2}$$

where $\phi_\pm$ and their derivatives are uniformly bounded in α, ν, and z, and converge exponentially fast to 0 at $z = +\infty$.

Proof. Following a semiclassical approach, we look for $\phi^{app}_{f,\pm}$ under the form

$$\phi^{app}_{f,\pm} = \exp\Big(\frac{\theta^{app}_\pm}{\sqrt{\nu}}\Big).$$

Let $\theta = \theta^{app}_\pm$ simplify the notation. We compute

$$\partial_z^2\phi^{app}_{f,\pm} = \Big(\frac{\theta'^2}{\nu} + \frac{\theta''}{\sqrt{\nu}}\Big)\phi^{app}_{f,\pm}$$

and

$$\nu\partial_z^4\phi^{app}_{f,\pm} = \Big(\frac{\theta'^4}{\nu} + 6\frac{\theta'^2\theta''}{\sqrt{\nu}} + 4\theta'\theta''' + 3\theta''^2 + \sqrt{\nu}\theta''''\Big)\phi^{app}_{f,\pm}.$$

We now expand θ in powers of $\sqrt{\nu}$; namely,

$$\theta = \sum_{i=0}^N \theta_j\nu^{j/2},$$

where the functions θ_j will themselves depend on α and λ. Putting the ansatz into the Orr–Sommerfeld equations, at leading order, we obtain

$$(\lambda+i\alpha U)(\theta_0'^2-\nu\alpha^2) - \Big(\theta_0'^4 - 2\nu\alpha^2\theta_0'^2 + \nu^2\alpha^4\Big) = 0.$$

Factorizing by $\theta_0'^2 - \nu\alpha^2$, we get

$$\theta_0'^2 = \lambda + \nu\alpha^2 + i\alpha U = \nu\mu_f^2(z),$$

which gives

$$\theta_0' = \pm\sqrt{\nu}\mu_f(z).$$

Note that θ_0' converges exponentially fast to $\pm\sqrt{\nu}\mu_f^+$ and θ_0'' converges exponentially fast to 0. To obtain θ_1, we equate the powers in $\sqrt{\nu}^{-1}$ and get

$$-4\theta_0'^3\theta_1' + 4\nu\alpha^2\theta_0'\theta_1' + 2(\lambda + i\alpha U)\theta_0'\theta_1' = S,$$

where the source term $S = 6\theta_0'^2\theta_0''$ only depends on θ_0' and its derivatives. This leads to

$$\theta_1' = \frac{S}{(-4\theta_0'^2 + 4\nu\alpha^2 + 2(\lambda + i\alpha U))\theta_0'} = -\frac{S}{2(\lambda + i\alpha U)\theta_0'}.$$

As θ_0' is bounded away from 0, θ_1' is correctly defined. Moreover, θ_1 converges exponentially at infinity, as well as all its derivatives, and as $\theta_0'' = O(\alpha)$, $\theta_1 = O(\alpha)$. This leads to

$$\theta_\pm^{app} = \theta_0 + O(\alpha\nu^{-1/2}). \tag{2.3}$$

We then obtain equations and similar estimates on the remaining θ_j by equaling successive powers of ν. The proposition follows. ∎

2.2. Slow modes.

Proposition 2.2. *There exist two solutions $\phi_{s,\pm}^{app}$ which approximately solve the Orr–Sommerfeld equations: precisely, for any N,*

$$|\mathrm{OS}_{\alpha,\lambda}(\phi_{s,\pm}^{app})| \le C_N\nu^N e^{\pm|\alpha|z-\eta z}$$

and behave like $e^{\pm|\alpha|z}$ as z goes to $+\infty$: for any n,

$$|\partial_z^n\phi_{s,\pm}^{app}(z)| \le C_n e^{\pm|\alpha|z}.$$

For the proof of Proposition 2.2, we shall distinguish three cases, bounded α, moderate α, and large α, that will be detailed in the next sections. We restrict ourselves to $\alpha > 0$, the opposite case being similar.

2.2.1. Approximate slow modes for bounded $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$.

As z goes to $+\infty$, the Rayleigh equation "converges" to $\Delta_\alpha\phi$. Therefore, the Rayleigh equation admits two particular equations, called $\phi_{\alpha,\pm}$, which behave like $e^{\pm|\alpha|z}$ as $z \to +\infty$. Moreover, $|\partial_z^n\phi_{\alpha,\pm}(z)| \le C_n e^{\pm|\alpha|z}$ for every positive n. Note that

$$\mathrm{OS}_{\alpha,\lambda}(\phi_{\alpha,\pm}) = -\nu\Delta_\alpha^2\phi_{\alpha,\pm}.$$

Using the Rayleigh equation, we compute

$$\Delta_\alpha\phi_{\alpha,\pm} = \frac{i\alpha U''\phi_{\alpha,\pm}}{\lambda + i\alpha U},$$

which gives

$$\mathrm{OS}_{\alpha,\lambda}(\phi_{\alpha,\pm}) = -\nu\Delta_\alpha\left(\frac{i\alpha U''\phi_{\alpha,\pm}}{\lambda + i\alpha U}\right)$$

$$= -\nu\left(\frac{i\alpha U''}{\lambda + i\alpha U}\right)^2\phi_{\alpha,\pm} - 2\nu\partial_z\phi_{\alpha,\pm}\partial_z\left(\frac{i\alpha U''}{\lambda + i\alpha U}\right) - \nu\phi_{\alpha,\pm}\partial_z^2\left(\frac{i\alpha U''}{\lambda + i\alpha U}\right).$$

Note that $\lambda + i\alpha U$ is bounded away from 0, and therefore

$$|\mathrm{OS}_{\alpha,\lambda}(\phi_{\alpha,\pm})| \le C\nu e^{\pm|\alpha|z-\eta z},$$

and similarly for all its derivatives.

We now look for approximate solutions of Orr–Sommerfeld solutions $\phi^{app}_{s,\pm}$ of the form

$$\phi^{app}_{s,\pm} = \sum_{j=0}^{N} \phi^j_{\alpha,\pm}$$

for arbitrarily large N, starting with $\phi^0_{\alpha,\pm} = \phi_{\alpha,\pm}$. We have

$$Ray_\alpha(\phi^{j+1}_{\alpha,\pm}) = -\mathrm{OS}_{\alpha,\lambda}(\phi^j_{\alpha,\pm}).$$

Note that

$$\mathrm{OS}_{\alpha,\lambda}(\phi^{app}_{s,\pm}) = -\nu\Delta^2_\alpha \phi^N_{\alpha,\pm}. \tag{2.4}$$

We will focus on the construction of $\phi^{app}_{s,-}$, the construction of $\phi^{app}_{s,+}$ being similar. To end the proof of Proposition 2.2, we need to bound the various $\phi^i_{\alpha,-}$, which is done through the iterative use of the following proposition.

Proposition 2.3. *There exist constants C_n such that the following assertion is true. For any $\beta > 0$ and any smooth function ψ, there exists a smooth solution ϕ of $Ray_\alpha(\phi) = \psi$ such that*

$$\sup_{k\le n} \|\partial^k_z \phi\|_\alpha + \sup_{k\le n} \|\partial^n_z \Delta_\alpha \phi\|_{\alpha+\beta} \le \frac{C_n}{E(\alpha,\lambda)} \sup_{k\le n} \|\partial^k_z \psi\|_{\alpha+\beta},$$

where $\|\phi\|_\eta = \sup_{z\ge 0} e^{\eta|z|}|\phi(z)|$.

Proof. We first construct the Green function of the Rayleigh operator. Let

$$\widetilde{\phi}_{\alpha,+}(z) = \phi_{\alpha,-}(0)\phi_{\alpha,+}(z) - \phi_{\alpha,+}(0)\phi_{\alpha,-}(z).$$

Then $\widetilde{\phi}_{\alpha,+}(0) = 0$ and the Wronskian of $\widetilde{\phi}_{\alpha,+}$ and $\phi_{\alpha,-}$ equals

$$W(\widetilde{\phi}_{\alpha,+}, \phi_{\alpha,-}) = \phi_{\alpha,-}(0) W(\phi_{\alpha,+}, \phi_{\alpha,-}) = 2\alpha\phi_{\alpha,-}(0),$$

evaluating this latest Wronskian at infinity. The Green function of the Rayleigh operator is therefore

$$G(x,z) = \frac{1}{2\alpha\phi_{\alpha,-}(0)} \phi_{\alpha,-}(x)\widetilde{\phi}_{\alpha,+}(z) \qquad \text{if } z < x,$$
$$G(x,z) = \frac{1}{2\alpha\phi_{\alpha,-}(0)} \widetilde{\phi}_{\alpha,+}(x)\phi_{\alpha,-}(z) \qquad \text{if } z > x.$$

We then have

$$\phi(z) = \int_0^{+\infty} G(x,z)\psi(x)dx.$$

Using the asymptotic behavior of $\phi_{\alpha,\pm}$, we get the claimed bounds on $\|\partial^n_z \phi\|_\alpha$, with $n = 0$ and $n = 1$, by a direct computation. Higher derivatives are obtained by differentiating

$$\partial^2_y \phi = \alpha^2\phi + \frac{i\alpha U''}{\lambda + i\alpha U}\phi + \psi,$$

keeping in mind that α is bounded and λ is away from the range of $-i\alpha U$. Next, we write

$$\Delta_\alpha \phi = \frac{i\alpha U''}{\lambda + i\alpha U}\phi + \psi,$$

which gives the desired bounds on $\Delta_\alpha \phi$. ◻

2.2.2. Approximate slow modes for $1 \ll |\alpha| \leq \nu^{-1/4}$ or large λ/α. For large α, or for large λ/α, the Rayleigh operator is a small perturbation of $\partial_z^2 - \alpha^2$ and we can construct approximate eigenmodes $\phi_{s,\pm}^{app}$ using a perturbative construction. Namely, the Rayleigh equation may be rewritten as

$$\Delta_\alpha \phi = \frac{i\alpha U''\phi}{\lambda + i\alpha U}.$$

Note that $\alpha^{-1}e^{-\alpha|x-z|}$ is a Green function for Δ_α. We therefore define the operator $\mathcal{T}$ by

$$\mathcal{T}[\phi](z) := \int_0^\infty \alpha^{-1}e^{-\alpha|x-z|}\frac{i\alpha U''\phi(x)}{\lambda + i\alpha U}\, dx.$$

We shall prove that for sufficiently large α, the map $\mathcal{T}$ is well defined and contractive from $L^\infty_{\alpha+\eta}$ to itself. Indeed, for $\phi \in L^\infty_{\alpha+\eta}$, as $\lambda + i\alpha U$ is bounded away from 0, we have

$$|\mathcal{T}[\phi](z)| \leq C_0 \int_0^\infty e^{-\alpha|x-z|}e^{-\eta x - \alpha x}\|\phi\|_{\alpha+\eta}\, dx \leq C_0 \alpha^{-1}\|\phi\|_{\alpha+\eta}e^{-\eta z - \alpha z}.$$

This proves that $\mathcal{T}[\phi] \in L^\infty_{\alpha+\eta}$. If α is large enough, then $\mathcal{T}$ is a contraction in this space. On the other hand, if λ/α is large enough, we rewrite

$$\frac{i\alpha U''\phi(x)}{\lambda + i\alpha U} = \frac{U''\phi(x)}{U - i\alpha^{-1}\lambda},$$

which is bounded by $C/(\alpha^{-1}\lambda)$. Hence $\mathcal{T}$ is a contraction if λ/α is large enough.

We now construct two independent solutions of the Rayleigh equation, which behaves like $e^{\pm\alpha z}$ for large z. Let us detail the "$_-$" case. We look for $\phi_{s,-}$ under the form

$$\phi_{s,-} = \sum_{n\geq 0}\phi_-^n,$$

with $\phi_-^0 = e^{-\alpha z}$ and $\phi_-^{n+1} = \mathcal{T}[\phi_-^n]$. As $\mathcal{T}$ is contractive, the previous sum converges in $L^\infty_{\alpha+\eta}$. Note that, in particular,

$$\phi_{\alpha,-} = e^{-\alpha z}(1 + O(\alpha^{-1})_{L^\infty_{\alpha+\eta}}),$$

and similarly for its derivatives. The construction of $\phi_{\alpha,+}$ is similar.

The construction of approximate solutions of Orr–Sommerfeld is similar to that of the previous section. We start with $\phi_{s,-}$ and note that

$$\nu\|\Delta_\alpha^2 \phi_{s,-}\|_{\alpha+\eta} \leq C\nu|\alpha|^2 \lesssim \nu^{1/2}.$$

We then introduce $\phi_{s,-}^1$, defined by

$$Ray_\alpha(\phi_{s,-}^1) = -\nu\Delta_\alpha^2\phi_{s,-},$$

which can be bounded using the $\mathcal{T}$ operator. To end the proof of Proposition 2.2, we iterate the construction as in the previous section.

2.2.3. Approximate slow modes for $\nu^{-1/4} \le |\alpha| \ll \nu^{-1/2}$. We look for eigenmodes of the form

$$\phi^{app}_{s,\pm} = \exp(\alpha\theta^{app}_{\pm}),$$

where $\theta^{app}_{\pm}$ may be expanded in powers of α^{-1}. As in section 2.1, we get

$$-\nu\alpha^4\theta_0'^4 + 2\nu\alpha^4\theta_0'^2 - \nu\alpha^4 + (\lambda + i\alpha U)(\alpha^2\theta_0'^2 - \alpha^2) = 0.$$

This time we choose $\theta_0 = \pm 1$ and iterate as in section 2.1 to prove Proposition 2.2. Note again that the leading order of $\Delta_\alpha \phi^{app}_{s,\pm}$ vanishes.

3. Approximate Green function. We now construct an approximate Green function H^{app} using the approximate solutions $\phi^{app}_{s,\pm}$ and $\phi^{app}_{f,\pm}$. We will decompose this Green function into two components:

$$H^{app} = G^{app} + \hat{G}^{app},$$

where G^{app} does not take into account the boundary conditions and focus on the discontinuity at $y = x$, and where $\hat{G}^{app}$ restores the proper boundary conditions.

Hence, first forgetting the boundary condition, we look for $G^{app}(x, y)$ of the form

$$\begin{aligned} G^{app}(x,y) &= a_+(x)\frac{\phi^{app}_{s,+}(y)}{c_2} + b_+(x)\frac{\phi^{app}_{f,+}(y)}{\phi^{app}_{f,+}(x)} \quad \text{for } y < x, \\ G^{app}(x,y) &= a_-(x)\frac{\phi^{app}_{s,-}(y)}{c_1} + b_-(x)\frac{\phi^{app}_{f,-}(y)}{\phi^{app}_{f,-}(x)} \quad \text{for } y > x, \end{aligned} \tag{3.1}$$

where the normalization constants c_1 and c_2 will be fixed later. Let

$$v(x) = (-a_-(x), a_+(x), -b_-(x), b_+(x)). \tag{3.2}$$

By definition, G^{app}, $\partial_y G^{app}$, $\sqrt{\nu}\partial_y^2 G^{app}$ are continuous at $x = y$ and $\nu\partial_y^3 G^{app}$ has a jump at $x = y$ of magnitude 1. Let

$$M = \begin{pmatrix} \phi_{s,-}/c_1 & \phi_{s,+}/c_2 & \phi_{f,-} & \phi_{f,+} \\ \partial_y\phi_{s,-}/c_1\mu_f & \partial_y\phi_{s,+}/c_2\mu_f & \partial_y\phi_{f,-}/\mu_f & \partial_y\phi_{f,+}/\mu_f \\ \partial_y^2\phi_{s,-}/c_1\mu_f^2 & \partial_y^2\phi_{s,+}/c_2\mu_f^2 & \partial_y^2\phi_{f,-}/\mu_f^2 & \partial_y^2\phi_{f,+}/\mu_f^2 \\ \partial_y^3\phi_{s,-}/c_1\mu_f^3 & \partial_y^3\phi_{s,+}/c_2\mu_f^3 & \partial_y^3\phi_{f,-}/\mu_f^3 & \partial_y^3\phi_{f,+}/\mu_f^3 \end{pmatrix}, \tag{3.3}$$

where the functions $\phi_{s,\pm} = \phi^{app}_{s,\pm}$ and $\phi_{f,\pm} = \phi^{app}_{f,\pm}$ and their derivatives are evaluated at $y = x$. Then

$$Mv = (0, 0, 0, 1/\nu\mu_f^3). \tag{3.4}$$

In the following sections, we will bound the solution v of (3.4). Let us define the four two-by-two matrices A, B, C, and D by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Note that, using (2.3),

$$D = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + O(\alpha\mu_f^{-1}).$$

Hence the matrix D is bounded and invertible, upon recalling that $\alpha \ll \mu_f$ in the range of α that we consider (see (1.17)). Moreover, its inverse is bounded and equals

$$D^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + O(\alpha\mu_f^{-1}).$$

We shall consider two cases: bounded α and unbounded α.

3.1. First case: Bounded α. We take $c_1 = c_2 = 1$. Note that $A = A_1 A_2$, where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \mu_f^{-1} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \phi_{s,-} & \phi_{s,+} \\ \partial_y\phi_{s,-} & \partial_y\phi_{s,+} \end{pmatrix}.$$

The determinant $E^{app}(\alpha,\lambda)$ of A_2 is a perturbation of the Evans function $E(\alpha,\lambda)$ in the sense that

$$E^{app}(\alpha,\lambda) = E(\alpha,\lambda) + O(\nu^\sigma)$$

for some positive σ. Hence if $E(\alpha,\lambda) \neq 0$, then A_2 and A are invertible provided ν is small enough, and A_2^{-1} is bounded. Moreover, the matrix M has an approximate inverse

$$\widetilde{M} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}$$

in the sense that $M\widetilde{M} = \mathrm{Id} + N$, where

$$N = \begin{pmatrix} 0 & 0 \\ CA^{-1} & -CA^{-1}BD^{-1} \end{pmatrix}.$$

Note that C is of order $O(\mu_f^{-2})$ since α is bounded, B is bounded, and $A^{-1} = A_2^{-1}A_1^{-1}$ is of order $O(\mu_f)$. Hence we have $N = O(\mu_f^{-1})$. Therefore, $(\mathrm{Id}+N)^{-1}$ is well defined and uniformly bounded for ν small enough provided $E(\alpha,\lambda) \neq 0$. As a consequence,

$$M^{-1} = \widetilde{M}(\mathrm{Id}+N)^{-1} = \widetilde{M}\sum_n N^n.$$

Note that the two first lines of N^n vanish. Therefore,

$$(\mathrm{Id}+N)^{-1}(0,0,0,1/\nu\mu_f^3) = \Big(0,0,O(1/\nu\mu_f^4), 1/\nu\mu_f^3 + O(1/\nu\mu_f^4)\Big).$$

As D^{-1} is bounded and $A^{-1}BD^{-1}$ is of order $O(\mu_f)$, we obtain that $a_\pm$ and $b_\pm$ are of orders $O(1/\nu\mu_f^2)$ and $O(1/\nu\mu_f^3)$, respectively. Note that α is bounded in this case, which give the desired bounds since

$$\nu\mu_f^2 = \lambda + \nu\alpha^2 + i\alpha U$$

and hence

$$|\nu\mu_f^2| \geq d(\alpha,\lambda),$$

which ends this first case.

3.2. Case 2: Large α. We take $c_1 = \phi_{s,+}^{app}(x)$ and $c_2 = \phi_{s,-}^{app}(x)$. In this case, A is of the form

$$A = \begin{pmatrix} 1 & 1 \\ -\alpha\mu_f^{-1} & \alpha\mu_f^{-1} \end{pmatrix} (1 + o(1)).$$

Its inverse A^{-1} equals

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -\alpha^{-1}\mu_f \\ 1 & \alpha^{-1}\mu_f \end{pmatrix} (1 + o(1)).$$

Note that D^{-1} and B are bounded and A^{-1} is order $O(\mu_f/\alpha)$. As C is of order $O(\alpha^2/\mu_f^2)$, N (defined in the previous section) is of order $O(\alpha/\mu_f)$. Hence, as $|\alpha| \ll \mu_f$ in view of (1.17), we have

$$(\mathrm{Id} + N)^{-1} = \sum_n (-1)^n N^n.$$

This leads to

$$(Id + N)^{-1}(0, 0, 0, 1/\nu\mu_f^4) = \Big(0, 0, O(\alpha/\nu\mu_f^4), O(1/\nu\mu_f^3)\Big). \tag{3.5}$$

It remains to evaluate the image of this vector by $\widetilde{M}$. As D^{-1} is bounded, we obtain that $b_\pm$ are of order $O(1/\nu\mu_f^3) = O(1/\mu_f d(\alpha, \lambda))$.

Moreover, we compute

$$D^{-1}(0, O(1/\nu\mu_f^3)) = \Big[(-1, 1) + O(\alpha\mu_f^{-1})\Big] O(1/\nu\mu_f^3).$$

As

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} (1 + O(\mu_f^{-1})),$$

we obtain

$$BD^{-1}(0, O(1/\nu\mu_f^3)) = \Big[(0, 1) + O(\alpha\mu_f^{-1})\Big] O(1/\nu\mu_f^3).$$

As a consequence, we obtain

$$A^{-1}BD^{-1}(0, O(1/\nu\mu_f^3)) = O(1/\alpha\nu\mu_f^2).$$

It remains to bound the images of the $O(\alpha/\nu\mu_f^4)$ term in (3.5). We have

$$D^{-1}(O(\alpha/\nu\mu_f^4), 0) = \Big[(1, 1) + O(\alpha\mu_f^{-1})\Big] O(\alpha/\nu\mu_f^4).$$

Hence

$$BD^{-1}(O(\alpha/\nu\mu_f^4), 0) = \Big[(1, 0) + O(\alpha\mu_f^{-1})\Big] O(\alpha/\nu\mu_f^4)$$

and $A^{-1}BD^{-1}(O(\alpha/\nu\mu_f^4), 0) = O(\alpha/\nu\mu_f^4)$. Using again $\alpha \ll \mu_f$, we obtain that $a_\pm$ are of order $O(1/\nu\mu_f^2\alpha) = O(1/\alpha d(\alpha, \lambda))$.

3.3. Boundary condition. We now add to G^{app} another approximate Green function $\hat{G}^{app}$ to handle the boundary conditions. We look for $\hat{G}^{app}$ under the form

$$\hat{G}^{app}(y) = d_s \frac{\phi_{s,-}(y)}{d_1} + d_f \frac{\phi_{f,-}(y)}{\phi_{f,-}(0)},$$

where the normalization constant d_1 will be fixed later, and look for d_s and d_f such that

$$G^{app}(x,0) + \hat{G}^{app}(0) = \partial_y G^{app}(x,0) + \partial_y \hat{G}^{app}(0) = 0. \tag{3.6}$$

Let

$$\hat{M} = \begin{pmatrix} \phi_{s,-}/d_1 & \phi_{f,-}/\phi_{f,-}(0) \\ \partial_y \phi_{s,-}/d_1 & \partial_y \phi_{f,-}/\phi_{f,-}(0) \end{pmatrix},$$

the functions being evaluated at $y = 0$. Then (3.6) can be rewritten as

$$\hat{M} d = -(G^{app}(x,0), \partial_y G^{app}(x,0)),$$

where $d = (d_s, d_f)$. Note that

$$(G^{app}(x,0), \partial_y G^{app}(x,0)) = Q(a_+, b_+),$$

where

$$Q = \begin{pmatrix} \phi_{s,+}(0)/c_2 & 1 \\ \partial_y \phi_{s,+}(0)/c_2 & \partial_y \phi_{f,+}(0)/\phi_{f,+}(0) \end{pmatrix}.$$

By construction,

$$d = -\hat{M}^{-1} Q(a_+, b_+). \tag{3.7}$$

Let us first consider bounded α. We take $d_1 = 1$. This leads to

$$\hat{M} = \begin{pmatrix} \phi_{s,-}(0) & 1 \\ \partial_y \phi_{s,-}(0) & -\mu_f + O(1) \end{pmatrix}.$$

Note that $\hat{M} = M_1 M_2$, with

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & \mu_f \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \phi_{s,-}(0) & 1 \\ \partial_y \phi_{s,-}(0)/\mu_f & -1 + O(1/\mu_f) \end{pmatrix}.$$

The determinant of M_2 equals $-E(\alpha,\lambda) = -\phi_{s,-}(0)$, up to a small term of order $\mu_f^{-1} \sim \sqrt{\nu}$, recalling that α is bounded. Hence M_2 is invertible, and M_2^{-1} is bounded if $E(\alpha,\lambda) \neq 0$, provided ν is small enough. Then

$$\hat{M}^{-1} Q = M_2^{-1} M_1^{-1} Q.$$

Note that $(a_+, b_+) = (O(1/\nu\mu_f^2), O(1/\nu\mu_f^3))$. Hence $Q(a_+, b_+) = O(1/\nu\mu_f^2)$. Therefore, $M_1^{-1} Q(a_+, b_+) = (O(1/\nu\mu_f^2), O(1/\nu\mu_f^3))$. Hence, as the second term of the first column of M_2 is of order $O(1/\mu_f)$, we get, as desired, that

$$d = (O(1/\alpha\nu\mu_f^2), O(1/\nu\mu_f^3)), \tag{3.8}$$

keeping in mind that α is bounded.

For large α, we choose $d_1 = \phi_{s,-}(0)$. Then

$$Q = \begin{pmatrix} 1 & 1 \\ \alpha + O(1) & \mu_f + O(1) \end{pmatrix},$$
$$\hat{M} = \begin{pmatrix} 1 & 1 \\ -\alpha + O(1) & -\mu_f + O(1) \end{pmatrix},$$

and

$$\hat{M}^{-1} = \frac{1}{\mu_f - \alpha + O(1)} \begin{pmatrix} \mu_f + O(1) & 1 \\ -\alpha + O(1) & -1 \end{pmatrix}.$$

In this case, $(a_+, b_+) = (O(1/\alpha\nu\mu_f^2), O(1/\nu\mu_f^3))$. A direct computation of $\hat{M}^{-1}Q(a_+, b_+)$ again gives (3.8). Combining all the previous estimates ends the proof.

4. Exact Green function. Let

$$H^{app} = G^{app} + \hat{G}^{app}$$

be the complete approximate Green function. By construction, H^{app} satisfies the zero boundary conditions (1.9). We now construct the exact Green function $G(x, z)$ as an infinite sum:

$$G(x, z) = \sum_{n \geq 0} G_n(x, z), \tag{4.1}$$

where $G_0 = H^{app}$,

$$G_1 = -H^{app} \star (\mathrm{OS}_{\alpha,\lambda}(H^{app}) - \delta_{y=x}),$$

and G_n is defined by iteration through

$$G_{n+1} = -H^{app} \star \mathrm{OS}_{\alpha,\lambda}(G_n).$$

Hence it suffices to prove that the series (4.1) converges in a suitable function space, which follows immediately from the following lemma. The stated bounds for $G(x, z)$ in Theorem 1.1 then follow from those on $H^{app}(x, z)$.

LEMMA 4.1. *For each x, assume that*

$$|f^x(y)| \leq e^{-\alpha'|x-y|}$$

for some α' such that $\alpha' < |\alpha|$ and $\alpha' < \Re\mu_f$. Then

$$|\mathrm{OS}_{\alpha,\lambda}(G^{app} \star f^x)(y)| \leq C\nu^{N-2} e^{-\alpha'|x-y|}.$$

Proof. Note that

$$\mathrm{OS}_{\alpha,\lambda}(G^{app} \star f^x)(y) = \int \mathrm{OS}_{\alpha,\lambda}(G^{app})(z, y) f^x(z) dz.$$

However, we recall that $\phi^{app}_{s,\pm}$ satisfy

$$|\mathrm{OS}_{\alpha,\lambda}(\phi^{app}_{s,\pm})| \leq C\nu^N e^{\pm|\alpha|z},$$
$$|\mathrm{OS}_{\alpha,\lambda}(\phi^{app}_{f,\pm})| \leq C\nu^N |\phi^{app}_{f,\pm}|.$$

Using the bounds on the coefficients on $G^{app}(z, y)$, this leads to

$$|\mathrm{OS}_{\alpha,\lambda}(G^{app}(z, y))| \leq C\nu^{N-2} e^{-\alpha|y-z|}.$$

The lemma follows by convolution. ∎

5. Construction of a pseudoinverse. We now focus on the case when λ is close to a simple eigenvalue λ_0.

THEOREM 5.1. *Let α be fixed. Let λ_0 be a simple eigenvalue of $Orr_{\alpha,\lambda}$ with corresponding eigenmode ϕ_{α,λ_0}. Then there exist a bounded family of linear forms l^ν and a family of pseudoinverse operators $Orr^{-1}_{\alpha,\lambda}$ such that for any stream function ϕ,*

$$Orr_{\alpha,\lambda}\Big(Orr^{-1}_{\alpha,\lambda}(\phi)\Big) = \phi - l^\nu(\phi)\phi_{\alpha,\lambda_0}$$

for λ near λ_0. Moreover, the pseudoinverse $Orr^{-1}_{\alpha,\lambda}$ may be defined through a Green function $\widetilde{G}_{\alpha,\lambda}(x,z)$ which satisfies the same bounds in (1.19).

5.1. Principle of the construction. Let us sketch the principle of the proof on a simplified case. Let A_0 be an $N\times N$ matrice of rank $N-1$ (which is a toy model for the Rayleigh operator when λ is a simple eigenvalue), and let $A(\varepsilon)$ be a bounded family of $N\times N$ matrices (a toy model for the Orr–Sommerfeld equation). We want to construct an inverse for

$$A^\varepsilon = A_0 + \varepsilon A(\varepsilon).$$

Let us first invert A_0. Let v be a unit vector, orthogonal to the image of A_0. Let P be the orthogonal projector on the image of A, namely

$$Pv = f - (f.v)v.$$

Let B be a pseudoinverse of A_0, namely a matrix such that, on the image of A_0, $A_0B = \mathrm{Id}$. Then $u = BPf$ solves

$$A_0 u = f - (f.v)v.$$

We now fulfill a similar construction for A^ε for small ε. Let $u_0 = BPf$. Then

$$A^\varepsilon u_0 = f - (f.v)v + \varepsilon A(\varepsilon)u_0.$$

We now define $u_1 = -BPA(\varepsilon)u_0$. Then $u_0 + u_1$ solves

$$A^\varepsilon(u_0+\varepsilon u_1) = f - (f.u_0)v + \varepsilon(A(\varepsilon)u_0.v)v - \varepsilon^2 A(\varepsilon)BPA(\varepsilon)u_0$$

and the construction follows by iteration.

5.2. Rayleigh equation. In this section, we fix α and investigate the Rayleigh operator $Ray_{\alpha,\lambda}$ when λ is near a simple eigenvalue λ_0 of $Ray_{\alpha,\lambda}$. We will also assume that $\mathrm{Ker}(Ray^2_{\alpha,\lambda_0}) = \mathbb{C}\phi_{\alpha,\lambda_0,\pm}$. At $\lambda=\lambda_0$, $\phi_{\alpha,\lambda_0,\pm}$ are colinear (that is, the Jacobian of $\phi_{\alpha,\lambda_0,\pm}$ vanishes). Up to a renormalization, we may assume that $\phi_{\alpha,\lambda_0,+} = \phi_{\alpha,\lambda_0,-}$. For $\lambda \neq \lambda_0$, the solution of $Ray_{\alpha,\lambda}(\phi) = \psi$ is explicitly given by

$$\phi(z) = \phi_{\alpha,\lambda,+}(z)\int_z^{+\infty}\frac{\phi_{\alpha,\lambda,-}(x)}{Jac(x)}\psi(x)dx + \phi_{\alpha,\lambda,-}(z)\int_0^z\frac{\phi_{\alpha,\lambda,+}(x)}{Jac(x)}\psi(x)dx, \tag{5.1}$$

where

$$Jac(x) := \phi_{\alpha,\lambda,-}(x)\partial_x\phi_{\alpha,\lambda,+}(x) - \phi_{\alpha,\lambda,+}(x)\partial_x\phi_{\alpha,\lambda,-}(x)$$

is the Jacobian of $\phi_{\alpha,\lambda,\pm}$. Note that, as λ_0 is a simple eigenvalue, $Jac(\lambda_0) = 0$ and that, for λ near λ_0,

$$Jac(\lambda) = (\lambda-\lambda_0)\widetilde{J}ac(\lambda),$$

where $\widetilde{J}ac(\lambda)$ is a smooth function with $\widetilde{J}ac(\lambda_0) \neq 0$ since λ_0 is a simple eigenvalue. Let us also define

$$\widetilde{\phi}_{\alpha,\lambda,\pm} = \frac{\phi_{\alpha,\lambda,\pm} - \phi_{\alpha,\lambda_0,\pm}}{\lambda - \lambda_0}.$$

Then it follows from (5.1) that

$$\phi(z) = \frac{\phi_{\alpha,\lambda_0,+}(z)}{\lambda - \lambda_0} \int_0^{+\infty} \frac{\phi_{\alpha,\lambda_0,+}(x)}{\widetilde{J}ac(x)} \psi(x)dx + \widetilde{\phi}(z),$$

where

$$\widetilde{\phi}(z) = \int_0^{+\infty} \widetilde{G}(x,z)\psi(x)dx, \tag{5.2}$$

with

$$\widetilde{G}(x,z) = \frac{\widetilde{\phi}_{\alpha,\lambda,+}(z)\phi_{\alpha,\lambda,-}(x) + \phi_{\alpha,\lambda,+}(z)\widetilde{\phi}_{\alpha,\lambda,-}(x) + (\lambda-\lambda_0)\widetilde{\phi}_{\alpha,\lambda,+}(z)\widetilde{\phi}_{\alpha,\lambda,-}(x)}{\widetilde{J}ac(x)}$$

if $x > z$ and a similar expression if $x < z$. This computation may be rewritten as follows. Let l be the linear form defined by

$$l(\psi) = \int_0^{+\infty} \frac{\phi_{\alpha,\lambda_0,+}(x)}{\widetilde{J}ac(x)} \psi(x)dx.$$

Then, for any ψ, if $l(\psi) = 0$, then $\widetilde{\phi}$ solves $Ray_{\alpha,\lambda}(\widetilde{\phi}) = \psi$. In particular, as the image of the Rayleigh operator $\mathrm{Im}(Ray_{\alpha,\lambda_0})$ is of codimension 1, $\mathrm{Ker}(l) = \mathrm{Im}(Ray_{\alpha,\lambda_0})$. Note that, as λ_0 is a simple eigenvalue, $\phi_{\alpha,\lambda_0,+}$ is not in $\mathrm{Im}(Ray_{\alpha,\lambda_0})$. Therefore, $l(\phi_{\alpha,\lambda_0,+}) \neq 0$. As a consequence,

$$\tilde{\psi} = \psi - \frac{l(\psi)}{l(\phi_{\alpha,\lambda_0,+})}\phi_{\alpha,\lambda_0,+} \in \mathrm{Im}(Ray_{\alpha,\lambda})$$

since the image by l of this function vanishes. We then have

$$Ray_{\alpha,\lambda}(\widetilde{\phi}) = \psi - \frac{l(\psi)}{l(\phi_{\alpha,\lambda_0,+})}\phi_{\alpha,\lambda_0,+}, \tag{5.3}$$

where

$$\tilde{\phi}(z) = \int_0^{+\infty} \tilde{G}(x,z)\tilde{\psi}(x)dx.$$

That is, $\widetilde{\phi}$ defines the pseudoinverse $Ray^{-1}_{\alpha,\lambda}$ of $Ray_{\alpha,\lambda}$ for λ near λ_0. We shall now fulfill a similar analysis for the $Orr_{\alpha,\lambda}$ operator.

5.3. Orr–Sommerfeld equation. Let us now prove Theorem 5.1. We follow the analysis in the previous section to construct the Green function $\widetilde{G}_{\alpha,\lambda}(x,z)$ for the pseudoinverse of $Orr_{\alpha,\lambda}$. Let λ_0^{app} be a simple eigenvalue of the approximate Evans function E^{app} of the $Ray_{\alpha,\lambda}$ operator. To simplify the notation, we drop the "app" and set $\lambda_0 = \lambda_0^{app}$. At $\lambda = \lambda_0$, the matrix M, defined by (3.3), is singular since its first two columns are colinear. Up to the multiplication by a constant of $\phi_{s,-}$, we may assume that $\phi_{s,\pm}$ coincide at $\lambda = \lambda_0$. To desingularize it, we introduce

$$\Lambda = \begin{pmatrix} (\lambda-\lambda_0)^{-1} & 1 & 0 & 0 \\ -(\lambda-\lambda_0)^{-1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, recalling (3.2) and defining $\widetilde{M} = M\Lambda$, with the notation of (3.4), we have

$$v = \Lambda \widetilde{M}^{-1}(0, 0, 0, 1/\nu\mu_f^3) \tag{5.4}$$

when $\lambda \neq \lambda_0$. The arguments applied to the matrix M in section 3 may now be applied to $\widetilde{M}$ since the corresponding matrix

$$\widetilde{A}_2 = A_2\Lambda = \begin{pmatrix} (\phi_{s,-} - \phi_{s,+})/(\lambda - \lambda_0) & \phi_{s,+} \\ (\partial_y\phi_{s,-} - \partial_y\phi_{s,+})/(\lambda - \lambda_0) & \partial_y\phi_{s,+} \end{pmatrix}$$

is nonsingular near $\lambda = \lambda_0$, keeping in mind that λ_0 is a simple eigenvalue.

Let $l_4 = (l_{4,1}, \dots, l_{4,4})$ be the fourth line of the inverse of $\widetilde{M}$. It follows from (5.4) that

$$v = \Lambda l_4(x)/\nu\mu_f^3.$$

The singular part v^s of v, namely the terms involving $(\lambda - \lambda_0)^{-1}$, is

$$v^s = \frac{1}{\nu\mu_f^3} l_{4,1}(x)(1, -1, 0, 0).$$

Let us now compute $l_{4,1}(x)$. We have to evaluate $\Lambda^{-1}A_2^{-1}A_1^{-1}BD^{-1}(0, 1)$ (see section 3.1). But, up to higher order terms, $A_1^{-1}BD^{-1} \sim (0, \mu_f)$. Note that

$$A_2^{-1} = \frac{1}{E^{app}(\alpha, \lambda)} \begin{pmatrix} \partial_y\phi_{s,+} & -\phi_{s,+} \\ -\partial_y\phi_{s,-} & \phi_{s,-} \end{pmatrix}.$$

Hence, when λ is close to λ_0,

$$A_2^{-1}A_1^{-1}BD^{-1}(0, 1) \sim \frac{\mu_f}{E^{app}(\alpha, \lambda)}\phi_{s,+}(-1, 1),$$

namely like $C(-\mu_f, \mu_f)\phi_{s,+}/(\lambda - \lambda_0)$. At leading order, the computation is exactly the same as in the previous section. Let

$$L(\psi) = -\int_0^{+\infty} l_{4,1}(x)\psi(x)dx.$$

Then, at leading order, $L = l$. Moreover, the regular part v^r of $v = v^r + v^s$ is

$$v^r = \frac{1}{\nu\mu_f^3}\Big(l_{4,2}, l_{4,2}, l_{4,3}, l_{4,4}\Big).$$

We now define $\widetilde{G}^{app}(x, z)$ to be the approximate Green kernel that corresponds to the regular part v^r, recalling the Green function construction in (3.1)–(3.2). Setting

$$\widetilde{\psi} = \psi - \frac{L(\psi)}{L(\phi_{\alpha,\lambda_0,+})}\phi_{\alpha,\lambda_0,+},$$

we have $L(\widetilde{\psi}) = 0$, and so

$$Orr_{\alpha,\lambda}(\widetilde{G}^{app} \star \psi) = \widetilde{\psi}.$$

The exact Green function $\widetilde{G}_{\alpha,\lambda}(x, z)$ then follows by iteration as in section 4.

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